BILIPSCHITZ ROUGH NORMAL COORDINATES FOR SURFACES WITH AN L^1 CURVATURE BOUND

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Our aim is to prove the following. Let $K^* = .51959...$ denote the smallest positive solution of $\frac{\pi}{4} - K^* - \tan\frac{K^*}{2} = 0$.

Theorem A. *Let Σ be a surface with a complete C^3 Riemannian metric and Gaussian curvature K, and $p_0 \in \Sigma$. Let $R > 0$. Suppose*

$$K_0 := \int_{B(p_0,3R/2)} |K|\, dA < K^*, \tag{0.0}$$

and that any simple loop of length $< 3R$ based at p_0 bounds a topological disc $D \subset \Sigma$ with

$$\int_D K dA < \pi. \tag{0.1}$$

Then there are constants $c(K_0), C(K_0) > 0$, depending only on K_0, and a homeomorphism Φ from the euclidean disc $D_R := \{x^2 + y^2 < R^2\}$ onto the metric ball $B(p_0, R) \subset \Sigma$ such that

$$c(K_0)|\xi - \eta| \le d(\Phi(\xi), \Phi(\eta)) \le C(K_0)|\xi - \eta| \tag{0.2}$$

for all $\xi, \eta \in D_R$.

This yields a proof of the celebrated theorem of Toro:

Corollary B (Toro's theorem). *If U is a domain in $\mathbb{R}^2$ and $f \in W^{2,2}(U)$ then the graph of f is a Lipschitz submanifold of $\mathbb{R}^3$.*

Proof of corollary. Approximate f in $W^{2,2}$ by a sequence f_i of C^4 functions. Then for every point $x_0 \in U$ there is a radius $R > 0$ such that

$$\int_{\{|x-x_0|<3R/2\}} |\det D^2 f_i| \le K^*/2.$$

It follows at once that the absolute curvature integral of the graph Γ_i of each f_i satisfies the bound (0.0) in the $3R/2$ ball around $p_0^i = (x_0, f_i(x_0))$, and consequently the bound (0.1) holds as well. Consider the sequence $\Phi_i, i = 1, 2, ...$ of biLipschitz

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coordinates for these graphs constructed by the Theorem. Since the intrinsic distance on Γ_i dominates the euclidean distance, as maps into $\mathbb{R}^3$ this sequence is uniformly Lipschitz. On the other hand, by results of Semmes [Se 1,2,3] (cf. also [MS], 5.1.5, for a short simple proof, and [T1], 3.2), the euclidean distance also dominates some positive multiple of the intrinsic distance, uniformly in i. In other words, in the inequalities (0.2) for the sequence Φ_i the intrinsic distance d may be replaced by the euclidean distance (if we change the constant c). By Arzela-Ascoli there is a convergent subsequence $\Phi_{i'} \to \Phi_0$; by the uniformity of the biLipschitz bounds for the Φ_i, this Φ_0 is a biLipschitz parametrization of a neighborhood of $(x, f(x))$ in the graph of f, $\square$.

Of course, a very different proof of Toro's theorem was given by [MS], who showed that in fact the ***conformal*** parametrization of the graph of a $W^{2,2}$ function is global and biLipschitz (a third proof is given in [T2]). Our point here, however, is to show that the simple existence of local biLipschitz parameters follows essentially from the much weaker condition of a local bound on the absolute curvature integral, together with the hypothesis that the intrinsic and extrinsic distances are comparable. As such the proof above should be applicable to a variety of situations where the Gauss curvature K of the graph is a measure.

We also obtain immediately

Corollary C. *If Σ is a simply connected Riemannian surface with total absolute curvature less than K^* then Σ is globally Lipschitz equivalent to the euclidean plane.*

Observe that such a Lipschitz equivalence cannot in general be given by conformal parameters: for instance, if Σ is a cone then the conformal factor is 0 or ∞ at the vertex, accordingly as the link of the vertex has length $b >$ or $< 2\pi$ (cf. [Re], (7.8)). In this case the Gauss curvature is a Dirac mass $2\pi - b$ at the vertex. Smoothing slightly a countable collection of such cones with absolutely summable curvatures $\sum_i |2\pi - b_i| < K^*$ and gluing the results together, it is easy to produce Riemannian examples satisfying the hypothesis of Corollary C and of any desired smoothness, where any global conformal parameters cannot be Lipschitz, nor have Lipschitz inverse.

