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THE ARITHMETIC MIRROR SYMMETRY AND CALABI-YAU MANIFOLDS

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ABSTRACT. We extend our variant of mirror symmetry for K3 surfaces [GN3] and clarify its relation with mirror symmetry for Calabi–Yau manifolds. We introduce two classes (for the models A and B) of Calabi–Yau manifolds fibrated by K3 surfaces with some special Picard lattices. These two classes are related with automorphic forms on IV type domains which we studied in our papers [GN1]–[GN6]. Conjecturally these automorphic forms take part in the quantum intersection pairing for model A, Yukawa coupling for model B and mirror symmetry between these two classes of Calabi–Yau manifolds. Recently there were several papers by physicists where it was shown on some examples. We propose a problem of classification of introduced Calabi–Yau manifolds. Our papers [GN1]–[GN6] and [N3] — [N14] give a hope that this is possible. They describe possible Picard or transcendental lattices of general K3 fibers of the Calabi–Yau manifolds.

0. INTRODUCTION

This paper was subject of the talk given by the second author at the conference “Mirror Symmetry and Calabi–Yau manifolds” held at RIMS Kyoto University on 2 – 6 December, 1996. It contains results by both authors.

In [GN3] we suggested a variant of mirror symmetry for K3 surfaces which is related with reflection groups in hyperbolic spaces and automorphic forms on IV type domains. This subject was developed in our papers [GN1]–[GN6], [N11] — [N14]. Some results of R. Borcherds [B1]–[B7] are also connected with this subject.

Recently several papers by physicists have appeared where our automorphic forms (and some automorphic forms constructed by R. Borcherds) were used in mirror symmetry for Calabi–Yau manifolds. Physicists have shown that automorphic forms on IV type domains which we considered for our variant [GN3] of mirror symmetry for K3 surfaces take part in the quantum intersection pairing and the Yukawa coupling for some Calabi–Yau manifolds. We only mention papers which are directly connected with this subject: Harvey – Moore [HM1] – [HM3]; Henningson – Moore [HeM1], [HeM2]; Kawai [Ka1],[Ka2]; Dijkgraaf – E. Verlinde – H. Verlinde [DVV]; Cardoso – Curio – Lüst [CCL], but there are many other papers which are connected indirectly.

For further study it is important to clarify relation between our variant of mirror symmetry for K3 surfaces, and the Calabi–Yau manifolds which were used by physicists. We

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want to give a definition of Calabi–Yau manifolds related with the variant [GN3] of mirror symmetry for K3 surfaces and propose a problem of their classification. Our papers [GN1]–[GN6] and [N3]–[N14] give a hope that this is possible. They describe possible Picard or transcendental lattices of general K3 fibers of these Calabi–Yau manifolds.

The preliminary variant of this paper was published as preprint [GN7].

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1. K3 SURFACES WITH SKELETON AS FIBERS OF CALABI–YAU MANIFOLDS FOR MODEL A

The main property of Calabi–Yau manifolds \mathcal{X} which are related with our variant of mirror symmetry for K3 surfaces and which were considered in papers of physicists we cited above is that *they are fibrated by K3 surfaces*. There is a morphism

$$\pi : \mathcal{X} \rightarrow B$$

with the general fiber which is a K3 surface X . Let S be the Picard lattice of the general fiber of π . Then \mathcal{X} is related with the moduli \mathcal{M}_S of K3 surfaces with the Picard lattice S . It is known that $\mathcal{M}_S = G \setminus \Omega(T)$ where $\Omega(T)$ is a Hermitian symmetric domain of type IV which is defined by the transcendental lattice $T = S_{L_{K3}}^\perp$. Here L_{K3} is the second cohomology lattice of K3 and $G \subset O(T)$ is a subgroup of finite index. Thus, K3 fibrated Calabi–Yau manifolds are related with arithmetic quotients $G \setminus \Omega(T)$ of IV type domains. One can pick up the family π above from a morphism $B \rightarrow \mathcal{M}_S = G \setminus \Omega(T)$.

Recently there was a big progress in studying mirror symmetry for Calabi–Yau complete intersections in toric varieties. We mention papers of Candelas – de la Ossa – Green – Parkers [COGP], Morrison [M], Batyrev [Bat], Kontsevich [Ko], Kontsevich – Manin [KoM], Ruan – Tian [RT] and Givental [Gi]. For these Calabi–Yau manifolds mirror symmetry comes from duality between polyhedra defining ambient toric varieties. Moreover, the Yukawa coupling and the quantum intersection pairing of Calabi–Yau complete intersections in toric varieties are strongly related with quantum cohomology of ambient toric varieties. One can ask about similar theory for K3 fibrated Calabi–Yau manifolds when one replaces toric varieties by arithmetic quotients of IV type domains defined by moduli of K3 surfaces with condition on Picard lattice. One can expect that *in some cases mirror symmetry for K3 fibrated Calabi–Yau manifolds is dominated by some variant of mirror symmetry for their K3 fibers*. This variant has in fact been suggested in [GN3]. Below we just extend it a little.

