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Abstract

We give an elementary introduction to lattice calculations of the QCD equation of state and briefly review results for the case of two light flavors [1, 2].

1 Introduction and Background

Lattice simulations have shown for some time that ordinary hadronic matter at zero temperature undergoes a dramatic crossover characterized by large increases in the entropy and energy densities of the system [3] when the temperature is raised to about 150 MeV. In

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the thermodynamic limit, this crossover may become a phase transition to a new state of matter, the quark-gluon plasma(QGP). The equation of state (EOS), or energy density and pressure as a function of the temperature, is important input for phenomenological models of upcoming heavy-ion collision experiments at RHIC that seek to detect the QGP. Because the phase transition occurs at relatively low temperature, a nonperturbative method is required for first principles calculations using QCD.

Generally, thermodynamic quantities are given by derivatives of the partition function. In particular, the energy density ε and pressure p are given by

$$\varepsilon = -\frac{1}{V}\frac{\partial \log Z}{\partial(1/T)}\bigg|_V \quad \text{and} \quad p = T\frac{\partial \log Z}{\partial V}\bigg|_T, \tag{1}$$

where V and T are the spatial volume and the temperature of the system. The partition function is given by a path integral over all possible field configurations of the Boltzmann weight,

$$Z = \int [d\mathcal{A}, d\bar{\psi}, d\psi] e^{-S_E}, \tag{2}$$

where the Euclidean space-time action is given by

$$S_E = \int_0^{\tau} d\tau' \int_V d^3\vec{x}\mathcal{L}_E(\mathcal{A}(\vec{x},\tau'), \bar{\psi}(\vec{x},\tau'), \psi(\vec{x},\tau')), \tag{3}$$

and the integral over Euclidean time is cut off at time τ. The above path integral corresponds to the thermodynamic partition function if τ is identified as the inverse temperature, and the boundary conditions on the (fermion) boson fields are chosen to be (anti-) periodic.

To regularize the theory nonperturbatively, S_E is discretized on a four dimensional space-time lattice (see Fig. 1) with spacing a, so the continuum derivatives become finite differences, and the integral over space-time becomes a sum over all lattice sites. In the limit that $a \to 0$, the classical continuum action is recovered. The lattice spacing disappears from the lattice action; the extra factors of a are absorbed into the fields and the quark mass to make them dimensionless. Therefore the lattice spacing, or cut off, is varied implicitly by changing the bare gauge coupling, $6/g^2$, and the bare quark mass am_q. Explicitly,

$$S_E = \int d\tau \int_V d^3\vec{x}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\not{D} + m)\psi\right) \to \sum_{sites}\sum_{\mu>\nu}\left(\frac{6}{g^2}\,\mathrm{Re\,Tr}\,\Box_{\mu\nu}\right) + \bar{\psi}\mathrm{M}\psi. \tag{4}$$

The gauge fields, $U_\mu(x)$, live on the links of the lattice to maintain exact gauge invariance. They are elements of the group SU(3) and are related to their continuum counterparts by a simple exponential relation,

$$U_\mu(x) \;=\; \exp\{iga_\mu A_\mu(x)\} \approx \exp\{ig\int_x^{x+a_\mu} dy\mathcal{A}_\mu(y)\}. \tag{5}$$

The quadratic part of the continuum gauge lagrangian then becomes the trace of the path ordered product of gauge links around an elementary plaquette,

$$\Box_{\mu\nu}(x) \;=\; \frac{1}{3}\,\mathrm{Tr}\,U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x), \tag{6}$$

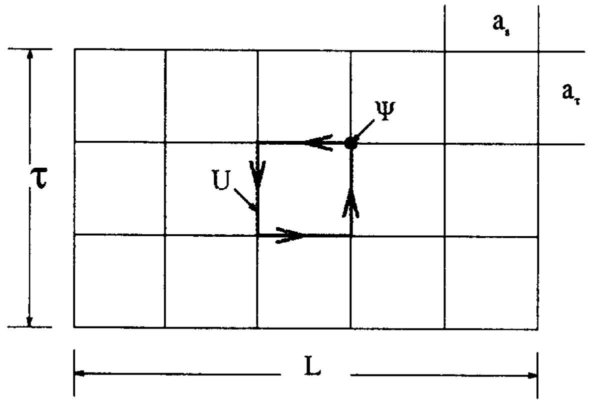

Figure 1: The four dimensional Euclidean space-time lattice.

