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Somnath Bharadwaj, Dipak Munshi and Tarun Souradeep

Inter-University Centre for Astronomy and Astrophysics

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Post : IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, India

Skewness in the Cosmic Microwave Background Anisotropy from Inflationary Gravity Wave Background

Somnath Bharadwaj
*Mehta Research Institute, 10 Kasturba Gandhi Marg, Allahabad 211 002, India.**
email: somnath@mri.ernet.in
Raman Research Institute, Bangalore 560080, India.

Dipak Munshi
Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India
email: munshi@iucaa.ernet.in

Tarun Souradeep
*Canadian Institute for Theoretical Astrophysics, University of Toronto, ON M5S 1A7, Canada**
email: tarun@cita.utoronto.ca
Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India.
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In the context of inflationary scenarios, the observed anisotropy in the Cosmic Microwave Background (CMB) probes the primordial metric perturbations from inflation. The perturbations from inflation are expected to be gaussian random fields, there remains the possibility that nonlinear processes at later epochs induce "secondary" non-gaussian features in the corresponding CMB anisotropy maps. The non-gaussianity induced by nonlinear gravitational instabilty of scalar (density) perturbations has been investigated in exisiting literature. In this paper, we highlight another source of non-gaussianity arising out of higher order scattering of CMB photons off the metric perturbations. We find that in a flat, $\Omega = 1$ universe the skewness in CMB is dominated by the contribution from the gravity waves via the multiple scattering effect for very small contribution of the gravity waves to the total *rms* CMB anisotropy. Consequently, we provide new estimates of total secondary CMB skewness expected in a broader class of inflationary scenarios where the power spectrum of initial perturbations is tilted (reddened). We find that skewness from gravity waves dominates over the skewness arising from density perturbations for small deviations from the scale invariant Harrison-Zeldovich (HZ) spectrum ($n_S < 0.92$). The total secondary skewness is found to be smaller than the cosmic variance leading to the conclusion that inflationary scenarios do predict gaussian CMB anisotropy.

98.80.-k, 98.70Vc, 04.30-w, 04.30Nk, 98.80Bp,

I. INTRODUCTION

The Cosmic Microwave Background (CMB) has proved to be an extremely significant observational guide in our quest towards understanding the universe since its discovery by Penzias and Wilson [1]. The detection of tiny anisotropies in the CMB by the COBE - DMR group [2] was an important milestone in the study of the universe and the understanding the large structure that we see around us. The COBE detection has opened up a fresh avenue of investigation and has been followed by a host of new developments both on the observational and theoretical fronts [3].

The idea of incorporating an inflationary phase in the early universe [4–7] has gained wide acceptance in the last decade and is perhaps the most prevalent scenario within which one attempts to understand the universe. Soon after the notion of an inflationary scenario was put forward , it was realised that besides resolving some long standing problems of the Big-Bang model of cosmology [5], inflationary models also predict the form of the power spectrum of the primordial scalar metric fluctuations which could seed the formation of the large scale structures observed in the present universe [8–11]. In fact, both gravity waves (GW), i.e., tensor metric fluctuations [12–16], as well as adiabatic density perturbations (related to the scalar metric fluctuations) arise as natural consequences of the inflationary scenario, due to the superadiabatic amplification of zero-point quantum fluctuations occuring during inflation. As gravity waves [17–19] and scalar density perturbations enter the horizon during matter domination, they induce distortions in the cosmic microwave background (CMB) through the Sach-Wolfe effect [20]. The relative contribution of the gravity and the adiabatic perturbations is linked to the specific model of inflation [21–26]. The spectral index of the power spectrum of initial perturbations can be inferred directly from the large angle CMB anisotropy maps.

The CMB anisotropy measurements have till date been found to be consistent with inflationary scenarios of the early universe. In particular, the spectral index infered from the COBE - 2year data (see for eg. [27,28]) is consistent with the near scale-invariant spectrum of fluctuations generically predicted by inflationary scenarios. Another generic prediction of inflation is that the metric perturbations generated are gaussian random fields. At the linear order, the CMB anisotropy produced would reflect the gaussian nature of initial perturbations. However, there always remains the possibility that non-linear corrections to the growth of perturbations and the Sach-Wolfe effect could induce some non-gaussian features in the CMB even for gaussian initial metric perturbations.

Non-zero skewness is a definite signature of non-gaussianity in a distribution. For gaussian initial perturbations the skewness in the CMB appears only beyond the linear approximation. At the leading order the skewness arises from two distinct effects. The first effect is that initially gaussian metric perturbations become non-gaussian when we include the lowest order nonlinearity in their evolution due to gravitational instability. The non-linear component of the metric perturbations are non-gaussian and introduce non-gaussian anisotropies in the CMB through a linear Sachs-Wolf relation at the corresponding order. The non-gaussianity of the CMB is reflected in a non-zero skewness of the statistical distribution of temperature fluctuations. The second effect is that the gaussian metric perturbations introduce non-gaussian anisotropies in the CMB due to a second order (double) scattering of the photon off the linear order metric perturbations. This gives rise to new terms in the Sachs-Wolfe relation calculated upto the second order. All previous discussions of skewness in the CMB, have been limited to the the estimation of only the first effect, i.e., nonlinearity (and consequent non-gaussianity) due to gravitational instability [29,30,32]. The tensor component of metric perturbations (GW) does not exhibit gravitational instability, consequently the possibility of non-gaussianity in the CMB caused by the GW background has been entirely ignored in the previous literature.