Definition 1. An algebraic K3 surface X over \mathbb{C} has *skeleton* if there exists an integral non-zero nef element $r \in \text{NEF}(X) \cap S$ such that r is invariant with respect to $\text{Aut}(X)$. The element r is called *canonical nef element*. Here we denote $\text{NEF}(X) \subset S \otimes \mathbb{R}$ the *nef cone* (equivalently, the clouser of the *Kähler cone*) of X and S the *Picard lattice* of X .

Let $W^{(2)}(S) \subset O(S)$ be the group generated by reflections in all elements with square -2 of S and $P(X)$ the set of all irreducible non-singular rational curves on X . From the global Torelli Theorem for K3 surfaces and description of the automorphism groups of

K3 surfaces [P-ŠŠ], it follows that the canonical map $\pi : \text{Aut}(X) \rightarrow \text{Aut}(\text{NEF}(X))$ has finite kernel and cokernel. Here $\text{Aut}(\text{NEF}(X)) = \{\phi \in O(S) \mid \phi(\text{NEF}(X)) = \text{NEF}(X)\}$. Moreover, the $\mathcal{M} = \text{NEF}(X)/\mathbb{R}_{++}$ is a fundamental chamber for the group $W^{(2)}(S)$ acting in the hyperbolic space $\mathcal{L}(S)$ defined by S with the set $P(X)$ of orthogonal vectors to faces (of highest dimension) of $\text{NEF}(X)/\mathbb{R}_{++}$. We then get the following description of K3 surfaces X with skeleton and their Picard lattices S .

a) *Elliptic type*. The set $P(X)$ is finite and generates $S \otimes \mathbb{Q}$. Then one can find a canonical nef element r with $r^2 > 0$. One can consider the finite set of non-singular rational curves on X as a “skeleton” of X . It essentially defines geometry of X . A K3 surface X has elliptic type if and only if the group $W^{(2)}(S)$ has finite index in $O(S)$, and for $\text{rk } S = 2$ the S contains at least 4 different elements with square -2 . Then S is called *2-reflective of elliptic type*.

a') *Special elliptic type*. The Picard lattice S is one-dimensional. Then $r^2 > 0$. The lattice S is called *2-reflective of special elliptic type*.

b) *Parabolic type*. There exists a *canonical* elliptic fibration $|r| : X \rightarrow \mathbb{P}^1$ which is preserved by a subgroup $G \subset \text{Aut}(X)$ of finite index. If $\text{Aut}(X)$ is infinite, then the canonical elliptic fibration $|r|$ is unique and is preserved by the $\text{Aut}(X)$ which is a finitely generated Abelian group up to finite index. For this case non-singular rational curves $P(X)$ on X have bounded degree with respect to r . The r together with $P(X)$ also may be considered as a “skeleton” of X . A K3 surface X has parabolic type if and only if the quotient group $O^+(S)/W^{(2)}(S)$ considered as a group $\text{Aut}(\mathcal{M})$ of symmetries of a fundamental polyhedron \mathcal{M} for $W^{(2)}(S)$ contains a subgroup $G \subset \text{Aut}(\mathcal{M})$ of finite index which fixes a non-zero element $r \in S$ with $r^2 = 0$ (it is unique and is fixed by $\text{Aut}(\mathcal{M})$ if $\text{Aut}(\mathcal{M})$ is infinite). Then S is called *2-reflective of parabolic type*.

Hyperbolic lattices S satisfying one of conditions above are called *2-reflective*. Thus, a K3 surface X has skeleton if and only if its Picard lattice is *2-reflective (of elliptic, including special elliptic, or parabolic type)*. All K3 surfaces X with skeleton are distributed on several families \mathcal{M}_S of K3 surfaces with condition on Picard lattice according to their Picard lattice S . Any S of rank one is 2-reflective. If $\text{rk } S = 2$, the lattice S is 2-reflective if and only if it has a non-zero element with norm -2 or 0 . It was proved in [N3], [N4], [N12] that for $\text{rk } S \geq 3$ the set of 2-reflective hyperbolic lattices S is finite. All 2-reflective hyperbolic lattices S of elliptic type were classified in [N3], [N7], [N8].

In practice, to construct K3 surfaces X with skeleton, one need to find a polyhedron (the fundamental polyhedron $\mathcal{M} = \text{NEF}(X)/\mathbb{R}_{++}$ for $W^{(2)}(S)$) in hyperbolic space with some condition of finiteness of volume and some condition of integrity (its Gram matrix should define a symmetric generalized Cartan matrix (e. g. see[K])). Thus, the theory of K3 surfaces with skeleton is in fact *similar to the theory of toric varieties* where one need to consider some polyhedra in Euclidean space with a lattice.