which yields the standard Wilson gauge action given by the RHS of Eq. 4. The Dirac action is constructed by replacing the the $\not{D}$ operator with a finite difference operator. For Kogut-Susskind (KS), or staggered fermions, the quark fields are transformed to a spin diagonal basis which mixes the spin and flavor degrees of freedom in a complicated way, and all spin components but one are thrown away. In the limit $a \to 0$, the continuum action for *four* degenerate Dirac fermions is recovered. The fermion matrix for KS quarks is

$$\mathrm{M} = 2(am_q)\delta_{x,y} + \sum_{\mu}(\eta_\mu(x)U_\mu(x)\delta_{x+a\hat{\mu},y} - \eta_\mu(y)U^\dagger_\mu(y)\delta_{x-\hat{\mu},y}), \quad (7)$$

where the $\eta_\mu' s$ are the KS phases which correspond to the Dirac $\gamma_\mu' s$ in the spin diagonal basis.

Once the lattice action has been constructed, it is straightforward to calculate observables. Integrating over the quark fields (which are Grassman variables), we obtain

$$\langle \mathcal{O} \rangle = Z^{-1} \int [dU] \mathcal{O} \det((\mathrm{M}(U)))^{n_f/4} e^{-S_g} \quad \text{where} \quad Z = \int [dU] \det((\mathrm{M}(U)))^{n_f/4} e^{-S_g}, \quad (8)$$

for any observable. Thus $\det((\mathrm{M}(U)))^{n_f/4} e^{-S_g}$ serves as a probability weight for n_f flavors of quarks, and the remaining path integral over gauge fields is done by Monte Carlo simulation. Using importance sampling to generate the gauge field configurations with the desired weight, observables become simple averages over the configurations.

Now, let's return to discussing thermodynamics on the lattice where the volume and inverse temperature are $V = N_s^3 a_s^3$ and $T^{-1} = N_t a_t$. V and T are varied by changing the number of lattice sites N_s, N_t, or the lattice spacings a_s, a_t, or both. To simulate at finite temperature in the continuum, the prescription was to cut off the Euclidean time integral in the action at $\tau = T^{-1}$. This is accomplished on the lattice by taking $N_t \ll N_s$ for the usual case when $a_t = a_s$. This sets the overall temperature scale. To vary the temperature around this scale, we vary the lattice spacing by adjusting $6/g^2$ and am_q. For small am_q increasing $6/g^2$ is roughly equivalent to raising the temperature. However one should keep in mind that varying am_q also changes the lattice spacing and thus the temperature. Derivatives with respect to T^{-1} and V are most easily obtained by adjusting the couplings. Then, the interaction measure, or energy density minus three times the pressure is

$$\frac{\varepsilon - 3p}{T^4} = \left(-\frac{1}{V}\frac{\partial}{\partial(1/T)} - 3T\frac{\partial}{\partial V}\right)\log Z = -N_t^4\left(\frac{\partial(6/g^2)}{\partial \ln(a)}\langle\Box\rangle + \frac{\partial(am_q)}{\partial \ln(a)}\left\langle\bar{\psi}\psi\right\rangle\right), \tag{9}$$

where the rightmost expression is for KS quarks and the Wilson action, and

$$\langle\Box\rangle = \frac{1}{N_s^3 N_t}\frac{\partial \log Z}{\partial(6/g^2)} \quad \text{and} \quad \left\langle\bar{\psi}\psi\right\rangle = \frac{1}{N_s^3 N_t}\frac{\partial \log Z}{\partial(m_q a)} = \frac{n_f}{4}\frac{2}{N_s^3 N_t}\langle \mathrm{Tr}\, M^{-1}\rangle \tag{10}$$

are the derivatives of $\log Z$ with respect to $6/g^2$ and am_q. The definition of $\Box$ is given in Eq. 6, and the form of $\bar{\psi}\psi$ results from exponentiating the quark determinant. The derivatives of $6/g^2$ and am_q with respect to $\log(a)$ are just the β function and anomalous dimension of the quark mass which have been calculated nonperturbatively from spectrum data in the literature [1]. It is important to use the nonperturbative β function since it differs by roughly a factor of two from the perturbative result in the region of $6/g^2$ used for present simulations.

The pressure is obtained from

$$\frac{p}{T} = \frac{\partial \log Z}{\partial V} = \frac{\log Z}{V}, \tag{11}$$

which is just a restatement of the fact that the free energy density is independent of volume for large volumes. The derivatives of $\log Z$, given by Eqs. 6 and 10, are calculated rather than Z itself. These are integrated numerically with respect to $6/g^2$ and am_q to obtain the pressure.

