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"HAMILTONIAN" MECHANICS WITH z AS INDEPENDENT VARIABLE. THIN LENS APPROXIMATION FOR AN ACCELERATING GAP AND CORRECTIONS TO PREVIOUS EQUATIONS.

by

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ABSTRACT

In the first part we introduce a kind of "Hamiltonian" Mechanics where z and not time t is the independent variable. We discuss the "canonically conjugate" variables of this scheme and list a number of useful formulae.

In the second part we apply this formalism to the motion of a proton in a linac gap. We introduce the Thin Lens Approximation for such a gap by Canonical Transformations leading to reduced variables. Reduced Variables are constant if no field is accelerating the particle. We thoroughly discuss the principles of the Thin Lens Approximation. In the one-dimensional case we derive by this method the Modified Panofsky equations for the change of phase and kinetic energy across a linac gap. When including transversal motion, we have to make further approximations. But we then succeed in giving a set of difference equations corresponding to those given by P. Lapostolle at the Frascati Conference (1965)[3].

In Appendix B we correct a mistake made in deriving an improved version[5] of Lapostolle's equations. Tables for the difference equations for the change of longitudinal kinetic energy, phase, and the transversal quantities across a linac gap for the non-relativistic and for the relativistic case may be found at the end of the paper.

1. INTRODUCTION

This paper consists of two parts somewhat differing in scope and presentation. In the first part we develop a kind of Hamiltonian Mechanics with z, i.e. with a space coordinate, as independent variable. In the second part this formalism is employed to derive the Thin Lens Approximation for an accelerating gap.

When using the usual Lagrangian or Hamiltonian formulation of mechanics where time t is the independent variable, in the theory of accelerators, one often hits on the following problem: By solving the equations of motions, one gets solutions giving the position and the velocity (or momentum) of a particle as a function of t. But one wants to know what are the values of some of these or other quantities (e.g. energy) when the particle arrives at a certain point, say $z = z_o$ (e.g. at the end of an accelerating gap). One must now solve the equation $z(t_o) = z_o$ for t_o and insert this t_o in the time dependent solutions describing the quantities one is looking for. In general, since the field of force will exhibit a harmonic or an even more complicated time behaviour, the equation $z(t_o) = z_o$ will be a nasty transcendental one. In this and similar applications one notes that it would be more convenient to have a space coordinate, say z, as independent variable.

Now, the game usually played - essentially a Legendre-transformation- for t as independent variable leading to canonically conjugate variables and to Hamilton's equations which then may be brought to new representations by canonical transformations, can be executed for any variational problem, provided the integrand does not depend on higher derivatives than the first one and is not homogeneous of degree one in these derivatives[1),11)]. Thus, introducing by a substitution in Hamilton's principle a certain variable, say z, as variable of integration, it becomes the independent variable of the

Euler equations, i.e. of the equations of motion in their Lagrangian or Hamiltonian form. Such formulations have been employed before in many cases e.g. in linac theory in refs.[2)3)]. Here we work out this scheme in a more systematic and general fashion and enumerate a number of formulae, hoping that such a list may be useful for reference.

In Section 2 we give the relations between electromagnetic field and potentials. In Section 3 we summarize Hamiltonian Mechanics with t as independent variable. In Section 4, we introduce z as the independent variable and state non-relativistic expressions for "canonical momenta" and the "Hamiltonian" in the z-scheme. Section 5 deals with "canonical" transformations. In Section 6 the formulae are specialized in the practically important case of axial symmetry. Section 7 contains relativistic expressions of "canonical momenta" and the "Hamiltonian".

In the second part we apply this formalism to the theory of the motion of a proton in an accelerating gap. Treating first (Section 8) the one dimensional problem, especially that of a time-harmonic spacially homogeneous electrical field, we introduce by a "canonical" transformation as new coordinates reduced time-angle: $\varphi - z\varphi' = \omega t - \omega z\,(dt/dz)$ and T/ω = kinetic energy divided by the angular frequency. Solving afterwards "Hamiltons equations" by iteration, we find again the parameter $\overline{E} = eE_o/(m\omega v_o)$ which plays an important rôle in the perturbation theoretic treatment of the motion of protons in the RF-fields of accelerators.[4)5)]. The first order solutions are the modified Panofsky equations[6)3)]. Simultaneously, we elaborate the principles of the Thin Lens Approximation, in which one seeks to replace the description of the real motion of a particle in a gap by that of an equivalent fictitious "motion" where the dynamical variables have jumps across a plane in the centre of the gap and behave before and after it as if the particle were moving in a region free from fields.

The principles how to conduct this approximation and how to derive a Thin Lens Hamiltonian by "canonical transformations" leading to reduced variables, are discussed in Section 9.

When pursuing this programme in Section 10 for a general axially symmetric TM-RF-field described by a Fourier integral in z, we meet some difficulties forcing us to make severe approximations concerning the radial dependence of the field. When we approximate it by a field which does not change radially, we can give the exact "canonical transformations", but the resulting new "Hamiltonian" is only linear in the transversal quantities. Taking into account the next power of radius r in the field expressions we must employ a different approach where we blend the foregoing results with a treatment using Newton's equations. We find only approximate expressions both for the "Hamiltonian" quadratic in transversal quantities and for the new transversal variables. As a consequence, the latter do not exactly fulfil the "Poisson brackets".

In appendix B we take the opportunity to correct an error committed in earlier papers [3)5)8)9)] in the course of the derivation of $(r'_+ - r'_-)$. The fault was that in the transformation $dr/dz = dr/d\varphi \cdot d\varphi/dz$ one wrongly replaced $d\varphi/dz$ by the constant $k = \omega/\dot{z}_0 = (d\varphi/dz)_0$.

In Table I we give the new set of non-relativistic difference equations for a linac gap, in Table II the relativistic one. In the latter we made, in addition to the improvement just mentioned, some corrections for computational errors and missprints in the expressions for $(\varphi_+ - \varphi_-)_r$ and for $(\bar{r}_+ - \bar{r}_-)_r$.

Remark concerning notation: Repeated indices have to be summed over their whole range: 1...f.

$$\dot{} = d/dt \qquad {}' = d/dz \qquad {}^{\grave{}} = d/d\varphi = d/d(\omega t)$$

2. REPRESENTATION OF THE FIELD

It is just for completeness that we here shall state a few well-known formulae relating the electromagnetic potentials to the electromagnetic field. The latter may be described by a scalar potential U and a vector potential $\vec{A}$: [7]

$$\vec{E} = -\nabla U - \vec{A}_t \qquad (2.1)$$
$$\vec{B} = \nabla \times \vec{A} \qquad (2.2)$$

U and $\vec{A}$ are only unique if we additionally pose a side condition. We chose the Lorentz gauge :

$$\nabla \cdot \vec{A} + \varepsilon\mu \; U_t = 0 \qquad (2.3)$$

The four-potential is connected to the electrical Hertz vector by :

$$\vec{A} = \mu\varepsilon \vec{\Pi}_t \qquad U = -\nabla \cdot \vec{\Pi} \qquad (2.4)$$

whose rectangular components are solutions of the Helmholtz equation. Any axially symmetric TM-field may be expressed by the following Hertz vector :

$$\vec{\Pi} = \vec{e}_z \Pi(r,z) \cos(\varphi + \varphi_0) = \vec{e}_z \Pi(r,z) \cos(\omega t + \varphi_0) \qquad (2.5)$$

The two non-zero potential functions are in this case :

$$U = -\cos(\varphi + \varphi_0)\Pi_z \qquad A_z = -k_0^2/\omega \, \Pi \sin(\varphi + \varphi_0) \qquad (2.6)$$

In our applications[3)5)8)9)] it is useful to represent Π by a Fourier integral :

$$\Pi(r,z) = E_1/(2\pi) \int_{-\infty}^{\infty} dk_z \; b(k_z) \; e^{ik_z z} J_0(\gamma r)/(\gamma^2 J_0(\gamma a)) \qquad (2.7)$$

with:

$$k_o^2 = \varepsilon\mu\omega^2 \qquad k_z^2 = k_o^2 - \gamma^2 \tag{2.8}$$

According to (1), (2) and (6) the fields are given by :

$$\vec{E} = \cos(\varphi + \varphi_o)\left[\vec{e}_r \Pi_{rz} + e_z(\Pi_{zz} + k_o^2 \Pi)\right] \tag{2.9}$$

$$\vec{B}_\varphi = \vec{e}_\varphi B_\varphi \qquad B_\varphi = k_o^2/\omega \ \sin(\varphi + \varphi_o) \Pi_r \tag{2.10}$$

