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Discrete time portfolio management with transaction costs

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Abstract

The majority of portfolio management strategies are based on continuous-time processes that are assumed to be the solution of stochastic differential equations. Stochastic differential equations are usually based on the first order truncations of the Taylor series expansion with the inclusion of selected second order terms using Ito's lemma. If there is a solution to the Taylor series expansion then, formally at least, it will be the Volterra functional expansion.

In this work a new approach to practical portfolio management is presented that is based on estimating the Volterra function values with novel time series analysis techniques. The approach is concerned with predicting the discrete time future behaviour of risky assets. A knowledge of the future value of the risky assets enables an optimal choice for portfolio selection to be made at each moment of time. The decision for any given portfolio selection can be made with the inclusion of transaction costs for the trades. In this work the transaction costs are assumed to be proportional to the value of the risky asset traded. However, the method presented can include the effect of the actual trading costs encountered in practice. The relationship between the risky assets and their temporal derivatives of a given portfolio is described as a set of coupled ordinary differential equations. The solution of the coupled equations is considered to be an evolutionary Volterra functional expansion. The values of the kernel, or response, functions are estimated using the moment hierarchy method. The estimated response function values are then used to predict the out of sample future behaviour of each risky asset. The two examples presented indicate that out of sample predictions obtained can consistently describe the behaviour of each of 100 common stock names to an accuracy of within the empirical volatility of each name. The out of sample predicted price values were used as the basis of decisions for trading the common stocks each day including the effects of transaction costs on the percentage return.
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Introduction

The continuous-time behaviour of a portfolio composed of risky assets is usually described by geometrical motion models, such as that popularised by Merton [1]. In continuous-time models, the portfolio to be managed is considered to contain n independent risky assets whose prices fluctuates according to, for example, an Ito process with jumps. Ito processes are generated by nonlinear diffusion equations, which themselves are obtained by truncating the Taylor series expansion to first order and including selective second order terms using Ito's lemma [2,1,3,4]. In Merton's model, transactions are costless, and the continuously arriving information can be used to identify optimal trading strategies. Recent research on continuous-time portfolio management has concentrated on the effect of transaction costs [5-14]. However, the models developed involve infinitesimal continuous trading or alternatively fixed fraction trading that are less than optimal [14].
In practice, investors trade discrete quantities of risky assets in discrete time. The trading process incurs costs which have both fixed and variable components that can be time dependent.

The purpose of this paper is to present a novel discrete time method that can be used to predict the future behaviour of a set of risky assets. Given the predicted behaviour of the portfolio the time dependent optimal strategy for portfolio selection can be made whenever it is required. In this work the incurred cost for each transaction is a fixed fraction of the current price of the underlying asset. However, it is straightforward to include the actual costings that are used in practice. The complex nature of the time series sequences of risky assets renders quantitative analysis extremely difficult in general. This is partially because the mapping between adjacent members of each time series sequence is nonlinear and also because the multivariate relationships are functions of many stochastic or chaotic variables.

The truncated Taylor series expansion is usually used to describe the relationship between financial sequences [4]. If there is a solution to the Taylor series then, formally at least, it is the Volterra functional expansion [15]. The behaviour of the price sequences contained in the portfolio are considered to be determined by a set of ascending order evolutionary coupled nonlinear differential equations and that the solution of these equations can be represented either as a Taylor or Volterra series expansion. The Volterra functional values extracted from the data are used to characterise the behaviour of common stock prices and their evolution into the future.

In this work the behaviour of the vector valued financial process is represented as a set of coupled ordinary differential equations of the form

$$\sum_{k=0}^{K} a_k \frac{d^k x_i(t)}{dt^k} = F\left(x_1,\ldots,x_M, \frac{dx_1}{dt},\ldots,\frac{dx_M}{dt},\ldots,\frac{d^T x_1}{dt^T},\ldots,\frac{d^T x_M}{dt^T} \right) \tag{1}$$

The coupled first order ordinary differential equation representation has been used to associate topological structures with experimental time series [16,17]. For example many physical laws are expressed in infinitesimal form with the corresponding differential equations being defined as a vector field. A vector field in this case is interpreted as the right hand side of the coupled ordinary differential equations given by equation (1) which are satisfied by the integral curves denoted by the time series trajectories $\{x(t)\}$. The form of equation (1) facilitates the 'reconstruction' of a set of differential equations in phase space which mimic some aspects of the phenomena. There have been a number of previous attempts to extract from time series data coupled differential equations which replicate aspects of the observed behaviour [18-25].

Particular emphasis is placed on characterising complex behaviour in the context of coupled differential equations. A new method for extracting coupled mixed order linear-nonlinear differential equations from time series data in the presence of noise is used [26]. The method employs the hierarchy moment formalism to characterise the mapping between time series values [27]. The coefficients of the coupled differential equations are estimated with the hierarchy moment method. The coefficients for the reversed time series sequence are then calculated and used to estimate the initial conditions of the problem being studied. The uncertainty of the coefficients and initial conditions are estimated and discussed in terms of the accuracy of the method and reproducibility in any given experimental situation.

The hope and expectation is that empirical forecasts can be used to provide realistic predictions of expected behaviour of financial processes. Not only will this increase the practical limit for out of sample predictions, but they may further the understanding of the fundamental mechanisms which produce the observed behaviour. The coupled trajectories of the common stock prices in the portfolio are considered to be a function of each price, its rate of change, acceleration and rate of acceleration. The common stock prices are coupled through self and cross forcing terms and the portfolio is driven by transactions which amend each individual price. The kernel, or response, functions which characterise the observed behaviour are estimated using the moment hierarchy method [26-30] and the future behaviour of the common stock prices in the portfolio are predicted.

