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Analytical formulae for Racah Coefficients and $6-j$ symbols of the quantum superalgebra $U_q(osp(1|2))$

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Abstract

Using the method of projection operators, analytical formulae for Racah coefficients and $6-j$ symbols of the quantum superalgebra $U_q(osp(1|2))$ are derived. The formulae obtained by this method are transformed by means of algebraic identities into symmetrical analytical formulae, the form of which are very similar to the classical formulae obtained by Racah and Regge for $su(2)$ Racah coefficients and $6-j$ symbols. Symmetry properties of $U_q(osp(1|2))$ Racah coefficients and $6-j$ symbols following from these analytical formulae are studied. In particular, it is shown that, similarly to the $su(2)$ classical case, in addition to the usual tetrahedral symmetry, $6-j$ symbols of the quantum superalgebra $U_q(osp(1|2))$ satisfy a Regge type symmetry.

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I Introduction

In this paper, we continue the analysis of Racah-Wigner calculus for the quantum superalgebra $U_q(osp(1|2))$, by the projection operator method. This very effective method already permitted us to derive the analytical formula for Clebsch-Gordan coefficients (denoted $sqCG$) of the quantum superalgebra $U_q(osp(1|2))$ and many properties of these coefficients[1]. In Ref.[2], we also defined the corresponding $3-j$ symbols (denoted $sq3-j$). The analytical formula obtained by this method coïncide with the formula derived by Kulish with the more conventional method of recursion relations [3].

In Ref. [4], we have defined Racah coefficients (denoted $sqRC$) and $6-j$ symbols (denoted $sq6-j$) for the quantum superalgebra $U_q(osp(1|2))$. Racah cofficients were defined as the coefficients that relate two reduced basis in two different reduction schemes of tensor product of three irreducible representations. As in the cases of $su(2)$ or $U_q(su(2))$, Racah coefficients defined in this way can be expressed in terms of Clebsch-Gordan coefficients. Due to this construction, using the properties of Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$, one can derive many properties of Racah coefficients, such as the symmetry properties and the pseudo-orthogonalty relations. It has been shown also that $sq6-j$ symbols can be defined from the $sq3-j$ symbols and that they are related to Racah coefficients in a way similar to the cases of $su(2)$ or $U_q(su(2))$. Then, symmetry properties and other properties of $sq6-j$ symbols follow from the properties of $sq3-j$ symbols. However, in order to know everything about $sqRC$ and $sq6-j$ symbols, in particular to analyse their full symmetry properties, analytical formulae are necessary.

In this paper, we use the projection operator method, to derive analytical formulae for $sqRC$ coefficients and $sq6-j$ symbols. This method was used earlier in [5] to derive the analytical formula for Racah coefficients of $U_q(su(2))$. The analytical formula for $sq6-j$ symbols that we obtain by this method is rather complicated and unsymmetrical. However, by means of algebraic identities, it is possible to transform this unsymmetrical expression into a formula which has a form very similar to the form obtained by Racah for $su(2)$ coefficients[6]. Moreover, it is possible to transform further the formula into a very symmetrical form similar to the expression given by Regge in his Nuovo Cimento letter in 1959 [7]. This symmetrical formula allows us to study the symmetry proprties of $sq6-j$ symbols. In particular, it follows readily from this expression that $sq6-j$ symbols have not only the usual tetrahedral symmetry but presents also an additional, conditional Regge symmetry. Here we have an interesting phenomenon: althought the analytical formula itself has full Regge symmetry without any condition, because of the fact that the highest wheights of $U_q(osp(1|2))$ are integers, we have to impose some condition on the values of the highest weights in $sq6-j$ symbol in order to preserve the integrity of all highest weights in the $sq6-j$ symbol obtained after Regge transformation.

Using the analytical formula we also derive the values of some particular $sq6-j$ symbols. For instance, we give the expression of $sq6-j$ symbols where one highest weight is the sum of the remainings highest weights in a triangular triplet.

This paper is organized in the following way: section II contains basic definitions and relations, which will be necessary later on. In section III, we derive the analytical formulae for $sqRC$ coefficients and $sq6-j$ symbols and we give some particular values of these symbols. Section IV is devoted to the analysis of the symmetry properties that follow from the analytical formulae. Finally, in Appendix A we give a Table of values for $sq6-j$ symbols where one highest weight is equal to one and in Appendix B we collect algebraic identities that are used for the transformation of analytical formulae in section III.

II Preliminaries

A The irreducible representations of the quantum superalgebra $U_q(osp(1|2))$

The quantum superalgebra $U_q(osp(1|2))$ is generated by 4 elements: $\mathbf{1}$, H (even) and $v_\pm$ (odd) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\frac{1}{2}v_\pm, \quad [v_+, v_-]_+ = -\frac{sh\,(\eta H)}{sh\,(2\eta)}, \tag{2.1}$$

where the deformation parameter η is real and $q = e^{-\frac{\eta}{2}}$ (we choose $\eta > 0$ so that $q < 1$). The quantum superalgebra $U_q(osp(1|2))$ is a Hopf algebra with the following coproduct

$$\Delta(v_\pm) = v_\pm \otimes q^H + q^{-H} \otimes v_\pm, \tag{2.2}$$

$$\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H, \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \tag{2.3}$$

A representation of a quantum superalgebra $U_q(osp(1|2))$ in a finite dimensional graded space V is a homomorphism $T:\ U_q(osp(1|2)) \rightarrow L(V,V)$ of the associative graded algebra $U_q(osp(1|2))$ into the associative graded algebra $L(V,V)$ of linear operators in V, such that

$$[T(H), T(v_\pm)] = \pm\frac{1}{2}T(v_\pm), \quad [T(v_+), T(v_-)]_+ = -\frac{sh\,(\eta T(H))}{sh\,(2\eta)}, \tag{2.4}$$

The irreducible representation space of highest weight l, $V = V^l(\lambda)$ is a graded vector space of dimension $2l+1$ with basis $e^{lq}_m(\lambda)$, where $-l < m < l$,and $\lambda = 0, 1$ is the parity of the highest weight vector $e^l_l(\lambda)$. The parity of the basis vectors $e^{lq}_m(\lambda)$ is determined by the values of l, m and λ:

$$deg(e^l_m(\lambda)) = l - m + \lambda \quad mod(2)\,. \tag{2.5}$$

The vectors $e^{lq}_m(\lambda)$ are pseudo-orthogonal with respect to an Hermitean form and their normalization is determined by the signature parameters φ, ψ

$$(e^{lq}_m(\lambda), e^{lq}_{m'}(\lambda)) = (-1)^{\varphi(l-m)+\psi}\delta_{mm'}\,, \tag{2.6}$$

where (,) denotes the Hermitean form in the representation space.

It has been shown in Ref.[2], that any finite dimensional grade star representation of $U_q(osp(1|2))$ is characterized by four parameters: the highest weight l (a non negative integer), the parity λ of the highest weight vector in the representation space and the signature parameters $\varphi, \psi = 0, 1$ of the Hermitean form in the representation space V. The parity λ and the signature φ define the class $\varepsilon = 0, 1$ of the grade star representation by the relation $\varepsilon = \lambda + \varphi + 1 \pmod 2$. In such a representation the operators satisfy the following relations

$$T(H)^\star = T(H), \quad T(v_\pm)^\star = \pm(-1)^\varepsilon T(v_\mp), \quad T(\mathbf{1})^\star = T(\mathbf{1}), \tag{2.7}$$

where $(\star)$ is the grade adjoint operation defined by

$$(T(X)^\star f, g) = (-1)^{deg(X)deg(f)}(f, T(X)g), \tag{2.8}$$

for any $X \in U_q(osp(1|2))$ and $f, g \in V$.

The operators $T(v_\pm)$ and $T(H)$ act on the basis $e_m^{lq}(\lambda)$ in the following way :

$$\begin{aligned} T(H)e_m^{lq}(\lambda) &= \frac{m}{2}\, e_m^{lq}(\lambda), \\ T(v_+)e_m^{lq}(\lambda) &= (-1)^{l-m}\sqrt{[l-m][l+m+1]\gamma}\; e_{m+1}^{lq}(\lambda), \\ T(v_-)e_m^{lq}(\lambda) &= \sqrt{[l+m][l-m+1]\gamma}\; e_{m-1}^{lq}(\lambda), \end{aligned} \tag{2.9}$$

where the symbol $[n]$ is the graded quantum symbol defined by

$$[n] = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} \tag{2.10}$$

and $\gamma = \dfrac{\cosh(\frac{\eta}{4})}{\sinh(2\eta)}$. The representation T of class ε which acts in the representation space $V^l(\lambda)$ with the Hermitean form characterized by the signature parameters φ and ψ is denoted by $T^{l\varepsilon}_{\varphi\psi}$. However, for simplicity, the indices $\varepsilon, \varphi, \psi$ will sometimes be omitted in the following.

For $q = 1$, the grade star representation $T^{l\varepsilon}_{\varphi\psi}$ of $U_q(osp(1|2))$ becomes a grade star representation of the superalgebra $osp(1|2)$ described in [8],[13].