In this paper, as an illustration of the second effect, we calculate the CMB skewness produced by a gaussian stochastic linear gravity wave background generated by inflation. In the context of $\Omega = 1$, flat FRW models, the magnitude of the effect considered here appears to be significantly larger than the corresponding estimate of CMB skewness arising from the gravitational instability of scalar metric perturbations. Although inflation predicts both scalar and tensor metric perturbations, and scalar perturbations do produce CMB skewness through both the effects, a very general argument shows that the CMB skewness arising from the double scattering of photons off scalar perturbations is expected to be subdominant to that arising from the gravitational instability of scalar perturbations (see §II).

In §II, we outline the basic formalism involved in estimating the skewness in the CMB and give a very general approach for obtaining the higher order corrections to the Sachs-Wolfe effect for a general cosmological perturbation. In the following section (§III) we estimate the skewness in the CMB anisotropy that would arise from a inflationary gravity wave background for a range of values of the spectral index.

II. FORMALISM

In this section, we outline the basic approach and results that are used in obtaining our result. The first part of the section contains a brief discussion of the perturbative approach used in estimating non-gaussianity in the CMB anisotropy. The second part gives a compact derivation of the CMB anisotropy from the Sachs-Wolfe effect upto second order in the primordial metric fluctuations.

A. Non-gaussianity and Nonlinearity in the CMB anisotropy

It is possible to address non-linear effects in the CMB within a perturbative framework by expanding the temperature fluctuations, $\Delta T/T$, in orders corresponding to the powers of the initial metric perturbation as :

$$\frac{\Delta T}{T} = \left(\frac{\Delta T}{T}\right)^{(1)} + \left(\frac{\Delta T}{T}\right)^{(2)} + \left(\frac{\Delta T}{T}\right)^{(3)} \dots \quad . \tag{2.1}$$

Given that the *initial* metric perturbations from inflation are *linear* and *gaussian*, any *non-gaussian feature* in the CMB maps can only arise from the *higher order temperature anisotropy* such as $(\Delta T/T)^{(2)}$. We shall call this higher order effect — *secondary* non-gaussianity. (The term "secondary" is used to denote the effects which take place after recombination. The effects prior to recombination are "primary".)

A non-vanishing skewness is a definite signature of non-gaussianity in a distribution. At the linear order, the mean CMB skewness $C_3^{(3)}(0) = \langle(\Delta T^{(1)}/T)^3\rangle = 0$ where $\langle\rangle$ denotes averaging with respect to different realizations of stochastic space-time metric perturbations of the FRW cosmological model which produce $\Delta T/T$. Substituting the expansion (2.1) into the expression $C_3(0) = \langle(\Delta T/T)^3\rangle$, it is clear that the leading order (in powers of the initial linear metric perturbations) contribution is at the fourth order,

$$C_3^{(4)}(0) = 3\left\langle \left(\frac{\Delta T}{T}\right)^{(1)} \left(\frac{\Delta T}{T}\right)^{(1)} \left(\frac{\Delta T}{T}\right)^{(2)} \right\rangle . \tag{2.2}$$

(In this paper, we deal only with $C_3^{(4)}(0)$, and hence the superscript denoting the order has been dropped in the rest of the text). It is clear from the above expression that the mean skewness depends not only on the magnitude of $\Delta T^{(2)}/T$ but also on the extent of correlation of this term with the linear order terms, $\Delta T^{(1)}/T$. For example, in the case of scalar perturbations the second order term grows linearly with the expansion of the universe and can attain values $\Delta T^{(2)}/T \approx 0.1\ \Delta T^{(1)}/T$ at late times [33]. However, in a flat, $\Omega = 1$ universe the linear order term contributes only close to the surface of recombination ($\eta \approx \eta_{rec}$) and the second order term attains its largest value only at late times ($\eta \approx \eta_0$). Consequently in the final result for the mean skewness, the decay of correlation between the linear and the second order term over this large physical separation ($\approx \eta_o - \eta_{rec}$) along the line of sight attenuates the effect of the growth of the second order term, leading to a very modest value for $C_3(0)$ [30]. It was also pointed out in the same paper that the mean skewness is expected to be somewhat larger in models (eg. CDM+Λ, $\Omega \neq 1$ models) where the gravitational potential changes at late times leading to a significant linear order integrated Sachs-Wolfe contribution at late times.

Even in a flat, $\Omega = 1$ universe, contributions to linear order integrated Sachs-Wolfe effect comes from inflationary tensor perturbations (gravity waves). Consequently, one expects that the correlation between the linear and second order terms is not attenuated in this case leading to larger values of $C_3(0)$. The second order $\Delta T^{(2)}/T$ in the case of gravity waves comes only from double scattering since GW do not exhibit any gravitational instability. Scalar perturbations also give rise to second order anisotropy through double scattering. However, for flat, $\Omega = 1$ models the contribution to the mean skewness is expected to be even smaller than that from gravitational instability considered in [30]. This can be seen from the fact that $\Delta T^{(2)}/T$ from double scattering too has contributions only at late times (implying attenuated correlation with the linear term) and, moreover, for scalar perturbations this second order effect is smaller than that from gravitational instability.