For the model A of mirror symmetry we consider Calabi–Yau manifolds fibrated by K3 surfaces with skeleton (equivalently, with a 2-reflective Picard lattice). Here and in what follows we always mean the K3 surface or the Picard lattice of the general K3 fiber of the general Calabi–Yau manifold. From this point of view, it is a very interesting problem to classify Calabi–Yau manifolds fibrated by K3 surfaces with skeleton.

Problem A. *Find all 2-reflective hyperbolic lattices S such that there exists a Calabi–*

Yau manifold of dimension ≥ 3 fibrated by K3 surfaces X with the Picard lattice S . Find for this S all Calabi–Yau manifolds fibrated by K3 surfaces with the Picard lattice S .

This problem looks much easier than the general problem of classification of Calabi–Yau manifolds fibrated by K3 surfaces because Picard lattices S of K3 surfaces with skeleton are very special (and actually known). The monodromy group on the Picard lattice is easy to control. These Calabi–Yau manifolds have very special divisors defined by non-singular rational curves in K3 fibers.

There are some examples of Calabi–Yau manifolds fibrated by K3 surfaces X with skeleton. For example hyperbolic lattices S with a 2-elementary discriminant group: $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a$, give a big part of 2-reflective hyperbolic lattices S of high rank and correspond to K3 surfaces with non-symplectic involutions (see [N3] and also [N8]). Using these involutions, C. Borcea [Bo] and Cl. Voisin [V] constructed Calabi–Yau 3-folds fibrated by K3 surfaces with these Picard lattices. One can construct some other examples as complete intersections in toric varieties. But even for $\text{rk } S \geq 3$ not for all 2-reflective lattices S we know existence of Calabi–Yau manifolds fibrated by K3 surfaces with the Picard lattice S .

We remark that the theory of reflective hyperbolic lattices of elliptic and parabolic type was extended to *hyperbolic type* in [N14]. One can introduce K3 surfaces with skeleton (or with 2-reflective Picard lattice) of hyperbolic type. Without any doubt they are also important for mirror symmetry.

2. 2-REFLECTIVE AUTOMORPHIC FORMS, AND TRANSCENDENTAL LATTICES FOR K3 FIBERS OF CALABI–YAU MANIFOLDS OF MODEL B

Now we consider the model B for mirror symmetry.

The mirror symmetric subject to 2-reflective hyperbolic lattices S is given by so called 2-reflective lattices T with 2 positive squares and by so called 2-reflective automorphic forms on the symmetric domain

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\}_0.$$

Here a holomorphic automorphic form Φ on $\Omega(T)$ with respect to a subgroup of $O(T)$ of finite index is called *2-reflective for T* if the divisor of Φ in $\Omega(T)$ is union of quadratic divisors $\mathcal{H}_\delta = \{\mathbb{C}\omega \in \Omega(T) \mid \omega \cdot \delta = 0\}$ (with some multiplicities) orthogonal to elements $\delta \in T$ with $\delta^2 = -2$. A lattice T having a 2-reflective automorphic form is called *2-reflective*.

Geometrical meaning of a 2-reflective lattice T with two positive squares and a 2-reflective automorphic form Φ of T is that Φ is equal to zero only on the discriminant of moduli \mathcal{M}_{T^\perp} of K3 surfaces with the Picard lattice T^\perp (or with the transcendental lattice T). Here we use global Torelli theorem [P-ŠS] and epimorphicity of the Torelli map [Ku]. All 2-reflective automorphic forms Φ corresponding to T define a semi-group which is very interesting. These automorphic forms take part in the mirror symmetry which we will consider.

We expect that like the set of 2-reflective hyperbolic lattices S the set of 2-reflective lattices T with two positive squares is very small. Here the main conjecture is (see [N13] and [GN5] for more general and exact formulation)

Arithmetic Mirror Symmetry Conjecture.

- a) The set of 2-reflective lattices T of $\text{rk } T \geq 5$ is finite.
 b) For any primitive isotropic $c \in T$ such that c_T^\perp contains an element with square -2 , the hyperbolic lattice $S = c_T^\perp / \mathbb{Z}c$ is 2-reflective.

We will explain why we suppose that this is true. For a lattice T we denote by $\Delta^{(2)}(T)$ the set of all elements of T with square -2 and by $\mathcal{H}_\delta \subset \Omega(T)$ a quadratic divisor \mathcal{H}_δ which is orthogonal to $\delta \in \Delta^{(2)}(T)$. By Koecher principle (e. g. see [Ba]), any automorphic form on a IV type domain $\Omega(T)$ has zeros if $\text{codim}_{\Omega(T)} \Omega(T)_\infty \geq 2$. Considering restriction of a 2-reflective automorphic form Φ on subdomains $\Omega(T_1)$ where $T_1 \subset T$, we get that

$$\left(\bigcup_{\delta \in \Delta^{(2)}(T)} \mathcal{H}_\delta \right) \cap \Omega(T_1) \neq \emptyset \quad (1)$$

for any primitive sublattice $T_1 \subset T$ with two positive squares such that $\text{codim}_{\Omega(T_1)} \Omega(T_1)_\infty \geq 2$. One can consider condition (1) as the *analog of condition of finiteness of volume* for a polyhedron in hyperbolic space. This condition is extremely strong (see [N13]), and we expect that lattices T satisfying (1) satisfy the arithmetic mirror symmetry conjecture. Using (1), it was shown in [N13]