The out of sample forecasts of the future behaviour of the risky assets provide a sound basis for optimal portfolio selection for trading. Results from an ongoing, in depth, empirical study are presented indicating the accuracy of the predictions up to one month into the future. The accuracy of approximately 20000 individual predictions is presented together with the daily and cumulative percentage return on two example portfolios.

A functional representation of an evolutionary financial portfolio

Financial processes are evolutionary phenomena. Although evolutionary financial processes display auto-regressive attributes, the evolution will normally cease when the external stimuli, i.e. the trading processes stop. Under typical trading conditions it is assumed that the evolutionary processes of the risky assets can be represented as a set of coupled differential equations. For example, a portfolio of common stock prices $\Pi(t)$ containing π individual names, can be considered to be a set of coupled nonlinear differential equations, one equation for each individual stock name. Consider the set of coupled simultaneous differential equations given by equation (2)

$$\sum_{k=0}^{K} a_k \frac{d^k x_i(t)}{dt^k} = F\left(x_1, \ldots, x_M, \frac{dx_1}{dt}, \ldots, \frac{dx_M}{dt}, \ldots, \frac{d^T x_1}{dt^T}, \ldots, \frac{d^T x_M}{dt^T} \right) \quad (2)$$

where it is assumed that the vector sequence has temporal derivatives that exist up to order T. Assume that the coupled equations have an inverse solution which determines the relationship between the stock price values $\{P_k(t)\}$ contained in the portfolio $\Pi(t)$, this solution can be defined as a multidimensional functional of the form

$$P_k(t) = r_k(t) + F_k\left(P_1, \ldots, P_\pi, \frac{\partial P_1}{\partial t}, \ldots, \frac{\partial P_\pi}{\partial t}, \ldots, \frac{\partial^M P_1}{\partial t^M}, \ldots, \frac{\partial^M P_\pi}{\partial t^M}, t \right) \quad (3)$$

where $\{\Xi_k(t)\}$ is a stochastic function to take account of external processes and 'noise'.

Generally speaking, when a financial observable $\{P_k(t)\}$ is evolutionary, and when it is not dependent on any other financial variables, then the temporal development can be written as a Taylor series expansion truncated to order M, of the form

$$P_k(t + \Delta t) = \sum_{m=0}^{M} \frac{\Delta t^m}{m!} \frac{d^m P_k(t)}{dt^m}$$

This is effectively an auto-regressive scheme, which will persist indefinitely and never forget any event in its history and is not dependent on external stimuli.

Equally, the evolution of the financial observables can be described by an expansion of functionals which characterises the behaviour as a mapping between functional spaces. If there is a unique solution to the Taylor series expansion then, formally at least, it is the inverse mapping. The emphasis of the inverse problem approach is to identify the form of relationship between the risky assets and hence establish the laws governing the process. In the absence of noise, a discrete form of the multidimensional Volterra expansion for a time invariant mapping is defined as [27]

$$P_k(t) = \sum_{m=0}^{M} \frac{1}{m!} \sum_{\tau_1=1}^{\mu} \cdots \sum_{\tau_m=1}^{\mu} g_{P_k P_k{}^m}(\tau_1, \ldots, \tau_m) \prod_{k=1}^{m} P_k(t - \tau_k) \tag{3}$$

where μ is equal to one for the differential equation approach [30].

The effects of 'noise' on the mapping can be considered by decomposing the price $\{P_k(t)\}$ into a stochastic component $\{\Xi_k(t)\}$, and a deterministic component $\{p_k(t)\}$, which are assumed to be statistically independent of each other.

Under this assumption each data value $\{P_k(t)\}$ in the sequence can be represented as [26-27]

$$\begin{aligned} P_k(t) &= \Xi_k(t) + p_k(t) \\ &= \Xi_k(t) + \sum_{m=0}^{M} \frac{1}{m!} \sum_{\sigma_1=1}^{\mu} \cdots \sum_{\sigma_m=1}^{\mu} g_{p_k p_k{}^m}(\sigma_1, \ldots, \sigma_m) \\ &\qquad \prod_{k=1}^{m} \{P_k(t - \sigma_k) - \Xi_k(t - \sigma_k)\} \end{aligned} \tag{4}$$

In effect the mapping is theoretical decomposed into a deterministic component and a stochastic component.

The value of the portfolio and its temporal derivatives is considered as a vector field which is a function of the time series histories of the component common stock prices and their temporal derivatives. The price, $\{P_k(t)\}$, for each common stock name will be an evolutionary function of the history of all the common stock prices and the external forces of current trading which act.

The inverse form of the evolutionary Volterra functional expansion for an evolutionary mapping of an individual price contained in the portfolio is defined by

$$P_k(t) = \Xi_k(t) + \sum_{n=1}^{\eta} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu}$$

$$\delta\left(i_j \neq i_k;\ \sigma_j > 0 \quad i_j = i_k \quad j = 1 \ldots, n\right)$$

$$d_r\left[g_{P_k P_{i_1} \ldots P_{i_n}}(\sigma_1, \ldots, \sigma_n) \prod_{j=1}^{n} \left\{ P_{i_j}(t - \sigma_j) - \Xi_{i_j}(t - \sigma_j) \right\} \right] \qquad (5)$$

where the differential operator $d_r[.] = \left\{ 1 + \frac{\partial}{\partial t} + \ldots + \frac{\partial^r}{\partial t^r} \right\}[.]$, and the delta function denotes the fact that the terms of the auto-regressive terms are identically equal to zero. The stochastic component values, $\{\Xi_k(t)\}$, are assumed to be drawn from Gaussian random processes so that estimates for their variance can be obtained, for example, by regression or maximum likelihood methods. This enables the kernel, or response, function values to be determined from the moment hierarchies [26-30]. The response function values are then used to predict the out of sample behaviour of the common stock close price values.
The form of equation (5) is ill posed, in the sense that there are many coefficients. In addition, they are ill conditioned because their dependent and independent variables are stochastic functions of time. When the data $\{P_k(t)\}$ are drawn from stochastic sequences, the form of the Volterra functional series is ill conditioned because the dependent and independent variables are stochastic functions of time.