B. Second order CMB anisotropy from the Sachs-Wolfe effect

In a perfectly isotropic universe the CMB would have the same temperature in all directions on the celestial sphere. If, however, the cosmological metric is perturbed away from isotropy, the temperature observed today fluctuates over the celestial sphere.

The dominant contribution at large angular scales ($\theta > 1°$) to the observed temperature fluctuations comes from the change in the frequency of any CMB photon as it travels from the surface of last scattering to us. In the case of an isotropic universe, the overall increase in the scale factor $a(\eta_0)/a(\eta_{rec})$ redshifts the entire Planckian distribution of photons leading to a Planckian distribution at a lower temperature given $T_{rec}/T_0 = a(\eta_0)/a(\eta_{rec})$. The presence of the perturbations $h_{ab}(\eta, \mathbf{x})$, produces an additional change in the frequency and direction (momentum) of a photon as it moves in and out of the fluctuating metric perturbations.

We consider the trajectory of a photon (or ray) in a perturbed flat FRW universe and work in a synchronous coordinate system where the line element has the form

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + (\delta_{ab} + h_{ab}(\eta, x))\ dx^a dx^b\right], \tag{2.3}$$

where $h_{ab} = h_{ab}^{(1)}(\sim \epsilon) + h_{ab}^{(2)}(\sim \epsilon^2)$ is the metric perturbation, and $\epsilon \ll 1$ is a small number characterizing the amplitude of deviations from the unperturbed background FRW universe.

The photon's trajectory can be obtained by perturbatively solving the eikonal equation for the optical path or phase $S(x,\eta)$. The eikonal equation for a photon propagating in a spacetime with metric, $g_{\mu\nu}$, is

$$\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\, g^{\mu\nu} = 0\,, \tag{2.4}$$

(analogous to the Hamilton-Jacobi equation for a massive particle). The frequency and the direction of the photon can be obtained from the the optical path, $S(x,\eta)$

$$\omega(x,\eta) = -\frac{1}{a(\eta)}S_{,0}(x,\eta), \qquad k_a(x,\eta) = S_{,a}(x,\eta)\,. \tag{2.5}$$

where we use the notation $\partial/\partial\eta \equiv {}_{,0}$ and $\partial/\partial x^i \equiv \nabla_i \equiv {}_{,i}$.

Keeping terms to order ϵ^2 when inverting the metric the eikonal equation becomes

$$-(S_{,0})^2 + (\delta^{ab} - h^{(1)ab} - h^{(2)ab} + h^{(1)a}_{\ c}h^{(1)cb})S_{,a}S_{,b} = 0\,. \tag{2.6}$$

where we use background spatial metric δ_{ab} to raise and lower the spatial indices. We perturbatively solve for the trajectory by expanding $S(x,\eta)$ in powers of ϵ

$$S = S + S^{(1)}(\sim\epsilon) + S^{(2)}(\sim\epsilon^2)\,. \tag{2.7}$$

The zeroth order equation is

$$\left(S^{(0)}{}_{,0}\right)^2 - S^{(0)}{}_{,a}S^{(0),a} = 0 \tag{2.8}$$

The first order equation is

$$S^{(1),0}S^{(1)}{}_{,0} + S^{(0),a}S^{(1)}{}_{,a} = \frac{1}{2}h^{(1)ab}S^{(0)}{}_{,a}S^{(0)}{}_{,b} \tag{2.9}$$

and the second order equation is

$$S^{(2),0}S^{(2)}{}_{,0} + S^{(0),a}S^{(2)}{}_{,a} = \frac{1}{2}\left(S^{(1)}{}_{,0}\right)^2 - \frac{1}{2}S^{(1)}{}_{,a}S^{(1),a} + h^{(1)ab}S^{(1)}{}_{,a}S^{(0)}{}_{,b}$$
$$+ \left[h^{(2)ab} - h^{(1)ac}h^{(1)b}_{\ \ c}\right]S^{(0)}{}_{,a}S^{(0)}{}_{,b}. \tag{2.10}$$

We have to solve these equations for rays arriving at the observer from different directions. This can be done by considering rays that leave one observer in all possible directions. We evolve (or trace back) these rays backwards in time until they reach the last scattering surface. This allows us to relate the observed frequency with the emitted frequency of the light. The zeroth order solution is

$$S^{(0)}(x,\eta) = k_a x^a - \eta + C\,. \tag{2.11}$$

We choose the constant C to get the photons zeroth order trajectory as

$$x^a(\lambda) = k^a(\eta_0 - \lambda) \tag{2.12}$$

and $\eta(\lambda) = \lambda$ with η_e(emitted) $\leq \lambda \leq \eta_o$(observed). For our purposes we consider this as a photon going in the direction $\vec{k}$ from the observer who is at the origin of the spatial coordinate system and we consider the photon to be propagating backward in time, i.e., from η_o to η_e. In reality this corresponds to a photon emitted at frequency $1/a(\eta_{rec})$ at the last scatterring surface and received from the direction $\vec{k}$ at the frequency $1/a(\eta_0)$ by the observer.