Theorem 1. *Let T_n be the transcendental lattice of a general (i.e. with the Picard number 1) algebraic K3 surface of degree n . For any $N > 0$ there exists $n > N$ such that the lattice T_n is not 2-reflective. In particular, the discriminant of moduli \mathcal{M}_n of general K3 surfaces of the degree n is not equal to zero set of any automorphic form on $\Omega(T_n)$.*

Arithmetic mirror symmetry Conjecture is very important for classification of 2 reflective lattices T . After classification of 2-reflective hyperbolic lattices S (their set is finite for $\text{rk } S \geq 3$), using arithmetic mirror symmetry Conjecture it would not be difficult to find all 2-reflective lattices T with 2 positive squares.

For the model B of mirror symmetry we consider Calabi–Yau manifolds fibred by K3 surfaces with a 2-reflective transcendental lattice T . Here and in what follows we always mean the transcendental lattice of the general K3 fiber of the general Calabi–Yau manifold. Similarly to Problem A we propose

Problem B. *Find all 2-reflective lattices T such that there exists a Calabi–Yau manifold fibred by K3 surfaces with the transcendental lattice T . Classify all Calabi–Yau manifolds fibred by K3 surfaces with the 2-reflective transcendental lattice T .*

Like for Problem A, we expect that classification of Calabi–Yau manifolds fibred by K3 surfaces with a 2-reflective transcendental lattice T is much simpler than the general problem of classification of K3 fibred Calabi–Yau manifolds because the set of 2-reflective transcendental lattices T is very small.

3. MIRROR SYMMETRY

Let T be a 2-reflective lattice and Φ a 2-reflective automorphic form of T . Considering product of $g^*\Phi$ over all $g \in G \setminus O^+(T)$, we can suppose that Φ is automorphic with respect

to $O^+(T)$. For this case it is expected that Φ has very special Fourier expansions at cusps $c \in T$. Here $c \in T$ is a primitive non-zero element with $c^2 = 0$.

For simplicity we suppose that $T = U(k) \oplus S$ and $c \in U(k)$ (general case can be treated like in [GN5, Sect. 2.3]). Here $\{c, e\}$ is a basis of $U(k)$ such that $c^2 = e^2 = 0$ and $c \cdot e = k \in \mathbb{N}$. We consider the *mirror symmetry coordinate* $z \in \Omega(V^+(S)) = S \otimes \mathbb{R} + iV^+(S)$ where $V^+(S)$ is the light cone of S . We correspond to $z \in V^+(S)$ the point

$$\mathbb{C}\omega_0 \in \Omega(T), \quad \omega_0 = ((-z^2/2)c + (1/k)e) \oplus z.$$

Here $\omega_0 \in \mathbb{C}\omega_0 \in \Omega(T)$ is chosen by the condition $\omega_0 \cdot c = 1$. It is the mirror symmetry normalization. We expect that after identification of S with the Picard lattice of a K3 surface X the *2-reflective automorphic form* Φ multiplied by some constant *could be written as*

$$\begin{aligned} \Phi(z) &= \\ &= \sum_{w \in W^{(2)}(S)} \epsilon(w) \left(\exp(2\pi i(w(\rho) \cdot z)) - \sum_{a \in \text{NEF}(S)} N(a) \exp(2\pi i(w(\rho + a) \cdot z) \right) \\ &= \exp(2\pi i(\rho \cdot z)) \prod_{\alpha \in \text{EF}(S)} (1 - \exp(2\pi i(\alpha \cdot z)))^{\text{mult } \alpha}. \end{aligned} \quad (2)$$

Here $\epsilon : W^{(2)}(S) \rightarrow \{\pm 1\}$ is some character and $\text{EF}(S)$ is the set of effective elements of X . All Fourier coefficients $N(a)$ and “multiplicities” $\text{mult } \alpha$ are integral. The element $\rho \in \text{NEF}(S) \cap (S \otimes \mathbb{Q})$, and the ρ is non-zero. It is called the *generalized lattice Weyl vector*. The infinite product in this formula is the product of Borcherds type [B5]. We remark that *if one has the Fourier expansion of type (2) with a non-trivial reflection group $W^{(2)}(S)$, then the lattice S is automatically 2-reflective* (see [N12], [GN5]). It is why we were previously forced to restrict by 2-reflective hyperbolic lattices S for the model A. To have the Fourier and the infinite product expansion of type (2) is important for physicists (e. g. see [CCL], [DVV], [HM1] – [HM3], [HeM1], [HeM2], [Ka1], [Ka2]).