B The projection operator for the quantum superalgebra $U_q(osp(1|2))$

The projection operator on the highest weight vector P^q acts linearly in the space V, the direct sum of all representation spaces V^l. It is defined by the following requirements

$$[T(H), P^q] = 0, \quad T(v_+)P^q = 0, \quad (P^q)^2 = P^q, \quad P^q\, e_l^{lq}(\lambda) = e_l^{lq}(\lambda). \tag{2.11}$$

It has been shown in Ref.[1],[2], that the operator P^q can be written in the form of a series

$$P^q = \sum_{r=0}^{\infty} c_r(T(H))(T(v_-))^r(T(v_+))^r, \tag{2.12}$$

where the coefficient $c_r(T(H))$ is an operator

$$c_r(T(H)) = \frac{[4T(H)+1]!}{[4T(H)+r+1]![r]!\gamma^r} \tag{2.13}$$

In the following we will consider the action of the operator (2.12) in the finite dimensional representation spaces, where only finite truncations of the series (2.12) will actually contribute, so in these cases the convergence of the formal series (2.12) presents no problem. General formulae for the projection operator of quantum orthosymplectic superalgebras have been derived by Koroshkin and Tolstoy [9]. In the limit $q \to 1$, the coefficient $c_r(T(H))$ and therefore the projection operator P^q are equal to the corresponding $osp(1|2)$ coefficient and projection operator P, cf. [10].

If we consider the space W_m of all vectors of weight m, i.e., $W_m = \{f | T(H)f = \frac{m}{2} f\}$, then the restriction of P^q to this space is denoted by P^{mq} and it has the form

$$P^{mq} = \sum_{r=0}^{\infty} c_r(m)(T(v_-))^r (T(v_+))^r, \tag{2.14}$$

where the coefficients $c_r(m)$ are now graded quantum numbers

$$c_r(m) = \frac{[2m+1]!}{[2m+r+1]![r]!\gamma^r}. \tag{2.15}$$

The operators P^q, P^{mq} are even and self adjoint with respect to the grade adjoint operation, i.e. we have

$$\deg(P^q) = \deg(P^q) = 0, \quad (P^q)^\star = P^q, \ (P^{mq})^\star = P^{mq} \tag{2.16}$$

C Tensor product of irreducible representations

The space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ is a representation space for the tensor product of two representations $T^{l_1\varepsilon}_{\varphi_1\psi_1} \otimes T^{l_2\varepsilon}_{\varphi_2\psi_2}$ of the same class ε. The bilinear Hermitean form in the tensor product space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ is defined in the following way :

$$((X_1 \otimes X_2), (Y_1 \otimes Y_2)) = (-1)^{deg(X_2)deg(Y_1)}(X_1, Y_1)(X_2, Y_2) \tag{2.17}$$

where X_1, Y_1 and X_2, Y_2 are homogeneous elements of $V^{l_1}(\lambda_1)$ and $V^{l_2}(\lambda_2)$ respectively. The generators $v_\pm$ and H are represented in the space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ by the operators

$$v^{\otimes}_{\pm}(1,2) \ = \ (T^{l_1} \otimes T^{l_2})(\Delta(v_\pm)) \ = \ T^{l_1}(v_\pm) \otimes q^{T^{l_2}(H)} + q^{-T^{l_1}(H)} \otimes T^{l_2}(v_\pm), \tag{2.18}$$

$$H^{\otimes}(1,2) = (T^{l_1} \otimes T^{l_2})\Delta(H) \ = \ T^{l_1}(H) \otimes T^{l_2}(\mathbf{1}) + T^{l_1}(\mathbf{1}) \otimes T^{l_2}(H). \tag{2.19}$$

The tensor product of three irreducible representations of the same class ε, $T^{l_1\varepsilon}_{\varphi_1\psi_1} \otimes T^{l_2\varepsilon}_{\varphi_2\psi_2} \otimes T^{l_3\varepsilon}_{\varphi_3\psi_3}$ act in the representation space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2) \otimes V^{l_3}(\lambda_3)$, the tensor product of the corresponding representation spaces.The bilinear Hermitean form in this space is defined with the bilinear Hermitean forms in each space $V^{l_i}(\lambda_i)$, $i=1,2,3$, and for the basis vectors we have

$$\begin{aligned}&(e^{l_1}_{m_1}(\lambda_1)\otimes e^{l_2}_{m_2}(\lambda_2)\otimes e^{l_3}_{m_3}(\lambda_3)\ ,\ e^{l_1}_{n_1}(\lambda_1)\otimes e^{l_2}_{n_2}(\lambda_2)\otimes e^{l_3}_{n_3}(\lambda_3))\\ &= (-1)^{(\sum_{i<j}(l_i-m_i+\lambda_i)(l_j-m_j+\lambda_j))}(-1)^{(\sum_{i=1}^{3}\varphi_i(l_i-m_i)+\psi_i)}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}. \quad (2.20)\end{aligned}$$

The operators $H, v_\pm$ are represented in the representation space by

$$\begin{aligned}v^{\otimes}_{\pm}(12,3) &= (T^{l_1}\otimes T^{l_2}\otimes T^{l_3})((\Delta\otimes id)\Delta(v_\pm)) = T^{l_1}(v_\pm)\otimes q^{T^{l_2}(H)}\otimes q^{T^{l_3}(H)}+\\ &+q^{-T^{l_1}(H)}\otimes T^{l_2}(v_\pm)\otimes q^{T^{l_3}}(H)+\ q^{-T^{l_1}(H)}\otimes q^{-T^{l_2}(H)}\otimes T^{l_3}(v_\pm), \quad (2.21)\end{aligned}$$

$$\begin{aligned}H^{\otimes}(12,3) &= (T^{l_1}\otimes T^{l_2}\otimes T^{l_3})((\Delta\otimes id)\Delta(H)) = T^{l_1}(H)\otimes T^{l_2}(\mathbf{1})\otimes T^{l_3}(\mathbf{1})+\\ &+T^{l_1}(\mathbf{1})\otimes T^{l_2}(H)\otimes T^{l_3}(\mathbf{1})+\ T^{l_1}(\mathbf{1})\otimes T^{l_2}(\mathbf{1})\otimes T^{l_3}(H). \quad (2.22)\end{aligned}$$

It has been shown in [2] and [4] that the tensor products of representations $T^{l_1\varepsilon}_{\varphi_1\psi_1}\otimes T^{l_2\varepsilon}_{\varphi_2\psi_2}$ and $T^{l_1\varepsilon}_{\varphi_1\psi_1}\otimes T^{l_2\varepsilon}_{\varphi_2\psi_2}\otimes T^{l_3\varepsilon}_{\varphi_3\psi_3}$ are representation of class ε with respect to the Hermitean forms (2.17) and (2.20) respectively.

The projection operator P^{lq} is represented in the spaces $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ and $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)\otimes V^{l_3}(\lambda_3)$ by the following operators

$$P^{lq\otimes}(1,2) = (T^{l_1}\otimes T^{l_2})(\Delta(P^{lq})) = \sum_{r=0}^{\infty} c_r(l)(v^{\otimes}_{-}(1,2))^r(v^{\otimes}_{+}(1,2))^r\,, \quad (2.23)$$

$$P^{lq\otimes}(1,2,3) = (T^{l_1}\otimes T^{l_2}\otimes T^{l_3})((\Delta\otimes id)\Delta(P^{lq})) = \sum_{r=0}^{\infty} c_r(l)(v^{\otimes}_{-}(1,2,3))^r(v^{\otimes}_{+}(1,2,3))^r\,, \quad (2.24)$$

D $\;U_q(osp(1|2))$ Clebsch-Gordan coefficients

By definition, the Clebsch-Gordan coefficients $(l_1m_1\lambda_1,l_2m_2\lambda_2|lm\lambda)_q$ relate the pseudo-normalized basis $e^{l_1q}_{m_1}(\lambda_1)\otimes e^{l_2q}_{m_2}(\lambda_2)$ of $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ to the reduced pseudo-orthogonal basis $e^{lq}_m(l_1,l_2,\lambda)$ of $V^l(\lambda)$ in the following way :

$$e^{lq}_m(l_1,l_2,\lambda) = \sum_{m_1,m_2}(l_1m_1\lambda_1,l_2m_2\lambda_2|lm\lambda)_q\ e^{l_1q}_{m_1}(\lambda_1)\otimes e^{l_2q}_{m_2}(\lambda_2)\,, \quad (2.25)$$

or equivalently

$$\sum_{l,m}(-1)^{(l-m)L}(l_1m_1\lambda_1,l_2m_2\lambda_2|lm\lambda)_q\ e^{lq}_m(l_1,l_2,\lambda) = (-1)^{(l_1-m_1)(l_2-m_2)}e^{l_1q}_{m_1}(\lambda_1)\otimes e^{l_2q}_{m_2}(\lambda_2) \quad (2.26)$$

where $m_1+m_2=m$, $L=l_1+l_2+l$ and l is an integer satisfying the conditions

$$|l_1-l_2|\le l\le l_1+l_2\,. \quad (2.27)$$

The reduced basis $e^{lq}_{m}(l_1, l_2, \lambda)$ is orthogonal but not positive definite i.e. we have

$$(e^{lq}_{m}(l_1, l_2, \lambda), e^{l'q}_{m'}(l_1, l_2, \lambda)) = \delta_{ll'}\ \delta_{mm'}(-1)^{\varphi(l-m)+\psi}$$

where

$$\varphi = L+\lambda_1+\varphi_2 \pmod 2,\ \psi = (L+\lambda_2)\lambda_1+\varphi_2 L+\psi_1+\psi_2 \pmod 2,\ \lambda = L+\lambda_1+\lambda_2 \pmod 2 \tag{2.28}$$

