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stress-energy tensor ***

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A simple prescription for computing the stress-energy tensor

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Abstract. A non-variational technique for computing the stress-energy tensor is presented. The prescription is used, among other things, to obtain the correct field equations for Prasanna's highly nonlinear electrodynamics. (Prasanna A R 1973 *Lett. Nuovo Cimento* **6** 420)

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As is well-known, the field equations related to a gravitational field generated by some matter field described by the variables Ψ^A , where A represents any tensorial indices, may be derived from the action

$$S = \int \sqrt{-g} \frac{R}{\kappa} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x , \quad (1)$$

where R is the Ricci scalar, $\kappa = 8\pi G$ is the Einstein constant, and \mathcal{L}_m is the Lagrangian density for the matter field. If $\mathcal{L}_m = \mathcal{L}_m(\Psi^A, g_{\mu\nu})$, i.e. \mathcal{L}_m contains g 's but not Γ 's (so that Γ 's are present only in R), the coupling between the g -field and the Ψ -field is called a gravitational minimal coupling. On the other hand, if \mathcal{L}_m contain both g 's and Γ 's, one has a gravitational non-minimally coupled theory. Treating the g 's and Ψ 's as independent variables one can show that $\delta S = 0$ leads to

$$G^{\mu\nu} = -\kappa T^{\mu\nu} , \quad (2)$$

$$\frac{\delta \mathcal{L}_m}{\delta \Psi^A} = 0 , \quad (3)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R , \quad (4)$$

is the Einstein tensor, and the stress-energy tensor is defined by

$$\int \frac{\delta (\mathcal{L}_m \sqrt{-g})}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d^4x \equiv \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d^4x .$$

In general, the passage from (1) to (2) involves the use of non-trivial variational techniques. Our goal here is to present a simple method for computing the stress-energy tensor from the equation for the Ψ -field which does not rely upon variational techniques.

In what follows we will work in natural units, where $\hbar = c = 1$. Throughout the letter our conventions are to use a metric of signature $(+---)$ and define the Riemann and Ricci tensors as $R^\rho{}_{\lambda\mu\nu} = -\partial_\nu \Gamma^\rho{}_{\lambda\mu} + \partial_\mu \Gamma^\rho{}_{\lambda\nu} - \Gamma^\sigma{}_{\lambda\mu} \Gamma^\rho{}_{\sigma\nu} + \Gamma^\sigma{}_{\lambda\nu} \Gamma^\rho{}_{\sigma\mu}$ and $R_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$, respectively.

Suppose we want to find the stress-energy tensor for the massive Klein-Gordon field. Generalizing the usual Klein-Gordon equation to curved spacetime by the ‘‘comma-goes-to-

semicolon" rule (minimal coupling) we obtain

$$(\square + m^2)\phi = 0 ,$$

where ϕ is a real scalar field and $\square \equiv \nabla_\mu \nabla^\mu$. Let us then tentatively multiply the previous equation by $\nabla_\alpha \phi$

$$\nabla_\alpha \phi \nabla_\beta \nabla^\beta \phi + (\nabla_\alpha \phi) m^2 \phi^2 = 0 .$$

The first term can be transformed in the following obvious way:

$$\begin{aligned} \nabla_\alpha \phi \nabla_\beta \nabla^\beta \phi &= \nabla_\beta (\nabla_\alpha \phi \nabla^\beta \phi) - (\nabla_\beta \nabla_\alpha \phi) \nabla^\beta \phi \\ &= \nabla_\beta \left[\nabla_\alpha \phi \nabla^\beta \phi - \delta_\alpha^\beta \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right] . \end{aligned} \quad (5)$$

Similarly, we obtain for the second term

$$(\nabla_\alpha \phi) m^2 \phi = \nabla_\beta \left[\frac{1}{2} \delta_\alpha^\beta m^2 \phi^2 \right] . \quad (6)$$

From (5) and (6) it follows that

$$\nabla_\beta T^{\alpha\beta} = 0 ,$$

where

$$T^{\alpha\beta} = \nabla^\alpha \phi \nabla^\beta \phi - \frac{1}{2} g^{\alpha\beta} (\nabla_\mu \phi \nabla^\mu \phi - m^2 \phi^2) \quad (7)$$

is the stress-energy tensor we are looking for. The above example allows us to outline the recipe which gives the stress-momentum tensor from the Ψ -field equation. Multiply the equation for the Ψ -field by a suitable covariant derivative of this field so that the resulting expression contains only one free spacetime index and then rewrite the expression in hand as a covariant four-divergence. To arrive at this prescription we have only reversed the argument used to show that $T^{\mu\nu}$ is divergenceless.