These problems are overcome with statistical averaging, and the resulting moment hierarchy, is likely to be well posed and have well behaved coefficients. Operating on the Volterra series with averaging operators [26-30] and re-arranging, the simultaneous moment hierarchies are defined by

$$\left\langle \prod_{u=1}^{v} P_{i_j}(t - \sigma_j) \frac{\partial^m P_k(t)}{\partial t^m} \right\rangle = \left\langle \prod_{u=1}^{v} P_{i_j}(t - \sigma_j) \Xi_k(t) \right\rangle + \sum_{n=1}^{\eta} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu}$$

$$\prod_{j=1}^{n} \delta\left(i_j \neq i_k;\ \sigma_j > 0 \quad i_j = i_k \quad j = 1 \ldots, n\right)$$

$$\sum_{s=0}^{r} \left\{ \frac{\partial^{s+m} g_{P_k P_{i_1} \ldots P_{i_n}}(\sigma_1, \ldots, \sigma_n)}{\partial t^{s+m}} \right\} M^{m,s}_{x_K, x_n}(\sigma_1, \ldots, \sigma_n)$$

$$(6)$$

Thus the multivariate Volterra functional expansion and its temporal derivatives given in equations (5) and (6) have been transformed into a tractable linear algebra expression that can be solved for the evolving response functions.
In equation (7) the auto-moment values denoted by $M^{m,s}_{X_K,X_n}(\sigma_1,\ldots,\sigma_n)$ are simple arithmetic sums of the usual absolute time series moments [30].

An approximation to equation (5) is used in the present work for the study of evolutionary financial processes. If the financial process responds in a finite time, say μ units, then the observed behaviour can be represented as a discrete Volterra functional expansion truncated to order N. The deterministic component can be analysed and used to predict the future behaviour of the phenomena. Each time series values of the financial observables is theoretical decomposed into a deterministic component and a stochastic component and this introduces additional variables into the representation. The discrete approximation to equation (5) used in the present is a set of coupled discrete Volterra functional expansion truncated to order N, which are defined by the simultaneous equations

$$P_k(t) = \Xi_k(t) + \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{P_k \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1,\ldots,\sigma_n) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_{i_j}(t-\sigma_j) \right\} \quad (7)$$

and

$$\frac{d^m P_k(t)}{dt^m} = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{\frac{d^m P_k}{dt^m} \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1,\ldots,\sigma_n) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_{i_j}(t-\sigma_j) \right\} \quad (8)$$

for m=1,...T.
Here the elements $\Phi_{i_j}(t)$ are drawn from the set of common stock price values and the finite difference approximations to the temporal derivatives with

$$\left\{\Phi_{i_j}(t)\right\} = \left\{ P_j(t), \frac{dP_j(t)}{dt}, \ldots, \frac{d^T P_j(t)}{dt^T} \right\}.$$

Operating on these equations with the averaging operator $\left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_r)* \right\rangle$ the R th order absolute moment hierarchy of $\{P_k(t)\}$ and its temporal derivatives is given by

$$\left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k)\, P_k(t) \right\rangle = \left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k) \Xi_u(t) \right\rangle$$
$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{P_k \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1, \ldots, \sigma_n)$$
$$\left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_u(t-\sigma_j) \right\} \right\rangle$$

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$$\left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k)\, \frac{d^T P_k(t)}{dt^T} \right\rangle = \left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k) \Xi_u(t) \right\rangle$$
$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{\frac{d^T P_k(t)}{dt^T} \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1, \ldots, \sigma_n)$$
$$\left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_u(t-\sigma_j) \right\} \right\rangle \quad (9)$$

The moment hierarchy form given by equation (9) can readily be solved for the unknown response function values using , for example, regressional or maximum likelihood methods [27,30]. That is, the application of averaging operators to the Volterra series generates a hierarchy of tractable moment equations and this moment hierarchy is then regressed to obtain the variance of the noise and the response function values.

Given that the choice of $\{\Xi_k(t)\}$ is not unique, it is assumed here that $\{\Xi_k(t)\}$ is drawn from a zero mean Gaussian white noise process which is statistically independent to the deterministic components $\{\varphi_u(t)\}$. Under this assumption, the moment hierarchy becomes

$$\underline{\psi} = \underline{\underline{\Psi}}\, \underline{h} \quad (10)$$

where the cross - moment elements are given by

$$\psi_R(\tau_1, \ldots, \tau_R) = \left\langle \prod_{r=1}^{R} \Phi_{i_r}(t-\tau_k)\, \phi_{u_v}(t) \right\rangle$$

the off block - diagonal auto - moment elements are given by

$$\Psi_{Rn}(\tau_1,\ldots,\tau_R,\sigma_1,\ldots,\sigma_n)=\alpha_{Rn}\left\langle\prod_{r=1}^{R}\Phi_{i_r}(t-\tau_k)\prod_{j=1}^{n}\Phi_{i_j}(t-\sigma_j)\right\rangle$$

the off block - diagonal auto - moment elements are given by

$$\Psi_{RR}(\tau_1,\ldots,\tau_R,\sigma_1,\ldots,\sigma_R)=\beta_{RR}\left\langle\prod_{r=1}^{R}\Phi_{i_r}(t-\tau_k)\prod_{j=1}^{R}\Phi_{i_j}(t-\sigma_j)\right\rangle-\gamma^R$$

and the response function elements are $h_{\phi_{u_v}\Phi_{i_1}\ldots\Phi_{i_M}}(\sigma_1,\ldots,\sigma_M)$.