For the present we have made no attempt to relax the C^3 smoothness condition, although the principal application clearly indicates the possibility of doing this quite drastically. Furthermore we have not tried to distinguish between the effects of positive and negative curvature, though it seems that the hypothesis of the theorem could possibly be weakened in this way.

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§0. Notation. We will use $|S|$ to refer to the length (or 1-dimensional Hausdorff measure) of a set S.

We exploit the ambiguity of the term "curve" to use the same symbol for a curve and (any of) its arclength parametrization(s).

§1. Basic facts.

1.0. We will use polar coordinates on $\mathbb{R}^2$. Note that the euclidean distance between points given in polar coordinates as $(r,\theta),(s,\psi)$ is bounded above and below by positive constant multiples of any expression

$$d((r,\theta),(s,\psi)) := A|r-s| + B\min\{r,s\}||\theta-\psi||$$

associated to any given positive constants A, B. Here $||\phi||$ is the circular distance from 0,

$$||\phi|| := \min_{k\in\mathbb{Z}} |\phi - 2\pi k|.$$

Recall that if M is a Riemannian manifold and $S \subset M$, then $q \in M$ is said to be a ***critical point*** of the distance function $\rho_S = \mathrm{dist}(S,.)$ iff there are minimizing geodesic segments $g_1,\ldots g_k$ from q to S (i.e. g_i starts at q, ends in S and $|g_i| = \rho_S(q)$ for each i) and the convex hull of $\{\dot g_1(0),\ldots,\dot g_k(0)\}$ contains $0 \in T_qM$. Based on this definition one speaks also of regular points, and of critical and regular values of ρ_S. Henceforth we will consider only the case when S is a singleton $\{p_0\}$, and abbreviate $\rho := \rho_{\{p_0\}}$.

1.1. Lemma. *Let Σ be a complete Riemannian surface with C^2 Riemannian metric. If there are no critical values of ρ in the interval $(0,r]$ then $\rho^{-1}(r)$ is a simple closed curve.*

Proof. The notion of critical point for ρ given above coincides with that of Clarke for general Lipschitz functions, so the lemma follows at once from his inverse function theorem 7.11, [Cl], and the fact that the conclusion certainly holds if r is less than the injectivity radius at p_0.

The following is basically a compendium of results of Hartman ([Ha]), adapted to the situation at hand. The results of [Ha] refer to the distance to a given smooth curve rather than to a point, but they apply here if we take the curve to be $\rho^{-1}(\epsilon)$, where $\epsilon > 0$ is smaller than the injectivity radius at p_0.

By a ***geodesic loop based at*** p_0 we mean a simple closed curve which is a geodesic segment, both endpoints of which lie at p_0.

1.2. Lemma (Hartman). *Let Σ be a connected surface with a complete Riemannian metric of class C^3 and $p_0 \in \Sigma$. Suppose there are no geodesic loops of length $\le 2s$ based at p_0 and that $\Sigma \not\subset \bar B(p_0,s)$. Then:*

1. *Each $\rho^{-1}(t)$, $t \in (0,s]$, is a single simple closed curve.*
2. *There is $N \subset [0,s]$, $|N| = 0$, such that if $t \in [0,s] - N$ then there are finitely many points $q_1,\ldots,q_k \in \rho^{-1}(t)$ such that if $q \in \rho^{-1}(t) - \{q_1,\ldots,q_k\}$ then there is a unique minimizing geodesic segment between p_0 and q, and there are exactly two minimizing segments between p_0 and q_i, $i = 1,\ldots,k$. Furthermore $\rho^{-1}(t) - \{q_1,\ldots,q_k\}$ is a finite pairwise disjoint union of simple C^2 arcs, and the angles between adjacent C^2 pieces is bounded below by a positive constant $\theta_0 > 0$, independent of $t \in [0,s] - N$.*
3. *Let $g_\pm$ be minimizing geodesic segments of length $t_0 < s$ starting at p_0. Let $U \subset B(p_0,t_0)$ be one of the components of $B(p_0,t_0) - (g_+ \cup g_-)$, and for $0 < t < t_0$ put $c_t := \rho^{-1}(t) \cap U$. Then the length $|c_t|$ is absolutely continuous*