3. USUAL HAMILTONIAN MECHANICS

In usual Hamiltonian Mechanics[10)11)] where time t is the independent variable, one starts from Hamilton's principle :

$$\int_{t_1}^{t_2} L(q_k, \dot{q}_k; t)dt = \text{Extr.} \tag{3.1}$$

the Lagrangian L being either :

$$L = T - U \tag{3.2}$$

where U is the Potential (not depending on velocities or time) or the generalized potential describing the force F_i according to :

$$F_i = - U_{q_i} \qquad F_i = d/dt\, U_{\dot{q}_i} - U_{q_i} \ ; \tag{3.3}$$

or the Lagrangian for an external electromagnetic field (e = charge of the particle) :

$$L = T + e\,(\vec{v}, \vec{A}) - eU \tag{3.4}$$

In the non-relativistic case, T is kinetic energy

$$T = m\,v^2/2 = m\,\dot{q}_i\dot{q}_i/2 \tag{3.5}$$

In the relativistic treatments we insert for T :

$$T = -mc^2(1 - \beta^2)^{1/2} \qquad (3.6)$$

(m = rest mass)(some use : $T = mc^2(1 - (1 - \beta^2)^{1/2})$ which does not change the equations of motion; but neither of them is relativistic kinetic energy =

$$mc^2 \left[(1 - \beta^2)^{-1/2} - 1 \right].$$

The Euler equation of the variational problem (1) are the equations of motion :

$$d/dt \; L_{\dot{q}_i} \quad - \quad L_{q_i} = 0 \qquad (3.7)$$

a system of f ordinary second order differential equations (f = number of degrees of freedom). Introducing canonical momenta (which in general differ from common momenta $m\dot{q}_i$);

$$p_i = L_{\dot{q}_i} \qquad (3.8)$$

one transforms the variational problem into a canonical one whose Euler equations are a system of 2f first order differential equations, Hamilton's equations :

$$H = p_i \dot{q}_i - L \qquad \dot{p}_i = - H_{q_i} \qquad \dot{q}_i = H_{p_i} \qquad (3.9)$$

Transformations of the dependent variables in these equations are called canonical if they preserve the canonical form of the variational problem and therewith of the equations of motion.

In the non-relativistic case, if kinetic energy is a quadratic form of the $\dot{q}$ and if U does not depend on the $\dot{q}$, then H equals total energy. The second condition is not fulfilled if a magnetic field is present (see (4)).

4. HAMILTONIAN MECHANICS WITH z AS AN INDEPENDENT VARIABLE

As explained in the introduction, we want z to be the independent and consequently time t to be a dependent variable. Instead of the latter we prefer the dimensionless (ω = circular frequency of the accelerating field) :

$$\varphi(z) = \omega t(z) \qquad \omega = \text{const.} \tag{4.1}$$

for which we suggest the name time-angle. It has a certain relation to phase which we do not know exactly, to a first approximation they may be equal.

We substitute

$$\begin{aligned} d\varphi &= \omega\, dt = (d\varphi/dz)\, dz = \varphi' dz \\ x &= x(t) \rightarrow x(z) \qquad\qquad y = y(t) \rightarrow y(z) \\ \dot{x} &= dx/dt = x'/t' = \omega x'/\varphi' \qquad \dot{y} = \omega y'/\varphi' \end{aligned} \tag{4.2}$$

in Hamilton's principle (3.1) :

$$\int_{t_1}^{t_2} L(x,y,z;\dot{x},\dot{y},\dot{z};t)\,dt = \int_{z_1}^{z_2} L(x,y,z;x'/t',y'/t',1/t';t)\,t'dt =$$

$$= \int_{z_1}^{z_2} \overline{L}(x,y,t;x'y';t';z)\,dz \quad = \text{Extr.}$$

We use all the names we know from the t-scheme for the mathematically corresponding quantities in the z-scheme except that we put them within quotation marks to avoid confusion, since the latter may have different physical dimension and meaning. Differentiating the "Lagrangian" $\overline{L}$:

$$\begin{aligned} \overline{L} &= m(x'^2 + y'^2 + 1)/(2t') + e(x'A_x + y'A_y + A_z) - eU\,t' \\ &= m\omega(x'^2 + y'^2 + 1)/(2\varphi') + e(x'A_x + y'A_y + A_z) - eU\,\varphi'/\omega \end{aligned} \tag{3.6}$$

we get the "canonical momenta":

$$p_x = \bar{L}_{x'} = m\, x'/t' + eA_x = L_{\dot{x}} \qquad p_y = m\, y'/t' + e\,A_y \tag{4.7}$$

$$p_t = \bar{L}_{t'} = -m(x'^2 + y'^2 + 1)/(2\,t'^2) \quad -eU = -E \tag{4.8}$$

$$p_\varphi = \bar{L}_{\varphi'} = -\,E/\omega = -\left[(x'^2 + y'^2 + 1)m\,\omega^2/(2\varphi'^2) - eU'\right]/\omega \tag{4.9}$$

The transversal "canonical momenta" p_x, p_y agree with the canonical momenta (3.8). The "momentum" canonically conjugate to time (time-angle) is the negative total energy (divided by ω). We solve (7) till (9) for t' :

$$t' = +\,(m/2)^{1/2}\left[-p_t \quad -eU \quad -((p_x - eA_x)^2 + (p_y - eA_y)^2)/(2m)\right]^{-1/2} \tag{4.10}$$

Since we are only concerned with particles moving in the positive z direction, $t' > 0$, we use throughout the positive branch of the square root. Therewith we form the "Hamiltonian":

$$\begin{aligned}\bar{H}(p_x, p_y, p_t; x, y, t; z) &= x'p_x + y'p_y + t'p_t - \bar{L} = \\ &= -(2m)^{1/2}\left[-p_t - eU - ((p_x + eA_x)^2 + (p_y + eA_y)^2)/(2m) - eA_z\right]^{1/2}\end{aligned} \tag{4.11}$$

$$\begin{aligned}\bar{H}(p_x, p_y, p_\varphi; x, y, \varphi; z) &= \\ &= -(2m)^{1/2}\left[-\omega p_\varphi - eU - ((p_x + eA_x)^2 + (p_y + eA_y)^2)/(2m) - eA_z\right]\end{aligned} \tag{4.12}$$

and we find the variational problem equivalent to (4.5) :

$$\int_{z_1}^{z_2}\left[p_k q_k' - H(p_k, q_k)\right] dz = \text{Extr.} \tag{4.13}$$

The transition from (5) to (13) is here exposed in a rather superficial fashion. A very careful treatment of it (for t instead of z as independent variable, but this does not strike the structure of the mathematical proof) is contained in ref.[10].

The Euler equations of (13) are "Hamiltons equations":

$$d\,{}_k/dz = H_{p_k} \qquad d\,{}_k/dz = -H_{q_k} \tag{4.14}$$

It is easy to find the physical meaning of $\bar{H}$ with the help of (10):

$$\bar{\bar{H}} = -m\ dz/dt - eA_z = -p_z \tag{4.15}$$

This foreshadows a bit the four- (or eight-) dimensional symmetry inherent to Hamilton mechanics[12]. Among the pairs x, p_x; y, p_y; z, p_z; $t, p_t = -E$, one may choose any one as independent variable, the other one being the "Hamiltonian" and the remaining pairs are the dependent "canonically conjugate" variables. The minus in (15) should not irritate us toomuch, for $t' < 0$ it would not be here.

5. Canonical Transformations in the z-scheme.

Canonical Transformations (C.T.) are such a choice of new dependent variables

$$P_i = P_i(p_k, q_k; z) \qquad p_k = p_k(P_i, Q_i; z) \tag{5.1}$$

$$Q_i = Q_i(p_k, q_k; z) \qquad q_k = q_k(P_i, Q_i; z)$$

which preserves the canonical form of the variational problem (4.5),[1),10)], therewith the form of "Hamilton's equations". This does not mean that the integral in the new variables P_i, Q_i

$$\int_{z_1}^{z_2} \left[P_k Q'_k - K(P_k, Q_k; z) \right] dz = \text{Extr.} \tag{5.2}$$

must be identic alwith that of (4.13), but only that both assume simultaneously their extremal values, i.e. if (4.13) assumes its extremal value for the $q_i(z)$, $p_i(z)$, then the same thing should happen to (2)for those $Q_i(z)$, $P_i(z)$ which arise from the $q_i(z)$, $p_i(z)$by the substitutions (1). A necessary and sufficient condition is that both integrands only differ by the total derivative of an otherwise arbitrary function $\phi(q_k, Q_k; z)$, because for fixed limits the integral $\int_{z_1}^{z_2} d\phi/dz = \phi(z_2) - \phi(z_1)$ is a constant which does not influence the extremal values of the integrals.

$$p_k q'_k - \bar{H} = P_k Q'_k - K + d\phi/dz \tag{5.3}$$

together with

$$d\phi/dz = \phi_{q_k} q'_k + \phi_{Q_k} Q'_k + \phi_z \tag{5.4}$$

gives by equating coefficients of q'_k and Q'_k :

$$p_k = \phi_{q_k} \qquad P_k = - \phi_{Q_k} \qquad K = \bar{H} + \phi_z \tag{5.5}$$

One solves the second set of equations for $q_k = q_k(Q_i, P_i; z)$; these are inserted in the first set to give $p_k = p_k(P_i, Q_i; z)$; with both we get the new "Hamiltonian" $K(P_k, Q_k; z)$.