We next use the zeroth order solution in equation (2.10) to obtain the first order solution. Using the notation $h^{(1)}(\lambda) \equiv h^{(1)}(x(\lambda),\eta(\lambda))$ the first order solution is

$$S^{(1)}(x,\eta) = \frac{1}{2}\int_0^{\eta_0} h^{(1)}_{ab}(\lambda)k^a k^b d\lambda\,. \tag{2.13}$$

Using the first and zeroth order solutions, the second order equation has the solution

$$S^{(2)}(x,\eta) = \frac{1}{2}\int_0^{\eta_o} d\lambda \left\{ \left[\frac{1}{2}\int_{\eta_o}^{\lambda} h^{(1)}{}_{ij,0}(\lambda')k^i k^j d\lambda'\right]\left[\frac{1}{2}\int_{\eta_o}^{\lambda} h^{(1)}{}_{lm,0}(\lambda')k^l k^m d\lambda'\right] \right.$$
$$- \delta^{ab}\left[\frac{1}{2}\int_{\eta_o}^{\lambda} h^{(1)}{}_{ij,a}(\lambda')k^i k^j d\lambda'\right]\left[\frac{1}{2}\int_{\eta_o}^{\lambda} h^{(1)}{}_{lm,b}(\lambda')k^l k^m d\lambda'\right]$$
$$+ h^{(1)ij}(\lambda)\ k_i \left[\frac{1}{2}\int_{\eta_o}^{\lambda} h^{(1)}{}_{lm,j}(\lambda')k^l k^m d\lambda'\right]$$
$$\left. + \left[h^{(2)ij}(\lambda) - h^{(1)il}(\lambda)h^{(1)}{}_l^j(\lambda)\right] k_i k_j \right\}. \tag{2.14}$$

Using these we can relate the frequency of the observed photon to the frequency of the emitted photon. Keeping terms upto order ϵ^2 we have

$$\omega_e = \frac{1}{a(\eta_e)}\left[1 - S^{(1)}{}_{,o}(x_e,\eta_e) - S^{(2)}{}_{,o}(x_e,\eta_e)\right] \tag{2.15}$$

where $\omega_o = 1/a(\eta_o)$. We consider a photon with unit frequency when it leaves the last scattering surface and write its observed frequency in terms of the emitted frequency obtained by inverting equation (2.15). We then have

$$\omega_o = \frac{1}{a(\eta_o)}\left[1 + S^{(1)}{}_{,o}(x_e,\eta_e) + S^{(2)}{}_{,o}(x_e,\eta_e) - (S^{(1)}{}_{,o}(x_e,\eta_e))^2\right] \tag{2.16}$$

and $\omega_e = 1/a(\eta_e)$. This relates the observed frequency to the emitted frequency and the metric perturbations. Using these we obtain expressions for the fractional change in the frequency of the observed photon relative to the frequency that would be observed if the universe were unperturbed. Since the CMB photons have a Planckian distribution, (frequency independent) fractional changes in frequency translates to fluctuations in the temperature characterising the distribution. At the linear order we recover the familiar (linear order) Sachs Wolfe effect

$$\frac{\Delta T^{(1)}}{T} = -\frac{1}{2}\int_{\eta_{rec}}^{\eta_o} \frac{\partial}{\partial\eta} h^{(1)}_{ab}(x(\lambda),\eta(\lambda))k^a k^b d\lambda\ . \tag{2.17}$$

The expression for temperature fluctuations at the second order can be split into two distinct set of terms as follows

$$\frac{\Delta T^{(2)}}{T} = -\frac{1}{2}\int_{\eta_{rec}}^{\eta_o} \frac{\partial}{\partial\eta} h^{(2)}_{ab}(x(\lambda),\eta(\lambda))k^a k^b d\lambda$$

(Effect I : Nonlinearity from Gravitational Instability)

$$+ \int_{\eta_o}^{\eta_e} d\lambda \left\{ \left[\frac{1}{2}\int_{\eta_o}^{\lambda} \frac{\partial^2}{\partial\eta^2} h^{(1)}{}_{ij}(\lambda')k^i k^j d\lambda'\right]\left[\frac{1}{2}\int_{\eta_o}^{\lambda} \frac{\partial}{\partial\eta} h^{(1)}{}_{lm}(\lambda')k^l k^m d\lambda'\right] \right.$$
$$- \delta^{ab}\left[\frac{1}{2}\int_{\eta_o}^{\lambda} \nabla_a \frac{\partial}{\partial\eta} h^{(1)}{}_{ij,}(\lambda')k^i k^j d\lambda'\right]\left[\frac{1}{2}\int_{\eta_o}^{\lambda} \nabla_b h^{(1)}{}_{lm}(\lambda')k^l k^m d\lambda'\right]$$
$$+ \frac{\partial}{\partial\eta} h^{(1)ij}(\lambda)k_i \left[\frac{1}{2}\int_{\eta_o}^{\lambda} \nabla_j h^{(1)}{}_{lm}(\lambda')k^l k^m d\lambda'\right]$$
$$+ h^{(1)ij}(\lambda)k_i \left[\frac{1}{2}\int_{\eta_o}^{\lambda} \nabla_j \frac{\partial}{\partial\eta} h^{(1)}{}_{lm}(\lambda')k^l k^m d\lambda'\right]$$
$$\left. - h^{(1)il}(\lambda)\frac{\partial}{\partial\eta} h^{(1)}{}_l^j(\lambda)k_i k_j \right\} + \left[\frac{1}{2}\int_{\eta_e}^{\eta_o} \frac{\partial}{\partial\eta} h^{(1)ij}(\lambda)k^i k^j d\lambda\right]^2 .$$

