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DEFORMED u(2) ALGEBRA AS THE SYMMETRY ALGEBRA OF THE PLANAR ANISOTROPIC QUANTUM HARMONIC OSCILLATOR WITH RATIONAL RATIO OF FREQUENCIES

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Abstract

The symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies is identified as a deformation of the u(2) algebra. The finite dimensional representation modules of this algebra are studied and the energy eigenvalues are determined using algebraic methods of general applicability to quantum superintegrable systems. For labelling the degenerate states an "angular momentum" operator is introduced, the eigenvalues of which are roots of appropriate generalized Hermite polynomials. The cases with frequency ratios 1:n correspond to generalized parafermionic oscillators, while in the special case with frequency ratio 2:1 the resulting algebra corresponds to the finite W algebra $W_3^{(2)}$.

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Abstract

The symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies is identified as a deformation of the u(2) algebra. The finite dimensional representation modules of this algebra are studied and the energy eigenvalues are determined using algebraic methods of general applicability to quantum superintegrable systems. For labelling the degenerate states an "angular momentum" operator is introduced, the eigenvalues of which are roots of appropriate generalized Hermite polynomials. The cases with frequency ratios 1:n correspond to generalized parafermionic oscillators, while in the special case with frequency ratio 2:1 the resulting algebra corresponds to the finite W algebra $W_3^{(2)}$.

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1 Introduction

Quantum algebras [1,2] (also called quantum groups) are nonlinear deformations of the corresponding Lie algebras, to which they reduce when the deformation parameter is set equal to unity. The interest in their possible applications in physics was triggered by the introduction of the q-deformed harmonic oscillator in 1989 [3-5] as a tool for providing a boson realization of the quantum algebra $su_q(2)$, although similar mathematical structures had already been known [6,7]. By now several kinds of generalized deformed oscillators (see [8-12] and references therein) and generalized nonlinear deformed su(2) algebras [13-19] have been introduced, finding applications in a variety of physical problems.

On the other hand the two-dimensional [20-25] and three-dimensional [26-32] anisotropic harmonic oscillator have been the subject of several investigations, both at the classical and the quantum mechanical level. These oscillators are examples of superintegrable systems [33]. The special cases with frequency ratios 1:2 [34,35] and 1:3 [36] have also been considered. While at the classical level it is clear that the su(N) or sp(2N,R) algebras can be used for the description of the N-dimensional anisotropic oscillator, the situation at the quantum level, even in the two-dimensional case, is not as simple.

In this paper we are going to prove that a generalized deformed u(2) algebra is the symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator, which is the oscillator describing the single-particle level spectrum of "pancake" nuclei, i.e. of triaxially deformed nuclei with $\omega_x >> \omega_y, \omega_z$ [37].

2 The deformed u(2) algebra

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2}\left({p_x}^2 + {p_y}^2 + \frac{x^2}{m^2} + \frac{y^2}{n^2}\right), \tag{1}$$

where m and n are two natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m,n) = 1$.

We define the creation and annihilation operators [20]

$$a^\dagger = \tfrac{x/m - ip_x}{\sqrt{2}}, \quad a = \tfrac{x/m + ip_x}{\sqrt{2}},$$
$$b^\dagger = \tfrac{y/n - ip_y}{\sqrt{2}}, \quad b = \tfrac{y/n + ip_y}{\sqrt{2}}. \tag{2}$$

These operators satisfy the commutation relations:

$$\left[a, a^\dagger\right] = \frac{1}{m}, \quad \left[b, b^\dagger\right] = \frac{1}{n}, \quad \text{other commutators} = 0. \tag{3}$$

Using Eqs (2) and (3) we can prove by induction that:

$$\left[a, (a^\dagger)^p\right] = \tfrac{p}{m}\,(a^\dagger)^{p-1}, \quad \left[b, (b^\dagger)^p\right] = \tfrac{p}{n}\,(b^\dagger)^{p-1},$$
$$\left[a^\dagger, (a)^p\right] = -\tfrac{p}{m}\,(a)^{p-1}, \quad \left[b^\dagger, (b)^p\right] = -\tfrac{p}{n}\,(b)^{p-1}. \tag{4}$$