One has much more freedom to choose on which variables the generating function ϕ of the canonical transformation should depend. The most general result is : Let $\phi(x_k, X_k; z)$ be an arbitrary function of the 2f+1 Variables x_k, X_k, z : x_k $(k=1,..f)$ being any of the p_k, q_k ; X_k any of the P_k, Q_k.

Then

$$y_k = \pm \phi_{x_k} \qquad Y_k = \mp \phi_{X_k} \qquad K = H + \phi_z \tag{5.6}$$

is a canonical transformation, y_k is "canonically conjugate" to x_y, Y_k to X_k. Take the upper (lower) sign when deriving with respect to a cordinate ("momentum").

Defining the "Poisson bracket" :

$$(u,v) = u_{q_k} v_{p_k} - u_{p_k} v_{q_k} \tag{5.7}$$

we can give an other set of necessary and sufficient conditions for the transformations (1) to be "canonical";

$$(Q_i, Q_k) = (P_i, P_k) = 0 \qquad (Q_i, P_k) = \delta_{ik} \tag{5.8}$$

Analogous conditions may be stated by use of the "Lagrange bracket":

$$[u, v] = \partial q_k/\partial u \cdot \partial p_k/\partial v - \partial p_k/\partial v \cdot \partial q_k/\partial u \tag{5.9}$$

and replacing in (8) the round by square brackets. These conditions are sometimes very useful : One often has to guess some of the new "canonical" variables as function of the old ones or vice-versa. The above relations permit to check whether these assumed functions are admissible and compatible as "canonical transformations". Of course, it is much better to guess or to find the generating function, because then you get at once transformations which are "canonical". The difficulty is only : You can assume a generating function, but will then the "canonical" transformation do what you would like to achieve?

6. Specialization to axial symmetry.

We employ cylindrical coordinates (r, θ, z) and describe the electromagnetic field by U and A_z (see (2.5)). On account of the assumed symmetry there is no force acting in the θ-direction. θ and θ′ are constant and we put them to zero. We simply restate equations (4.6) to (4.12):

$$\bar{L} = \frac{m}{2}(r'^2 + 1)\frac{1}{t'} + eA_z - eUt' = m\omega\,(r'^2 + 1)/(2\varphi') + eA_z - eU\varphi'/\omega \tag{6.1}$$

$$p_r = \frac{\partial \bar{L}}{\partial r'} = \frac{\partial L}{\partial \dot{r}} = m\omega r'/\varphi' = mr'/t' \tag{6.2}$$

$$p_t = \bar{L}_{t'} = -E = -(r'^2/t'^2 + 1/t'^2)\, m/2 + eU \tag{6.3}$$

$$p_\varphi = \bar{L}_{\varphi'} = -E/\omega = -\left[(r'^2/\varphi'^2 + 1/\varphi'^2)\, m\omega^2/2 + eU\varphi'\right]/\omega \tag{6.4}$$

$$t' = \sqrt{\frac{m}{2}}\left[-p_t - eU - p_r^2/(2m)\right]^{-1/2};\ \varphi' = \omega(m/2)^{1/2}\left[-\omega p_\varphi - eU - p_r^2/(2m)\right]^{-1/2} \quad (6.5)$$

$$\tilde{H} = -(2m)^{1/2}\left[-\omega p_\varphi + eU(r,z,\varphi) - p_r^2/(2m)\right]^{1/2} - eA_z \quad (6.6)$$

The "Poisson bracket" (5.7) becomes:

$$(u,\ v) = u_r v_{p_r} - u_{p_r} v_r + u_\varphi v_{p_\varphi} - u_{p_\varphi} v_\varphi \quad (6.7)$$

It's this one in particular that we shall use extensively throughout the later sections.

7. Relativistic case.

At the one hand we give the related formulae for the general case in Cartesian coordinates, at the other hand those for the axially symmetric problem in cylindrical coordinates under the assumption $\theta = \dot{\theta} = o$. ($m$ = rest mass.)

The "Lagrangians" are (see 3.4), (3.6) and (4.3)):

$$\bar{L} = -mc^2\left[1-(x'^2+y'^2+1)/(c^2\ t'^2)\right]^{1/2} t' - eUt' + eA_x x' + eA_y y' + eA_z \quad (7.1)$$

$$\bar{L} = -mc^2\left[1-(r'^2+1)\omega^2/(c^2\varphi'^2)\right]^{1/2} \varphi'/\omega - eU\varphi'/\omega + eA_z \quad (7.2)$$

From these we derive the "canonical momenta":

$$p_x = \bar{L}_{x'} = mx'/t'\left[1-(x'^2+y'^2+1)/(c^2\ t'^2)\right]^{-1/2} + eA_x = L_{x'}\quad p_y = L_{y'} \quad (7.3)$$

$$p_r = \bar{L}_{r'} = mr'/t'\left[1-(r'^2+1)/(c^2\ t'^2)\right]^{-1/2} = m\omega r'/\varphi'\left[1-(r'^2+1)\omega^2/(c^2\varphi'^2)\right]^{-1/2} \quad (7.4)$$

$$p_t = \bar{L}_{t'} = -E = -mc^2\left[1-(x'^2+y'^2+1)/(c^2\ t'^2)\right]^{-1/2} - eU \quad (7.5)$$

$$\omega p_\varphi = \omega\bar{L}_{\varphi'} = -E = -mc^2\left[1-(x'^2+y'^2+1)\omega^2/(c^2\varphi'^2)\right]^{-1/2} - eU \quad (7.6)$$

$$\omega p_\varphi = -E = -mc^2\left[1-(r'^2+1)\omega^2/(c^2\varphi'^2)\right]^{-1/2} - eU \tag{7.7}$$

For particles moving in the positive z-direction:

$$ct' = -(p_t+eU)\left[(p_t+eU)^2-c^2(p_x-eA_x)^2-c^2(p_y-eA_y)^2-m^2c^4\right]^{-1/2} \tag{7.8}$$

$$\bar{H} = -eA_z - \left[(eU+p_t)^2c^{-2}-m^2c^2-(p_x-eA_x)^2-(p_y-eA_y)^2\right]^{1/2} \tag{7.9}$$

$$\bar{H} = -eA_z - \left[(eU+p_t)^2c^{-2}-m^2c^2-p_r^2\right]^{1/2} \tag{7.10}$$

$$= -eA_z - \left[(eU+\omega p_\varphi)^2c^{-2}-m^2c^2-p_r^2\right]^{1/2}$$

8. <u>One-Dimensional Case. Derivation of the Modified Panofsky Equations.</u>

As a first application of the general formalism elaborated before, we treat the motion of a proton in a time harmonic spacially homogeneous electrical field [13]. We neglect in the formulae of Sect. 6 all radial quantities and drop the subscript φ in $p_\varphi = p$ We have only a scalar potential:

$$U(z,\varphi) = -eE_o z \cos(\varphi+\varphi_o) \tag{8.1}$$

and "Hamiltons equations" read:

$$\varphi' = \omega(m/2)^{1/2}\left[-\omega p - eU(z,\varphi)\right]^{-1/2} = \omega(m/(2T))^{1/2} \tag{8.2}$$

$$p' = \omega(m/2)^{1/2}\left[-\omega p - eU(z,\varphi)\right]^{-1/2}U_\varphi = -\omega(m/(2T))^{1/2}U_\varphi \tag{8.3}$$

As data we are given:

$$z = 0 : \varphi = \varphi_o \qquad p = -W/\omega = -mv_o^2/(2\omega) = -m\omega/(2k^2) \tag{8.4}$$

We know that $\varphi(z)$ cannot be expressed in closed form [4]. Therefore it is not possible to solve these equations of motion exactly. We intend to solve them by perturbation theory starting from free particle motion ($E_o = 0$) and taking into account the influence of the field in iterative steps. To zeroth order (3) becomes:

$$p^{(o)\prime} = 0 \quad p^{(o)} = - m\omega/(2k^2) = \text{const.} \quad k = \omega/v_o \tag{8.5}$$

(v_o = velocity at z = 0). This inserted in (2) yields :

$$\varphi^{(o)} = k \qquad \phi^{(o)} = kz + \varphi_o \tag{8.6}$$

One could put these zero order solutions into the right hand sides on (2) and (3), get the first order solutions and so on.