In the following we will need the particular values of Clebsch-Gordan coefficients when $l = m$. This Clebsch-Gordan coefficient can be presented as follows

$$\begin{aligned}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l l \lambda)_q \ &= \ (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(\sum_{i=1}^{2}\varphi_i(l_i-m_i)+\psi_i)} \times \\ &\frac{(e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2), P^{lq\otimes}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l-l_1}(\lambda_2)))}{(l_1 l_1 \lambda_1, l_2 l - l_1 \lambda_2 | l l \lambda)_q}\,. \end{aligned} \tag{2.29}$$

and its analytical formula takes the following form[2]

$$\begin{aligned}&(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l l \lambda)_q = \\ &(-1)^{\lambda_1(l-l_1-m_2)}(-1)^{(l_1-m_1)(l_2-m_2)+\frac{(l_1-m_1)(l_1-m_1+1)}{2}} q^{\frac{(l_1+l_2-l)(l+l_2-l_1+1)}{4}} q^{-\frac{(l_1-m_1)(l+1)}{2}} \times \\ &\left(\frac{[2l+1]![l_2+m_2]![l_1+m_1]![l_1+l_2-l]!}{[l_1-m_1]![l_2-m_2]![l_2-l_1+l]![l_1-l_2+l]![l_1+l_2+l+1]!}\right)^{\frac{1}{2}}\,. \end{aligned} \tag{2.30}$$

For more details on Clebsch-Gordan coefficients of the quantum superalgebra $U_q(osp(1|2)$ see ref.[2]

E Racah coefficients and $6-j$ symbols for $U_q(osp(1|2))$

The reduction of the tensor product $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)\otimes V^{l_3}(\lambda_3)$ of representation spaces can be done, as in the classical case, in two different schemes. In the first scheme, one couples first the representations T^{l_1} and T^{l_2} and then the result is coupled to the representation T^{l_3} in order to give as a final result the representation T^{l}. In the second scheme, one couples T^{l_1} with the result $T^{l_{23}}$ of the coupling of representations T^{l_2} and T^{l_3} in order to yield T^{l}. These schemes can expressed in the short way

$$T^{l}\subset((T^{l_1}\otimes T^{l_2})_q\otimes T^{l_3})_q,\ \ T^{l}\subset(T^{l_1}\otimes(T^{l_2}\otimes T^{l_3})_q)_q. \tag{2.31}$$

The reduced bases corresponding to these schemes are given by the expressions

$$\begin{aligned}e^{lq}_{m}(l_{12}, l_3, \lambda) \ &= \ \sum_{m_i}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l_{12} m_{12} \lambda_{12})_q \times \\ &(l_{12} m_{12} \lambda_{12}, l_3 m_3 \lambda_3 | l m \lambda)_q\ e^{l_1}_{m_1}(\lambda_1)\otimes e^{l_2}_{m_2}(\lambda_2)\otimes e^{l_3}_{m_3}(\lambda_3), \end{aligned} \tag{2.32}$$

where $i = 1, 2, 3, 12$, and

$$e_m^{lq}(l_1, l_{23}, \lambda) = \sum_{m_i} (l_2 m_2 \lambda_2, l_3 m_3 \lambda_3 | l_{23} m_{23} \lambda_{23})_q \times$$
$$(l_1 m_1 \lambda_1, l_{23} m_{23} \lambda_{23} | l m \lambda)_q \; e_{m_1}^{l_1}(\lambda_1) \otimes e_{m_2}^{l_2}(\lambda_2) \otimes e_{m_3}^{l_3}(\lambda_3) , \quad (2.33)$$

where $i = 1, 2, 3, 23$ and we have

$$m = m_1 + m_2 + m_3, \quad \lambda = \sum_{i=1}^{3} (\lambda_i + l_i) + l. \quad (2.34)$$

The bases $e_m^{lq}(l_1, l_{23}, \lambda)$ and $e_m^{lq}(l_{12}, l_3, \lambda)$ are orthogonal and normalized in the following way

$$(e_m^{lq}(l_{12}, l_3, \lambda) \; , \; e_{m'}^{l'q}(l'_{12}, l_3, \lambda)) = (-1)^{\varphi_{12,3}(l-m)+\psi_{12,3}} \; \delta_{ll'} \; \delta_{mm'} \; \delta_{l_{12}l'_{12}} \; , \quad (2.35)$$

$$(e_m^{lq}(l_1, l_{23}, \lambda) \; , \; e_{m'}^{l'q}(l_1, l'_{23}, \lambda)) = (-1)^{\varphi_{1,23}(l-m)+\psi_{1,23}} \; \delta_{ll'} \; \delta_{mm'} \; \delta_{l_{23}l'_{23}} \; , \quad (2.36)$$

where

$$\varphi_{1,23} = \varphi_{12,3} = \mathcal{L} + \lambda_1 + \lambda_2 + \varphi_3 \pmod 2, \quad \mathcal{L} = l_1 + l_2 + l_3 + l,$$
$$\psi_{12,3} = (l_1 + l_2 + l_{12})(\mathcal{L} + 1) + (\lambda_1 + \lambda_3 + \varphi_2)\mathcal{L} + \sum_{i<j} \lambda_i \lambda_j + \sum_{i=1}^{3} \psi_i \pmod 2, \quad (2.37)$$
$$\psi_{1,23} = (l_2 + l_3 + l_{23})(\mathcal{L} + 1) + (\lambda_1 + \lambda_3 + \varphi_2)\mathcal{L} + \sum_{i<j} \lambda_i \lambda_j + \sum_{i=1}^{3} \psi_i \pmod 2.$$

The sqRacah Coefficients $U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q)$ of the quantum superalgebra $U_q(osp(1|2))$ are defined in the standard way as the coefficients that relate two reduced bases in two different reduction schemes [4]

$$e_m^{lq}(l_{12}, l_3, \lambda) = \sum_{l_{23}} (-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)} \; U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q) \; e_m^{lq}(l_1, l_{23}, \lambda), \quad (2.38)$$

or equivalently

$$e_m^{lq}(l_1, l_{23}, \lambda) = \sum_{l_{12}} (-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)} U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q) e_m^{lq}(l_{12}, l_3, \lambda). \quad (2.39)$$

In order to have better symmetry properties one can define the parity-dependent $6-j$ symbols for $U_q(osp(1|2))$ (denoted $sq6-j\lambda$) which are related to sqRacah coefficients in the following way

$$U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q) = (-1)^{\lambda_1(l_2+l_3+l_{23})+\lambda_3(l_1+l_2+l_{12})} (-1)^{(l_1+l_3+l_{12}+l_{23})(\mathcal{L}+1)} \times$$
$$(-1)^{\frac{\mathcal{L}(\mathcal{L}+1)}{2}} \sqrt{[2l_{12}+1][2l_{23}+1]} \begin{Bmatrix} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{Bmatrix}_q^s \quad (2.40)$$

The parity dependence of $sq6-j\lambda$ symbols can be factored out in a phase factor, so that it is possible to define parity independent $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$ (denoted $sq6-j$) which are related to $sq6-j\lambda$ symbols by

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}_q^s = (-1)^{(\sum_{i=1}^6 \lambda_i)(\sum_{i=1}^6 l_i)+(\sum_{i=1}^6 l_i\lambda_i)} \left\{ \begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{array} \right\}_q^s \tag{2.41}$$

Using equation (2.40), we get the relation between $sq6-j$ symbols and sqRacah coefficients

$$\begin{aligned} \left\{ \begin{array}{ccc} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{array} \right\}_q^s = & (-1)^{\lambda_1(l_2+l_3+l_{23})+\lambda_3(l_1+l_2+l_{12})}(-1)^{(l_1+l_3+l_{12}+l_{23})(\mathcal{L}+1)}(-1)^{\frac{\mathcal{L}(\mathcal{L}+1)}{2}} \times \\ & \frac{(-1)^{(\sum_{i=1}^6 \lambda_i)(\sum_{i=1}^6 l_i)+(\sum_{i=1}^6 l_i\lambda_i)}}{\sqrt{[2l_{12}+1][2l_{23}+1]}} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \end{aligned} \tag{2.42}$$

In order for the $sq6-j\lambda$ and $sq6-j$ to exist, the four superspin triplets

$$\{l_1,l_2,l_3\}, \quad \{l_3,l_4,l_5\}, \quad \{l_1,l_5,l_6\}, \quad \{l_2,l_4,l_6\}, \tag{2.43}$$

must satisfy triangular constraints of the form (2.27).
For more information about sqRC, $sq6-j$ symbols and on bases in representation spaces, see Ref. [4].

