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# Interpolating black holes

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## Abstract

We describe an unusual three parameter family of static spherically symmetric black hole solutions of general relativity. The solutions arise from gravitational coupling to a one parameter non-linear generalization of the electromagnetic field. This parameter determines how long range the matter field is. One class of the black hole metrics 'lies between' the Schwarzschild and Reissner-Nordstrom solutions, while the other class 'lies beyond' the latter, in the sense of radial fall off of the metric.

For vacuum general relativity, the Kerr metrics, which are parametrized by mass and angular momentum, are the unique stationary black hole solutions. For general relativity coupled to matter fields, black hole metrics may have additional parameters which are matter charges, such as the electric charge.

If the matter fields associated with a black hole solution fall to zero with radial distance $r$ faster than $1/r^2$, then the matter charges will not be captured by surface integrals at spatial infinity, otherwise they will be. If the matter field is the electromagnetic field, the electric charge on the black hole appears as a conserved surface integral at spatial infinity.

It is of interest to ask whether there are other black hole solutions in general relativity, or in other gravitational theories derived from string theory, and what parameters characterize them. There continues to be much work on this question.

In this essay we describe an unusual three parameter family of spherically symmetric black hole solutions which carry a matter charge other than the electric charge. The matter arises as a one parameter non-linear generalization of electromagnetism, and its charge reduces to the electric charge in the electromagnetic limit. Depending on the energy condition imposed, the matter charge may be written as a conserved surface integral at spatial infinity; This is the case for the dominant energy condition, but not for the weak one.

The coupled Einstein-Maxwell equations, without electromagnetic sources, are

$$G_{ab} = 8\pi\ (F_{ac}F_b{}^c - \frac{1}{4}\ g_{ab}F_{cd}F^{cd})\frac{1}{4\pi}. \qquad (1)$$

The Reissner-Nordstrom metric is the unique two parameter static spherically symmetric solution of these equations parametrized by the mass $M$ and electric charge $Q$ [1].

The standard static spherically symmetric metric may be written using an advanced time coordinate $-\infty \leq v \leq \infty$ and the proper radial coordinate $0 \leq r \leq \infty$ as

$$ds^2 = g_{ab}dx^a dx^b = -f(r)\ dv^2 + 2\ dvdr + r^2 d\Omega^2, \qquad (2)$$

where $d\Omega^2$ is the metric on the unit two-sphere. In these coordinates the timelike Killing vector field is $(\partial/\partial v)^a$, and has norm $-f$.

With the electromagnetic source given by $F_{ab} = E(r)(dv \wedge dr)_{ab}$, the Reissner-Nordstrom solution is

$$E(r) = \frac{Q}{r^2}, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{3}$$

The stress-energy tensor for this source free Maxwell field may be rewritten in a form resembling that of the perfect fluid, for which the stress-energy tensor is

$$T_{ab} = \rho\ u_a u_b + P\ (g_{ab} + u_a u_b), \tag{4}$$

where $u_a$ is timelike, and $\rho$ and $P$ have interpretations as the energy density and pressure of the perfect fluid.

Using the two future pointing null vectors $v_a = (1, 0, 0, 0)$ and $w_a = (f/2, -1, 0, 0)$ ($v_a w^a = -1$) of the metric (2) (in the coordinates $(v, r, \theta, \phi)$ ), the stress-energy tensor for the Reissner-Nordstrom solution may be rewritten as

$$T_{ab} = \frac{Q^2}{8\pi r^4}\ (v_a w_b + v_b w_a) + \frac{Q^2}{8\pi r^4}\ (g_{ab} + v_a w_b + v_b w_a). \tag{5}$$

Like the perfect fluid, the coefficients $Q^2/8\pi r^4 = E^2/8\pi$ have the interpretation of electric energy density and pressure. One can therefore view this stress-energy tensor as describing a 'fluid' with the equation of state $P = \rho$. It is however not a perfect fluid because the fluid flow lines are not timelike. The stress-energy tensor is in fact degenerate in the sense that it has one null eigenvector and two spacelike eigenvectors. (It is a Type II stress-energy tensor [2].) This is unlike the perfect fluid tensor which has one timelike and three spacelike eigenvectors (Type I).

We now ask what the metric is for the source given by the stress-energy tensor

$$T_{ab} = \rho(r)\ (v_a w_b + v_b w_a) + P(r)\ (g_{ab} + v_a w_b + v_b w_a), \tag{6}$$

and with the equation of state $P = k\rho$. This tensor is a one parameter ($k$) generalization of (5), which gives the Reissner-Nordstrom metric. Furthermore, the matter satisfies the dominant energy condition, (which means positive energy density and timelike or null energy fluxes) for $0 \le k \le 1$, and the weak energy condition, (which means positive energy density), for $k > 1$. We will restrict attention to the case $k > 1/2$ because it will turn out that for this range the metric is asymptotically flat at spatial infinity.