But we would prefer to have instead of (6.6), (2) and (3) a "Hamiltonian" with something like a power series in the potential of the field, since then we could solve the equations of motion by assuming for the solutions power series in the same perturbation parameter and separate the orders of approximation in the same way as we did earlier [5]. We try to achieve this by "canonical transformations" and for a first attempt we choose as new variables:

$$Q = \varphi_o + \varphi - z\omega \, (-m/(2\omega P))^{1/2} \tag{8.7}$$

$$P = - T/\omega = P + eU \, (z,\varphi)/\omega \tag{8.8}$$

which are "canonically conjugate" (Q,P) = 1. The main reason for this choice is that similar variables have been used for a kind of thin lens approximation [3)8)] (see Sec. 9). We suggest for Q the name: reduced time-angle. T is kinetic energy. From $\varphi = -\emptyset_p$, $Q = \emptyset_p$ (see (5.6)) we find the generating function:

$$\emptyset(p,P,z) = z(-2m\omega P)^{1/2} + \varphi_o\, p + (P-p)\arccos(-\omega(P-p)/(eE_o\, z)) + ((eE_o z/\omega)^2 - (P-p)^2)^{1/2} \tag{8.9}$$

and we get the new "Hamiltonian": $K = \tilde{H} + \emptyset_z$

$$K(P,Q,z) = (eE_o/\omega)\ \sin(Q + \omega z(-m/(2\omega P))^{1/2}) \tag{8.10}$$

$$Q' = K_P \quad P' = -K_Q$$

This "Hamiltonian" has exactly the shape we are looking for. Solving (10) by iteration we begin with free particle motion ($E_o = 0$):

$$Q^{(o)} = \varphi_o = \text{const.} \quad P^{(o)} = -W/\omega = \text{const.} \tag{8.11}$$

Inserting these into the right sides of (10) we find the next approximation by integration w.r.t. z:

$$Q = \varphi_o + \bar{E}\left[kz \sin(\varphi_o + kz) + \cos(\varphi_o + kz) - \cos\varphi_o\right] = Q^{(o)} + \bar{E}\, Q^{(1)} \tag{8.12}$$

$$P = -W/\omega\left[1 + \bar{E}\, 2\, (\sin(\varphi_o + kz) - \sin\varphi_o)\right] = P^{(o)} + \bar{E}\, P^{(1)} \tag{8.13}$$

Thus this method leads us straightforward to the perturbation parameter:

$$\bar{E} = eE_o/(m\omega v_o) = (eE_o/\omega)/(mv_o) \tag{8.14}$$

=(impuls exerted upon a proton during one period)/(free particle momentum),

$\bar{E} < .1$ in a proton machine above .5 MeV. This parameter has been found in an earlier paper [4] and has been shown to work in cases where one assumes more realistic fields.[5] It is of importance in the theory of electron linacs; there ($v_o = c$) it may assume values

> 1 and it cannot be employed as a perturbation parameter[14].

We now describe an accelerating gap by a strip $(g/2 \geq z \geq -g/2)$ of a homogeneous electric field (1). To first order in $\bar{E}$ the effect of the field integrated along the whole gap is

$$Q_+ - Q_- = \varphi_{n+1} - \varphi_n = \int_{-g/2}^{g/2} \partial K/\partial P \, dz = \partial H_{TL}/\partial P^{(o)} = (k^3/m\omega)\partial H_{TL}/\partial k \quad (8.15)$$

$$P_+ - P_- = -(W_{n+1} - W_n)/\omega = -\partial H_{TL}/\partial Q^{(o)} = -\partial H_{TL}/\partial \varphi. \quad (8.16)$$

with the "Thin Lens Hamiltonian":

$$H_{TL} = \int_{-g/2}^{g/2} K(Q = \varphi_o, P = -W/\omega; z)dz = (eV_o/\omega)\, T(k) \sin\varphi \quad (8.17)$$

where $T(k)$ is the transit time factor $T(k) = (\sin(kg/2))/(kg/2)$. These are the modified Panofsky equations for the change of reduced time-angle and kinetic energy as given by Carne, Lapostolle and Promé [3].

Panofski[6] put the right hand side of (15) to zero, i.e. he set $\varphi_{n+1} = \varphi_n$. For this very reason the equations (15) and (16) violated Liouville's theorem. J.S. Bell[2] then gave a Hamiltonian formulation of difference equations for an accelerating gap, but employing different variables (cf. (22) to (24)). One is permitted to replace in (15) and (16) the derivatives $\partial/\partial P$, $\partial/\partial Q$ by $\partial/\partial P^{(o)}$, $\partial/\partial Q^{(o)}$ and to interchange these derivations with the integration w.r.t. z, since to first order in $\bar{E}$ $Q = Q^{(o)}$ and $P = P^{(o)}$ in the right side of (10), (15) and (16) and the only thing which remains variable is z. Of course, this is no longer possible in higher orders of approximation.

A more detailed discussion on the thin lens approximation will follow at the beginning of the next section. For this purpose it may be useful to give a few further examples. One might take a different generating function, for instance:

$$\emptyset\ (\varphi, P;\ z) = (\varphi + \varphi_o)\ P + z\ (-\ 2m\omega P)^{1/2} \tag{8.18}$$

which again leads to reduced variables:

$$\begin{aligned} \tilde{Q} &= Q = \varphi + \varphi_o - z\omega\ (-m/(2\omega P))^{1/2} \\ \bar{P} &= p = -\ E/\omega \end{aligned} \tag{8.19}$$

$\tilde{Q}$ is not reduced time-angle and differs from it already to first order. The new "Hamiltonian":

$$\bar{K} = (-\ 2m\omega\bar{P})^{1/2} - (\ (2m)(-\ \omega\bar{P} - eU)\ )^{1/2} \tag{8.20}$$

is zero if the field is zero. Therefore it can be employed for the derivation of a Thin Lens "Hamiltonian":

$$\bar{K}_{TL} = \int_{-g/2}^{g/2} \bar{K}^{(1)}\ dz = (eV_o/\omega)\quad T(k)\ -\ \cos(kg/2)\ \sin\varphi_o \tag{8.21}$$

though the procedure and the result are less simple.

Yet another set of variables has been introduced by J.S. Bell[2], namely time t and relativistic kinetic energy. In close analogy to that we may choose time-angle and nonrelativistic kinetic energy for Q and P :

$$Q = \varphi\ ,\ P = -\ T/\omega = p + eU(z,\varphi)\ /\omega \tag{8.22}$$

$$\emptyset(\varphi,P) = \varphi P - (e/\omega) \int^{\varphi} U\ (z,\tau)\ d\tau \tag{8.23}$$

$$K = (-2m\omega P)^{1/2} - (e/\omega) \int^{Q} U_z(z,\tau)\, d\tau \tag{8.24}$$

But in this case Q is not a reduced variable. Therefore the new "Hamiltonian" does not vanish if the field is switched off and it is unsuited for the derivation of a Thin Lens Hamiltonian. Finally we point out that the C.T. leading to the variables (7), (8) can be set up quite generally for any potential $U(z, \varphi)$. We solve (8) for φ :

$$\varphi = \Psi\,(z,\, P-p) \tag{8.25}$$

The generating function is:

$$\emptyset\,(p,P;z) = \int^{P-p} \Psi\,(z,\tau)\, d\tau + z\,(-2m\omega P)^{1/2} + \varphi_o p \tag{8.26}$$

and the new "Hamiltonian" is:

$$K\,(p,Q;z) = \int^{P-p} \Psi_z(z,\tau)\, d\tau \tag{8.27}$$

where one has to eliminate p with the help of (8) after having thrown out φ from this equation by use of (7). Though we do not know for certain that this transformation will lead for every z-dependence of the potential to a "Hamiltonian" suitable for the derivation of a Thin Lens Hamiltonian, we expect this from the reduced character of Q and P .

The relativistic problem can be handled in a very similar manner. Unfortunately, it is not a realistic approximation to take into account the mass variation while neglecting the magnetic field which necessarily accompanies a time dependent electric field. In short, the "Hamiltonian" may be found by specializing (7.10), the reduced time-angle is:

$$Q = \varphi+\varphi_o - z\varphi' = \varphi+\varphi_o + (z/c)(\omega p_\varphi + cU)\,((\omega p_\varphi + cU)^2 - m^2c^2)^{-1/2} \tag{8.28}$$

while P and K are the same as in (8) and (27) :

$$K\,(P,Q;z) = (cE_o/\omega)\ \sin(Q + z(\omega P/c)((\omega P)^2 - m^2c^4)^{-1/2}) \qquad (8.29)$$

9. Principles of the Thin Lens Approximation for a Realistic Field.

From what has been learned in the preceding section, and anticipating some of the insight gained when working out the realistic field case which will follow, we state what we believe to be the principles of the Thin Lens Approximation. The derivation of such a "Hamiltonian" H_{TL} essentially involves two steps:

a) At first one has to introduce by a C.T. "reduced Variables" (R.V.), P_i, Q_i. By definition we regard as R.V. such ones which are constant outside the region where the field is acting on the particle. (In some cases this can be only achieved for the limits $z = \pm\infty$.) The change of P_i, Q_i across the gap then characterizes the integrated effect of the field on the particle. The Thin Lens approximation then consists in replacing the continous change of a R.V. through the gap by a step function whose initial and final values are equal to the real initial and final values of the r.V. and whose jump in the centre of the gap equalsthe total change of the R.V. across the gap. Kinetic Energy is an example of such a R.V. (see Fig. 1). Time angle is not a R.V., it increases linearly when there is no acceleration. This increment is cancelled by the term $-z\varphi'$ in the reduced time angle (8.7). This reduction of time-angle has been aplied earlier, either for the total phase change across the gap [4], Fig. 2, or in a continous way [8] as indicated in Fig. 3. Similar transformations must be and have been applied to the radial variables [3)5)8)9)].