We consider the families \mathcal{M}_{T^\perp} and \mathcal{M}_S of K3 surfaces with condition on Picard lattice as *mirror symmetric* if for T and $S = c_T^\perp/\mathbb{Z}c$ a 2-reflective automorphic form Φ satisfying (2) does exist. Remark that here both lattices T and S (if the group $W^{(2)}(S)$ is non-trivial) are 2-reflective. This definition of mirror symmetry for K3 surfaces we used in [GN3] for some more narrow class of Φ .

Geometrically existence of the form Φ is very nice. On the one hand, Φ is an “algebraic function” on the moduli \mathcal{M}_{T^\perp} with zeros only on the discriminant, and the identity (2) reflects geometry of moduli \mathcal{M}_{T^\perp} in the neighborhood of the cusp c . On the other hand, the identity (2) reflects geometry of curves on general K3 surfaces $X \in \mathcal{M}_S$ of the mirror symmetric family. It is why we considered in [GN3] this type of mirror symmetry for K3 surfaces as a very natural and beautiful one. It seems that the case when Φ has zeros of multiplicity one is the most important. Then one can correspond to (2) the *generalized Lorentzian Kac-Moody superalgebra with the denominator function* (2) (this case was considered in [GN3]). Moreover, it is seems important to have Φ which is equal

to zero along all quadratic divisors orthogonal to elements of T with square -2 and with multiplicities which are as small as possible.

Like for toric geometry, we can introduce *K3 surfaces X with reflexive 2-reflective Picard (or transcendental) lattice* when X takes part in our mirror symmetry on the (A) or (B) side respectively, or both sides. Finding of all these cases is especially interesting.

Problems A', B'. Find all reflexive 2-reflective lattices S (respectively T) such that there exists a Calabi–Yau manifold fibrated by K3 surfaces with the Picard lattice S (respectively with the transcendental lattice T). Classify all Calabi–Yau manifolds fibrated by K3 surfaces with the reflexive 2-reflective Picard lattice S (respectively with the reflexive 2-reflective transcendental lattice T).

The first multi-dimensional automorphic form of type (2) was found by R. Borcherds [B2]. He found it for the even unimodular lattice T of rank 28. Then S is the even unimodular hyperbolic lattice of rank 26. It is 2-reflective of parabolic type. It is expected that this case is the most multi-dimensional. This case does not correspond to K3 surfaces but it is important. Considering a primitive sublattice $T_1 \subset T$ with two positive squares and restriction of Borcherds form on $\Omega(T_1)$, one can construct other examples. But one should be very careful because this restriction may have additional zeros to quadratic divisors orthogonal to elements with square -2 of T_1 (e. g. it might be identically 0). Considering this restriction, R. Borcherds found the form Φ with (2) for $T = U(2) \oplus U \oplus E_8(-2)$ (we denote $K(t)$ a lattice K with the form multiplied by $t \in \mathbb{Q}$). This case corresponds to moduli of K3 surfaces which cover twice Enriques surfaces. Considering orthogonal complement to $c \in U(2)$, we get the mirror symmetric family with $S = U \oplus E_8(-2)$. This corresponds to K3 surfaces with involution having the set of fixed points equals to union of two elliptic curves (see [N3] or [N8]). Both these cases are parabolic.

In our papers [GN1]—[GN6] we mainly considered the case when $\text{rk } S = 3$. Respectively, $\text{rk } T = 5$. In particular, for

$$T = 2U \oplus \langle -2t \rangle, \quad t = 1, 2, 3, 4, \quad (3)$$

and

$$T = 2U(k) \oplus \langle -2 \rangle, \quad k = 1, \dots, 8, 10, 12, 16, \quad (4)$$

we found 2-reflective automorphic forms Φ with expansions of type (2) for an isotropic c in the first summand U or $U(k)$ respectively. Then

$$S = c_T^{\perp} / \mathbb{Z}c = \begin{cases} U \oplus \langle -2t \rangle & \text{for } T = 2U \oplus \langle -2t \rangle, \\ U(k) \oplus \langle -2 \rangle & \text{for } T = 2U(k) \oplus \langle -2 \rangle. \end{cases} \quad (5)$$

Thus, families \mathcal{M}_S and $\mathcal{M}_{T \perp}$ are mirror symmetric families of K3 surfaces for our mirror symmetry. It seems, for many lattices S and T from (5) existence of Calabi–Yau manifolds with K3 fibers having these Picard and transcendental lattices is not known. Problems A, B and A', B' are very interesting for these S and T .

Physicists (we have mentioned them in Introduction) considered several examples when 2-reflective automorphic forms Φ take part in calculation of the Yukawa coupling (for

model B) and the quantum intersection pairing (for model A) when existence of Calabi–Yau manifolds fibred by the corresponding K3 surfaces was known.