Next we consider the most general theory describing a real scalar field ϕ non-minimally coupled to Einstein gravity. The corresponding equation for the ϕ -field is given by [1]

$$\square \phi + V'(\phi) - R f'(\phi) = 0 , \quad (8)$$

where $f'(\phi) = df/d\phi$ and $V'(\phi) = dV/d\phi$. The arbitrary functions $f(\phi)$ and $V(\phi)$ distinguish the different scalar-tensor theories. Particular examples are provided by Zee's broken-symmetric theory of gravity [2], gravity theories with a conformally invariant scalar field [3], the theory of gravitation of Callan *et al* [4], Madsen's theory of gravity [5], the gravity theory of Schmidt *et al* [6] and so on. Note that $[f(\phi)] = [\kappa^{-1}] = L^{-2}$. It is worth mentioning that these theories, contrary to what is widely believed, do obey the equivalence principle, i.e. the universality of free-fall trajectories [1,7].

To find $T^{\mu\nu}$ for this general scalar-tensor theory we multiply (8) by $\nabla_\alpha\phi$. Using (7) we obtain:

$$\nabla_\beta \left[\nabla_\alpha\phi \nabla^\beta\phi - \frac{1}{2}\delta_\alpha^\beta \nabla_\mu\phi \nabla^\mu\phi \right] + \nabla_\alpha V(\phi) - \delta_\alpha^\beta R \nabla_\beta f(\phi) = 0 .$$

Introducing in the third term of this equation the expression for the Einstein tensor, Eq.(4), we find

$$-\delta_\alpha^\beta R \nabla_\beta f(\phi) = \nabla_\beta \left[2G_\alpha^\beta f(\phi) \right] - 2R_\alpha^\beta \nabla_\beta f(\phi) .$$

Now, taking into account the definition of the curvature tensor,

$$[\nabla_\mu, \nabla_\nu] a_\lambda = -R^\rho_{\lambda\mu\nu} a_\rho ,$$

where a_λ is some covariant vector field, we may write

$$-2R_\alpha^\beta \nabla_\beta f(\phi) = \nabla_\beta \left[\left(-2\delta_\alpha^\beta \square + 2\nabla_\alpha \nabla^\beta \right) f(\phi) \right] .$$

Putting all the pieces together, we come to the conclusion that

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2} \nabla^\mu\phi \nabla^\nu\phi - \frac{1}{4} g^{\mu\nu} \nabla_\alpha\phi \nabla^\alpha\phi + \frac{1}{2} g^{\mu\nu} V(\phi) \\ &+ G^{\mu\nu} f(\phi) + [\nabla^\mu \nabla^\nu - g^{\mu\nu} \square] f(\phi) . \end{aligned}$$

Thus, the field equations for this non-minimally coupled scalar-tensor theory are

$$G^{\mu\nu} = -\kappa T^{\mu\nu} ,$$

$$\square\phi + V'(\phi) - Rf'(\phi) = 0 .$$

Note that for $f = -\frac{1}{2\kappa}$ and $V = \frac{1}{2}m^2\phi^2$, these equations reduce to

$$G^{\mu\nu} = -\kappa \left[\nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} g^{\mu\nu} (\nabla_\alpha \phi \nabla^\alpha \phi - m^2 \phi^2) \right] ,$$

$$(\square + m^2) \phi = 0 ,$$

which are nothing but the field equations for the minimally coupled theory previously discussed.

To conclude we consider a theory involving a more sophisticated gravitational coupling. In the early seventies Prasanna proposed a highly nonlinear electrodynamics generated via gravitational non-minimal coupling whereupon the equation for $F_{\mu\nu}$ -field, where $F_{\mu\nu}$ is the antisymmetric electromagnetic field tensor, are

$$\nabla_\nu F^{\mu\nu} - 2\lambda \nabla_\nu (R^{\rho\theta\mu\nu} F_{\rho\theta}) = 0 , \quad (9)$$

$$F_{[\mu\nu;\alpha]} = 0 . \quad (10)$$

Here λ is a suitable coupling constant with dimension of (length)². To construct $T^{\mu\nu}$ for that theory we multiply (9) by $F_{\alpha\mu}$:

$$F_{\alpha\mu} \nabla_\nu F^{\mu\nu} - 2\lambda F_{\alpha\mu} \nabla_\nu (R^{\rho\theta\mu\nu} F_{\rho\theta}) = 0 .$$