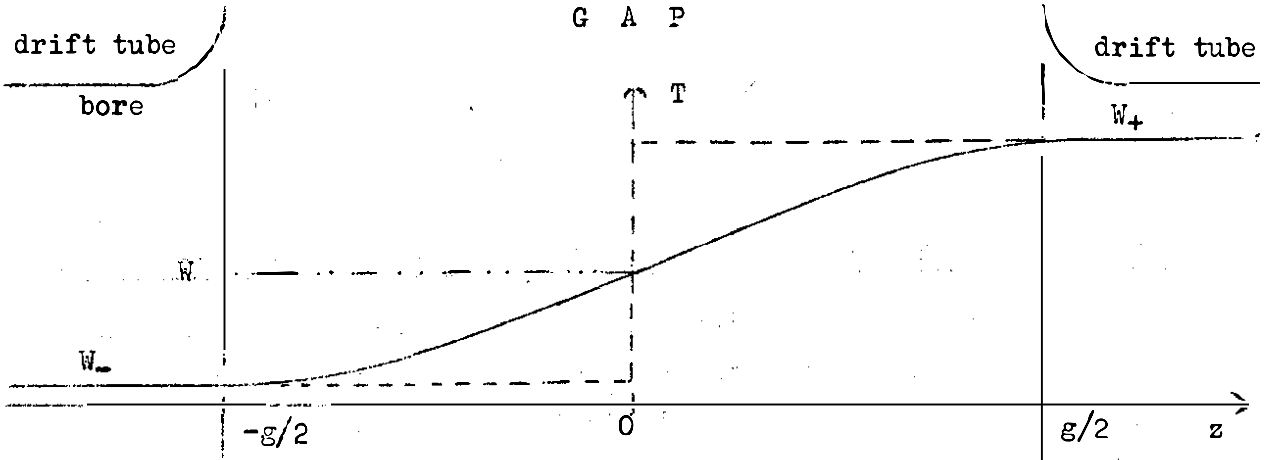

Fig.1: The behaviour of a reduced variable, e.g. of kinetic energy.
- - - Thin lens approximation.

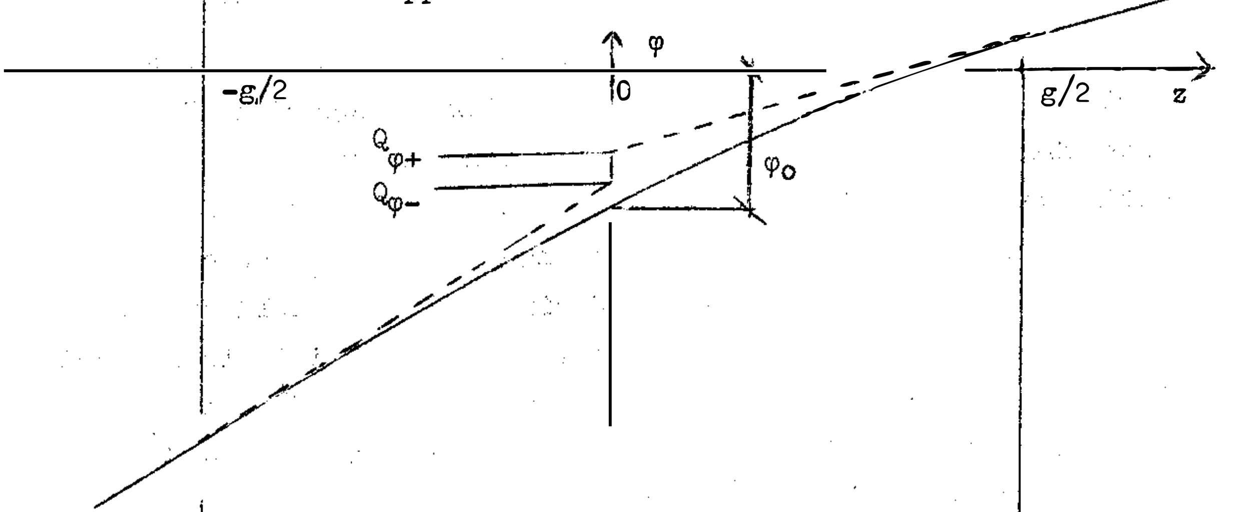

Fig.2: Reduction of the change of time-angle (phase) across the whole gap.
- - - Thin lens approximation.

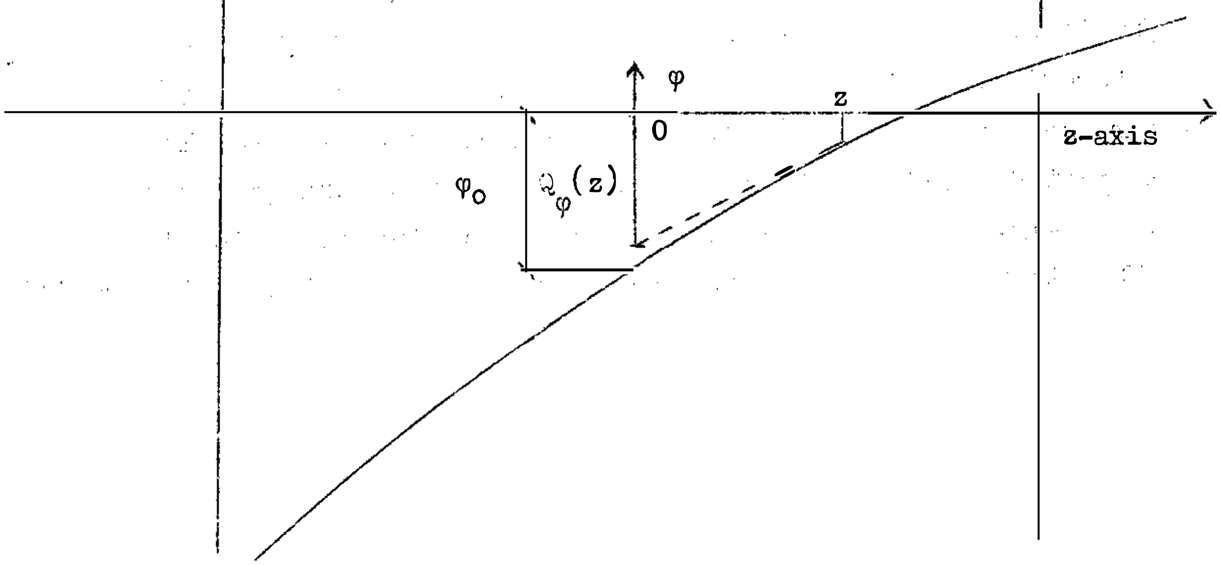

Fig.3: Continous reduction of time-angle.

b) The second step is only possible if one restricts oneself to first order perturbation theory (i.e. first iteration starting from free particle motion or, what is equivalent, first order in $\bar{E}$.) Only then one is allowed to substitute the constants P_{io}, Q_{io} for P_i, Q_i in the right hand sides of "Hamiltons equations":

$$Q' = K_P = K_{P_o} \qquad P' = - {}_Q = - K_{Q_o} \tag{9.1}$$

and as indicated, derivations w.r.t. the "canonical" variables are equivalent with those w.r.t. their zero order constants. The only thing which remains variable in (1) is z, one integrates w.r.t. z and interchanges this with the derivations to get the Thin Lens "Hamiltonian":

$$H_{TL}\,(P_{io}, Q_{io}) = \int_{-L}^{L} dz\; K^{(1)}(P_{io}, Q_{io}; z) \tag{9.2}$$

We neglect in $K = K^{(1)} + K^{(3)}$ terms nonlinear in the potentials (= $K^{(3)}$). At present we cannot evaluate them in a satisfactory manner. Moreover when one solves them by iteration starting from free-particle motion (= $K^{(3)}(Q^{(o)})$), they are expected to give contributions which are of the same order of magnitude as those coming from the second iteration of the linear terms (= $K^{(1)}(Q^{(1)})$). For the last ones the step b) is no longer possible. Linear contributions are roughly of the order of $\bar{E} = eE_o/(m\omega v_o)$ ($<$ .1 in a proton machine above .5 MeV), nonlinear ones at least of the order of $\bar{E}^2$.

The example (8.19) till (8.21) shows that it is not absolutely necessary that the new "Hamiltonian" be a power series in the potentials. But comparing the derivations leading to the two Thin Lens Hamiltonians (8.17) and (8.21), it seems that the power series Hamiltonian is easiest to handle.

This Then Lens Approximatin should be regarded as a purely mathematical procedure which is very useful for numerical applications. But it seems not possible to build a physical system i.e. a small accelerating gap which performs like a real accelerating gap with velocity dependent energy gain as e.g. (8.15) - (8.17).