Together, the interaction measure and the pressure form the equation of state. The energy density (and pressure) can be calculated directly from an expression similar to Eq. 9, but with derivatives of the couplings with respect to the temporal (spatial) lattice spacing at fixed spatial (temporal) lattice spacing. These derivatives are much harder to measure, and in fact have not been calculated nonperturbatively for QCD with $n_f \neq 0$.

It is important to note that the physical energy and pressure are given after subtracting off their divergent vacuum, or $T = 0$, contributions. In practice this is done by subtracting the same quantities measured on a symmetric lattice ($N_t = N_s$) from quantities measured on the finite temperature, or asymmetric, lattice ($N_t < N_s$). These are often referred to (unfortunately) as cold and hot lattices, respectively. This fact more than doubles the cost of calculating the EOS since each point in the phase diagram requires two lattices.

Finally, to determine the temperature from $T^{-1} = N_t a$, the lattice spacing is given by by setting one observable to its physical value. For example, in the results discussed below the (zero temperature) rho mass measured in units of a is used, or $am_\rho = a \times 770$. Since simulations are done in the strong coupling regime, there are scaling violations. In other words, using the nucleon mass results in a different a, and therefore a different temperature. While no one has done a detailed analysis, the conventional wisdom is that these scaling violations are on the order of ten percent. Also, typically thermodynamic quantities are normalized by T^n by simply multiplying by N_t^n; for example, $pa^4 \times N_t^4 = p/T^4$. While this also yields physical numbers, the values will show scaling violations depending on N_t (see below). All of these systematic errors disappear in the continuum limit.

2 Results for $n_f = 2$

Below we briefly review recent calculations of the two flavor QCD EOS at zero chemical potential by the MILC collaboration [1, 2]. Most simulations with dynamical fermions have used two light flavors because doing so cuts the computational burden in half over simulations with two light and one strange flavor. Since previous simulations have shown the critical temperature T_c to be roughly 150 MeV, it is reasonable to assume that the dynamics near T_c is governed mainly by pions, and thus two flavor simulations should capture the bulk of the physics. Fine details, like the order and universality class of the transiton may of course depend on the presence of the strange quark. Also, presently there is no viable method of simulating QCD at finite chemical potential; however at RHIC, the baryon number density in the central rapidity region is expected to be small (see the contribution from J. Harris). Simulations for the EOS are also limited by computational resources to relatively small lattices ($N_t = 4, 6$). Results for the pure SU(3) gauge theory and preliminary calculations with four flavors of quarks have been obtained by the Bielefeld group following a similar approach, and are also discussed in this volume by Karsch.

In Fig. 2(a) the interaction measure for $N_t = 6$ is shown as a function of $6/g^2$. The two curves in the figure correspond to quark masses $am_q = 0.0125$ and $am_q = 0.025$, and up to an overall shift in the coupling, appear quite similar. The interaction measure rises sharply through the crossover region (it is normalized to zero at $T = 0$) and then decreases towards zero at large $6/g^2$ (high temperature).

In Fig. 2(b) we show the $N_t = 6$ EOS as a function of the temperature. Again, there is a rapid rise in ε/T^4 which levels off around 160 MeV. The energy density is about 1 GeV/fm^3 at this point. The pressure rises smoothly through the crossover region but has not leveled off at the highest temperature simulated. Also shown in Fig. 2(b) is an earlier calculation on $N_t = 4$ lattices. There is a large finite size correction when N_t is increased from 4 to 6. This is expected from the lattice Stefan-Boltzmann results (see Fig. 2(b)). It appears that the approach to these expected asymptotic values is quite slow.

From the location of the maximum in the slope of $\langle\Box\rangle$ or $\langle\bar{\psi}\psi\rangle$ with respect to $6/g^2$, the pseudocritical temperature of the transition is roughly 140 MeV (for both $am_q = 0.025$ and 0.0125) [2]. Figure 2(b) shows that the energy density is already substantial at this point.