T. Kawai in [Ka2] considered $S = U \oplus \langle -2 \rangle$ and $T = 2U \oplus \langle -2 \rangle$. For the model (A) he took Calabi–Yau 3-folds of degree 20 in the weighted projective space $\mathbb{P}(10, 3, 3, 2, 2)$. They are naturally fibred by K3 surfaces with the Picard lattice S . There are two 2-reflective automorphic forms Φ on $\Omega(T)$ with respect to $O^+(T)$. One of them is the classical Siegel modular form Δ_5 which is the product of even theta-constants. Another one is the famous Igusa modular form Δ_{35} which is the first Siegel modular form of odd weight. For both these forms we found infinite product expansions of type (2) in [GN1], [GN2], [GN4]. T. Kawai used combination of these forms for the quantum intersection pairing of the Calabi–Yau 3-folds above.

The Borcherds automorphic form for $T = U \oplus U(2) \oplus \langle -2 \rangle$ (we discussed it above) was recently used by J. Harvey and G. Moore in [HM3]. For model (B) they used Calabi–Yau 3-folds constructed by C. Borcea [Bo] and Cl. Voisin [V]. They are fibred by K3 surfaces with the transcendental lattice T .

One can suggest that: *The 2-reflective automorphic forms of type (2) used for the described above variant of mirror symmetry for K3 surfaces always take part in the quantum intersection pairing or Yukawa coupling of Calabi–Yau manifolds fibred by the corresponding K3 surfaces if the Calabi–Yau manifolds do exist.* Certainly, one need to make this conjecture much more concrete.

4. AN EXAMPLE OF ONE OF THE MOST BEAUTIFUL 2-REFLECTIVE AUTOMORPHIC FORMS IN DIMENSION 3

We finish with the most beautiful example of an automorphic form of type (2) from [GN6]. We give it for the lattice $T = 2U(12) \oplus \langle -2 \rangle$ (it is one of the lattices (4)).

In [GN6] we constructed an automorphic form Δ_1 of weight 1 with respect to the orthogonal group of $T = 2U(12) \oplus \langle -2 \rangle$. It has a character of order 6. We use basis f_2, \hat{f}_3, f_{-2} of the lattice $S = U(12) \oplus \langle -2 \rangle$ with the Gram matrix

$$\begin{pmatrix} 0 & 0 & 12 \\ 0 & -2 & 0 \\ 12 & 0 & 0 \end{pmatrix}$$

and corresponding coordinates z_1, z_2, z_3 . Then

$$\begin{aligned} \Delta_1(z_1, z_2, z_3) &= \sum_{M \geq 1} \sum_{\substack{m > 0, l \in \mathbb{Z} \\ n, m \equiv 1 \pmod{6} \\ 4nm - 3l^2 = M^2}} \begin{pmatrix} -4 \\ l \end{pmatrix} \begin{pmatrix} 12 \\ M \end{pmatrix} \sum_{a | (n, l, m)} \left(\frac{6}{a} \right) q^{n/6} r^{l/2} s^{m/6} \\ &= q^{1/6} r^{1/2} s^{1/6} \prod_{\substack{n, l, m \in \mathbb{Z} \\ (n, l, m) > 0}} (1 - q^n r^l s^m)^{f_3(n, m, l)} \end{aligned} \quad (6)$$

where $q = \exp(24\pi iz_1)$, $r = \exp(4\pi iz_2)$, $s = \exp(24\pi iz_3)$ and

$$\left(\frac{-4}{l}\right) = \begin{cases} \pm 1 & \text{if } l \equiv \pm 1 \pmod{4} \\ 0 & \text{if } l \equiv 0 \pmod{2} \end{cases}, \quad \left(\frac{12}{M}\right) = \begin{cases} 1 & \text{if } M \equiv \pm 1 \pmod{12} \\ -1 & \text{if } M \equiv \pm 5 \pmod{12} \\ 0 & \text{if } (M, 12) \neq 1 \end{cases}$$

$$\left(\frac{6}{a}\right) = \begin{cases} \pm 1 & \text{if } a \equiv \pm 1 \pmod{6} \\ 0 & \text{if } (a, 6) \neq 1 \end{cases}$$

The multiplicities $f_3(nm, l)$ of the infinite product are defined by a weak Jacobi form $\phi_{0,3}(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} f_3(n, l) q^n r^l$ of weight 0 and index 3 with integral Fourier coefficients:

$$\phi_{0,3}(\tau, z) = \left(\frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)}\right)^2 = r^{-1} \left(\prod_{n \geq 1} (1 + q^{n-1}r)(1 + q^n r^{-1})(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2})\right)^2$$

where $q = \exp(2\pi i\tau)$, $r = \exp(2\pi iz)$. The divisor of Δ_1 is sum with multiplicities one of all quadratic divisors orthogonal to elements of T with square -2 . The Δ_1 defines the generalized Lorentzian Kac–Moody superalgebra with the denominator function (6) (see [GN3] and [GN1]–[GN6] for details and other examples). Conjecturally the algebra is related with symmetries of some physical theory.