10. Practical Realization of the Thin Lens Approximation.

We apply the program established above to the theory of the motion of a proton in the axially symmetric field (2.5) of a linac gap. The "Hamiltonian" is given in (6.6). We begin by looking for the reduced variables. We choose for Q_y and P_φ again reduced time angle and negative longitudinal kinetic energy, divided by ω, because they were good in the one-dimensional problem and have a physical behaviour as we expect it from R.V.:

$$Q_\varphi = \varphi_0 + \varphi - z\varphi' = \varphi_0 + \varphi + z\tilde{H}_{P_\varphi} = \varphi - z\omega(-m/(2\omega P_\varphi))^{1/2} \tag{10.1}$$

$$P_\varphi = p_\varphi + eU(r,z,\varphi)/\omega + p_r^2/(2m\omega) \tag{10.2}$$

They fulfill the "Poisson bracket" (6.7):

$$(Q_\varphi, P_\varphi) = 1 \tag{10.3}$$

For each of the two new radial coordinates we have the two conditions:

$$0 = (X,P_\varphi) = L(X) - X_{p_\varphi} eU_\varphi/\omega \tag{10.4}$$

$$0 = (X,Q_\varphi) = -z(m\omega/2)^{1/2}(-P_\varphi)^{-3/2}\left[L(X) - X_{p_\varphi}\left(eU_\varphi/\omega - ((2/m\omega)^{1/2}(-P_\varphi)^{3/2})/z\right)\right] \tag{10.5}$$

with

$$L(X) = X_r \, p_r/(m\omega) - X_{p_r} \, eU_r/\omega + X_\varphi \tag{10.6}$$

They are only compatible if the new radial coordinates are solutions of

$$L(X) = 0 \tag{10.7}$$

and do not depend on p_φ :

$$(Q_r)_{p_\varphi} = (P_r)_{p_\varphi} = 0 \tag{10.8}$$

This forbids to employ $r - r'z = r + z\bar{H}_{p_r}$ for Q_r, which would closely resemble Q_φ.

We try to solve the partial differential equation (6), (7), but this appears prohibitevely difficult for a general potential (2.6), (2.7). Therefore we expand:

$$U(r,z,\varphi) = - \cos(\varphi + \varphi_o) \Pi_z (r,z) \tag{10.9}$$

$$= - \cos(\varphi+\varphi_o) \left[\Pi_z(r_1,z) + (r-r_1) \Pi_{r_1 z}(r_1,z) + (r-r_1)^2/2 \, \Pi_{r_1 r_1 z}(r_1,z) \right]$$

$$= - \cos(\varphi+\varphi_o) \left[A + (r-r_1) \cdot D + (r-r_1)^2/2 \cdot B/2 \right]$$

where r_1 may be given convenient numerical values including r_o and 0. A_z will be developed in the same way. We are not happy about this approximation because it splits the dynamical variable r into a parameter r_1 and a new dynamical variable $r-r_1$.

Taking into account only A and D one approximates the real field by a radially homogeneous field whose strength can be adjusted by prescribing r_1, i.e. particles will experience the same force irrespective wether their zero order orbit is parallel or not to the axis $r = 0$. The transformed "Hamiltonian" is only linear in the transversal quantities P_r and Q_r . If B is present, things become much more difficult, we must proceed differently and make further approximations from the begining.

a) Linear Approximation, B = 0.

We use the two solutions of (6), (7) derived in appendix A, to introduce as new radial variables:

$$P_r = p_r + (eD/\omega) \sin(\varphi + \varphi_o) \tag{10.10}$$

$$Q_r = r - \varphi p_r/(m\omega) - (eD/(m\omega^2)(\varphi \sin(\varphi+\varphi_o) + \cos(\varphi+\varphi_o)) \tag{10.11}$$

They fulfill the last "Poisson bracket":

$$(Q_r , P_r) = 1 \tag{10.12}$$

and look like R.V. Outside the gap region the field, i.e. D tends to zero. p_r is a R.V. and in (11) the second term removes the part of r which linearly increases with $\varphi \approx kz$, again making Q_r a R.V. It is interesting and gratifying that Q_φ and P_φ impose their reduced character upon the radial variables by means of the Poisson brackets.

In the old variables we assume the data:

$$z = 0 : Q_\varphi = \varphi_o, P_\varphi = - W/\omega \longrightarrow (-m/(2\omega P_\varphi))^{-1/2} = k = \dot{z}_o^{-1} \cdot \omega$$
$$Q_r = r_o + \ldots \qquad P_r = m \dot{r}_o + \ldots \tag{10.14}$$

where we dropped terms proportional to D. The new "Hamiltonian" is at least of order one in the potentials (see (20), (21)). Therefore the terms we neglected, would produce quadratic contributions which we do not take into account.

The generating function leading to the four new variables Q_φ, P_φ, Q_r and P_r of (1), (2), (10) and (11) is:

$$\emptyset\ (z; p_\varphi, P_\varphi, p_r, P_r) =$$

$$z\ (-2m\omega P_\varphi)^{1/2} + \varphi_o p_\varphi + eZ\ (r_1 D - A)/\omega + (P_\varphi - p_\varphi)\ \arcsin Z - \qquad (10.15)$$

$$- (eD/(m\omega^2))^2 \left[y^2/2 + 1/4 + (x + 3Z/4)(1 - Z^2)^{1/2} - \varphi\ y^2/2 \right]$$

with

$$x = \omega p_r/(eD) \quad y = \omega P_r/(eD) \quad Z = y - x \qquad (10.16)$$

and

$$q_i = - \emptyset_{p_i} \quad Q_i = \emptyset_{P_i} \qquad (10.17)$$

For the evaluation of the new "Hamiltonian" K one needs $\emptyset_z$. The necessary calculations are lengthy. It is convenient to proceed in the following way:

$$\emptyset_z = \emptyset_{z\ vis} - (D_z/D)(x\emptyset_x + y\emptyset_y) \qquad (10.18)$$

where $\emptyset_{z\ vis}$ denotes the partial derivations of D, A, $z(-2m\omega P_\varphi)$, i.e. the derivatives of those functions which do not owe their z-dependence to x or y. We split $K = \bar{H} + \emptyset_z$ into a part $K^{(1)}$ linear in the potentials and one of second order $K^{(3)}$:

$$K(P_\varphi, P_r, Q_\varphi, Q_r; z) = K^{(1)} + K^{(3)} \qquad (10.19)$$

$$K^{(1)} = (c/\omega)\ \sin(\varphi+\varphi_o) \Big[(k_o^2 \Pi + \Pi_{zz}) + (Q_r - r_1 + \varphi P_r/(m\omega))(k_o^2 \Pi_{r_1} - \Pi_{zzr_1})$$

$$+ (c/\omega)\ \cos(\varphi + \varphi_o)\ P_r/(m\omega)\ \Pi_{zzr_1}$$

The argument (r_1, z) of all Π's has been suppressed. For ease of writing $\varphi + \varphi_o$ has been used instead of:

$$\varphi + \varphi_o = Q_\varphi + \omega z(-m/(2\omega P_\varphi))^{1/2}$$

$$K^{(3)} = - e\, DD'/(m\omega^3) \left[(\varphi + \varphi_o)/2 + (1/4) \sin(2\varphi + 2\varphi_o) \right] + e^2\, k_o^2\, D\, \Pi_r /(2m\omega^3) \sin(2\varphi + 2\varphi_o) \qquad (10.21)$$

In

$$\varphi P_r/(m\omega) = P_r/(\omega m)\ (Q_\varphi - \varphi_o + \omega z(-m/(2\omega P_\varphi))^{1/2}\ ) \qquad (10.22)$$

it is not admissible to interchange the derivation $\partial/\partial Q_\varphi$ with the replacement of Q_φ by φ_o and the subsequent derivation with respect to φ_o. Since we want to do exactly that thing in the Thin Lens Approximation, we employ a little trick. We write:

$$\varphi P_r/(m\omega) = P_r/(m\omega)\ (Q_\varphi - \varphi_1 + \ldots) \qquad (10.23)$$

and set $\varphi_1 = \varphi_o$ at the end of all manipulations. The same difficulty would have appeared, if we had used r_o instead of r_1 in the expansion (9).

After this change $K^{(1)}$ reads as:

$$K^{(1)} = (e/\omega) \sin(\varphi + \varphi_o) \Big[(k_o^2 \Pi + \Pi_{zz}) + (Q_\varphi - r_1 + P_r/(m\omega) \cdot (Q_\varphi - \varphi_1 + \omega z(-m/(2\omega P_\varphi))^{1/2})(k_o^2 \Pi_{r_1} + \Pi_{zzr_1}) + (e/\omega) \cos(\varphi + \varphi_o)\ (P_r/m\omega)\, \Pi_{zzr_1} \qquad (10.24)$$

$K^{(1)}$ alone is not unique; in fact, with the help of $y - x = (eD/\omega) \sin(\varphi + \varphi_o)$, (10), and similar relations one can change terms in $K^{(1)}$ and afterwards shift to $K^{(3)}$ the expressions non-linear in the potentials. This ambiguity is a wellknown feature of perturbations theoretic series. But anyway, the new " Hamiltonian" is a power series in the potentials, so the new variables are reduced and render the service we want them to do.