The first term can be expanded to

$$F_{\alpha\mu} \nabla_\nu F^{\mu\nu} = \nabla_\nu (F_{\alpha\mu} F^{\mu\nu}) - (\nabla_\nu F_{\alpha\mu}) F^{\mu\nu} ,$$

and, using (10), we get

$$F_{\alpha\mu} \nabla_\nu F^{\mu\nu} = \nabla_\nu \left[F_{\alpha\mu} F^{\mu\nu} + \frac{1}{4} \delta^\nu_\alpha F_{\rho\theta} F^{\rho\theta} \right] .$$

Similarly, we obtain for the second term

$$-2\lambda F_{\alpha\mu} \nabla_\nu (R^{\rho\theta\mu\nu} F_{\rho\theta}) = -\nabla_\nu \left[2\lambda R^{\rho\theta\mu\nu} F_{\alpha\mu} F_{\rho\theta} + \frac{\lambda}{2} R^{\rho\theta\mu\nu} F_{\rho\theta} F_{\mu\nu} \right]$$

$$+ \frac{\lambda}{2} F^{\rho\theta} F^{\mu\nu} \nabla_\alpha R_{\rho\theta\mu\nu} .$$

But,

$$\begin{aligned} \frac{\lambda}{2} F^{\rho\theta} F^{\mu\nu} \nabla_{\alpha} R_{\rho\theta\mu\nu} &= -\nabla_{\nu} \left[\lambda F_{\rho\theta} F_{\mu\nu} R^{\rho\theta\alpha\mu} \right] \\ &\quad - 2\lambda R_{\rho\mu\theta\alpha} \nabla_{\nu} \left(F^{\rho\theta} F^{\mu\nu} \right) . \end{aligned}$$

In deriving the above equation we have made use of the identities

$$\begin{aligned} R_{\alpha\beta[\gamma\delta;\rho]} &= 0 , \\ R_{\alpha[\beta\gamma\delta]} &= 0 . \end{aligned}$$

On the other hand,

$$[\nabla_{\rho}, \nabla_{\theta}] \nabla_{\nu} \left(F^{\rho}_{\alpha} F^{\theta\nu} \right) = -R_{\rho\mu\theta\alpha} \nabla_{\nu} \left(F^{\rho\theta} F^{\mu\nu} \right) .$$

Hence, the stress-energy tensor is given by

$$\begin{aligned} T^{\alpha\nu} &= F^{\alpha\mu} F_{\mu}^{\nu} + \frac{1}{4} g^{\alpha\nu} F_{\rho\theta} F^{\rho\theta} + \lambda \left\{ -\frac{1}{2} g^{\alpha\nu} R^{\rho\theta\beta\gamma} F_{\rho\theta} F_{\beta\gamma} \right. \\ &\quad \left. + 3R^{\rho\theta\mu(\alpha} F_{\mu}^{\nu)} F_{\rho\theta} - 2\nabla_{\rho} \nabla_{\theta} \left[F^{\rho(\alpha} F^{\nu)\theta} \right] \right\} . \end{aligned}$$

It follows then that the correct field equations for Prasanna's electrodynamics are

$$\begin{aligned} \nabla_{\nu} \left[F^{\mu\nu} - 2\lambda R^{\rho\theta\mu\nu} F_{\rho\theta} \right] &= 0 , \\ F_{[\mu\nu;\alpha]} &= 0 , \\ G^{\alpha\nu} + k \left(F^{\alpha\mu} F_{\mu}^{\nu} + \frac{1}{4} g^{\alpha\nu} F_{\rho\theta} F^{\rho\theta} \right) &= \lambda k \left\{ \frac{1}{2} g^{\alpha\nu} R^{\rho\theta\beta\gamma} F_{\rho\theta} F_{\beta\gamma} \right. \\ &\quad \left. - 3R^{\rho\theta\mu(\alpha} F_{\mu}^{\nu)} F_{\rho\theta} + 2\nabla_{\rho} \nabla_{\theta} \left[F^{\rho(\alpha} F^{\nu)\theta} \right] \right\} . \end{aligned}$$

Note that in both of the non-trivial examples (the non-minimally coupled scalar field, and Prasanna's theory) there are steps in the derivation such as the use of the definition of the Einstein tensor to eliminate the Ricci scalar, or the use of symmetry properties of the Riemann tensor, which are decidedly non-algorithmic, i.e. which require some amount of intuition about the form of the final expression of the stress-energy tensor.

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