as a function of $t \in [0,s]$. *If* $t \in [0,s] - N$, *and the* C^2 *pieces of* $\rho^{-1}(t)$ *are* $b_1, \ldots, b_k$, *with* b_i *meeting* b_{i+1} *in the point* q_i *at an angle* κ_i, $i = 1, \ldots, k$ *(taking indices mod k; see Figure 1.2), then*

$$\frac{d|c_t|}{dt} = \sum_{i=1}^{k} \int_{b_i \cap c_t} k_i - \sum_{q_i \in c_t} \cot \frac{\kappa_i}{2}.$$

Here k_i *is the geodesic curvature of* b_i, *with orientation induced from that of* $\rho^{-1}(t)$ *as the boundary of* $B(p_0, t)$.

Proof of (1). We show that there are no critical values of ρ in $(0,s)$. Assume the contrary. Since all small positive values of ρ are regular we may find a smallest such value $T \in (0,s)$, corresponding to the critical point q. ¿From the definition of critical points, either q is a strict local maximum of ρ, or there are two minimizing geodesics from p_0 to q, with mutually antipodal velocities at q. In the second case the concatenation of the two geodesics is a geodesic loop of length $2d(p_0,q)$ based at p_0. Such a geodesic loop has length $\geq 2s$, so $d(p_0,q) > s$. Therefore the critical point q must be of the former type. For $t < T$ the curve $\rho^{-1}(t)$ is a single simple closed curve. But as $t \uparrow T$, some component of $\rho^{-1}(t)$ converges to q, since q is a strict local maximum. Thus q is a global maximum and the entire surface $\Sigma \subset B(p_0,s)$, contradicting the hypothesis.

The conclusion now follows from 1.1 and the fact that $\rho^{-1}(t)$ is a C^2 simple closed curve for all small $t > 0$.

Proof of (2). Except for the last assertion, this is contained in Lemma 5.2 and Prop. 6.1 of [Ha]. To prove the last assertion, assume the contrary. By compactness there is a convergent sequence of points $q_i \to q$, $\rho(q_i) \in (0,s] - N$, $\rho(q) \in (0,s]$, such that for each i there are two distinct minimizing geodesic segments $g_i^{\pm}$ from p_0 to q_i, making an angle ψ_i at q_i, with $\psi_i \to \pi$. Taking a subsequence if necessary, $g_i^{\pm} \to g^{\pm}$, a pair of minimizing geodesics from p_0 to q, making an angle of π at q. In other words the concatenation of g^+ and g^- is a geodesic loop of length $\leq 2s$ based at p_0, which is a contradiction.

Proof of (3). Referring to the remarks following equation (6.9) of [Ha], the absence of geodesic loops of length $\leq 2s$ implies that the function J defined in (6.10), op. cit., is identically zero. A simple modification of Theorem 6.2, op. cit., now implies that $|c_t|$ is absolutely continuous for $t \in [0,s]$. Our formula for the derivative is now a rewriting of equation (6.2), op. cit.

With respect to the representation of the level curves $\rho^{-1}(t)$, $t \in (0,s] - N$ as a union of C^2 arcs in Lemma 1.2 (2),we refer to the points $q_1, \ldots, q_k \in \rho^{-1}(t)$, $t \in (0,s] - N$ as the ***corners*** of $\rho^{-1}(t)$, and to the other points of $\rho^{-1}(t)$ as ***smooth points.*** It is convenient to define the geodesic curvature measure k_t by

$$k_t(U) := \sum_{i=1}^{k} \int_{b_i \cap U} k_i + \sum_{q_i \in U} (\kappa_i - \pi)$$

for Borel sets $U \subset \Sigma$. This is the appropriate quantity for applications of the Gauss-Bonnet theorem. Note that the atoms of the curvature measure are all negative, corresponding to the fact that the corners are all re-entrant. Let $k_t^\pm$ denote the positive and negative parts of this measure (so $k_t = k_t^+ - k_t^-$) and put $|k_t| := k^+(t) + k^-(t)$. The following is immediate from Lemma 1.2.