We shall not make here tho final step leading to the Thin Lens "Hamiltonian", H_{TL}, since $K^{(1)}$ is only linear in the radial variables. We shall do this, after having derived terms quadratic in this quantities.

b) Quadratic Approximation, $B \neq 0$.

If $B \neq 0$, it seems pretty hard, if not even practically impossible (see appendix), to solve the partial differential equation (10.6), (10.7) which P_r and Q_r are to obey. We employ a different way of approaching the problem.

The exact solutions P_r, Q_r of (10.6), (10,7) will be transcendental functions of B. However, at the end we only use those terms which are linear in the potentials. Here we shall make this approximation right from the beginning. The transformed "Hamiltonian" will then be:

$$K = K^{(1)} + K^{(2)} \tag{10.25}$$

where $K^{(1)}$ is given in (20). $K^{(2)}$ should be linear in B and of second order in Q_r and P_r. Q_r and P_r will consist of two parts:

$$P_r = P_{r1} + P_{r2} = P_{r1} + (eB/\omega)\, g(r,\varphi) \tag{10.26}$$

$$Q_r = Q_{r1} + Q_{r2} = Q_{r1} + eB/(m\omega^2) f(r,\varphi) \tag{10.27}$$

where P_r and Q_r equal to P_r and Q_r of (10), (11) resp. The definition of Q_φ and P_φ ((1) and (2)) remains unchanged.

"Hamiltons equations" then read:

$$P'_{\varphi} = - K^{(1)}_{Q_{\varphi}} - K^{(2)}_{Q_{\varphi}} \qquad (10.28)$$

$$Q'_{\varphi} = K^{(1)}_{P_{\varphi}} + K^{(2)}_{P_{\varphi}} \qquad (10.29)$$

$$P'_{r} = P'_{r1} + P'_{r2} = - K^{(1)}_{Q_{r}} - K^{(2)}_{Q_{r}} \qquad (10.30)$$

$$Q'_{r} = Q'_{r2} + Q'_{r2} = K^{(1)}_{P_{r}} + K^{(2)}_{P_{r}} \qquad (10.31)$$

In these equations we feed some of the results we obtained in an earlier paper [5] where $\varphi = \omega t$ was the independent variable. (Grave accents will denote derivations w.r.t. φ: $` = d/d\varphi$). We showed in this ref. that the equations of motion in Newtonian form:

$$m\omega^2 r`` = eE_r - e\omega B_{\varphi} z` = e \cos(\varphi+\varphi_o)\Gamma_{rz} - ek_o^2 z` \sin(\varphi+\varphi_o)\Gamma_r \qquad (10.32)$$

$$m\omega^2 z`` = eE_z + e\omega B_{\varphi} r` = e \cos(\varphi+\varphi_o)(k_o^2 \Gamma + \Gamma_{zz}) + ek_o^2 r` \sin(\varphi+\varphi_o)\Gamma_r \qquad (10.33)$$

can be split up into sets of perturbation equations by assuming perturbation theoretic series:

$$z = z^{(0)} + \bar{E} z^{(1)} + \ldots \qquad (10.34)$$

$$r = r^{(0)} + \bar{E} r^{(1)} + \ldots \qquad (10.35)$$

$$\bar{E} = eE_o/(m\omega v_o) \quad k = \omega/v_o \qquad (10.36)$$

which correspond to the different steps one reaches in solving (32) and (33) by iteration starting from free particle motion.

$$r^{(o)''} = 0,\ r^{(o)} = r_o'\ \varphi + r_o,\ z^{(o)''} = 0,\ z^{(o)} = \varphi/k \tag{10.37}$$

$$\bar{E}m\omega^2 r^{(1)''} = e\ \cos(\varphi+\varphi_o)\left[\Pi_{zr_1} + (r^{(o)} - r_1)\Pi_{zr_1r_1}\right] \tag{10.38}$$

$$- (ek_o^2/k)\ \sin(\varphi+\varphi_o)\left[\Pi_{r_1} + (r^{(o)} - r_1)\Pi_{r_1r_1}\right]$$

$$\bar{E}m\omega^2 z^{(1)''} = e\ \cos(\varphi+\varphi_o)\left[1+(r^{(o)} - r_1)\ \partial/\partial r_1 + (r^{(o)} - r_1)^2/2\ \ \partial/\partial r_1^2\right](k_o^2\Pi + \Pi_{zz})$$

$$+ ek_o^2\ r_o'\ \sin(\varphi+\varphi_o)\left[\Pi_{r_1} + (r^{(o)} - r_1)\ \Pi_{r_1r_1}\right] \tag{10.39}$$

where we expanded:

$$\Pi(r,z) = \Pi(r_1,z) + (r - r_1)\Pi_{r_1}(r_1,z) + (r - r_1)^2/2\ \Pi_{r_1r_1}(r_1,z) + \dots \tag{10.40}$$

From (1) we find:

$$Q_\varphi' = -z\varphi'' = \bar{E}\ k\varphi\ kz^{(1)''}(\varphi) \tag{10.41}$$

The last expression follows from (34) if we solve the series for φ by iteration

$$kz = \varphi + \bar{E}\ kz^{(1)}(\varphi) + \dots \quad \varphi = kz - \bar{E}\ kz^{(1)}(\varphi)$$

$$\varphi = \varphi^{(o)} + \bar{E}\ \varphi^{(1)} + \dots \tag{10.42}$$

$$\varphi^{(o)} = kz,\quad \varphi^{(1)} = -\bar{E}\ kz^{(1)}(\varphi^{(o)}) = -\bar{E}\ kz^{(1)}(kz)$$

and take into account the following fact: If a term already contains $\bar{E}$, then to this order one may use $\varphi = kz$, $d/dz = k^{-1}\ d/d\varphi$ in all functions accompanying $\bar{E}$.

Similarly one may calculate:

$$P'_\varphi = - m/(2\omega)\ d/dz\ (dz/dt)^2 = -(m\omega/2)\ d/dz\ (z^{`})^2$$
$$= - \bar{E}\ m\omega\ z^{(1)``}(\varphi) \qquad (10.43)$$

We insert (32) ((33)) into (41) ((43)), then (41) ((43)) into the left hand side of (28) ((29)). On the right hand side we use (20) for $K^{(1)}$ and put into it the zero order quantities (14). We get expressions for the partial derivatives

$K^{(2)}_{P_\varphi}(P_{io}, Q_{io}; z)$, $K^{(2)}_{Q_\varphi}$ and from these we evaluate $K^{(2)}(P_{io},Q_{io};z)$.

The derivation w.r.t. k is more convenient than that w.r.t. $P^{(o)}_\varphi$:

$$\partial/\partial P^{(o)}_\varphi = (k^3/(m\omega))\ \partial/\partial k \qquad (10.44)$$

which follows from (14). We integrate:

$$Q'_\varphi - K^{(1)}_{P_\varphi}\Big|_{data} = K^{(2)}_{P_\varphi}\Big|_{data} = (k^3/(m\omega))\ K_k^{(2)}(P_{io},Q_{io};z) \qquad (10.45)$$

$$K_k^{`(2)} = (ek_o^2/\omega)\Big\{ z\ \cos(kz+\varphi_o)\Big[(r_o-r_1)^2 + 2r^{`}_o(r_o-r_1) + r^{`2}_o(kz)^2\Big]/2$$
$$+ z\ \sin(kz+\varphi_o)\Big[\ r^{`}_o(r_o-r_1) + r^{`2}_o kz\ \Big]\Big\}\Big\|_{r_1r_1}$$
$$+(e/\omega)\ z\ \cos(kz+\varphi_o)\Big[(r_o-r_1)^2 + 2r^{`}_o\ kz(r_o-r_1) + r^{`2}_1(kz)^2\Big]/2\Big\|_{zzr_1r_1}$$

$$K^{(2)}(P_{io},Q_{io};z) =$$
$$= (ek_o^2/\omega)\Big\{\sin(kz+\varphi_o)\Big[(r_o-r_1)^2 + 2r^{`}_o\ kz(r_o-r_1) + r^{`2}_o\ (kz)^2\Big]\ /2\Big\}\Big\|_{r_1r_1}$$
$$+ (e/\omega)\Big\{\ \sin\ (kz+\varphi_o)\Big[(r_o-r_1+r^{`}_o\ kz)^2/2 - r^{`2}_o\Big]$$
$$+ \cos\ (kz+\varphi_o)\Big[\ r^{`}_o\ (r_o-r_1 + r^{`}_o\ kz)\Big]\Big\}\Big\|_{zzr_1r_1} \qquad (10.46)$$