where the variance of the stochastic component is denoted as γ, and the coefficients α_{Rn} and β_{RR} reflect the fact that the symmetry of the equations has been used to reduce the order of the matrix [27]. The moment hierarchy given by equation (7) can be rewritten in the obvious matrix form $\underline{C}=\underline{\underline{M}}\underline{g}$. Here $\underline{\underline{M}}$ is a square matrix whose elements are the time series moments of data. For each moment hierarchy $\underline{C}$ is a column vector whose elements are the cross moments between the common stock prices and their temporal derivatives and the set of variables $\left\{\Psi_{i_r}(t)\right\}$. The elements of the column vector $\underline{g}$ are the unknown kernel function values.

If the matrix $\underline{\underline{M}}$ is non-singular then $\underline{g}=(\underline{\underline{M}})^{-1}\underline{C}$ has a unique solution. If however $\underline{\underline{M}}$ is singular, then $\underline{\underline{M}}$ is rank deficient and some of its rows will be linearly dependent on the others. If the same relationship holds between the corresponding elements of the column vector $\underline{C}$, the solution will not be unique and an infinity of solutions will exist. If this is not the case then the matrix expression is not consistent and there will not be any solution. Thus, in general, there may be a unique solution, an infinite number of solutions or no solution. However, given the construction of the moment values used in the moment hierarchy, the rows of $\underline{\underline{M}}$ will be linearly independent of each other, thus the matrix will usually be non-singular and have a unique solution.

Now consider the effect of the uncertainty on the experimental measurments. In accordance with standard uncertainty analysis of experimental data, assume that it is possible for the analysis proceedure to yeild a unique set of empirical coefficients, that are physically meaningful, and where each coefficient has an uncertainty value associated with it. Including the uncertainty terms the moment hierarchy retains a linear algebraic form with

$$(\underline{\psi}\pm\underline{\Delta\psi})=(\underline{\underline{\Psi}}\pm\underline{\underline{\Delta\Psi}})(\underline{h}\pm\underline{\Delta h}) \tag{11}$$

Using the theory of the proparation of errors, then the fractional error on the estimated mixed order linear-nonlinear response function values will be given by

$$\frac{|\Delta \underline{h}|}{|\underline{h}|} \leq C(\underline{\underline{\Psi}})\left[\frac{\|\Delta \underline{\underline{\Psi}}\|}{\|\underline{\underline{\Psi}}\|} + \frac{\|\Delta \underline{\psi}\|}{\|\underline{\psi}\|}\right] \tag{12}$$

where $\| \ \|$ denotes the norm and the conditioning number is defined as

$$C(\underline{\underline{\Psi}}) = \|\underline{\underline{\Psi}}\| \bullet \|\underline{\underline{\Psi}}^{-1}\| .$$

These uncertainty values can be used to estimate the standard error, and multiples thereof, of the forecasted stock price values. Thus providing a value for the future behaviour of each stock name together with a probability distribution associated with it.

Results from an empirical study of Japanese high-volume common stock prices

The results from two example analyses are considered in this section. Firstly, on a daily basis, the time histories of a portfolio containing three hundred Japanese common stock prices are statistically analysed over a thirty day period. This portfolio was split into ten roughly equal baskets, with the names in each basket being drawn from the same area of the market. Secondly, the historical close price values of a single basket, containing thirty names, were statistically analysed over a period of four hundred days.

The time series moments of each basket were used to estimated those response functions which give the best out of sample predictions for that basket. In addition the names of common stocks which could be accurately predicted on a consistent basis were identified. The response function estimates were then used predict the future behaviour of the close price for each name up to fifty trading days ahead.

The predicted future behaviour of each common stock within the portfolio was then statistically compared with the actual prices that occurred. In a preliminary study it was found that 30% of the names used were predicted with an accuracy that captured approximately 98% of the actual movement each day. These names were then used as the basis for trading in the following examples. In each example some ten thousand predictions are analysed so that the accuracy and consistency of the results obtained are likely to be statistically significant. The predictions were also used as the basis of a simple trading strategy and the percentage return values of the common stock prices that result from the use of the simple strategy are presented.

The simple strategy being, that if a prediction indicated a likely change in a common stock price of greater than 3 %, for a name whose predictions are consistently accurate, then a trade in that name would be initiated on that day and completed within a sixteen day period after the initiation.

The effect of transaction costs were also considered. However, it is not possible to determine exact transition costs for every case. Institutional investors typically have brokerage charges amounting to approximately 0.06% of each stock value traded. Given that the typical commissions are usually negligible the transaction costs usually are likely to be in the range of 0.1 % to 0.15% of the stock price. In the examples presented below the effect of 0.0 %, 0.1%, 0.15% and 0.2% transaction charges are shown.

Because the future values of the external forces, trades in each name, cannot be known *a priori*; only the equilibrium behaviour can be accurately predicted. The predictions when these response function values are used are representative of the local equilibrium behaviour.

The ascending order time series moments of the data are estimated and the value of the variance of each stochastic component is determined so that the values of the response functions provide the best out of sample predictor. The self and cross coupling forces are assumed to have a finite memory up to a duration of μ days in the present examples.