It seems, for the automorphic form Δ_1 existence of Calabi–Yau manifolds fibrated by K3 surfaces with the corresponding Picard lattice $S = U(12) \oplus (-2)$ or the transcendental lattice $T = 2U(12) \oplus (-2)$ are not known.

REFERENCES

- [Ba] W.L. Baily, *Fourier–Jacobi series*, Algebraic groups and discontinuous subgroups. Proc. Symp. Pure Math. Vol. IX (A. Borel and G.D. Mostow, eds.), Amer. Math. Soc., Providence, Rhode Island, 1966, pp. 296–300.
- [Bat] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, Alg. Geom. **3** (1994), 493–535.
- [Bo] C. Borcea, *K3-surfaces with involution and mirror pairs of Calabi–Yau manifolds*, Preprint (1992).
- [B1] R. Borcherds, *Generalized Kac–Moody algebras*, J. of Algebra **115** (1988), 501–512.
- [B2] R. Borcherds, *The monster Lie algebra*, Adv. Math. **83** (1990), 30–47.
- [B3] R. Borcherds, *The monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.
- [B4] R. Borcherds, *Sporadic groups and string theory*, Proc. European Congress of Mathem. 1992, pp. 411–421.
- [B5] R. Borcherds, *Automorphic forms on $O_{s+2,2}$ and infinite products*, Invent. Math. **120** (1995), 161–213.
- [B6] R. Borcherds, *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology **35** (1996), no. 3, 699–710.
- [B7] R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Preprint (1996); alg-geom/9609022.
- [COGP] P. Candelas, X.C. de la Ossa, P.S. Green and L. Parkers, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21–74.

- [CCL] G.L. Cardoso, G. Curio and D. Lüst, *Perturbative coupling and modular forms in $N = 2$ string models with a Wilson line*, Preprint (1996); hep-th/9608154.
- [DVV] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Counting dyons in $N = 4$ string theory*, Preprint (1996); hep-th/9607026.
- [Gi] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices (1996), no. 13, 613-663.
- [G1] V.A. Gritsenko, *Irrationality of the moduli spaces of polarized Abelian surfaces*, Int. Math. Res. Notices **6** (1994), 235-243, In full form in "Abelian varieties", Proc. of the Egloffstein conference (1993) de Gruyter, Berlin, 1995, pp. 63-81.
- [G2] V.A. Gritsenko, *Modulformen zur Paramodulgruppe und Modulräume der Abelschen Varietäten*, Mathematica Gottingensis Schrift. des SFB "Geometrie und Analysis", Heft **12** (1995), 1-89.
- [G3] V.A. Gritsenko, *Arithmetical lifting and its applications*, Number Theory. Proceedings of Paris Seminar 1992-93 (S. David, eds.), Cambridge Univ. Press, 1995, pp. 103-126.
- [G4] V.A. Gritsenko, *Modular forms and moduli spaces of Abelian and $K3$ surfaces*, Algebra i Analiz **6:6** (1994), 65-102; English transl. in St.Petersburg Math. Jour. **6:6** (1995), 1179-1208.
- [G5] V.A. Gritsenko, *Jacobi functions of n -variables*, Zap. Nauk. Sem. LOMI **168** (1988), 32-45; English transl. in J. Soviet Math. **53** (1991), 243-252.
- [G6] V.A. Gritsenko, *Induction in the theory of zeta-functions*, Algebra i Analiz **6:1** (1994), 2-60; English transl. in St.Petersburg Math. Jour. **6:1** (1995), 1-50.
- [G7] V.A. Gritsenko, *Jacobi functions and Euler products for Hermitian modular forms*, Zap. Nauk. Sem. LOMI **183** (1990), 77-123; English transl. in J. Soviet Math. **62** (1992), 2883-2914.
- [GN1] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of some Lorentzian Kac-Moody Lie algebras*, Amer. J. Math. (1997) (to appear); alg-geom/9504006.
- [GN2] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of a Lorentzian Kac-Moody algebra*, C. R. Acad. Sci. Paris Sér. A-B **321** (1995), 1151-1156.
- [GN3] V.A. Gritsenko, V.V. Nikulin, *$K3$ surfaces, Lorentzian Kac-Moody algebras and mirror symmetry*, Math. Res. Let. **3** (1996), no. 2, 211-229; alg-geom/9510008.
- [GN4] V.A. Gritsenko, V.V. Nikulin, *The Igusa modular forms and "the simplest" Lorentzian Kac-Moody algebras*, Matem. Sbornik **187** (1996), no. 11; alg-geom/9603010.
- [GN5] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part I*, Preprint RIMS Kyoto Univ. **RIMS-1116** (1996); alg-geom/9610022.
- [GN6] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part II*, Preprint RIMS Kyoto Univ. **RIMS-1122** (1996); alg-geom/9611028.
- [GN7] V.A. Gritsenko, V.V. Nikulin, *A lecture on arithmetic mirror symmetry and Calabi-Yau manifolds*, Preprint (1996); alg-geom/9612002.
- [HM1] J. Harvey, G. Moore, *Algebras, BPS-states, and strings*, Nucl. Physics. **B463** (1996), 315; hep-th/9510182.
- [HM2] J. Harvey, G. Moore, *On the algebras of BPS-states*, Preprint (1996); hep-th/9609017.
- [HM3] J. Harvey, G. Moore, *Exact gravitational threshold correction on the FHSV model*, Preprint (1996); hep-th/9611176.
- [HeM1] M. Henningson, G. Moore, *Counting curves with modular forms*, Preprint (1996); hep-th/9602154.
- [HeM2] M. Henningson, G. Moore, *Threshold corrections in $K3 \times T^2$ heterotic string compactifications*, Preprint (1996); hep-th/9608145.
- [K] V. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, 1990.
- [Ka1] T. Kawai $N = 2$ Heterotic string threshold correction, $K3$ surfaces and generalized Kac-Moody superalgebra, Phys. Lett. **B371** (1996), 59; hep-th/9512046.
- [Ka2] T. Kawai, *String duality and modular forms*, Preprint (1996); hep-th/9607078.
- [Ko] M. Kontsevich, *Enumeration of rational curves via toric actions*, The moduli spaces of curves (R. Dijkgraaf, C. Farber, G. van der Geer, eds.), Progress in Math. 129, Birkhäuser, 1995, pp. 335-368.
- [Ko-M] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525-562.