III Analytical formulae for Racah coefficients and $6-j$ symbols of the quantum superalgebra $U_q(osp(1|2))$

In order to derive the analytical formula for Racah coefficients, we consider the matrix elements of the following operatotor $\mathcal{P}$ acting in the representation space $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)\otimes V^{l_3}(\lambda_3)$, the tensor product of three representations

$$\mathcal{P} = (id\otimes P^{l_{23}q\otimes}(2,3))P^{lq\otimes}(1,2,3)(P^{l_{12}q\otimes}(1,2)\otimes id) \tag{3.1}$$

i.e., we consider the elements

$$\begin{aligned} Y = & \Phi(e^{l_1}_{l-l_{23}}(\lambda_1)\otimes e^{l_2}_{l_2}(\lambda_2)\otimes e^{l_3}_{l_{23}-l_2}(\lambda_3), (id\otimes P^{l_{23}q\otimes}(2,3))P^{lq\otimes}(1,2,3)\times \\ & (P^{l_{12}q\otimes}(1,2)\otimes id)e^{l_1}_{l_1}(\lambda_1)\otimes e^{l_2}_{l_{12}-l_1}(\lambda_2)\otimes e^{l_3}_{l-l_{12}}(\lambda_3)) \end{aligned} \tag{3.2}$$

where Φ is the phase factor

$$\Phi = (-1)^{(l_1+l_{23}-l+\lambda_1)\lambda_2+(l_3+l_2-l_{23}+\lambda_3)\lambda_2+(l_1+l_{23}-l+\lambda_1)(l_3+l_2-l_{23}+\lambda_3)}(-1)^{\varphi_1(l_1+l_{23}-l)+\varphi_3(l_2+l_3-l_{23})+\sum_{i=1}^3 \varphi_i} \tag{3.3}$$

Acting with the operator $id \otimes P^{l_{23}q\otimes}(2,3)$ on the left and with the operator $P^{l_{12}q\otimes}(1,2) \otimes id$ and using relations (2.16), (2.29) we find that

$$\begin{aligned} Y &= \Phi(l_1 l_1 \lambda_1, l_2 l_{12} - l_1 \lambda_2 | l_{12} l_{12} \lambda_{12})_q (l_2 l_2 \lambda_2, l_3 l_{23} - l_2 \lambda_3 | l_{23} l_{23} \lambda_{23})_q \\ & (e^{l_1}_{l-l_{23}}(\lambda_1) \otimes e^{l_{23}}_{l_{23}}(l_2, l_3, \lambda_{23}), P^{lq\otimes}(1,2,3) e^{l_{12}}_{l_{12}}(l_1, l_2 \lambda_{12}) \otimes e^{l_3}_{l-l_{12}}(\lambda_3)). \end{aligned} \tag{3.4}$$

Then, using relations (2.26), (2.37) , (2.39), we obtain that the matrix element Y is equal to

$$\begin{aligned} Y &= (-1)^{(l_1+l_{23}-l)(l_3+l_2-l_{23})} (l_1 l_1 \lambda_1, l_2 l_{12} - l_1 \lambda_2 | l_{12} l_{12} \lambda_{12})_q (l_2 l_2 \lambda_2, l_3 l_{23} - l_2 \lambda_3 | l_{23} l_{23} \lambda_{23})_q \\ & (l_{12} l_{12} \lambda_{12}, l_3 l - l_{12} \lambda_3 | l l \lambda)_q (l_1 l - l_{23} \lambda_1, l_{23} l_{23} \lambda_{23} | l l \lambda)_q U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q) \end{aligned} \tag{3.5}$$

On the other hand, introducing in the operator

$$P^{lq\otimes}(1,2,3) = \sum_{r=0}^{\infty} c_r(l) (v^{\otimes}_{-}(1,2,3))^r (v^{\otimes}_{+}(1,2,3))^r \tag{3.6}$$

the relations, which follow from (2.4)

$$v^{\otimes}_{+}(1,2,3)^r = \sum_{k=0}^{r} \left[\begin{array}{c} r \\ k \end{array} \right] (q^{-1}) (v^{\otimes}_{+}(1,2)^{r-k} q^{-kH^{\otimes}(1,2)} \otimes v^k_{+}(3) q^{(r-k)H(3)} \tag{3.7}$$

$$v^{\otimes}_{-}(1,2,3)^r = \sum_{k=0}^{r} \left[\begin{array}{c} r \\ k \end{array} \right] (q) (v^{r-k}_{-}(1) q^{-kH(1)} \otimes (v^{\otimes}_{-}(2,3))^k q^{(r-k)H^{\otimes}(2,3)} \tag{3.8}$$

where $\left[\begin{array}{c} r \\ k \end{array} \right] (q) = \frac{[r]!}{[r-k]!\,[k]!}$ is the graded quantum Newton symbol, and using relations (2.11) we get

$$\mathcal{P} = \sum_{r=0}^{\infty} c_r(l) (id \otimes P^{l_{23}q\otimes}(2,3)) (v^r_{-}(1) q^{-rH(1)} \otimes id \otimes q^{rH(3)} v^r_{+}(3)) (P^{l_{12}q\otimes}(1,2) \otimes id) \tag{3.9}$$

Therefore Y can be written as the sum of products of matrix elements of three operators $(id \otimes P^{l_{23}q\otimes}(2,3))$, $(v^r_{-}(1) q^{-rH(1)} \otimes id \otimes q^{rH(3)} v^r_{+}(3))$ and $(P^{l_{12}q\otimes}(1,2) \otimes id)$:

$$\begin{aligned} Y &= \sum_{r=0}^{\infty} (-1)^{r(l_1+l_3+l_{23}+l_{12})} c_r(l) \Phi \times \\ & (e^{l_1}_{l-l_{23}}(\lambda_1) \otimes e^{l_2}_{l_2}(\lambda_2) \otimes e^{l_3}_{l_{23}-l_2}(\lambda_3), (id \otimes P^{l_{23}q\otimes}(2,3)) \\ & e^{l_1}_{l-l_{23}}(\lambda_1) \otimes e^{l_2}_{l_{12}+l_{23}-l-r}(\lambda_2) \otimes e^{l_3}_{l-l_{12}+r}(\lambda_3)) \times \\ & (e^{l_1}_{l-l_{23}}(\lambda_1) \otimes e^{l_2}_{l_{12}+l_{23}-l-r}(\lambda_2) \otimes e^{l_3}_{l-l_{12}+r}(\lambda_3), (v^r_{-}(1) q^{-rH(1)} \otimes id \otimes q^{rH(3)} v^r_{+}(3)) \\ & e^{l_1}_{l-l_{23}+r}(\lambda_1) \otimes e^{l_2}_{l_{12}+l_{23}-l-r}(\lambda_2) \otimes e^{l_3}_{l-l_{12}}(\lambda_3)) \times \\ & (e^{l_1}_{l-l_{23}+r}(\lambda_1) \otimes e^{l_2}_{l_{12}+l_{23}-l-r}(\lambda_2) \otimes e^{l_3}_{l-l_{12}}(\lambda_3), (P^{l_{12}q\otimes}(1,2) \otimes id) \\ & e^{l_1}_{l_1}(\lambda_1) \otimes e^{l_2}_{l_{12}-l_1}(\lambda_2) \otimes e^{l_3}_{l-l_{12}}(\lambda_3)) \end{aligned} \tag{3.10}$$

Where we have written only terms which give nonvanishing contribution in the products of matrix elements. Using relations (2.9), (2.15), (2.29) we may rewrite Y in the form

$$\begin{aligned} Y &= \sum_{r=0}^{\infty}(-1)^{(l_3-l_{12}+l)(l_{23}+l_2+l_{12}+l)}(-1)^{r(l_1+l_{23}-l+\lambda_1+\lambda_2)}(-1)^{\frac{r(r+1)}{2}}q^{\frac{r}{2}(l_{23}-l_{12})}\frac{[2l+1]!}{[2l+r+1]!\,[r]!}\times \\ & \left(\frac{[l_1-l_{23}+l-r]!\,[l_1+l_{23}-l]!\,[l_3+l_{12}-l]!\,[l_3-l_{12}+l+r]!}{[l_1+l_{23}-l-r]!\,[l_1-l_{23}+l]!\,[l_3-l_{12}+l]!\,[l_3+l_{12}-l-r]!}\right)^{\frac{1}{2}}\times \\ & (l_1l_1\lambda_1,l_2l_{12}-l_1\lambda_2|l_{12}l_{12}\lambda_{12})_q(l_1l-l_{23}+r\lambda_1,l_2l_{12}+l_{23}-l-r\lambda_2|l_{12}l_{12}\lambda_{12})_q\times \\ & (l_2l_2\lambda_2,l_3l_{23}-l_2\lambda_3|l_{23}l_{23}\lambda_{23})_q(l_2l_{12}+l_{23}-l-r\lambda_2,l_3l-l_{12}+r\lambda_3|l_{23}l_{23}\lambda_{23})_q \end{aligned} \tag{3.11}$$