The metric is most easily determined by starting with the ansatz (2), finding the stress-energy tensor from it, (which turns out to be *exactly* of the form (6)), and then imposing the equation of state $P = k\rho$ on its eigenvalues. This gives a simple equation for $f(r)$. We find

$$ds^2 = -(\ 1 - \frac{2M}{r} + \frac{Q^2}{(2k-1)r^{2k}}\ )\ dv^2 + 2\ dvdr + r^2 d\Omega^2, \qquad (k \neq \frac{1}{2}) \tag{7}$$

and

$$P = k\ \frac{Q^2}{8\pi r^{2k+2}} = k\rho, \tag{8}$$

where $Q$ is now the charge associated with this 'fluid', and $M$ retains its usual meaning as the Arnowitt-Deser-Misner mass.

The metric on static three-surfaces of this spacetime may be found by making the coordinate transformation $t = v - r$ and looking at the constant $t$ three-surfaces. The metric on such static spatial slices is

$$ds^2 = (\ 1 + \frac{2M}{r} - \frac{Q^2}{(2k-1)r^{2k}}\ )\ dr^2 + r^2 d\Omega^2. \tag{9}$$

From this we see that the metrics (7) are asymptotically flat at spatial infinity for $k > 1/2$. For $k < 1/2$ the metrics are cosmological because the leading order behaviour of the metric function is slower than $1/r$ as $r \to \infty$. For $k = 1/2$, which must be treated separately, we find $f = 1 - 2M/r + C\ \ln r$, ($C$ = constant), so here also the metric is cosmological.

The asymptotically flat solutions give a three (real) parameter family $(M, Q, k)$ of black hole metrics that 'fall between' the Schwarzschild and Reissner-Nordstrom black holes in the sense that $1 < 2k \leq 2$ in the metric (7). The event horizons of the black holes, defined as usual by $f(r) = 0$, are given by the polynomial equation

$$(2k-1)(r^{2k} - 2M\ r^{2k-1}) + Q^2 = 0. \tag{10}$$

We would now like to ask what is the field theory for the matter that gives rise to these black holes. By noting that the stress-energy tensor (6) is a one parameter generalization of the Maxwell one, we see that (6) may be rewritten as

$$T_{ab} = \frac{1}{4\pi}\ (F_{ac}F_b{}^c - \frac{\alpha}{4}\ g_{ab}F_{cd}F^{cd}), \tag{11}$$

where $\alpha$ is related to the parameter $k$ in the equation of state $P = k\rho$ by $\alpha = 2k/(k+1)$. The corresponding field equation is the one parameter non-linear generalization

$$F_{ab}\nabla_c F^{bc} + (1-\alpha)\ F^{bc}\nabla_c F_{ab} = 0, \tag{12}$$

of the vacuum Maxwell equation. The solution of this equation associated with the metric (7) is

$$F_{ab} = \sqrt{\frac{k+1}{2}}\ \frac{Q}{r^{k+1}}\ (dv \wedge dr)_{ab}, \tag{13}$$

which reduces to the ordinary Maxwell one for $k = 1$. The parameter $k$ (or $\alpha$) is thus a measure of how long range this matter field is.

When $k > 1$ in the fluid equation of state, the stress-energy tensor (6) satisfies both the weak and strong energy conditions, (which are identical for Type II stress-energy tensors). The charge term in the black hole metrics (7) now falls to zero faster than $1/r^2$, and the charge is therefore no longer captured by a surface integral at spatial infinity. If 'hairs' are considered to be those parameters characterizing black holes which do not appear as conserved surface integrals at spatial infinity, then the black hole solutions with $k > 1$ have hair. Thus, for this type of (Type II) matter, when the dominant energy condition is satisfied the black holes have no hair, whereas when only the weak (or strong) energy condition is satisfied the black holes do have hair.

Also, for $k > 1$, the matter field has a range shorter than electromagnetism, and exerts pressures that exceed its energy density. It appears that it is these larger pressures that keep the matter hovering above the hole, in much the same way that sufficienly high pressures can prevent gravitational collapse in a star. This is perhaps an 'intuitive explanation' for the presence of 'hair', and suggests that black hole solutions may always have hair for short range matter fields for which pressures exceed the energy density.

There is recent work suggesting that hair on spherically symmetric black holes must extend beyond a certain critical value for all matter fields satisfying the weak energy condition. This suggests a 'no short hair' conjecture [3]. It would be of interest to examine this conjecture for the class of solutions presented here.

The basic spherically symmetric ansatz (2) can give a variety of black hole solutions depending on the equation of state imposed in (6). For example, one might look for solutions with $P = k\rho^a$, and attempt to find the corresponding non-linear field theory.

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