Defining

$$U = \frac{1}{2}\left\{a, a^\dagger\right\}, \qquad W = \frac{1}{2}\left\{b, b^\dagger\right\}, \tag{5}$$

one can easily prove that:

$$\left[U, (a^\dagger)^p\right] = \tfrac{p}{m}\,(a^\dagger)^p, \quad \left[W, (b^\dagger)^p\right] = \tfrac{p}{n}\,(b^\dagger)^p,$$
$$\left[U, (a)^p\right] = -\tfrac{p}{m}\,(a)^p, \quad \left[W, (b)^p\right] = -\tfrac{p}{n}\,(b)^p. \tag{6}$$

Using the above properties we can define the enveloping algebra generated by the operators:

$$S_+ = (a^\dagger)^m\,(b)^n, \quad S_- = (a)^m\,(b^\dagger)^n,$$
$$S_0 = \tfrac{1}{2}(U - W), \quad H = U + W. \tag{7}$$

These genarators satisfy the following relations:

$$[S_0, S_\pm] = \pm S_\pm, \quad [H, S_i] = 0, \quad \text{for} \quad i = 0, \pm, \tag{8}$$

and

$$S_+S_- = \prod_{k=1}^{m}\left(U - \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(W + \frac{2\ell-1}{2n}\right), \tag{9}$$

$$S_-S_+ = \prod_{k=1}^{m}\left(U + \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(W - \frac{2\ell-1}{2n}\right). \tag{10}$$

The fact that the operators S_i, $i = 0, \pm$ are integrals of motion has been already realized in [20].

The above relations mean that the harmonic oscillator of Eq. (1) is described by the enveloping algebra of the generalization of the u(2) algebra formed by the generators S_0, S_+, S_- and H, satisfying the commutation relations of Eq. (8) and

$$[S_-, S_+] = F_{m,n}(H, S_0 + 1) - F_{m,n}(H, S_0),$$
$$\text{where} \quad F_{m,n}(H, S_0) = \prod_{k=1}^{m}\left(H/2 + S_0 - \tfrac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(H/2 - S_0 + \tfrac{2\ell-1}{2n}\right). \tag{11}$$

In the case of $m = 1$, $n = 1$ this algebra is the usual u(2) algebra, and the operators $S_0, S_\pm$ satisfy the commutation relations of the ordinary u(2) algebra, since in this case one easily finds that

$$[S_-, S_+] = -2S_0. \tag{12}$$

In the rest of the cases, the algebra is a deformed version of u(2), in which the commutator $[S_-, S_+]$ is a polynomial of S_0 of order $m + n - 1$. In the case with $m = 1$, $n = 2$ one has

$$[S_-, S_+] = 3S_0^2 - HS_0 - \frac{H^2}{4} + \frac{3}{16}, \tag{13}$$

i.e. a polynomial quadratic in S_0 occurs, while in the case of $m = 1$, $n = 3$ one finds

$$[S_-, S_+] = -4S_0^3 + 3HS_0^2 - \frac{7}{9}S_0 - \frac{H^3}{4} + \frac{H}{4}, \tag{14}$$

i.e. a polynomial cubic in S_0 is obtained.

3 The representations

The finite dimensional representation modules of this algebra can be found using the concept of the generalized deformed oscillator [8], in a method similar to the one used in [38] for the study of quantum superintegrable systems. The operators:

$$\mathcal{A}^\dagger = S_+, \quad \mathcal{A} = S_-, \quad \mathcal{N} = S_0 - u, \quad u = \text{constant}, \tag{15}$$

where u is a constant to be determined, are the generators of a deformed oscillator algebra:

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^\dagger\mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A}\mathcal{A}^\dagger = \Phi(H, \mathcal{N}+1). \tag{16}$$

The structure function Φ of this algebra is determined by the function $F_{m,n}$ in Eq. (11):

$$\Phi(H,\mathcal{N}) = F_{m,n}(H,\mathcal{N}+u) = \\ = \prod_{k=1}^{m}\left(H/2+\mathcal{N}+u-\tfrac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(H/2-\mathcal{N}-u+\tfrac{2\ell-1}{2n}\right). \tag{17}$$

The deformed oscillator corresponding to the structure function of Eq. (17) has an energy dependent Fock space of dimension $N+1$ if

$$\Phi(E,0)=0,\quad \Phi(E,N+1)=0,\quad \Phi(E,k)>0,\quad \text{for}\quad k=1,2,\ldots,N. \tag{18}$$