(Effect II : Nonlinearity from double scattering of linear terms.) (2.18)

As indicated above, the CMB temperature fluctuations at the second order arise from two distinct physical effects. Consequently, the expression (2.2) for the leading order contribution to the mean skewness, $C_3(0)$, will in general consist of two distinct effects :

I. The *first effect* arises due to the ***non-linear evolution of metric perturbation***, i.e., gravitational instability. The initially gaussian metric perturbations, h_{ab}, become non-gaussian when we include the lowest order nonlinearity, $h^{(2)}_{ab}$ in their evolution due to gravitational instability. The non-gaussian metric perturbation, $h^{(2)}_{ab}$, induces a second order CMB fluctuation $(\Delta T/T)^{(2)}$ through the first term of (2.18) which in turn leads to a non-zero skewness, $C_3^{(4)}(0)$ in the CMB anisotropy map [29,30,32].

II. The *second effect* relates to double scattering of CMB photons from gaussian linear order metric perturbations which introduce non-gaussian anisotropies in the CMB owing to the terms that arise purely from the initial linear metric perturbations, but are non-gaussian and non-linear since they depend on the product of two h_{ab}'s. We shall refer to this as the *double scattering effect.*

The expression for temperature fluctuations upto second order in the metric perturbations has been recently obtained by explicitly solving the geodesic equations [34]. The authors have used it to estimate the correction to the variance of CMB anisotropy. Our calculation based on the eikonal equation is much simpler than explicity solving the geodesic equation. The corresponding Boltzman equations correct upto second order effects can be found in [36], however, the equations have not been solved to explicitly obtain the second order temperature fluctuation.

III. SKEWNESS FROM GRAVITY WAVE

For an isotropic stochastic background of primordial GW, the metric perturbation can be written as

$$\hat{h}_{ab}(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{A_G(k)}{k^{\frac{3}{2}}}h_k(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}}e^{\alpha}_{ab}\hat{a}^{\alpha}_{k} \tag{3.1}$$

where the gaussian random variable, $\hat{a}^{\alpha}_{k}$, satifies the relation

$$\langle \hat{a}^{\alpha}_{k}\hat{a}^{\prime\beta}_{k}\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta^{\alpha\beta}\ , \tag{3.2}$$

the temporal evolution of the modes of the gravity waves in a $\Omega = 1$, dust dominated FRW universe is given by

$$h_k(\eta) = \frac{3}{k\eta}j_1(k\eta)\ , \tag{3.3}$$

and α, β take two values $+$ and $\times$ refering to the two polarization states of gravity waves. The quantity $A^2_G(k)$ is the power spectrum of the GW which has been factored out from the gravity wave modes for notational convenience.

We decompose a gravitational wave traveling in the $\hat{z}$ direction into two polarization states e^{+}_{ab} and $e^{\times}_{ab}$ For a gravitational wave in some arbitrary direction $\hat{n}$ we choose the two polarization states to be the ones obtained by rotating the gravitational wave traveling in the $\hat{z}$ direction so that it travels in the direction $\hat{n}$. If $R_{ab}(\hat{n})$ be this matrix that performs this rotation. we have

$$\hat{h}_{ab}(\eta,\mathbf{x}) = \int \frac{dk d\Omega(\hat{n})}{(2\pi)^3}k^{\frac{1}{2}}h_k(\eta)e^{i\mathbf{k}\mathbf{x}}R_{ac}(\hat{n})R_{db}(\hat{n})[a^{+}_{k}e^{+}_{cd} + a^{\times}_{k}e^{\times}_{cd}]A_G(k)\ . \tag{3.4}$$

At the lowest order the skewness is

$$C_3(0) \equiv \left\langle \left(\frac{\Delta T}{T}\right)^3 \right\rangle = 3\left\langle \left(\frac{\Delta T^{(1)}}{T}\right)^2\left(\frac{\Delta T^{(2)}}{T}\right)\right\rangle\ , \tag{3.5}$$

where the angular bracket denotes an ensemble average over different realizations of the stochastic GW background. The CMB observations can at the best provide an average over all directions for one realisation.

viz. the observed sky. The value obtained by taking an angular average over one sky would generally differ from the ensemble average over all realisations by a cosmic variance which is discussed later. We also invoke the isotropy of the background spacetime to assume equal contribution from the two polarisations.