The discrete approximation to the multidimensional convolution expansions given by equation (9) given by the discrete Volterra functional expansion truncated to order N, defined by the simultaneous equations

$$P_k(t) = \Xi_k(t) + \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{P_k \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1, \ldots, \sigma_n) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_{i_j}(t-\sigma_j) \right\} \tag{13}$$

and

$$\frac{d^m P_k(t)}{dt^m} = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{M} \cdots \sum_{i_n=i_n-1}^{M} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} g_{\frac{d^m P_k}{dt^m} \Phi_{i_1} \ldots \Phi_{i_n}}(\sigma_1, \ldots, \sigma_n) \prod_{j=1}^{n} \left\{ \Phi_{i_j}(t-\sigma_j) - \Xi_{i_j}(t-\sigma_j) \right\} \tag{14}$$

for k=1,2,3,4. That is, the moment hierarchy is operated on with a series of averaging operators and the resulting moment hierarchies are solved for the unknown response function values.

The estimated response function values are then used to predict the future behaviour of each member of the portfolio. Figure 1, shows samples of common stock name price histories together with the predictions made each day. The accuracy of the predictions were determined by considering the percentage difference between the prediction and the actual common stock price value divided by the percentage volatility of that name.

The results for the accuracy of the 20000 predictions used in the paper trading are presented as probabilities (normalised to 1.0 for each value of time delay considered) in Table 1 and in more detail in Figure 2a and the logarithm of the probabilities are shown in Figure 2b.

The findings from the statistical analysis of several weeks of such portfolio predictions indicate that approximately 30% of the stock names in the basket can be consistently predicted to within the variance, or spread, of the actual common stock price in the future. A preliminary study of a two month period of daily predictions enabled the names which were being accurately predicted on a consistent basis. The names identified where then used in the above mentioned simple strategy although no changes were made to the composition of the baskets used in the analysis.

The daily percentage return on the investment for the two examples are shown in Figures 3 and 5, and the cumulative percentage return on the investment for each of the two examples are shown in Figures 4 and 6. In each figure there are four curves which refer to the 0.0, 0.1, 0.15 and 0.2 % transaction costs incurred in each trade. Clearly these have a significant impact on the return on the investment in each case. However, for each example the cumulative return at the 0.15 % transaction cost level is approximately five times greater than that which may be expected when compared with alternative time series forecasting methods [32]. These results are very encouraging but are only the first of a series of studies of forecasting financial instruments.

Table 1: Probability of prediction being within a given number of standard deviations of the actual stock price in the future

	Time in the future				
number of standard deviations	1 day	5 days	10 days	15 days	20 days
6	0.0051	0.0051	0.0051	0.0051	0.0034
5	0.0034	0.0034	0.0051	0.0034	0.0051
4	0.0017	0.0017	0.0034	0.0034	0.0017
3	0.0068	0.0051	0.0051	0.0051	0.0068
2	0.0136	0.0153	0.0119	0.0153	0.0136
1	0.0153	0.0153	0.0153	0.0136	0.0153
0	0.9197	0.9163	0.9095	0.9044	0.9061
1	0.0221	0.0272	0.0306	0.0289	0.0272
2	0.0218	0.0204	0.0204	0.0238	0.0238
3	0.0034	0.0034	0.0034	0.0051	0.0051
4	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0

Conclusions

The results presented in this paper can be summarised as follows. A general theoretical representation for the time dependent behaviour of a set of risky assets in terms of coupled differential equations was developed. The representation being in terms of a suitably truncated multivariate evolutionary Volterra functional expansion. The order of truncation is determined by analyses of the predicted behaviour, and is limited by the ability to numerically invert the moment hierarchy. An approximation of the general representation was used to analyse and predict the out of sample future behaviour of a number of portfolios composed of common stock price sequences. Ascending order time series moments of the data were estimated and used to solve a moment hierarchy for the unknown response function values. The response function values in this case are equal to the coefficients of the coupled differential equations. The statistical analyses of some 20000 individual predictions indicate that the moment hierarchy can correctly predict the future behaviour of common stock prices. It was found that approximately 30 % of the names considered consistently have predictions that lie within the spread up to 20 days in the future. These findings indicate that there is a useful level of forecast ability in financial sequences.

A subset of out of sample accurately predictable common stock names were identified and then used in the two examples. The daily percentage and cumulative return on the investment were shown as a function of incurred transaction cost. The transaction cost has a significant impact on the return on the investment in each case. However, for each example the cumulative return at the 0.15 % transaction cost level is approximately five times greater than that which may be expected if the best alternative time series forecasting method were used.

References

[1] Merton R. C., Optimum consumption and portfolio rules in a continuous-time model, ***J. Econ. Theory***, 3, 373-413.

[2] Ito K., ***On stochastic differential equations***, Memoirs, American Mathematical Society, No. 4, 1951, 1-51.

[3] Ito K. and McKean H. P., ***Dffusion processes and their sample paths***, Academic Press, New York, 1964.

[4] Hull J. C., ***Options, Futures and other derivative securities***, Prentice Hall Int. Edition, London, 1993.

[5] Assaf D., Taksar M. and Klass M. J., A diffusion model of optimal portfolio selection in the presence of brokerage fees, ***Math. Oper. Res.***, 13, 277-294.

[6] Atkinson C. and Wilmott P., Portfolio management with transaction costs: an asymptotic analysis of the Morton and Pilska model, ***Math. Finance***, Vol. 5, No. 4, 1995, 357-367.

[7] Constantinides G., Capital market equilibrium with transaction costs, ***J. Pol. Econ.***, 94, 842-862.

[8] Davis M. and Norman A., Portfolio selection with transaction costs, ***Math. Oper. Res.***, 15, 676-713.