The Fock space is defined by:

$$H|E,k>=E|E,k>,\quad \mathcal{N}|E,k>=k|E,k>,\quad a|E,0>=0, \tag{19}$$

$$\mathcal{A}^\dagger|E,k>=\sqrt{\Phi(E,k+1)}|E,k+1>,\quad \mathcal{A}|E,k>=\sqrt{\Phi(E,k)}|E,k-1>. \tag{20}$$

The basis of the Fock space is given by:

$$|E,k>=\frac{1}{\sqrt{[k]!}}\left(\mathcal{A}^\dagger\right)^k|E,0>,\quad k=0,1,\ldots N, \tag{21}$$

where the "factorial" $[k]!$ is defined by the recurrence relation:

$$[0]!=1,\quad [k]!=\Phi(E,k)[k-1]! \ . \tag{22}$$

Using the Fock basis we can find the matrix representation of the deformed oscillator and then the matrix representation of the algebra of Eqs (8), (11). The solution of Eqs (18) implies the following pairs of permitted values for the energy eigenvalue E and the constant u:

$$E = N+\frac{2p-1}{2m}+\frac{2q-1}{2n}, \tag{23}$$

where $p=1,2,\ldots,m$, $q=1,2,\ldots,n$, and

$$u=\frac{1}{2}\left(\frac{2p-1}{2m}-\frac{2q-1}{2n}-N\right), \tag{24}$$

the corresponding structure function being given by:

$$\Phi(E,x)=\Phi^N_{(p,q)}(x)= \\ =\prod_{k=1}^{m}\left(x+\frac{2p-1}{2m}-\frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(N-x+\frac{2q-1}{2n}+\frac{2\ell-1}{2n}\right) \\ =\frac{1}{m^m n^n}\frac{\Gamma(mx+p)}{\Gamma(mx+p-m)}\frac{\Gamma((N-x)n+q+n)}{\Gamma((N-x)n+q)}, \tag{25}$$

where $\Gamma(x)$ denotes the usual Gamma-function. In all these equations one has $N=0,1,2,\ldots$, while the dimensionality of the representation is given by $N+1$. Eq. (23) means that there are $m\cdot n$ energy eigenvalues corresponding to each N value, each eigenvalue having degeneracy $N+1$. (Later we shall see that the degenerate states corresponding to the same eigenvalue can be labelled by an "angular momentum".)

It is useful to show at this point that a few special cases are in agreement with results already existing in the literature.

i) In the case $m=1$, $n=1$ Eq. (13) gives

$$\Phi(E,x)=x(N+1-x), \tag{26}$$

while Eq. (12) gives

$$E=N+1, \tag{27}$$

in agreement with Sec. IV.A of [38].

ii) In the case $m=1$, $n=2$ one obtains for $q=2$

$$\Phi(E,x)=x(N+1-x)\left(N+\frac{3}{2}-x\right),\qquad E=N+\frac{5}{4}, \tag{28}$$

while for $q=1$ one has

$$\Phi(E,x)=x(N+1-x)\left(N+\frac{1}{2}-x\right),\qquad E=N+\frac{3}{4}. \tag{29}$$

These are in agreement with the results obtained in Sec. IV.F of [38] for the Holt potential (for $\delta=0$).

iii) In the case $m=1$, $n=3$ one has for $q=1$

$$\Phi(E,x)=x(N+1-x)\left(N+\frac{1}{3}-x\right)\left(N+\frac{2}{3}-x\right),\qquad E=N+\frac{2}{3}, \tag{30}$$

while for $q=2$ one obtains

$$\Phi(E,x)=x(N+1-x)\left(N+\frac{2}{3}-x\right)\left(N+\frac{4}{3}-x\right),\qquad E=N+1, \tag{31}$$

and for $q=3$ one gets

$$\Phi(E,x)=x(N+1-x)\left(N+\frac{4}{3}-x\right)\left(N+\frac{5}{3}-x\right),\qquad E=N+\frac{4}{3}. \tag{32}$$

These are in agreement with the results obtained in Sec. IV.D of [38] for the Fokas–Lagerstrom potential.