1.3. Lemma. *With the assumptions of 1.2, suppose that*

$$A \leq k_t^+(c_t) \leq B,$$
$$k_t^-(c_t) \leq C$$

for all $t \in (0, s] - N$, where $A, B \geq 0$ and $\pi > C \geq 0$. Then $|c_t|$ is a Lipschitz function of $t \in [0, s]$, with

$$A - 2\tan\frac{C}{2} \leq \frac{d|c_t|}{dt} \leq B$$

for a.e. $t \in [0, s]$.

§2. Various lemmata. We assume henceforth that the hypotheses of Theorem A are satisfied.

2.1. Lemma. *There is no geodesic loop of length $< 3R$ based at p_0.*

Proof. This is a trivial consequence of the Gauss-Bonnet theorem and the hypothesis (0.1).

Thus 1.2 applies with $s = 3R/2$. As in 1.2, we denote by N the set of values in $(0, 3R/2]$ at which the conclusion of 1.2(2) fails.

Let $q_\pm \in \rho^{-1}(t), 0 < t < 3R/2, t \notin N$ be smooth points, bounding an arc $c \subset \rho^{-1}(t)$, and let $g_\pm$ be the minimizing geodesics from p_0 to the $q_\pm$. By ***the angle subtended by** c* we mean the angle at p_0 between the $g_\pm$ — more precisely, the interior angle at p_0 of the curve $c \cup g_+ \cup g_-$, considered as the boundary of the unique domain $\subset B(p_0, t)$ that it bounds.

2.2 Lemma. *For an open arc $c \subset \rho^{-1}(t), 0 < t < 3R/2 t \notin N$, bounded by smooth points, the total absolute curvature, the total positive curvature, and the total negative curvature of c satisfy*

$$||k_t|(c) - \theta|, |k_t^+(c) - \theta|, k_t^-(c) \leq K_0.$$

where θ is the angle subtended by c.

Proof. Subdivide c arbitrarily into subarcs $c_1, c_2, \ldots, c_n$ bounded by smooth points, and let $g_- = g_0, g_1, g_2, \ldots, g_n = g_+$ be the minimizing geodesics from p_0 to the endpoints, with everything ordered in the obvious way (see figure 2.2). Then $e_i := g_{i-1} \cup g_i \cup c_i$, $i = 1, \ldots, n$ are simple piecewise C^2 loops bounding mutually disjoint domains D_i, of total geodesic curvature $2\pi - \theta_i + k_t(c_i)$, where θ_i is the angle between g_{i-1} and g_i at p_0. Applying the Gauss-Bonnet theorem,

$$\begin{aligned}\theta - \sum_{i=1}^{n} |k_t(c_i)| &= \sum_{i=1}^{n} (\theta_i - |k_t(c_i)|) \\ &\leq \sum_{i=1}^{n} |\theta_i - k_t(c_i)| \\ &= \sum_{i=1}^{n} |\int_{D_i} K| \\ &\leq \int_{B(p_0, 3R/2)} |K| = K_0.\end{aligned}$$

The conclusions now follow from a simple covering argument.

2.3 Proposition. *Let $g_\pm$ be minimizing geodesics from p_0 of length $t_0 < 3R/2$, let U be a component of $B(p_0, t_0) - (g_+ \cup g_-)$, and let θ be the interior angle of U at p_0 (i.e."the" angle between $g_\pm$). Put $c_t := U \cap \rho^{-1}(t)$, $0 < t < t_0$. Then the length function $|c_t|$ is Lipschitz, with*

$$\theta - 2 \tan \frac{K_0}{2} \leq \frac{d|c_t|}{dt} \leq \theta + \frac{K_0}{2}.$$

In particular, the length function $\lambda(t) := |\rho^{-1}(t)|$, $0 < t < 3R/2$, satisfies

$$2\pi - 2 \tan \frac{K_0}{2} \leq \frac{d\lambda}{dt} < 2\pi + K_0.$$

Proof. Apply 1.3 and 2.2.