Comparing this expression with (3.5), we obtain the formula for sqRacah coefficients

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) &= (-1)^{(l_3-l_{12}+l)(l_{23}+l_2+l_{12}+l)+(l_3+l_2-l_{23})(l_{23}+l_1-l)}\times \\ \sum_r(-1)^{r(l_1+l_{23}-l+\lambda_1+\lambda_2)}&(-1)^{\frac{r(r+1)}{2}}q^{\frac{r}{2}(l_{23}-l_{12})}\frac{[2l+1]!}{[2l+r+1]!\,[r]!}\times \\ \left(\frac{[l_1-l_{23}+l-r]!\,[l_1+l_{23}-l]!\,[l_3+l_{12}-l]!\,[l_3-l_{12}+l+r]!}{[l_1+l_{23}-l-r]!\,[l_1-l_{23}+l]!\,[l_3-l_{12}+l]!\,[l_3+l_{12}-l-r]!}\right)^{\frac{1}{2}}&\times \\ \frac{(l_1l-l_{23}+r\lambda_1,l_2l_{12}+l_{23}-l-r\lambda_2|l_{12}l_{12}\lambda_{12})_q(l_2l_{12}+l_{23}-l-r\lambda_2,l_3l-l_{12}+r\lambda_3|l_{23}l_{23}\lambda_{23})_q}{(l_{12}l_{12}\lambda_{12},l_3l-l_{12}\lambda_3|ll\lambda)_q(l_1l-l_{23}\lambda_1,l_{23}l_{23}\lambda_{23}|ll\lambda)_q}& \end{aligned} \tag{3.12}$$

Substituting here the explicit expression (2.30) of Clebsch-Gordan coefficients, we obtain a general analytical formula for sqRacah coefficients in the form of a single sum of factorial fractions

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) &= (-1)^{\lambda_2(l_{23}+l_2+l_{12}+l)+(l_3+l_{12}+l)(l_{23}+l_1+l)}\times \\ (-1)^{\frac{1}{2}(l_2-l_{23}-l_{12}+l)(l_2-l_{23}-l_{12}+l+1)}&([2l_{12}+1]\,[2l_{23}+1])^{\frac{1}{2}}\times \\ \frac{\Delta(l_1,l_2,l_{12})\Delta(l_2,l_3,l_{23})\Delta(l_{12},l_3,l)\Delta(l_1,l_{23},l)\,[l_1+l_{23}+l+1]!\,[l_3+l_{12}+l+1]!}{[l_1-l_{23}+l]!\,[l_3-l_{12}+l]!\,[l_2-l_1+l_{12}]!\,[l_1-l_2+l_{12}]!\,[l_2-l_3+l_{23}]!\,[l_3-l_2+l_{23}]!}& \\ \sum_r\frac{(-1)^{r(l_1+l_2+l_3+l+1)}(-1)^{\frac{r(r+1)}{2}}\,[l_1+l-l_{23}+r]!\,[l_3+l-l_{12}+r]!\,[l_2+l_{23}+l_{12}-l-r]!}{[2l+r+1]!\,[r]!\,[l_1-l+l_{23}-r]!\,[l_3-l+l_{12}-r]!\,[l_2-l_{23}-l_{12}+l+r]!}& \end{aligned} \tag{3.13}$$

where the integer summation index r runs over all values for which the arguments of all the factorials are non-negative.

This formula shows that sqRacah coefficients, similarly to Clebsch-Gordan coefficients for $U_q(osp(1|2))$, depend on the parities λ_i but depend neither on the class ε nor on the signature parameters φ_i, ψ_i $i = 1,2,3$. In Ref.[5] a similar general analytical expression for Racah coefficients of the quantum algebra $U_q(su(2))$ has been derived through the same method of projection operators.

Using relation (2.40) we get the following analytical formula for the parity dependent $sq6-j\lambda$ symbols

$$\left\{ \begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{array} \right\}_q^s = (-1)^{\frac{1}{2}(l_1+l_3+l_{23}+l_{12})(l_1+l_3+l_{23}+l_{12}+1)} \times$$

$$(-1)^{\lambda_1(l_2+l_3+l_{23})+\lambda_2(l_{23}+l_2+l_{12}+l)+\lambda_3(l_1+l_2+l_{12})+(l_1+l_{23})(l_3+l_{12})+l(l_1+l_3+l_{23}+l_{12}+1)} \times \tag{3.14}$$
$$\frac{\Delta(l_1,l_2,l_{12})\Delta(l_2,l_3,l_{23})\Delta(l_{12},l_3,l)\Delta(l_1,l_{23},l)\,[l_1+l_{23}+l+1]!\,[l_3+l_{12}+l+1]!}{[l_1-l_{23}+l]!\,[l_3-l_{12}+l]!\,[l_2-l_1+l_{12}]!\,[l_1-l_2+l_{12}]!\,[l_2-l_3+l_{23}]!\,[l_3-l_2+l_{23}]!} \times$$
$$\sum_r \frac{(-1)^{r(l_1+l_2+l_3+l+1)}(-1)^{\frac{r(r+1)}{2}}\,[l_1+l-l_{23}+r]!\,[l_3+l-l_{12}+r]!\,[l_2+l_{23}+l_{12}-l-r]!}{[2l+r+1]!\,[r]!\,[l_1-l+l_{23}-r]!\,[l_3-l+l_{12}-r]!\,[l_2-l_{23}-l_{12}+l+r]!}$$

Factorization of the parity dependence (2.42) yields the analytical formula for the parity independent $sq6-j$ symbols

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}_q^s = (-1)^{(l_2+l_5)(\sum_{i=1}^6 l_i)+(l_1+l_6)(l_3+l_4)+l_2l_5+l_3l_4+l_1l_6} \times$$

$$(-1)^{\frac{1}{2}(l_1+l_3+l_4+l_6)(l_1+l_3+l_4+l_6+1)}\Delta(l_1,l_2,l_3)\Delta(l_2,l_4,l_6)\Delta(l_3,l_4,l_5)\Delta(l_1,l_5,l_6) \times \tag{3.15}$$
$$\frac{[l_1+l_6+l_5+1]!\,[l_3+l_4+l_5+1]!}{[l_1-l_6+l_5]!\,[l_4-l_3+l_5]!\,[l_2-l_1+l_3]!\,[l_1-l_2+l_3]!\,[l_2-l_4+l_6]!\,[l_4-l_2+l_6]!} \times$$
$$\sum_r \frac{(-1)^{r(l_1+l_2+l_4+l_5+1)}(-1)^{\frac{r(r+1)}{2}}\,[l_1+l_5-l_6+r]!\,[l_4+l_5-l_3+r]!\,[l_2+l_6+l_3-l_5-r]!}{[2l_5+r+1]!\,[r]!\,[l_1-l_5+l_6-r]!\,[l_4-l_5+l_3-r]!\,[l_2-l_6-l_3+l_5+r]!}$$

This formula is rather complicated and unsymmetrical. However, using repeatedly the algebraic identities (B.2-B.4) given in Appendix B, it is possible to transform this formula into the following more symmetrical analytical expression

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}_q = (-1)^{(l_2+l_5)(\sum_{i=1}^6 l_i)+(l_1+l_6)(l_3+l_4)+l_2l_5+l_3l_4+l_1l_6} \times$$

$$(-1)^{\frac{1}{2}(l_1+l_3+l_4+l_6)(l_1+l_3+l_4+l_6+1)}\Delta(l_1,l_2,l_3)\Delta(l_2,l_4,l_6)\Delta(l_3,l_4,l_5)\Delta(l_1,l_5,l_6) \times$$
$$\sum_v \frac{(-1)^{v(l_1+l_3+l_4+l_6)}(-1)^{\frac{v(v+1)}{2}}\,[l_1+l_3+l_4+l_6+1-v]!}{[l_4-l_2+l_6-v]!\,[l_3-l_5+l_4-v]!\,[l_1-l_5+l_6-v]!} \times \tag{3.16}$$
$$\frac{1}{[v]!\,[l_5+l_2-l_1-l_4+v]!\,[l_5+l_2-l_3-l_6+v]!\,[l_1+l_3-l_2-v]!}$$

where v runs on integer values such that all arguments of the factorials are non-negative.This formula is very similar in form to the corresponding well known formula given by Racah for the classical case of $su(2)$[6]. Finally, performing the summation index substitution

$$z = l_1+l_3+l_4+l_6-v, \tag{3.17}$$

we obtain the simplest and very symmetrical analytical formula for $sq6-j$ symbols

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}_q^s = (-1)^{\frac{1}{2}(\sum_{i=1}^6 l_i)(\sum_{i=1}^6 l_i+1)+\sum_{i=1}^6 \frac{1}{2}l_i(l_i-1)} \times$$

$$\Delta(l_1,l_2,l_3)\Delta(l_2,l_4,l_6)\Delta(l_3,l_4,l_5)\Delta(l_1,l_5,l_6) \times \tag{3.18}$$
$$\sum_z \frac{(-1)^{\frac{1}{2}z(z-1)}\,[z+1]!}{[z-l_1-l_2-l_3]!\,[z-l_4-l_2-l_6]!\,[z-l_3-l_4-l_5]!\,[z-l_1-l_5-l_6]!} \times$$
$$\frac{1}{[l_1+l_3+l_4+l_6-z]!\,[l_5+l_2+l_3+l_6-z]!\,[l_1+l_2+l_4+l_5-z]!}$$

It is quite remarquable that, except for the phase factors, this formula has exactly the same form as Regge formula for $6-j$ symbols for the algebra $su(2)$[7].