In all of the above cases we remark that the structure function has the form

$$\Phi(x) = x(N+1-x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots), \tag{33}$$

which corresponds to a generalized deformed parafermionic algebra [39] of order N, if λ, μ, ν, ρ, σ, …, are real constants satisfying the conditions

$$\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \qquad x \in \{1, 2, \ldots, N\}. \tag{34}$$

These conditions are indeed satisfied in all cases. It is easy to see that the obtained algebra corresponds to this of the generalized parafermionic oscillator in all cases with frequency ratios $1 : n$.

The energy formula can be corroborated by using the corresponding Schrödinger equation. For the Hamiltonian of Eq. (1) the eigenvalues of the Schrödinger equation are given by:

$$E = \frac{1}{m}\left(n_x + \frac{1}{2}\right) + \frac{1}{n}\left(n_y + \frac{1}{2}\right), \tag{35}$$

where $n_x = 0, 1, \ldots$ and $n_y = 0, 1, \ldots$. Comparing Eqs (23) and (35) one concludes that:

$$N = [n_x/m] + [n_y/n], \tag{36}$$

where $[x]$ is the integer part of the number x, and

$$p = \text{mod}(n_x, m) + 1, \quad q = \text{mod}(n_y, n) + 1. \tag{37}$$

The eigenvectors of the Hamiltonian can be parametrized by the dimensionality of the representation N, the numbers p, q, and the number $k = 0, 1, \ldots, N$. k can be identified as $[n_x/m]$. One then has:

$$H \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \left(N + \frac{2p-1}{2m} + \frac{2q-1}{2n}\right) \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle, \tag{38}$$

$$S_0 \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \left(k + \frac{1}{2}\left(\frac{2p-1}{2m} - \frac{2q-1}{2n} - N\right)\right) \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle, \tag{39}$$

$$S_+ \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k+1)} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k+1 \right\rangle, \tag{40}$$

$$S_- \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k)} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k-1 \right\rangle. \tag{41}$$

4 The "angular momentum" quantum number

It is worth noticing that the operators $S_0, S_\pm$ do not correspond to a generalization of the angular momentum, S_0 being the operator corresponding to the Fradkin operator $S_{xx} - S_{yy}$ [40,41]. The corresponding "angular momentum" is defined by:

$$L_0 = -i\,(S_+ - S_-). \tag{42}$$

The "angular momentum" operator commutes with the Hamiltonian:

$$[H, L_0] = 0. \tag{43}$$

Let $|\ell>$ be the eigenvector of the operator L_0 corresponding to the eigenvalue ℓ. The general form of this eigenvector can be given by:

$$|\ell> = \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle. \tag{44}$$

In order to find the eigenvalues of L_0 and the coefficients c_k we use the Lanczos algorithm [42], as formulated in [43]. From Eqs (40) and (41) we find

$$\begin{aligned} L_0|\ell> &= \ell|\ell> = \ell \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle = \\ &= \frac{1}{i} \sum_{k=0}^{N-1} \frac{i^k c_k \sqrt{\Phi^N_{(p,q)}(k+1)}}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k+1 \right\rangle - \frac{1}{i} \sum_{k=1}^{N} \frac{i^k c_k \sqrt{\Phi^N_{(p,q)}(k)}}{\sqrt{[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k-1 \right\rangle \end{aligned} \tag{45}$$

From this equation we find that:

$$c_k = (-1)^k 2^{-k/2} H_k(\ell/\sqrt{2})/\mathcal{N}, \quad \mathcal{N}^2 = \sum_{n=0}^{N} 2^{-n} H_n^2(\ell/\sqrt{2})/n! \tag{46}$$

where the function $H_k(x)$ is a generalization of the "Hermite" polynomials (see also [44,45]), satisfying the recurrence relations:

$$H_{-1}(x) = 0, \quad H_0(x) = 1, \tag{47}$$

$$H_{k+1}(x) = 2x H_k(x) - 2\Phi^N_{(p,q)}(k) H_{k-1}(x), \tag{48}$$

and the "angular momentum" eigenvalues ℓ are the roots of the polynomial equation:

$$H_{N+1}(\ell/\sqrt{2}) = 0. \tag{49}$$

Therefore for a given value of N there are $N+1$ "angular momentum" eigenvalues ℓ, symmetric around zero (i.e. if ℓ is an "angular momentum" eigenvalue, then $-\ell$ is also an "angular momentum" eigenvalue). In the case of the symmetric harmonic oscillator ($m/n = 1/1$) these eigenvalues are uniformly distributed and differ by 2. In the general case the "angular momentum" eigenvalues are non-uniformly distributed. For small values of N analytical formulae for the "angular momentum" eigenvalues can be found [44]. Remember that to each value of N correspond $m \cdot n$ energy levels, each with degeneracy $N+1$.