Put

$$\Delta := \frac{\pi}{4} - K_0 - \tan \frac{K_0}{2} \in (0, 1).$$

2.4. Lemma. *If $0 < t < 3R/2$, $t \notin N$, then any open subarc $c \subset \rho^{-1}(t)$ of length*

$$|c| \leq t\Delta$$

has total absolute curvature

$$|k_t|(c) \leq \frac{\pi}{2} - K_0 - \Delta.$$

Proof. Any such c may be expressed as the nested union of a family of subarcs all bounded by smooth points of $\rho^{-1}(t)$. Approximating in this way we may assume that c is itself bounded by smooth points. Put θ for the angle subtended by c. Using 2.2, 2.3 and the mean value theorem,

$$\begin{aligned}|k_t|(c) &\leq \theta + K_0 \\ &\leq |c|t^{-1} + 2 \tan \frac{K_0}{2} + K_0 \\ &\leq \Delta + 2 \tan \frac{K_0}{2} + K_0 \\ &= \frac{\pi}{2} - K_0 - \Delta.\end{aligned}$$

The next lemma is key. For $q_1, q_2 \in \rho^{-1}(t)$ let us put $d_t(q_1, q_2)$ for the length of the smallest subarc of $\rho^{-1}(t)$ containing both q_1 and q_2.

2.5. Lemma. *If* $0 < t < R$ *then, for every* $q_1, q_2 \in \rho^{-1}(t)$,

$$d_t(q_1, q_2) \le (\pi + \frac{K_0}{2})\Delta^{-1} \csc\Delta\, d(q_1, q_2).$$

Proof. By continuity of the lengths $|c_t|$ we may assume that $t \notin N$.

Suppose first that $d_t(q_1, q_2) \le t\Delta$. Let $c \subset \rho^{-1}(t)$ be a subarc realizing this distance, and s an arc length parameter for c, $0 \le s \le |c|$, $c(0) = q_1, c(|c|) = q_2$. If $c(s)$ is a smooth point of $\rho^{-1}(t)$ then

$$\frac{d}{ds}\,\mathrm{dist}\,(q_1, c(s)) = \inf_v \langle \dot c(s), v\rangle, \tag{2.5.1}$$

where v ranges over all velocity vectors at $c(s)$ of minimizing geodesic segments from q_1 to $c(s)$.

We claim that all of these inner products are $\ge \sin\Delta$. For, let g be such a minimizing geodesic segment, $g(\delta) = c(s)$. Put

$$\delta' := \sup\{t < \delta : g(t) \in c\}$$

(see figure 2.5(a)). If $\delta' = \delta$ then $\dot c(s) = \dot g(\delta)$ and the relevant inner product is 1. Otherwise, the union of $g|[\delta', \delta]$ with the portion c' of c between $g(\delta')$ and $g(\delta)$ forms a simple piecewise C^2 loop C lying within distance $t + \frac{t\Delta}{2} < \frac{3R}{2}$ of p_0; since $B(p_0, 3R/2)$ is simply connected (as follows from 2.1 and 1.2), C bounds a topological disk $D \subset B(p_0, 3R/2)$. Let θ, ψ denote the interior angles of C at the junctions $g(\delta'), g(\delta)$. By Gauss-Bonnet,

$$k_t(c') + \int_D K - \theta - \psi = 0,$$

so

$$\begin{aligned}\psi &\le \theta + \psi \\ &\le |k_t|(c') + \int_D |K| \\ &\le |k_t|(c) + K_0 \\ &\le \frac{\pi}{2} - \Delta\end{aligned}$$

by 2.4, which establishes our claim.

Now we may integrate (2.5.1) to obtain

$$d_t(q_1, q_2) = |c| \le \csc\Delta\, d(q_1, q_2),$$

establishing the required bound *a fortiori* in this case.

In the complementary case, observe first that $d_t(q_1, q_2) \le \frac{1}{2}|\rho^{-1}(t)| \le (\pi + \frac{K_0}{2})t$ by 2.3. It is therefore enough to show that if $\rho(q_1) = \rho(q_2) = t$ and $d_t(q_1, q_2) \ge t\Delta$, then