[9] Duffie D. and Sun T., transaction costs and portfolio choice in a discrete-continuous time setting, ***J. Econ. Dyn. Control***, 14, 1990, 577-597.

[10] Dumas B. and Luciano E., An exact solution to a dynamical portfolio choice problem under transaction costs, ***J. of Finance***, 46, 1991, 577-595.

[11] Eastham J. F. and Hastings K. J., Optimal impulse control of portfolios, ***Math. Oper. Res.***, 13, 1988, 588-605.

[12] Karatzas I, optimisation problems in the theory of continuous trading, ***SIAM J. Cont. Opt.***, 27, 1989, 1221-1259.

[13] Magill M and Constantitnides G., portfolio selection with transaction costs, ***J. of Econ. Theory.***, 3, 1976, pp 264-271.

[14] Morton A. J. and Pilska R. S., Optimal portfolio management with fixed transaction costs, ***Math. Finance***, Vol. 5, No. 4, 1995, 337-356.

[15] Volterra V., ***Theory of Functionals and of Integral and Integro-differential Equations***, Dover, New York, 1959.

[16] Broomhead D. S., Jones R. and King G. P., ***J. Phys. A***, 20, 1987, L563.

[17] Broomhead D. S. and King G. P., Extracting quantitative dynamics from experimental data, ***Physica D***, 20, 1986, pp 217-236.

[18] Casdagli M., Nonlinear prediction of chaotic time series, ***Physica D***, 35, 1989, pp 335-356.

[19] Farmer J. D. and Sidovowich J. J., Predicting chaotic time series, ***Phys. Rev. Lett.,*** 59, 1987, pp 845-848.

[20] Crutchfield J. P. and McNamra B. S., Equations of motion from a data series, ***Complex Systems***, 1, 1987, pp 417-452.

[21] Cremers J. and Hubler A., Construction of differential equations from experimental data, ***Z. Naturforsch A***, 42, 1986, pp 797-802.

[22] Gonesbet G., Rceonstruction of vector fields of continuous dynamical systems from scaler time series, ***Phys. Rev. A***, 43, 1991, pp 5321-5331.

[23] Rowlands G. and Sprott J. C., Extraction of dynamical equations from chaotic data, ***Physica D***, 58, 1992, pp 251-259.

[24] Gonesbet G. and Maquet J., Construction of phenomenological models from numerical scalar time series, ***Physica D***, **58**, 1992, pp 202-215.

[25] Bunimovich L. A., Coupled map lattices: one step forward and two steps back, ***Physica D***, 86, 1995, pp 248-255.

[26] Dewson T. and Irving A. D., Determining mixed linear-nonlinear coupled differential equations from multivariate time series sequences, ***Submitted to Physica D***.

[27] Irving A. D. and Dewson T., The analysis of complex times series data, ***Applied Mathematical Modelling***, Vol. 20, 1996, pp 35-47.

[28] Irving A. D., Dewson T., Hong G. and Cunliffe N. H., General nonlinear response of a single input system to stochastic excitations, ***Applied Mathematical Modelling***, Vol. 19, 1995, pp 45-56.

[29] Irving A. D., Clayton B. R. and Dewson T., Nonlinear thermoviscoelastic behaviour in complex materials , ***IUTAM Symposium on Inhomogenity, Anisotropy and Nonlinearity in Solid Mechanics***, Elsevier, 1995, p127-132.

[30] Irving A. D. and Dewson T., General mixed linear-nonlinear response of evolutionary physical systems, ***In Preparation***.

[31] ***Neural Networks in the Capital Markets***, Ed. A. N. Refenes, Proceedings of the first international workshop, London Business School, November, 1993.

Figure captions

1. Four samples of the time series sequences of common stock prices with superimposed daily predictions 30 trading days ahead.

2a. The probability of a given prediction as a function of standard deviation and time delay for the predictions used in the paper trading.

2b. The logarithm of the probability of a given prediction as a function of standard deviation and time delay for the predictions used in the paper trading.