In order to have a formalism corresponding to the one of the isotropic oscillator, let us introduce for every N and (m,n,p,q) an ordering of the "angular momentum" eigenvalues

$$\ell_\mu^{L,m,n,p,q}, \quad \text{where} \quad L = N \quad \text{and} \quad \mu = -L, -L+2, \ldots, L-2, L, \tag{50}$$

by assuming that:

$$\ell_\mu^{L,m,n,p,q} \leq \ell_\nu^{L,m,n,p,q} \quad \text{if} \quad \mu < \nu, \tag{51}$$

the corresponding eigenstate being given by:

$$\begin{aligned} |L,\mu;m,n,p,q\rangle &= \sum_{k=0}^{L} \frac{(-i)^k H_k(\ell_\mu^{L,m,n,p,q}/\sqrt{2})}{\mathcal{N}\sqrt{2^{k/2}[k]!}} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle \\ &= \sum_{k=0}^{L} d_{k+1} \left| \begin{matrix} N \\ (p,q) \end{matrix}, k \right\rangle \end{aligned} \tag{52}$$

The above vector elements constitute the analogue corresponding to the basis of "spherical harmonic" functions of the usual oscillator. The calculation of the "angular momentum" eigenvalues of Eq. (50) and the coefficients $d_1, d_2, \ldots, d_{L+1}$ in the expansion of Eq. (52) is a quite difficult task. The existence of general analytic expressions for these quantities is not obvious. The first few "angular momentum" eigenvalues are given by:

$$\ell_{\pm 1}^{1,m,n,p,q} = \pm\sqrt{\frac{1}{m^m n^n} \frac{\Gamma(m+p)}{\Gamma(p)} \frac{\Gamma(n+q)}{\Gamma(q)}}, \tag{53}$$

and

$$\begin{aligned} \ell_0^{2,m,n,p,q} &= 0, \\ \ell_{\pm 2}^{2,m,n,p,q} &= \pm\sqrt{\frac{1}{m^m n^n} \left(\frac{\Gamma(m+p)}{\Gamma(p)} \frac{\Gamma(2n+q)}{\Gamma(n+q)} + \frac{\Gamma(2m+p)}{\Gamma(m+p)} \frac{\Gamma(n+q)}{\Gamma(q)} \right)} \end{aligned} \tag{54}$$

For $L > 2$ the analytic expressions of the angular momentum eigenvalues and the coefficients d_k are longer, but their calculation is a straightforward task.

5 Multisections of the isotropic oscillator

In [46] the concept of bisection of an isotropic harmonic oscillator has been introduced. One can easily see that multisections (trisections, tetrasections, ...) can be introduced in a similar way. The degeneracies of the various anisotropic oscillators can then be obtained from these of the isotropic oscillator by using appropriate multisections.

Using the Cartesian notation (n_x, n_y) for the states of the isotropic harmonic oscillator we have the following list:

N=0: (00)
N=1: (10) (01)
N=2: (20) (02) (11)
N=3: (30) (03) (21) (12)
N=4: (40) (04) (31) (13) (22)
N=5: (50) (05) (41) (14) (32) (23),

where $N = n_x + n_y$. The corresponding degeneracies are 1, 2, 3, 4, 5, 6, ..., i.e. these of u(2).

A bisection can be made by choosing only the states with n_y=even. Then the following list is obtained:

N=0: (00)
N=1: (10)
N=2: (20) (02)
N=3: (30) (12)
N=4: (40) (04) (22)
N=5: (50) (14) (32).