$$d(q_1, q_2) \ge t\Delta \sin\Delta.$$

Supposing this to be false, choose such q_1, q_2 so that their distance is a minimum. In particular, $d(q_1, q_2) < t$. Then $d_t(q_1, q_2) > t\Delta$, for the case $d_t = t\Delta$ is covered by the analysis of the first case above. In other words the solution (q_1, q_2) of this minimization problem is ***interior***. Let g be a minimizing geodesic segment between q_1 and q_2; in particular, $|g| < t$. If q_i is a smooth point then g must meet $\rho^{-1}(t)$ in a right angle at q_i, while if q_i is a corner then g must meet each tangent ray of $\rho^{-1}(t)$ at q_i in an angle no less than $\frac{\pi}{2}$. In particular, in either case some initial segment of g near q_i lies either entirely inside or entirely outside of the ball $B(p_0, t)$. If inside, then the tangent vector to g at q_i lies in the convex hull of the tangent rays at q_i to the minimizing geodesics $g_i^{\pm}$ from p_0 to q_i. Since g is minimizing, it cannot cross the $g_i^{\pm}$ and so must pass through p_0 — but this is absurd since $|g| < t = d(p_0, q_1)$.

Thus g must lie initially outside of $B(p_0, t)$ near each q_i. Since $\rho^{-1}(t)$ has only re-entrant corners , each q_i must be a smooth point of $\rho^{-1}(t)$, and there is a unique geodesic segment from q_i to p_0. Therefore g, g_1, and g_2 piece together to form a geodesic loop based at p_0, of length $< 3t < 3R$ (see figure 2.5(b)). This contradicts 2.1, qed.

§3. **The coordinates.** Our basic impulse is to use geodesic normal coordinates about p_0. However, the Lipschitz constants of such coordinates are controlled by the geodesic curvature ***function***. By means of the lemmata of §2 above, we are able to bound only the geodesic curvature ***measure***. Thus some modification is necessary.

Referring to standard polar coordinates on the euclidean plane, we set the radial coordinate of a point $q \in B(p_0, R)$ equal to the distance $\rho(q)$ from p_0. For the angular coordinate, we select an orientation of $B(p_0, R)$ and let g_0 be any geodesic segment of length $> R$ from p_0. We orient each $\rho^{-1}(t)$ as the boundary of $B(p_0, t)$, and for $q \in \rho^{-1}(t)$, we set $\Theta(q)$ equal to $\frac{2\pi}{\lambda(t)}$ times the length of the positively oriented subarc of $\rho^{-1}(t)$ connecting $g_0(t)$ to q (recall that $\lambda(t) := |\rho^{-1}(t)|$). (See figure 3.)

Let $\Phi := (\rho, \Theta)^{-1} : [0, R) \times [0, 2\pi) \to B(p_0, R)$ denote the inverse map. We must show that the corresponding map $\{x^2 + y^2 < R^2\} \to B(p_0, R)$ is biLipschitz, with bounds of the form (0.2).

3.1. Lemma. *Let $g : [0, r] \to \Sigma$ be a minimizing geodesic segment from p_0. Then the composition $\Theta \circ g$ is locally Lipschitz on $(0, r]$, with*

$$\left|\frac{d}{dt}\Theta \circ g(t)\right| \leq t^{-1}\left(4\pi(2\pi - 2\tan\frac{K_0}{2})^{-2}(2\pi + K_0)^2\right).$$

Proof. Put $\mu(t)$ for the length of the positively oriented subarc of $\rho^{-1}(t)$ connecting $g_0(t)$ to $g(t)$. Then $\Theta \circ g = 2\pi\frac{\mu}{\lambda}$. The estimate now follows at once from the quotient rule and the bounds 2.3.

3.2. Lemma. *For each $\theta \in [0, 2\pi)$, consider the curve*

$$c_\theta(t) := \Phi(t, \theta).$$

Then c_θ is Lipschitz, with constant

$$Lip(c_\theta) \leq \alpha_2 := \alpha_1 \frac{2\pi + K_0}{2\pi} + 1.$$

where α_1 is the coefficient of t^{-1} in 3.1.

Proof. Given $0 < s < t < R$, let g be a minimizing geodesic from p_0 to $c_\theta(t)$. By 3.1,

$$|\Theta \circ g(s) - \Theta \circ g(t)| \leq \alpha_1 \frac{t-s}{s}.$$

Therefore

$$\begin{aligned} d(c_\theta(s), c_\theta(t)) &\leq d(c_\theta(s), g(s)) + d(g(s), c_\theta(t)) \\ &\leq \alpha_1 \frac{(t-s)\lambda(s)}{2\pi s} + t - s \\ &\leq \left(\alpha_1 \frac{2\pi + K_0}{2\pi} + 1\right)(t-s), \end{aligned}$$

qed.