We substitute the expression for an arbitrary GW (3.4) in the expressions for the temperature fluctuation i.e. equations (2.17) and (2.18) and then put them in equation (3.5) to calculate the skewness. Integrating over the solid angles corresponding to different directions of propagation of the gravitational waves along the line of sight we obtain a rather lengthy expression for the skewness.

$$
\begin{aligned}
C_3(0) = \frac{31104}{(2\pi)^4} & \int_0^\infty dk_1 \int_0^\infty dk_2 \int_{\eta_{rec}}^{\eta_0} d\eta_1 \int_{\eta_{rec}}^{\eta_0} d\eta_2 \int_{\eta_{rec}}^{\eta_0} d\eta\; A^2(k_1)\; A^2(k_2) \\
& \left\{ \frac{9\pi^2}{256}\, k_2\, \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_1(k_1\eta)}{k_1\eta} \frac{j_2(k_2\eta)}{k_2\eta} \frac{J_3(k_1(\eta-\eta_1))}{k_1^2(\eta-\eta_1)^2} \frac{J_3(k_2(\eta-\eta_2))}{k_2^2(\eta-\eta_2)^2} \right. \\
& - k_2\, \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_1(k_1\eta)}{k_1\eta} \frac{j_2(k_2\eta)}{k_2\eta} \frac{j_2(k_1(\eta-\eta_1))}{k_1^2(\eta-\eta_1)^2} \frac{j_2(k_2(\eta-\eta_2))}{k_2^2(\eta-\eta_2)^2} \\
& - k_1 k_2^2 \int_\eta^{\eta_0} d\zeta \int_\eta^{\eta_0} d\zeta' \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_2(k_1\zeta)}{k_1\zeta} \frac{j_2(k_1(\zeta-\eta_1))}{k_1^2(\zeta-\eta_1)^2} \frac{j_2(k_1(\zeta'-\eta_2))}{k_1^2(\zeta'-\eta_2)^2} \frac{j_2(k_2\zeta')}{(k_2\zeta')^2} \\
& + \frac{1}{4}\, k_1 k_2^2 \int_\eta^{\eta_0} d\zeta \int_\eta^{\eta_0} d\zeta' \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_2(k_1\zeta)}{k_1\zeta} \frac{j_2(k_1(\zeta-\eta_1))}{k_1^2(\zeta-\eta_1)^2} \frac{j_2(k_1(\zeta'-\eta_2))}{k_1^2(\zeta'-\eta_2)^2} \frac{j_1(k_2\zeta')}{k_2\zeta'} \\
& - \frac{1}{2}\, k_1\, k_2 \int_\eta^{\eta_0} d\zeta \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_2(k_1\eta)}{k_1\eta} \frac{j_1(k_2\zeta)}{k_2\zeta} \frac{j_2(k_1(\eta-\eta_1))}{k_1^2(\eta-\eta_1)^2} \frac{j_2(k_2(\eta-\eta_2))}{k_2^2(\eta-\eta_2)^2} \\
& - \frac{1}{2}\, k_2^2 \int_\eta^{\eta_0} d\zeta \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_1(k_1\eta)}{k_1\eta} \frac{j_2(k_2\zeta)}{k_2\zeta} \frac{j_2(k_1(\eta-\eta_1))}{k_1^2(\eta-\eta_1)^2} \frac{j_2(k_2(\eta-\eta_2))}{k_2^2(\eta-\eta_2)^2} \\
& \left. + \frac{1}{4}\, k_1^2\, k_2 \int_\eta^{\eta_0} d\zeta \int_\eta^{\eta_0} d\zeta' \frac{j_2(k_1\eta_1)}{k_1\eta_1} \frac{j_2(k_2\eta_2)}{k_2\eta_2} \frac{j_2(k_1\zeta)}{k_1\zeta} \frac{j_1(k_1\zeta')}{k_1\zeta'} \frac{j_3(k_1(\zeta-\eta_1))}{k_1^2(\zeta-\eta_1)^2} \frac{j_3(k_2(\zeta'-\eta_2))}{k_2^2(\zeta'-\eta_2)^2} \right\} \qquad (3.6)
\end{aligned}
$$

We numerically evaluated the above expression to compute the skewness for different spectral indices of GW power spectrum. A dimensionless number S_3 can be constructed by dividing the skewness, $C_3(0)$ by the square of the variance, $C_2(0)$. The variance of $\Delta T/T$ that arises due to relic gravity waves from inflation [21] is given by

$$
C_2(0) = \frac{9}{16\pi^2} \int_0^\infty \frac{dk}{k}\, A_G^2(k) \int_{\eta_{rec}}^{\eta_0} \frac{d\eta}{\eta} \int_{\eta_{rec}}^{\eta_0} \frac{d\eta\prime}{\eta\prime}\, j_2(k\eta)\, j_2(k\eta\prime)\, j_0[k(\eta-\eta\prime)]. \qquad (3.7)
$$

The power spectrum of the initial gravitational wave perturbation is assumed to be a scale-free power law, $A_G^2(k) = A\, k^{n_T}$, where $n_T = 0$ corresponds to a scale-invariant spectrum. Power law models of inflation can produce gravity waves with $n_T < 0$. We compute the CMB skewness for a broad range in n_T ($-0.5 \le n_T \le 0$). We find that the value of S_3 varies between -61 to -74 for n_T varying from the scale invariant spectrum to $n_T = -0.5$. It is interesting to note that the skewness arising from gravitational instability of scalar perturbations is much smaller for the same range of tilt (~ -2.2 for scale invariant ($n_s = 1$) spectrum [30] and lesser in magnitude than -2.5 for $0.5 \le n_s \le 1$).