3. The daily % return on the portfolio of 300 common stock names for 30 days.

4. The cumulative % return on the portfolio of 300 common stock names for 30 days.

5. The daily % return on the portfolio of 30 common stock names for 377 days.

6. The cumulative % return on the portfolio of 30 common stock names for 377 days.

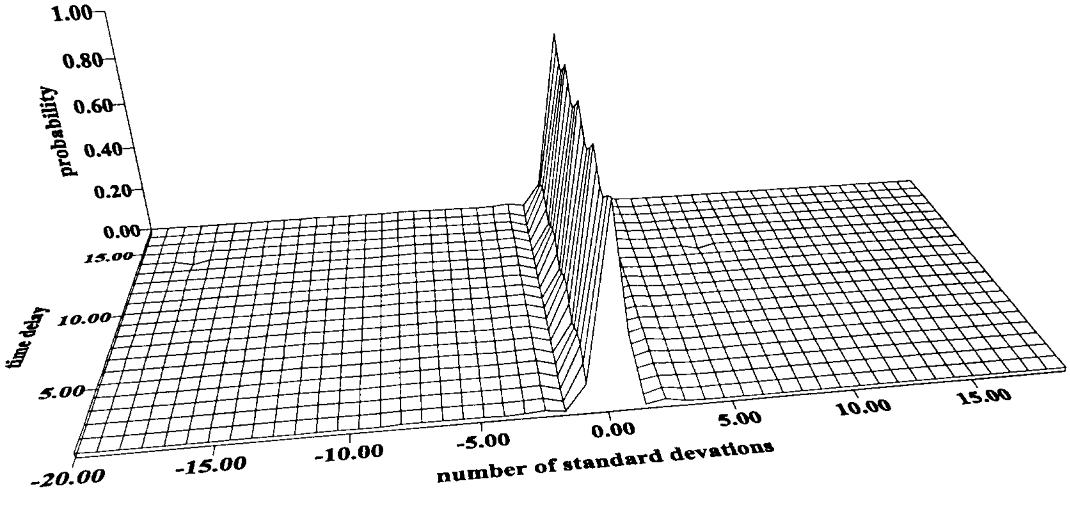


Figure 2a

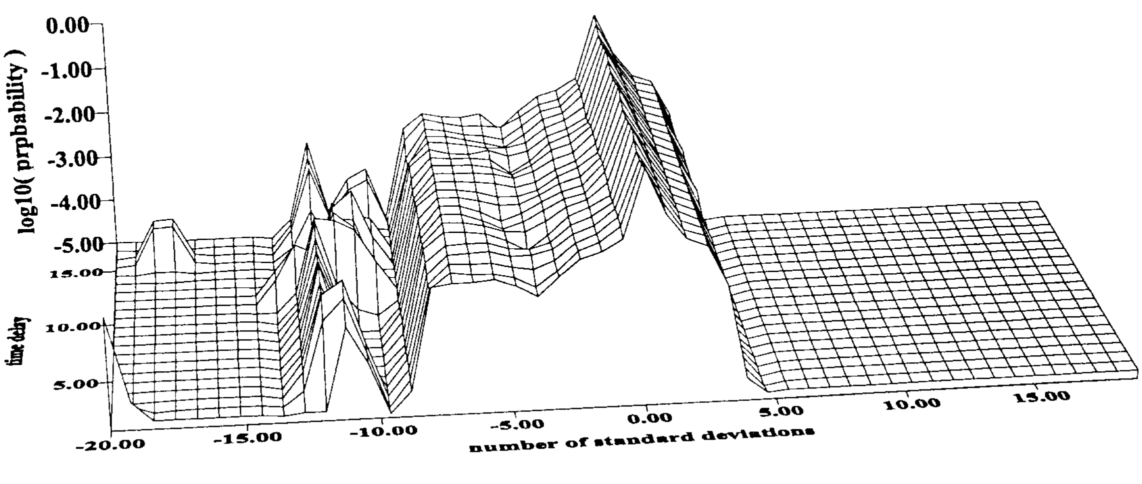


Figure 2b

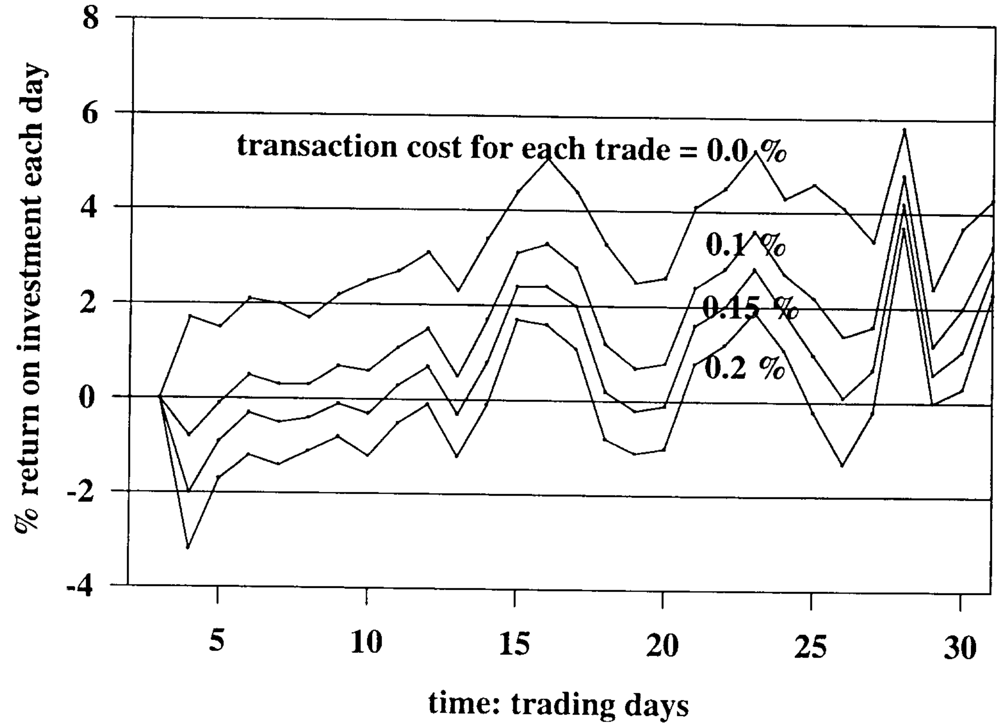
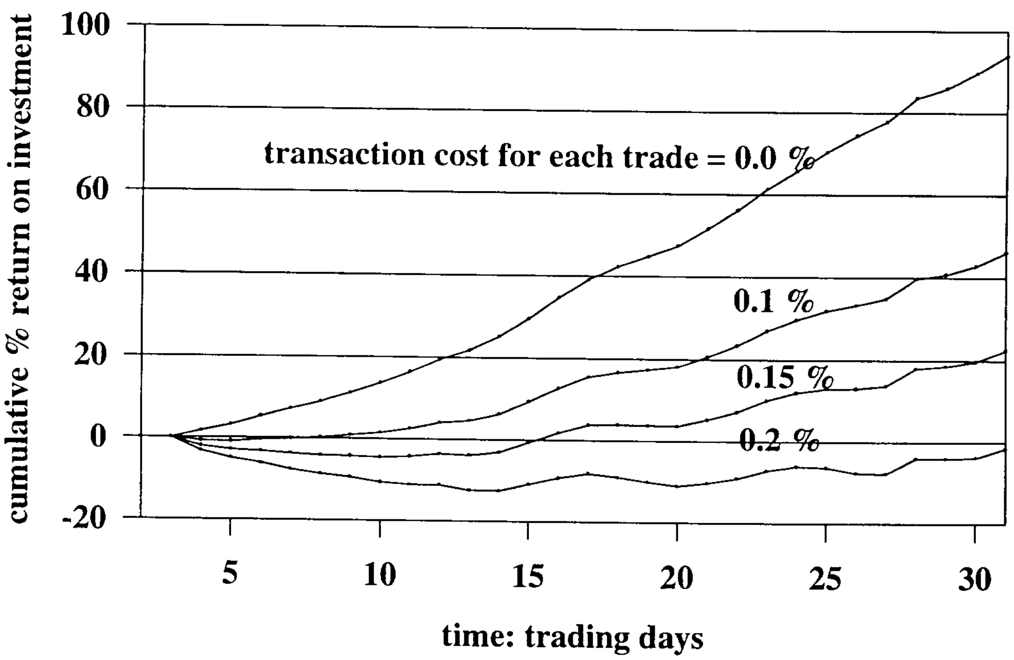