3.3. Theorem.

$$\alpha_3 \min\{r,s\}||\theta - \psi|| + \alpha_4|r-s| \leq d(\Phi(r,\theta), \Phi(s,\psi)) \leq \alpha_5 \min\{r,s\}||\theta-\psi|| + \alpha_2|r-s|,$$

where $\alpha_2, \alpha_3, \alpha_4, \alpha_5 > 0$ are constants depending only on K_0.

Proof. Assume $r \leq s$.

For the second estimate,

$$\begin{aligned} d(\Phi(r,\theta), \Phi(s,\psi)) &\leq d(\Phi(r,\theta), \Phi(r,\psi)) + d(\Phi(r,\psi), \Phi(s,\psi)) \\ &\leq \frac{2\pi + K_0}{2\pi} r||\theta - \psi|| + \alpha_2(s-r), \end{aligned}$$

by 3.2.

For the first estimate,

$$\begin{aligned} d := d(\Phi(r,\theta), \Phi(s,\psi)) &\geq d(\Phi(r,\theta), \Phi(r,\psi)) - d(\Phi(r,\psi), \Phi(s,\psi)) \\ &\geq \Delta \sin \Delta \,(\pi + \frac{K_0}{2})^{-1} d_r(\Phi(r,\theta), \Phi(r,\psi)) - \alpha_2(s-r) \\ &\text{(by 2.5 and 3.2)} \\ &= \Delta \sin \Delta \,(\pi + \frac{K_0}{2})^{-1} \frac{\lambda(r)}{2\pi} ||\theta - \psi|| - \alpha_2(s-r) \\ &\geq \alpha_6 r||\theta - \psi|| - \alpha_2(s-r), \end{aligned}$$

by 2.3, where

$$\alpha_6 := \Delta \sin \Delta (\pi + \frac{K_0}{2})^{-1} \frac{2\pi - 2\tan\frac{K_0}{2}}{2\pi}.$$

Thus if

$$\frac{2\alpha_6 r||\theta - \psi||}{3} \geq (1 + \alpha_2)(s - r) \tag{3.3.1}$$

then $d \geq \frac{\alpha_6}{3}(r||\theta - \psi|| + |r - s|)$. On the other hand, if (3.3.1) fails, then

$$d \geq |r - s| \geq \frac{1}{2}|r - s| + \frac{\alpha_6}{3(1 + \alpha_2)} r||\theta - \psi||.$$

This yields our estimate with

$$\alpha_3 := \min\{\frac{1}{2}, \frac{\alpha_6}{3}\}$$

and

$$\alpha_4 := \min\{\frac{\alpha_6}{3(1 + \alpha_2)}, \frac{\alpha_6}{3}\}.$$

With 1.0 this completes the proof of Theorem A.