In a general inflationary scenario, the relative contribution of scalar and tensor contributions to the CMB anisotropy can be related to the spectral index of the power spectrum of gravity waves. The ratio of the contribution to the quadrupole of the CMB anisotropy due to gravity waves to that due to scalar perturbations is given by, $Q_T^2/Q_S^2 \approx -7n_T$ (for a more precise relation, see [31]). In a very broad class of models, the power spectra of scalar and tensor perturbations are (nearly) scale free (i.e., power law over the range of astrophysical scales) and the spectral indices are related by $n_s \approx 1 + n_T$ [26,31]. We restrict our discussion to this class of models alone. The ratio $C_2^{(T)}(0)/C_2^{(S)}(0)$ can then be obtained using the predicted values of the CMB anisotropy at higher multipoles from scalar as well as the tensor perturbations for a given spectral index [37].

It is then possible to combine the skewness from gravity waves (through double scattering) and from scalar perturbation (through nonlinear gravitaional instabity) to estimate the total secondary skewness in CMB. The solid curve in Fig. 1 gives the value of the skewness parameter for various values of the spectral index. In Fig. 2 we present a plot of $C_3^{(T)}(0)/C_3^{(S)}(0)$. We find that the ratio $C_3^{(T)}(0)/C_3^{(S)}(0)$ exceeds unity for

$n_s < 0.92$, implying that the GW background from inflation dominates the secondary skewness in the CMB for very modest tilts. It should be noted that inflation never predicts a perfectly scale invariant spectrum and an effective spectral index $n_s \sim 0.9 - 0.95$ is quite generic for inflationary scenarios. (In the context of cosmological observations, the spectral indices of scalar and tensor perturbations from simple inflationary models can be taken to be a constant "effective value" and the relation $n_s \approx n_T + 1$ holds.)

The corresponding observable quantity for the skewness, $C_3(0)$ is the sky-average $\bar{C}_3(0) = (4\pi)^{-1} \int (\Delta T(\theta,\varphi)/T)^3 \, d\Omega$ of one particular realization. The value obtained by taking an angular average over one sky would generally differ from the ensemble average over all realisations by a cosmic variance for the skewness. The cosmic variance for the skewness can be expressed in terms of an angular integral over the two-point correlation function, $C_2(\theta)$, and is roughly of the order of $\langle(\Delta T/T)^2\rangle^{3/2}$ [35]. The skewness originating due to any effect would have an observable significance if the predicted signal stand above the cosmic variance. This is a fundamental limitation and a minimal requirement. In practice, a detectable signal has to stand above additional variances such as instrumental noise, finite beam width of antennas, incomplete sky coverage etc.

The Cosmic variance can be expressed in terms of an angular integral over the two-point correlation function, $C_2(\theta)$ [35]. Assuming a gaussian approximation for the two-point correlation function, we express

$$C_2(\theta) = C_2(0) \; \exp[-\frac{l_c(l_c+1)\,\theta^2}{2}], \quad C_2(0) \approx 3 \times 10^{-5} \tag{3.8}$$

where the cut-off, l_c, in the $\Delta T/T$ angular spectrum at large values of the spherical harmonic eigenvalue, l ($l_c \approx \eta_0/\eta_{rec} \sim 49$ for GW [21] and $l_c \approx 250$ for scalar perturbations). Using equation (3.8), the Cosmic variance, δS_3, for the case of a CMB anisotropy arising from gravity waves is given by $\delta S_3 \approx 1/(C_2(0)\; l_c) \approx 670$. The Cosmic variance for the scalar case is around 5 times smaller [30]. The Cosmic variance is larger as the power spectrum tilted away (reddened) from scale invariance for both scalar perturbations and gravity waves. It is clear that *in principle* the secondary skewness in the CMB for a CDM model ($\Omega = 1, \Omega_b = 0.05$ and $H_0 = 50 kms/s/Mpc.$) is *unobservable* since it is below the cosmic variance.

IV. CONCLUSIONS

In this work we investigated the possibility of secondary non-gaussianity in the CMB, i.e., whether initial gaussian metric perturbations (as expected from inflation) should lead to a gaussian CMB anisotropy. We point out that besides nonlinear gravitional instability, secondary non-gaussianity can be induced in the CMB maps due to multiple scattering of CMB photon off metric perturbations and estimate the skewness in the CMB that is expected to arise from a relic inflationary gravity wave background. By including the efffect of gravity waves (in tilted models), our work compliments and extends previous work on this issue where only the contribution of nonlinear evolution of scalar perturbations to the skewness in the CMB has been considered [30,32].

The gravity wave background is a generic prediction of inflation, the extent of its contribution to the CMB anisotropy being fixed by the spectral index of the metric perturbations. We find that the secondary skewness from the inflationary gravity wave background dominates the effect of nonlinear gravitational instability of scalar perturbations even for very small contribution of the gravity waves to the *rms* CMB anisotropy.

Combining the skewness from both scalar and tensor perturbations, we give new estimates of the secondary skewness expected in a flat, $\Omega = 1$ model. However, the skewness parameter, S_3, is shown much smaller than the corresponding Cosmic variance. Consequently, for the class of models we have investigated the CMB anisotropy is expected to retain the gaussian nature of inflationary metric perturbations.

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* present address.