The light quark mass is too large in the above simulations, in the sense that the pion mass is two to four times its physical value (simulations corresponding to the physical pions would require extremely large amounts of computer time on present computers). Thus results must be extrapolated to $m_q \approx 0$. Such an extrapolation for the EOS is shown in Fig. 3(a), where a second order phase transition in the limit $m_q \to 0$ has been assumed. Until recently, lattice simulations and universality arguments [5] have indicated that in the limit $m_q \to 0$ two flavor QCD probably exhibits a second order phase transition in the same universality class as the 3d O(4) spin model [5]. However, new lattice simulations on large volumes have cast some doubt on both conclusions (see A. Ukawa's contribution to this volume). The results displayed in Fig. 3 are from a fit to an O(4) universal scaling function [6] (plus polynomial terms), although the data are fit equally well to the mean field scaling function. The appearence of the bump in the energy density just after the transition is likely an artifact of the extrapolation, although it need not be [4]. Ignoring the bump, the extrapolated EOS is quantitatively similar to the $am_q = 0.025$ and 0.0125 results: the results depend weakly

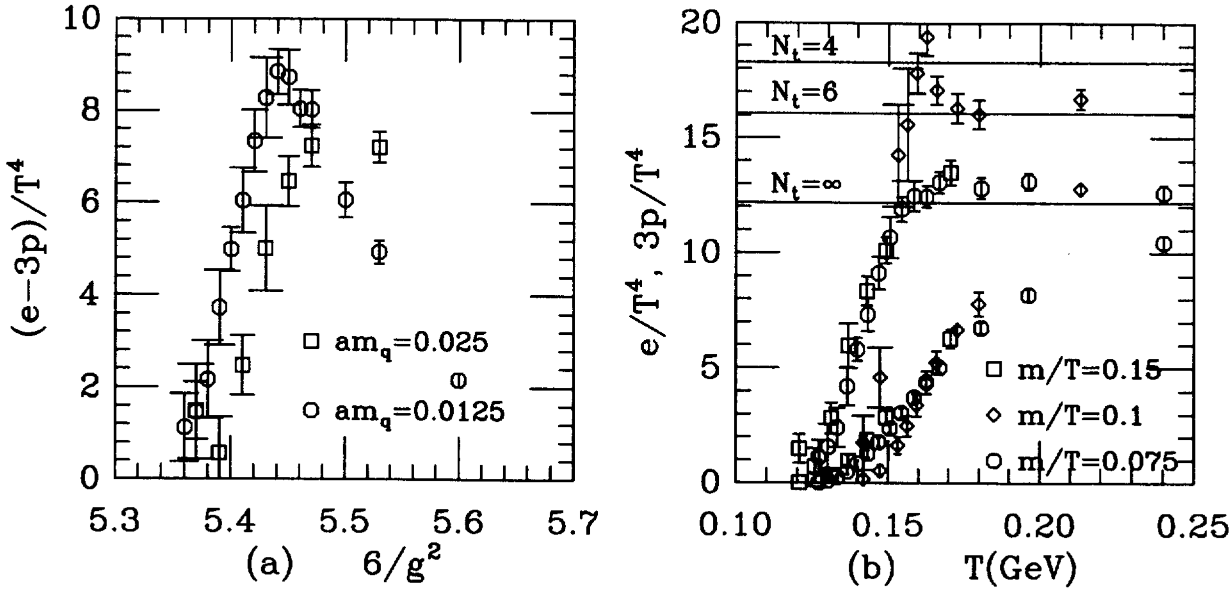

Figure 2: (a) The interaction measure on $N_t = 6$ lattices. (b) The EOS. The horizontal lines are Stefan-Boltzmann values for ε/T^4. The diamonds indicate an earlier result on $N_t = 4$ lattices.