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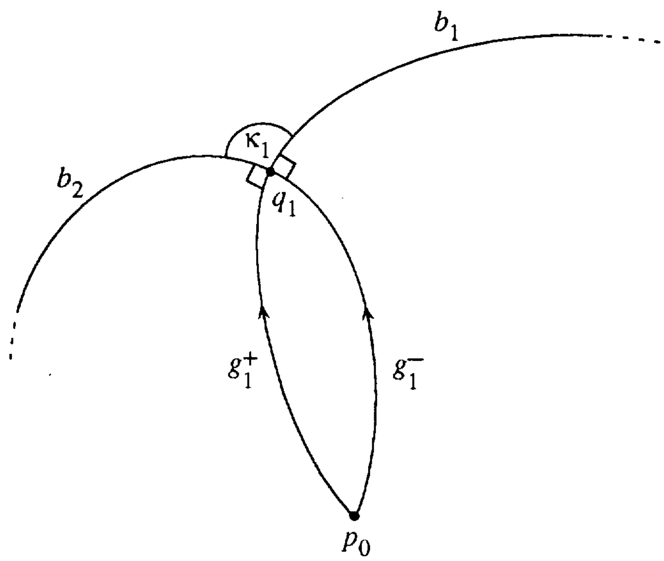
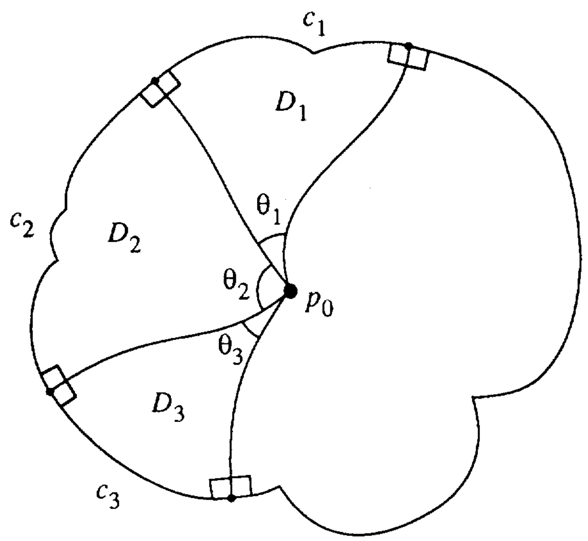
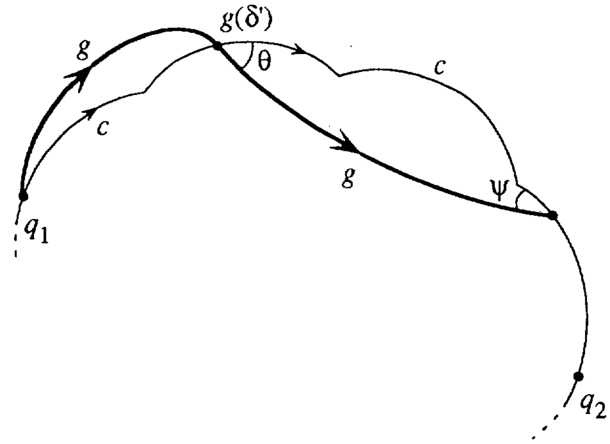


1.2



2.2



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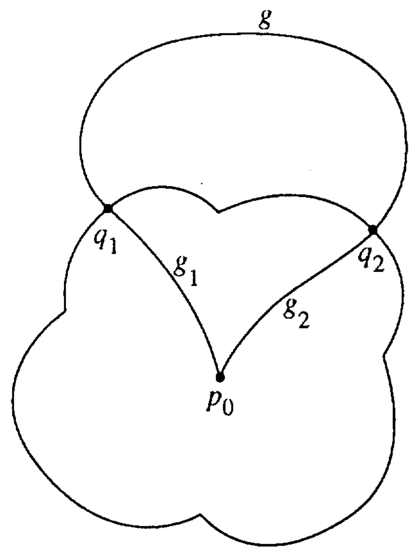

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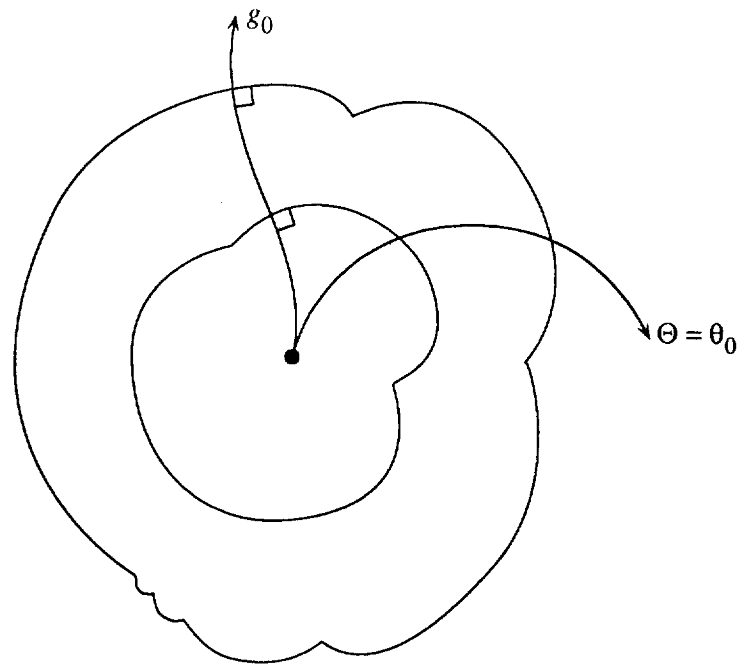
g_0
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