From these analytical formulae one can calculate particular values of $sq6-j$ symbols. For example, when one argument vanishes, the value of the $sq6-j$ symbols is

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_3 & 0 & l_1 \end{array} \right\}_q^s = \frac{(-1)^{\frac{1}{2}(l_1+l_2-l_3)(l_1+l_2+l_3+1)}}{\sqrt{[2l_1+1][2l_3+1]}}, \tag{3.19}$$

In the same way, if $l_5 = l_3+l_4$, only one value of the summation index z is possible in formula (3.14), and we obtain a simple expression for the corresponding $sq6-j$ symbols

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_3+l_4 & l_6 \end{array} \right\}_q^s =$$

$$= (-1)^{l_2(l_1+l_2+l_6)+l_1l_6+l_3l_4}(-1)^{\frac{1}{2}(l_1+l_3+l_4+l_6)(l_1+l_3+l_4+l_6+1)} \times \tag{3.20}$$
$$\left(\frac{[2l_3]!\,[2l_4]!\,[l_3+l_4+l_6-l_1]!\,[l_1+l_2-l_3]!\,[l_2+l_6-l_4]!}{[l_1-l_2+l_3]!\,[-l_1+l_2+l_3]!\,[l_1+l_2+l_3+1]!\,[l_4+l_2+l_6+1]!}\right)^{\frac{1}{2}} \times$$
$$\left(\frac{[l_1+l_3+l_4-l_6]!\,[l_1+l_3+l_4+l_6+1]!}{[l_4-l_2+l_6]!\,[l_1-l_3-l_4+l_6]!\,[l_4+l_2-l_6]!\,[2l_3+2l_4+1]!}\right)^{\frac{1}{2}}$$

It is also possible, for small fixed values of one highest weight, to derive from the general formula simple algebraic expressions for particular $sq6-j$ symbols. For instance, in Appendix A , the analytic expressions of $sq6-j$ symbols in which $l_4=1$ are given.

IV Properties of sqRacah coefficients and $sq6-j$ symbols

A Symmetry properties

We start our analysis of symmetry properties that follow from the analytical formulae, by considering the symmetry related to the substitution $q \to q^{-1}$. The invariance of sqRacah coefficients and $sq6-j$ symbols with respect to this operation is not obvious since the symbol $[n]$ itself is not invariant when $q \longrightarrow q^{-1}$. Namely, we have

$$[n] \equiv [n]_q = (-1)^{n+1}\,[n]_{q^{-1}} \tag{4.1}$$

By a direct calculation, one can easily check that, althought the symbol $[n]$ itself is not invariant, the analytical formulae for sqRacah coefficients, $sq6-j\lambda$ and $sq6-j$ symbols remain globally invariant with respect to the substitution $q \longrightarrow q^{-1}$,that is we have

$$U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q) = U^s(l_1, l_2, l_3, l, l_{12}, l_{23}, q^{-1}) \tag{4.2}$$

and

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}^s_q = \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}^s_{q^{-1}}, \quad \left\{ \begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{array} \right\}^s_q = \left\{ \begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{array} \right\}^s_{q^{-1}} \tag{4.3}$$

From the formula (3.18), it follows immediately that the $sq6-j\lambda$ symbols satisfy the same symmetry properties as $su(2)$ and $osp(1|2)$ $6-j$ symbols. Namely, they are invariant under any permutation of columns and they are invariant under interchange of upper and lower arguments in each pair of columns, i.e., they possess the tetrahedral S_4 symmetry. This result was already obtained in Ref.[4] where it had been derived, in a rather laborious way, from the the symmetry properties of $sq3-j$ symbols, whereas here it is straightforward. Indeed, the analytical formula (3.18) exhibits in the best way all symmetries of $sq6-j$ symbols. In particular it follows from it that $sq6-j$ symbols possess additional symmetries of Regge type. Let us consider the following Regge tranformations of highest weights l_i, $(i = 1, 2, .., 6)$ in $sq6-j$ symbol

$$\begin{aligned} l_1^{'} &= l_1, \quad l_2^{'} = \frac{1}{2}(l_2 + l_3 + l_5 - l_6), \quad l_3^{'} = \frac{1}{2}(l_2 + l_3 + l_6 - l_5), \\ l_4^{'} &= l_4, \quad l_5^{'} = \frac{1}{2}(l_2 + l_6 + l_5 - l_3), \quad l_6^{'} = \frac{1}{2}(l_3 + l_6 + l_5 - l_2) \end{aligned} \tag{4.4}$$

$$\begin{aligned} l_1^{''} &= \frac{1}{2}(l_1 + l_2 + l_4 - l_5), \quad l_2^{''} = \frac{1}{2}(l_2 + l_1 + l_5 - l_4), \quad l_3^{''} = l_3, \\ l_4^{''} &= \frac{1}{2}(l_4 + l_1 + l_5 - l_2), \quad l_5^{''} = \frac{1}{2}(l_4 + l_2 + l_5 - l_1), \quad l_6^{''} = l_6 \end{aligned} \tag{4.5}$$

$$\begin{aligned} l_1^{'''} &= \frac{1}{2}(l_1 + l_3 + l_4 - l_6), \quad l_2^{'''} = l_2, \quad l_3^{'''} = \frac{1}{2}(l_1 + l_3 + l_6 - l_4), \\ l_4^{'''} &= \frac{1}{2}(l_4 + l_1 + l_6 - l_3), \quad l_5^{'''} = l_5, \quad l_6^{'''} = \frac{1}{2}(l_4 + l_6 + l_3 - l_1). \end{aligned} \tag{4.6}$$

We remind that for the quantum superalgebra $U_q(osp(1|2))$ the highest weights are all integers. This is very important for the above tranformations since one can check that in general

$l'_i, l''_i, l'''_i (i = 1, 2, ..., 6)$ in the above relations need not be necessarily integers. For example if $l_i, (i = 1, 2, ..., 6)$ are the following

$$l_1 = 10, \quad l_2 = 6, \quad l_3 = 5, \quad l_4 = 3, \quad l_5 = 4, \quad l_6 = 6, \tag{4.7}$$

then for the third transformation (4.6) we get

$$l'''_1 = 6, \quad l'''_2 = 6, \quad l'''_3 = 9, \quad l'''_4 = 7, \quad l'''_5 = 4, \quad l'''_6 = 2, \tag{4.8}$$

but for the first transformation (4.4) we obtain

$$l'_1 = 10, \quad l'_2 = \frac{9}{2}, \quad l'_3 = \frac{13}{2}, \quad l'_4 = 3, \quad l'_5 = \frac{11}{2}, \quad l'_6 = \frac{9}{2}. \tag{4.9}$$

So, in this case, $l'_i, (i = 1, 2, .., 6)$ are no longer highest weights for the quantum superalgebra $U_q(osp(1|2))$. Therefore, it appears that some conditions have to be satisfied in order that Regge transformations be true symmetries of $sq6 - j$ symbols.

Let us introduce the following notation

$$L_{i,j,k} = l_i + l_j + l_k \tag{4.10}$$

for $i, j, k = 1, 2, ..., 6$. In this notation we have the following conditions for integrity of l'_i, l''_i, l'''_i in relations (4.4-4.6) :

$$\text{if } L_{1,2,3} = L_{1,5,6} \pmod 2, \quad \text{then} \quad l'_i, (i = 1, 2, .., 6) \text{ are integers} \tag{4.11}$$

$$\text{if } L_{1,2,3} = L_{3,4,5} \pmod 2, \quad \text{then} \quad l''_i, (i = 1, 2, .., 6) \text{ are integers} \tag{4.12}$$

$$\text{if } L_{1,2,3} = L_{2,4,6} \pmod 2, \quad \text{then} \quad l'''_i, (i = 1, 2, .., 6) \text{ are integers} \tag{4.13}$$

Note that, because of the relation

$$L_{1,2,3} + L_{1,5,6} + L_{2,4,6} + L_{3,4,5} = 0 \pmod 2, \tag{4.14}$$

conditions (4.11-4.13) could have been written in a different form; for instance, condition (4.11) is equivalent to the condition $L_{2,4,6} = L_{3,4,5} \pmod 2$.