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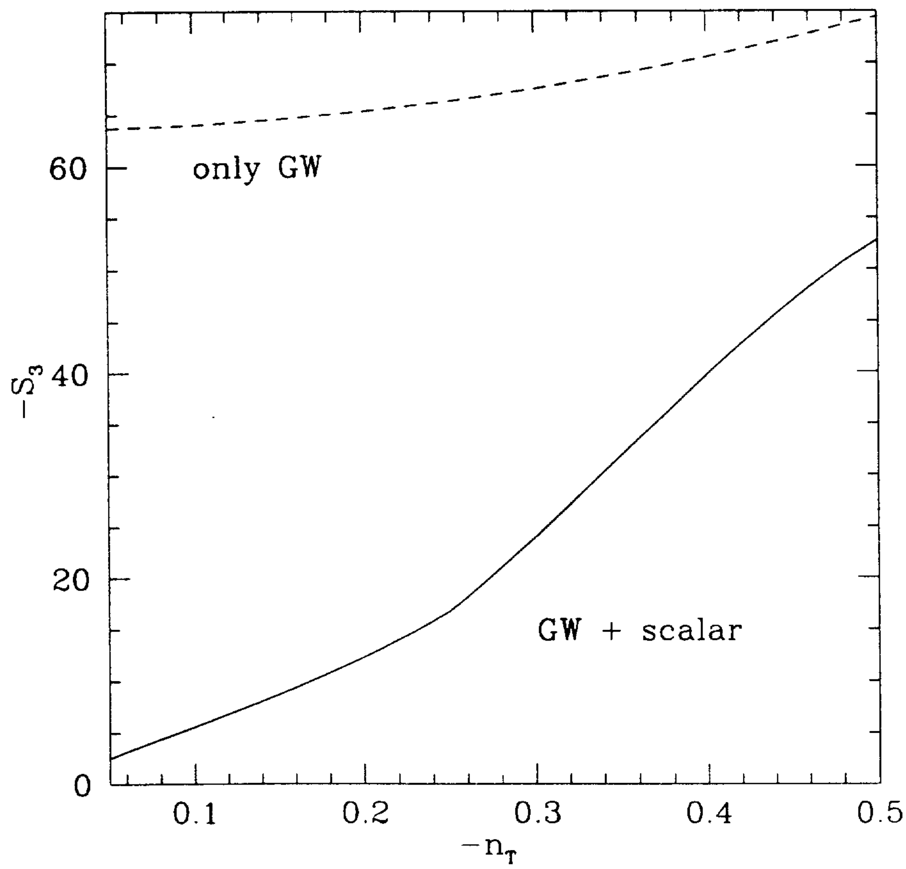

FIG. 1. The skewness parameter, S_3, expected in the CMB anisotropy from power law models of inflation with a range of tilts, $n_T(= n_s - 1)$. The dashed line shows the skewness parameter (from the double scattering effect) assuming the CMB anisotropy to arise from GW alone.

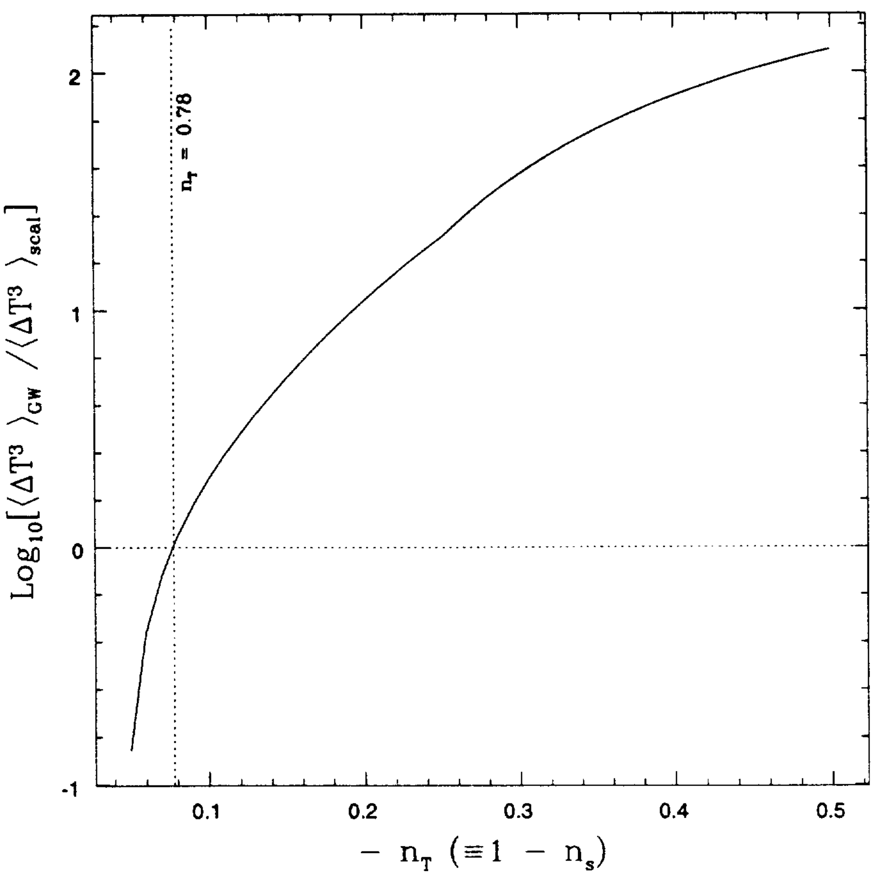

FIG. 2. The ratio of contribution to the secondary skewness from GW to that from scalar perturbation is plotted against the spectral index of the GW perturbation for Power law models of inflation.