Figure 3.



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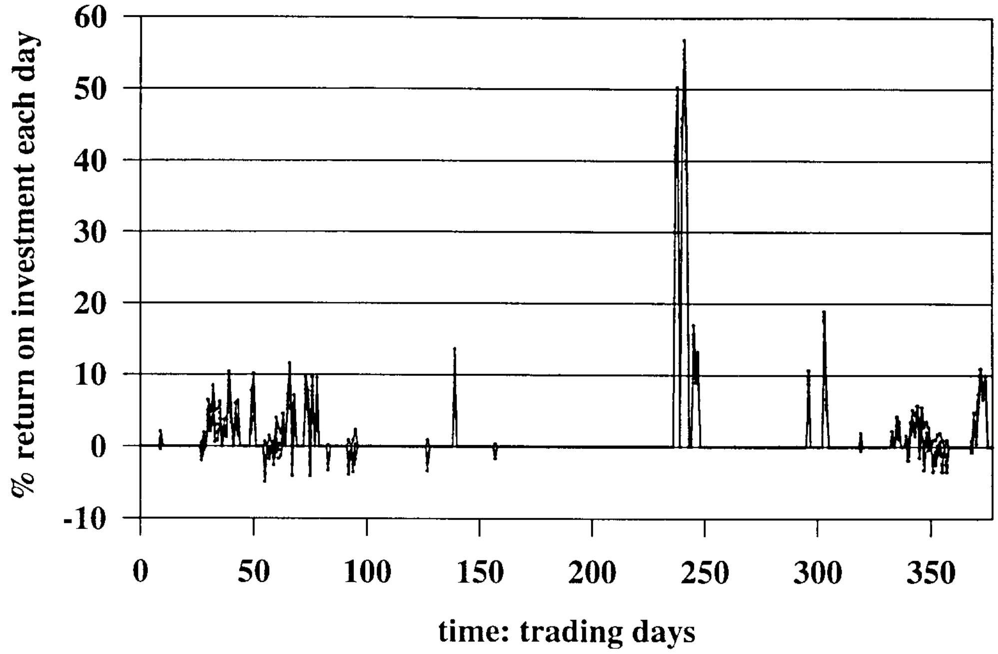

Figure 5.

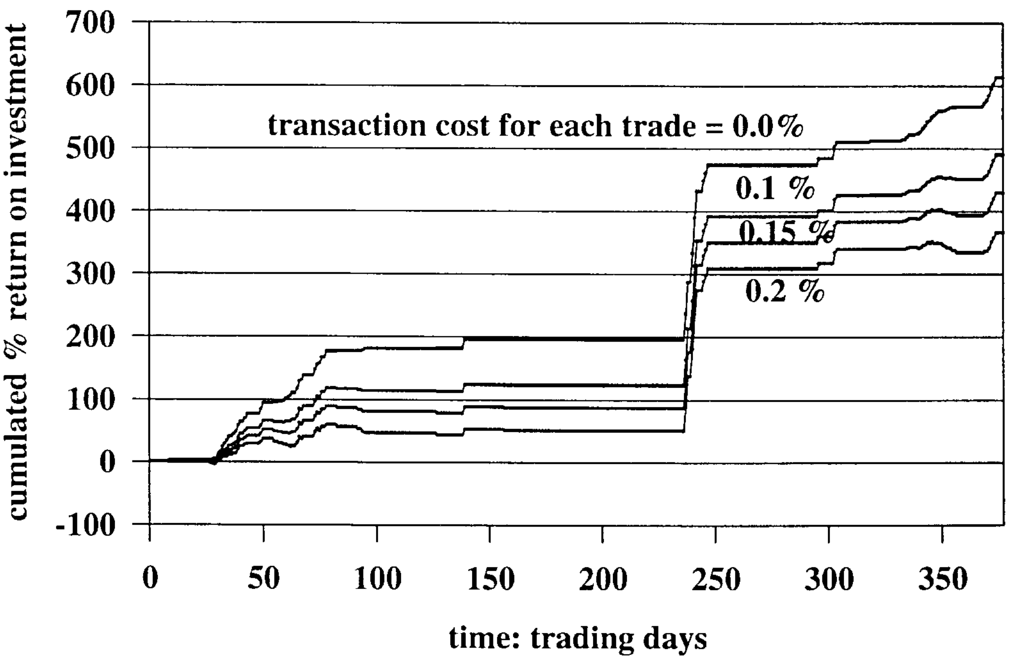

Figure 6.