The degeneracies are 1, 1, 2, 2, 3, 3, ..., i.e. these of the anisotropic oscillator with ratio of frequencies 1:2. The same degeneracies are obtained by choosing the states with n_y=odd. Therefore a bisection of the isotropic oscillator, distinguishing states with $\text{mod}(n_y, 2) = 0$ and states with $\text{mod}(n_y, 2) = 1$, results in two interleaving sets of levels of the 1:2 oscillator.

By analogy, a **trisection** can be made by distinguishing states with $\mathrm{mod}(n_y, 3) = 0$, or $\mathrm{mod}(n_y, 3) = 1$, or $\mathrm{mod}(n_y, 3) = 2$. One can easily see that in this case three interleaving sets of states of the 1:3 oscillator, having degeneracies 1, 1, 1, 2, 2, 2, 3, 3, 3, ..., occur.

Similarly a **tetrasection** can be made by distinguishing states with $\mathrm{mod}(n_y, 4) = 0$, or $\mathrm{mod}(n_y, 4) = 1$, or $\mathrm{mod}(n_y, 4) = 2$, or $\mathrm{mod}(n_y, 4) = 3$. The result is four interleaving sets of states of the 1:4 oscillator, having degeneracies 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, ...

By bisecting n_x and trisecting n_y one is left with six interleaving sets of states with degeneracies 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 3, ..., i.e. degeneracies of the 2:3 oscillator.

By bisecting (or trisecting, tetresecting, etc) both n_x and n_y one is obtaining the original u(2) degeneracies of the isotropic oscillator.

It is therefore clear that the degeneracies of all $m : n$ oscillators can be obtained from these of the isotropic oscillator by appropriate multisections. In particular:

i) The degeneracies of the $1 : n$ oscillator can be obtained from these of the 1:1 (isotropic) oscillator by n-secting n_y or n_x.

ii) The degeneracies of the $m : n$ oscillator can be obtained from these of the 1:1 oscillator by m-secting n_x and n-secting n_y.

6 Connection to $W_3^{(2)}$

For the special case $m = 1$, $n = 2$ it should be noticed that the deformed algebra received here coincides with the finite W algebra $W_3^{(2)}$ [47–50]. The commutation relations of the $W_3^{(2)}$ algebra are

$$[H_W, E_W] = 2E_W, \qquad [H_W, F_W] = -2F_W, \qquad [E_W, F_W] = H_W^2 + C_W,$$
$$[C_W, E_W] = [C_W, F_W] = [C_W, H_W] = 0, \tag{55}$$

while in the $m = 1$, $n = 2$ case one has the relations

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \qquad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \qquad [\mathcal{A}, \mathcal{A}^\dagger] = 3S_0^2 - \frac{H^2}{4} - HS_0 + \frac{3}{16},$$
$$[H, \mathcal{A}^\dagger] = [H, \mathcal{A}] = [H, S_0] = 0, \tag{56}$$

with $S_0 = \mathcal{N} + u$ (where u a constant). It is easy to see that the two sets of commutation relations are equivalent by making the identifications

$$F_W = \sigma\mathcal{A}^\dagger, \qquad E_W = \rho\mathcal{A}, \qquad H_W = -2S_0 + kH, \qquad C_W = f(H), \tag{57}$$

with

$$\rho\sigma = \frac{4}{3}, \qquad k = \frac{1}{3}, \qquad f(H) = -\frac{4}{9}H^2 + \frac{1}{4}. \tag{58}$$

7 Discussion

In conclusion, the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies equal to m/n, is described dynamically by a deformed version of the u(2) Lie algebra, the order of this algebra being $m + n - 1$. The representation modules of this algebra can be generated by using the deformed oscillator algebra. The energy eigenvalues are calculated by the requirement of the existence of finite dimensional representation modules. An "angular momentum" operator useful for labelling degenerate states has also been constructed. The algebras obtained in the special cases with $1 : n$ ratios are shown to correspond to generalized parafermionic oscillators. In the special case of $m : n = 1 : 2$ the resulting algebra has been identified as the finite W algebra $W_3^{(2)}$. Finally, it is demonstrated how the degeneracies of the various $m : n$ oscillators can be obtained from these of the isotropic oscillator by appropriate multisections.

The extension of the present method to the three-dimensional anisotropic quantum harmonic oscillator is already receiving attention, since it is of clear interest in the study of the symmetries underlying the structure of superdeformed and hyperdeformed nuclei [51].

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