If we introduce relations (4.4-4.6) into the analytical formula (3.18), we obtain the following symmetries of the $sq6 - j$ symbols

$$\text{if } L_{1,2,3} = L_{1,5,6} \pmod 2, \text{ then}$$

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}^s_q = \left\{ \begin{matrix} l_1 & \frac{1}{2}(l_2 + l_3 + l_5 - l_6) & \frac{1}{2}(l_2 + l_3 + l_6 - l_5) \\ l_4 & \frac{1}{2}(l_2 + l_6 + l_5 - l_3) & \frac{1}{2}(l_3 + l_6 + l_5 - l_2) \end{matrix} \right\}^s_q \tag{4.15}$$

$$\text{if } L_{1,2,3} = L_{3,4,5} \pmod 2, \text{ then}$$

$$\left\{\begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array}\right\}_q^s = \left\{\begin{array}{ccc} \frac{1}{2}(l_1+l_2+l_4-l_5) & \frac{1}{2}(l_2+l_1+l_5-l_4) & l_3 \\ \frac{1}{2}(l_4+l_1+l_5-l_2) & \frac{1}{2}(l_4+l_2+l_5-l_1) & l_6 \end{array}\right\}_q^s \qquad (4.16)$$

if $L_{1,2,3} = L_{2,4,6}\ (\mathrm{mod}\, 2)$, then

$$\left\{\begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array}\right\}_q^s = \left\{\begin{array}{ccc} \frac{1}{2}(l_1+l_3+l_4-l_6) & l_2 & \frac{1}{2}(l_1+l_3+l_6-l_4) \\ \frac{1}{2}(l_4+l_1+l_6-l_3) & l_5 & \frac{1}{2}(l_4+l_6+l_3-l_1) \end{array}\right\}_q^s \qquad (4.17)$$

We shall stress here that the right hand side of the analytical formula (3.18) is invariant with respect to the Regge transformations (4.4-4.6) without any condition. Conditions (4.11-4.13) are necessary to preserve the integrity of highest weights on the left hand side of the analytical formula, i.e., in the $sq6-j$ symbol.

One can verify that any $sq6-j$ symbol always satisfy one of the conditions (4.11-4.13) and therefore any $sq6-j$ symbol has one of the Regge symmetries (4.15-4.17). This means that the symmetry group of any $sq6-j$ symbol is at least $S_4 \times S_2$ and contains at least 48 elements.

If an $sq6-j$ symbol satisfies two of three conditions (4.11-4.13), then, because of the relation (4.14), the third one is also satisfied and we have

$$L_{1,2,3} = L_{1,5,6} = L_{2,4,6} = L_{3,4,5}\ (\mathrm{mod}\, 2) \qquad (4.18)$$

In this particular case the $sq6-j$ symbol possesses the full set of Regge symmetries, that is besides symmetries (4.15-4.17) we have also

$$\left\{\begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array}\right\}_q^s = \left\{\begin{array}{ccc} \frac{1}{2}(l_2+l_5+l_3-l_6) & \frac{1}{2}(l_1+l_3+l_6-l_4) & \frac{1}{2}(l_1+l_2+l_4-l_5) \\ \frac{1}{2}(l_2+l_5+l_6-l_3) & \frac{1}{2}(l_4+l_6+l_3-l_1) & \frac{1}{2}(l_4+l_1+l_5-l_2) \end{array}\right\}_q^s \qquad (4.19)$$

$$= \left\{\begin{array}{ccc} \frac{1}{2}(l_2+l_3+l_6-l_5) & \frac{1}{2}(l_1+l_3+l_4-l_6) & \frac{1}{2}(l_2+l_1+l_5-l_4) \\ \frac{1}{2}(l_3+l_6+l_5-l_2) & \frac{1}{2}(l_4+l_1+l_6-l_3) & \frac{1}{2}(l_4+l_2+l_5-l_1) \end{array}\right\}_q^s \qquad (4.20)$$

and together with the 24 tetrahedral symmetries, Regge symmetries form a group of rank 144, isomorphic to the group $S_4 \times S_3$. Let us remark that in this case, similarly as in the classical case of $su(2)$, in the set of five Regge symmetries (4.15-4.17,4.19,4.20) only one (4.15) is essentially new, the other ones can be obtained from it and the tetrahedral symmetry.

For the classical case of $su(2)$, in order to exhibit all symmetries of $6-j$ symbols, Bargmann [11], [12] proposed to associate a 3×4 array to a $6-j$ symbol. With this description, each of the 144 symmetries of $6-j$ symbol is represented by some combinations of permutations of rows and columns of the array. This Bargmann representation of the full symmetries of a $6-j$ symbols, can be extended in a natural way to the case of $sq6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. Let us associate to an $sq6-j$ symbol an array in the following way

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}^s_q = \left[\begin{array}{ccc} (-1)^{L_{2,4,6}}(l_6+l_4-l_2) & (-1)^{L_{1,5,6}}(l_6+l_5-l_1) & (-1)^{L_{3,4,5}}(l_4+l_5-l_3) \\ (-1)^{L_{1,5,6}}(l_6+l_1-l_5) & (-1)^{L_{2,4,6}}(l_6+l_2-l_4) & (-1)^{L_{1,2,3}}(l_1+l_2-l_3) \\ (-1)^{L_{3,4,5}}(l_4+l_3-l_5) & (-1)^{L_{1,2,3}}(l_3+l_2-l_1) & (-1)^{L_{2,4,6}}(l_4+l_2-l_6) \\ (-1)^{L_{1,2,3}}(l_3+l_1-l_2) & (-1)^{L_{3,4,5}}(l_5+l_3-l_4) & (-1)^{L_{1,5,6}}(l_5+l_1-l_6) \end{array} \right] \tag{4.21}$$

This array differs from the original one of Bargmann by the presence of phase factors in the entries of the array. The 24 tetrahedral symmetries of the $sq6-j$ symbol are represented by permutations of rows and columns of this array and the phases in the entries of the array transform in the same way as the remaining part of the entries. It is not the case for Regge tranformations, where the phases tranform in a different way than the remaining parts of the entries. For instance, for the third Regge tranformation (4.6) we have

$$\left\{ \begin{array}{ccc} \frac{1}{2}(l_1+l_3+l_4-l_6) & l_2 & \frac{1}{2}(l_1+l_3+l_6-l_4) \\ \frac{1}{2}(l_4+l_1+l_6-l_3) & l_5 & \frac{1}{2}(l_4+l_6+l_3-l_1) \end{array} \right\}^s_q =$$

$$= \left[\begin{array}{ccc} (-1)^{L_{1,2,3}}(l_6+l_4-l_2) & (-1)^{L_{3,4,5}}(l_6+l_5-l_1) & (-1)^{L_{1,5,6}}(l_4+l_5-l_3) \\ (-1)^{L_{3,4,5}}(l_4+l_3-l_5) & (-1)^{L_{2,4,6}}(l_3+l_2-l_1) & (-1)^{L_{1,2,3}}(l_4+l_2-l_6) \\ (-1)^{L_{1,5,6}}(l_6+l_1-l_5) & (-1)^{L_{1,2,3}}(l_6+l_2-l_4) & (-1)^{L_{2,4,6}}(l_1+l_2-l_3) \\ (-1)^{L_{1,2,3}}(l_3+l_1-l_2) & (-1)^{L_{1,5,6}}(l_5+l_3-l_4) & (-1)^{L_{3,4,5}}(l_5+l_1-l_6) \end{array} \right] \tag{4.22}$$

Comparing (4.21) and (4.22), we see that the right hand side of (4.22) is related to that one of (4.21) by a permutation of the second and third rows only if condition (4.13) is satisfied, i.e., if

$$L_{1,2,3} = L_{2,4,6} \pmod 2 \Leftrightarrow L_{1,5,6} = L_{3,4,5} \pmod 2 \tag{4.23}$$

Therefore, Regge transformations (4.4-4.6) are represented by permutations of rows of the array only if their respective conditions (4.11-4.13) are satisfied and then they are symmetries of an $sq6-j$ symbol. If condition (4.14) holds, then all phases in the array (4.21) are identical and the corresponding $sq6-j$ symbol possesses the full $S_4 \times S_3$ symmetry.

In [2] it has been shown that Clebsch-Gordan coefficients possess also a Regge type symmetry. It seems, however, that similarly to the classical case of $su(2)$, there is no connection between Regge symmetries of sqClebsch-Gordan coefficients and Regge symmetry of $sq6-j$ symbols.

B The limit $q = 1$

In the limit $q = 1$, Racah coefficients, $sq6-j\lambda$ symbols and $sq6-j$ symbols become coefficients and symbols of the non deformed Lie superalgebra $osp(1|2)$ and present similar symmetry properties. This follows from the fact that, in the limit under consideration, Clebsch-Gordan coefficients, projection operators P^q, P^{mq} and bases $e^{lq}_m(\lambda)$, $e^{lq}_m(l_1, l_{23}, \lambda)$, $e^{lq}_m(l_{12}, l_3, \lambda)$ become in a continuous way Clebsch-Gordan coefficients, projection operators and bases for the Lie superalgebra $osp(1|2)$ (for more explicit formulae see [1], [2] and [4]).

In particular, for $q = 1$, the $sq6-j$ symbols become identical, up to a phase factor $(-1)^{\Psi}$, to the $s6-j$ symbols of the superalgebra $osp(1|2)$ defined in Ref.[13]

$$(-1)^{\Psi} \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{array} \right\}^{s}_{q=1} = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}^{S}_{osp(1|2)} , \tag{4.24}$$

where $j_i = \dfrac{l_i}{2}$ is the superspin and the phase Ψ is

$$\Psi = \sum_{i=1}^{6} l_i + (l_1 + l_4)(l_2 + l_5) + (l_2 + l_5)(l_3 + l_6) + (l_3 + l_6)(l_1 + l_4) . \tag{4.25}$$

The phase difference between the two symbols in (4.24) derives from the fact that in Ref.[13] a different basis in the representation space had been used.