- [Ku] Vic. S. Kulikov, *Degenerations of K3 surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 1008–1042; English transl. in Math. USSR Izv. **11** (1977).
- [M] D.R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: A guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), no. 1, 223–247.
- [N1] V.V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, Trudy Moskov. Mat. Obshch. **37** (1979), 73–137; English transl. in Trans. Moscow Math. Soc. **38** (1980), no. 2.
- [N2] V.V. Nikulin, *Integral symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 111–177; English transl. in Math. USSR Izv. **14** (1980).
- [N3] V.V. Nikulin, *On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections*, Algebraic-geometric applications, Current Problems in Math. Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow (1981), 3–114; English transl. in J. Soviet Math. **22** (1983), 1401–1476.
- [N4] V.V. Nikulin, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 637–669; English transl. in Math. USSR Izv. **16** (1981).
- [N5] V.V. Nikulin, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 113–142; English transl. in Math. USSR Izv. **18** (1982).
- [N6] V.V. Nikulin, *Involutions of integral quadratic forms and their applications to real algebraic geometry*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 1; English transl. in Math. USSR Izv. **22** (1984), 99–172.
- [N7] V.V. Nikulin, *Surfaces of type K3 with finite automorphism group and Picard group of rank three*, Trudy Inst. Steklov **165** (1984), 113–142; English transl. in Proc. Steklov Math. Inst. **3** (1985).
- [N8] V.V. Nikulin, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proc. Int. Congr. Math. Berkeley 1986, vol. 1, pp. 654–669.
- [N9] V.V. Nikulin, *Basis of the diagram method for generalized reflection groups in Lobachevsky spaces and algebraic surfaces with nef anticanonical class*, Intern. J. of Mathem. **7** (1996), no. 1, 71–108; alg-geom/9405011.
- [N10] V.V. Nikulin, *Diagram method for 3-folds and its application to Kähler cone and Picard number of Calabi-Yau 3-folds. I*, Higher dimensional complex varieties: Proc. of Intern. Confer. held in Trento, Italy, June 15–24, 1994. (M. Andreatta, Th. Peternell, eds.), de Gruyter, 1996, pp. 261–328; alg-geom/9401010.
- [N11] V.V. Nikulin, *A lecture on Kac–Moody Lie algebras of the arithmetic type*, Preprint Queen’s University, Canada #1994-16, (1994); alg-geom/9412003.
- [N12] V.V. Nikulin, *Reflection groups in Lobachevsky spaces and the denominator identity for Lorentzian Kac–Moody algebras*, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 2, 73–106; English transl. in Izvestiya Math. **60** (1996), no. 2, 305–334; alg-geom/9503003.
- [N13] V.V. Nikulin, *The remark on discriminants of K3 surfaces moduli as sets of zeros of automorphic forms*, Preprint (1995); alg-geom/9512018.
- [N14] V.V. Nikulin, *K3 Surfaces with interesting groups of automorphisms*, Preprint (1997); alg-geom/9701011.
- [P-ŠŠ] I.I. Pjatetckij-Šapiro, I.R. Šafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572; English transl. in Math. USSR Izv. **5** (1971).
- [RT] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, Math. Res. Lett. **1** (1994), no. 1, 269–278.
- [V] Cl. Voisin, *Miroirs et involutions sur les surfaces K3*, Journées de géométrie algébrique d’Orsay, juillet 1992 (A. Beauville, O. Debarre, Y. Laslo, eds.), Astérisque 218, 1993.

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