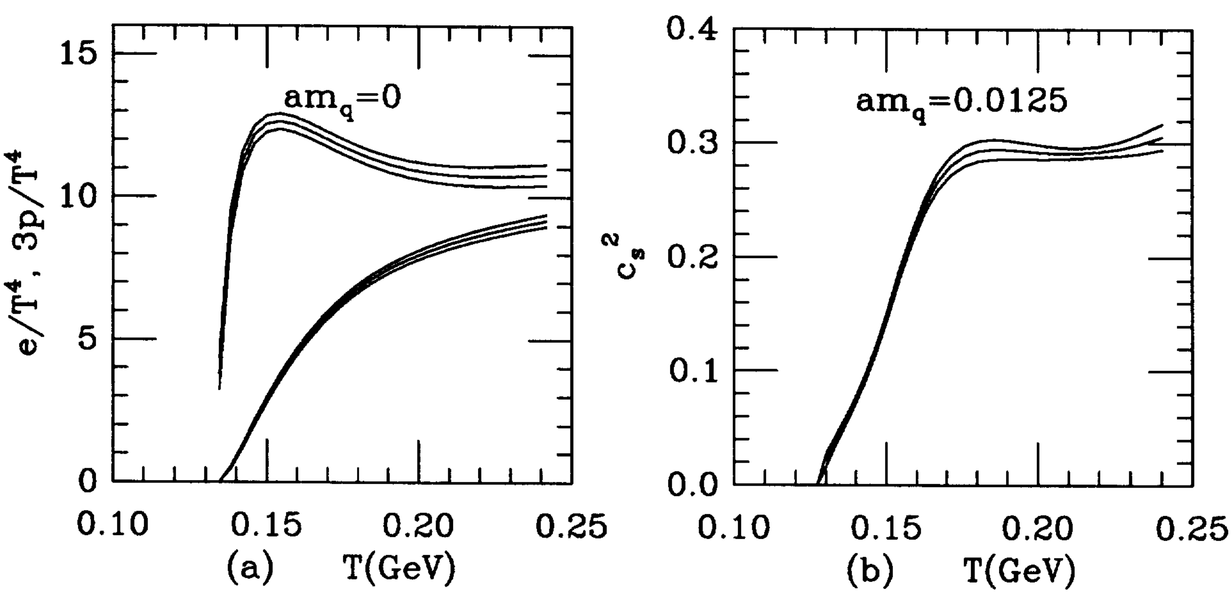

Figure 3: (a) The EOS extrapolated to $am_q = 0.0$ from a fit to the O(4) universal scaling function plus polynomial terms. Each set of curves indicates the central value and a one standard deviation spread resulting from the statistical uncertainty in the fit. (b) The speed of sound for $am_q = 0.0125$ from the same fit. Both figures are taken from Ref. [2].

on the quark mass. The gap in ε/T^4 at low temperature is due to a breakdown in the fit which, for $am_q = 0$, corresponds to an extrapolation in $6/g^2$.

The smooth interpolation of the data as a function of T also allows for a determination of the speed of sound, $c_s^2 = (dp/dT)/(d\varepsilon/dT)$, shown in Fig. 3(b). The speed of sound has important experimental implications for the detection of the quark-gluon plasma (see M. Gyulassy, these proceedings). At the transition we expect c_s to be small since $d\varepsilon/dT \gg dp/dT$. However, just below T_c, c_s should approach 1/3, the value for a relativistic free gas of massless pions (for massive interacting pions the value will be less than 1/3), and above T_c, c_s should again approach 1/3 if the system is a weakly interacting relativistic plasma. From Fig. 3(b), the second expectation is more or less borne out from the simulations. Unfortunately, while c_s is small near T_c, in the hadronic phase there is no indication of an increase, and thus no dip. Again, this is because of the difficulty in measuring ε and p, and thus their derivatives, in the low temperature hadronic phase.

References

[1] T. Blum, S. Gottlieb, L. Kärkkäinen, and D. Toussaint, Nucl. Phys. B (PS) **42** (1995) 460; T. Blum, S. Gottlieb, L. Kärkkäinen, and D. Toussaint, Phys. Rev. D. **51** (1995) 5153.

[2] C. Bernard *et al*, Nucl. Phys. B (PS) **47** (1995) 503; C. Bernard *et al*, "Thermodynamics for two flavor QCD", to appear in Nucl. Phys. B (PS), hep-lat 9608026.

[3] See annual reviews in the Proceedings of the International Symposium on Lattice Field Theory, Nucl. Phys. B (PS), and references therein.

[4] M. Asakawa and T. Hatsuda, hep-ph/9508360.

[5] For universality arguements, see R.D. Pisarski and F. Wilczek, Phys. Rev. D **29** (1984) 338; K. Rajagopal and F. Wilczek, Nucl. Phys. B**399** (1993) 395; and K. Rajagopal, this workshop. For lattice results see E. Laermann and F. Karsch, Phys. Rev. D **50** (1994) 6954; C. DeTar, in Quark Gluon Plasma 2, R. Hwa, Ed., World Scientific, 1995; Y. Iwasaki, K. Kanaya, S. Kaya, T. Yoshie, "Scaling of Chiral Order Parameter in Two-Flavor QCD", hep-lat/9609022.

[6] D. Toussaint, University of Arizona preprint AZPH-TH/96-13 (1996).