Thus, in the limit $q \longrightarrow 1$, the functions defined by the analytical formula (3.18) of $sq6-j$ symbols, tend continuously towards the numerical values of the corresponding $s6-j$ symbol. However, when $q \longrightarrow 1$, the analytical formula (3.18) loses its compact and symmetrical form. This phenomenon follows from the fact that, in the limit $q \longrightarrow 1$, the value of the symbol $[n]$ depends on the parity of its argument n. Indeed, from definition (2.10) of the symbol $[n]$, it follows that

$$[n] = \left\{ \begin{array}{ll} \dfrac{\sinh(\frac{\eta}{4})n}{\cosh(\frac{\eta}{4})}, & \text{if} \quad n \quad \text{even} \\ \dfrac{\cosh(\frac{\eta}{4})n}{\cosh(\frac{\eta}{4})}, & \text{if} \quad n \quad \text{odd} \end{array} \right. \quad \Rightarrow \quad \lim_{q\to 1} [n] = \left\{ \begin{array}{ll} 0, & \text{if} \quad n \quad \text{even} \\ 1, & \text{if} \quad n \quad \text{odd.} \end{array} \right. \tag{4.26}$$

Thus we have, for instance,

$$\lim_{q\to 1} \frac{[l_1 + l_2 + l_3]}{[l_1 + l_2 - l_3]} = \left\{ \begin{array}{ll} \dfrac{l_1 + l_2 + l_3}{l_1 + l_2 - l_3}, & \text{if} \quad l_1 + l_2 + l_3 \quad \text{even} \\ 1, & \text{if} \quad l_1 + l_2 + l_3 \quad \text{odd} \end{array} \right. . \tag{4.27}$$

Therefore, it is impossible to write down the analytical formula (3.18) for $q = 1$ without specifying the parities of arguments of all symbols $[n]$ in the formula, i.e., for $q = 1$, one cannot write an analytical formula for $s6-j$ symbols in a compact, symmetrical form similar to Eq.(3.18).

However, in order to calculate the numerical value of a given $s6-j$ symbol of the non deformed superalgebra $osp(1|2)$, it is always possible to calculate from Eq.(3.18) the corresponding $sq6-j$ symbol, which is a function of q, and then calculate the limit $q \longrightarrow 1$. The non deformed case of the superalgebra $osp(1|2)$ has been studied separately in Ref.[13] where, due to the inclusion $sl(2) \subset osp(1|2)$, it was possible to analyze all properties of $s6-j$ symbols and to compute their numerical values without knowledge of their analytical formula. For all $sq6-j$ symbols with $l_i \leq 4$,we have checked [14] that their limit $q \longrightarrow 1$ is equal to the numerical values of the corresponding $s6-j$ symbols tabulated in Ref.[13]

Appendices

A Analytic expression of $sq6-j$ symbols with $l_4=1$

TABLE. The symbols $\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ 1 & l_5 & l_6 \end{matrix} \right\}_q^s$

$l_6 \backslash l_3$	$l_3 = l_5+1$
l_2+1	$\Lambda \left(\frac{[l_1+l_2-l_5]\,[l_1+l_2-l_5+1]\,[l_1-l_2+l_5]\,[l_1-l_2+l_5+1]}{[2l_2+1]\,[2l_2+2]\,[2l_2+3]\,[2l_5+1]\,[2l_5+2]\,[2l_5+3]} \right)^{\frac{1}{2}}$
l_2	$(-1)^{l_1+l_2+1}\Lambda \left(\frac{[l_1+l_2+l_5+2]\,[l_1-l_2+l_5+1]\,[-l_1+l_2+l_5+1]\,[l_1+l_2-l_5]\,[2]}{[2l_2]\,[2l_2+1]\,[2l_2+2]\,[2l_5+1]\,[2l_5+2]\,[2l_5+3]} \right)^{\frac{1}{2}}$
l_2-1	$\Lambda \left(\frac{[-l_1+l_2+l_5]\,[-l_1+l_2+l_5+1]\,[l_1+l_2+l_5+2]\,[l_1+l_2+l_5+1]}{[2l_2-1]\,[2l_2]\,[2l_2+1]\,[2l_5+1]\,[2l_5+2]\,[2l_5+3]} \right)^{\frac{1}{2}}$
	$l_3 = l_5$
l_2+1	$(-1)^{l_1+l_5+1}\Lambda \left(\frac{[l_1+l_2+l_5+2]\,[l_1-l_2+l_5]\,[l_1+l_2-l_5+1]\,[-l_1+l_2+l_5+1]\,[2]}{[2l_2+3]\,[2l_2+1]\,[2l_2+2]\,[2l_5]\,[2l_5+1]\,[2l_5+2]} \right)^{\frac{1}{2}}$
l_2	$(-1)^{l_1+1}\Lambda \frac{[l_1+l_2-l_5]\,[l_1-l_2+l_5]+(-1)^{l_1+l_5+l_2}\,[-l_1+l_2+l_5]\,[l_1+l_2+l_5+2]}{([2l_2]\,[2l_2+1]\,[2l_2+2]\,[2l_5]\,[2l_5+1]\,[2l_5+2])^{\frac{1}{2}}}$
l_2-1	$(-1)^{l_2+1}\Lambda \left(\frac{[l_1+l_2+l_5+1]\,[l_1+l_2-l_5]\,[-l_1+l_2+l_5]\,[l_1-l_2+l_5+1]\,[2]}{[2l_2-1]\,[2l_2]\,[2l_2+1]\,[2l_5]\,[2l_5+1]\,[2l_5+2]} \right)^{\frac{1}{2}}$
	$l_3 = l_5-1$
l_2+1	$\Lambda \left(\frac{[l_1+l_2+l_5+1]\,[l_1+l_2+l_5+2]\,[-l_1+l_2+l_5]\,[-l_1+l_2+l_5+1]}{[2l_2+1]\,[2l_2+2]\,[2l_2+3]\,[2l_5-1]\,[2l_5]\,[2l_5+1]} \right)^{\frac{1}{2}}$
l_2	$(-1)^{l_5+1}\Lambda \left(\frac{[l_1+l_2+l_5+1]\,[l_1+l_2-l_5+1]\,[l_1-l_2+l_5]\,[-l_1+l_2+l_5]\,[2]}{[2l_2+2]\,[2l_2]\,[2l_2+1]\,[2l_5]\,[2l_5+1]\,[2l_5-1]} \right)^{\frac{1}{2}}$
l_2-1	$-\Lambda \left(\frac{[l_1+l_2-l_5]\,[l_1+l_2-l_5+1]\,[l_1-l_2+l_5]\,[l_1-l_2+l_5+1]}{[2l_2-1]\,[2l_2]\,[2l_2+1]\,[2l_5-1]\,[2l_5]\,[2l_5+1]} \right)^{\frac{1}{2}}$

where $\Lambda = (-1)^{\frac{1}{2}(l_1+l_2+l_5)(l_1+l_2+l_5+1)}$.

B Algebraic identities

From the equation

$$\left[\begin{array}{c} n+r \\ k \end{array}\right] = \sum_{i=0}^{r} \left[\begin{array}{c} n \\ k-i \end{array}\right] \left[\begin{array}{c} r \\ i \end{array}\right] (-1)^{(k-i)(r-i)} q^{\frac{(r-i)(n+r)}{2}} q^{-\frac{r(n-k+r)}{2}} \tag{B.1}$$

one can derive the following factorial sum rules

$$\frac{[a]!}{[b]!\,[c]!} = \sum_s \frac{(-1)^{(a-b)(a-c)+s(c+b+1)} q^{\frac{1}{2}(a-b)(a-c)} q^{-\frac{1}{2}as} \, [a-b]!\,[a-c]!}{[a-b-s]!\,[a-c-s]!\,[b+c-a+s]!\,[s]!} \tag{B.2}$$

$$\sum_s (-1)^{\frac{s(s+1)}{2}} q^{\frac{1}{2}s(b+c-a-1)} \frac{[a-s]!}{[b-s]!\,[c-s]!\,[s]!} = (-1)^{\frac{c(c+1)}{2}+ac} q^{\frac{1}{2}bc} \frac{[b+c-a-1]!\,[a-c]!}{[b-a-1]!\,[b]!\,[c]!} \tag{B.3}$$

for $b > a \geq c \geq 0$.

$$\sum_s (-1)^{\frac{s(s+1)}{2}} q^{\frac{1}{2}s(b+c-a-1)} \frac{[a-s]!}{[b-s]!\,[c-s]!\,[s]!} = (-1)^{bc} q^{\frac{1}{2}bc} \frac{[a-b]!\,[a-c]!}{[a-b-c]!\,[b]!\,[c]!} \tag{B.4}$$

for $a \geq b \geq 0$, $a \geq c \geq 0$.

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