Guided by experiences made in the linear theory a), we guess that we can find the full "Hamiltonian" $K^{(2)}(P_i,Q_i;z)$ by the substitutions:

$$
\begin{aligned}
r_o + r_o' \, kz &\longrightarrow Q_r \\
r_o' &\longrightarrow P_r/(m\omega) \\
kz &\longrightarrow Q_\varphi - \varphi_1 + \omega z(-m/(2\omega P_\varphi))^{1/2}
\end{aligned}
\tag{10.47}
$$

$$
\begin{aligned}
&K^{(2)}(P_\varphi,P_r,Q_\varphi,Q_r;z) = \\
&= (ek_o^2/\omega)\Big\{ \sin(\varphi+\varphi_o)\Big[Q_r-r_1+(P_r/m\omega)(Q_\varphi-\varphi_1+z\omega(-m/(2\omega P_\varphi))^{1/2})\Big]^2 \Big\|_{r_1r_1} \\
&+ (e/\omega)\Big\{ \sin(\varphi+\varphi_o)\Big[\Big[Q_r-r_1 +(P_r/m\omega)(Q_\varphi- \varphi_1+ z\omega(-m/(2\omega P_\varphi))^{1/2}\Big]^2 -(P_r/m\omega)^2\Big] \\
&+ \cos(\varphi+\varphi_o)(P_r/m\omega)\Big[Q_r-r_1+ (P_r/m\omega)(Q_\varphi-\varphi_1 + z\omega(-m/(2\omega P_\varphi))^{1/2}\Big]\Big\}\Big\|_{zzr_1r_1}
\end{aligned}
\tag{10.48}
$$

One easily verifies that this "Hamiltonian" is compatible with (29), or more accurately with:

$$
P_\varphi' + K^{(1)}_{Q_\varphi}\Big|_{data} = - \bar{E}m\omega \, z^{(1)''}(\varphi) + K^{(1)}_{Q_\varphi}\Big|_{data} = - K^{(2)}_{Q_\varphi}\Big|_{data}
\tag{10.49}
$$

The only unknown quantities in (30 and (31) are P_{r2}' and Q_{r2}' if we use (10) and (11) for:

$$
P_{r1}'\Big|_{data} = \bar{E} \, \omega mkr^{(1)''}(\varphi)-(e/\omega)\sin(kz+\varphi_o)\Big\|_{zzr_1} -(ek/\omega)\cos(kz+\varphi_o)\Big\|_{zr_1}
\tag{10.50}
$$

$$
\begin{aligned}
Q_{r1}'\Big|_{data} &= -\bar{E} \, \varphi k \, r^{(1)''}(\varphi)+e/(m\omega^2)(kz \sin(kz+\varphi_o)+\cos(kz+\varphi_o))\Big\|_{zzr_1} \\
&+ ek/(m\omega^2) \, kz \cos(kz+\varphi_o)\Big\|_{zr_1} ,
\end{aligned}
\tag{10.51}
$$

(30) for $\bar{E}r^{(1)''}$ and $K^{(1)}$, $K^{(2)}$ from (20), (48) resp.

With these relations we land the final blow:

$$Q'_{r2} = K_{P_r}^{(2)} - Q'_{r1} + K_{P_r}^{(1)}\Big|_{data} = (e/(m\omega^2))\ d/dz(Bf)$$

$$= e/(m\omega^2)\left\{ \sin(kz+\varphi_o)\left[(r_o-r_1+r_o^{`}\ kz)kz+2r_o^{`}\right] + \cos(kz+\varphi_o)\left[(r_o-r_1+2r_o^{`}kz)\right]\right\} \times$$

$$\prod_{zzr_1r_1} \qquad (10.52)$$

$$+ ek/(m\omega^2)\ \cos(kz+\varphi_o)\left[(r_o-r_1)kz + r_o^{`}(kz)^2\right]\prod_{zr_1r_1}$$

$$P'_{r2} = - K_{Q_r}^{(2)} - P'_{r1} = K_{Q_r}^{(1)}\Big|_{data} = (e/\omega)\ d/dz(Bg) \qquad (10.53)$$

$$= -(e/\omega)\left\{ \sin(kz+\varphi_o)(r_o - r_1 + r_o^{`}\ kz) + \cos(\varphi_o + kz)\ r_o^{`}\right\}\prod_{zzr_1r_1}$$

$$-(ek/\omega)\ \cos(kz+\varphi_o)\left[(r_o - r_1 + r_o^{`}\ kz)\right]\prod_{zr_1r_1}$$

The coefficients of $\prod_{zzr_1r_1}$ are f, g respectively. The coefficients of $\prod_{zr_1r_1}$ really are f' and g'; so that the overdetermined system for f and g contains no contradiction. Finally we substitute

$$r_o - r_1 + r_o^{`}\ kz \rightarrow r - r_1\ ,\quad r_o^{`}\ kz \rightarrow p_r/(m\omega) \qquad (10.54)$$

and get:

$$P_{r2} = (eB/\omega)\left[(r - r_1)\ \sin(\varphi + \varphi_o) + p_r/(m\omega)\ \cos(\varphi+\varphi_o)\right] \qquad (10.55)$$

$$Q_{r2} = -(eB/m\omega^2))\left\{\left[(r - r_1)\varphi - 2p_r/(m\omega)\right]\sin(\varphi + \varphi_o) \qquad (10.56)\right.$$

$$\left.\left[(r - r_1) + \varphi p_r/(m\omega)\right]\cos(\varphi + \varphi_o)\right\}$$

We evaluate the "Poisson brackots" for the transvorsal variables (26) and (27):

$$(Q_r, P_r) = 1 - (eB/(m\omega^2))^2\ (1 + \sin^2\ (\varphi + \varphi_o)) \qquad (10.57)$$

$$L(Q_r) = (e/(m\omega^2))^2\ B(D+(r-r_1)B)\ \cos(\varphi+\varphi_o)\ (-2\sin(\varphi+\varphi_o)+\ \varphi\cos(\varphi+\varphi_o)) \qquad (10.58)$$

$$L(P_r) = -\ e^2B/(m\omega)\ (D + (r - r_1)B)\ \cos^2\ (\varphi+\varphi_o) \qquad (10.59)$$

Due to the approximate nature of (26), (27) the "Poisson brackets" are only satisfied up to linear order in tho potentials. But we may express the following hope: The equation $L(X) = 0$ (6), (7) has two independent particular solutions and any arbitrary function of these two is again a solution. If we were able to get the exact expressions for such two solutions (E.g. if we could solve exactly the equations (A.2) to (A.4) for the characteristic curves), then it should be possible to find expressions which also fulfill the "Poisson bracket" $(P_r, Q_r) = 1$. To this set of new canonical variablos Q_r, P_r and P_φ, Q_φ belongs a generating function $\emptyset$ which loads to the new "Hamiltonian" $K = \bar{H} + \emptyset_z$. If one then expands in powers of B, retaining only terms linear in it, then one should get tho P_r, Q_r (26), (55), (27), (56) and $K = K^{(1)} + K^{(2)}$ with $K^{(1)}$ (24) and $K^{(2)}$ (48). But we have no proof of these stipulations. Howevor, we are convinced that the variables and tho "Hamiltonian" describe dynamics exactly to linear order in the potentials and transversal coordinates, since we derived them from dynamical equations exact to that order.

c) Computation of the Thin Lons Hamiltonian.

We evaluate tho Thin Lens "Hamiltonian" only in tho limit $L \to \infty$, since this expression is already complicated enough:

$$H_{TL}(P_{io}, Q_{io}) = \lim_{L\to\infty} \int_{L}^{L} K\ (P_\varphi^{(o)}, P_r^{(o)}, Q_\varphi^{(o)}, Q_r^{(o)}; z)dz \qquad (10.60)$$

K consists of $K^{(1)}$ (24) and $K^{(2)}$ (48). The method how to calculate the integrals has been described elsewhere[5]. We indent the path of integration in the complex k_z-plane upwards (downwards) at

$$k_z = -k(+k) \text{ with } k = \omega/v_o = \omega/\dot{z}_o. \tag{10.61}$$

Then one integrates w.r.t. z. One closes the path in the k_z-plane by a semi-circle of infinite radius in the upper (lower) half of this plane for $z = L(-L)$. One employs Cauchy's residue theorem. If L tends to infiniy the contributions due to the simple poles $J_o(\gamma a) = 0$ which contain a factor $e^{-L}\ldots$ vanish and only the residues of the poles $k_z = \pm k$ remain. From these we get[*]:

$$H_{TL} = H_{TL}^{(1)} + H_{TL}^{(2)} \tag{10.62}$$

$$H_{TL}^{(1)} = (eV_o/\omega)\sin\varphi_o \left\{ T_o I_o + \left[(r_o - r_1) + \dot{r}_o(\varphi_o - \varphi_1) \right] T_o k_r I_1 \right\}$$
$$-(eV_o/\omega)\,\dot{r}_o \cos\varphi_o \left\{ k\, d/dk(T_o k_r I_1) - T_o k^2 I_1/k_r \right\} \tag{10.63}$$

$$H_{TL}^{(2)} = (eV_o/\omega)\sin\varphi_o \Big\{ \left[(r_o - r_1) + \dot{r}_o(\varphi_o - \varphi_1) \right]^2 /2\; T_o k_r^2 I_1'$$
$$-(\dot{r}_o k)^2 \left[(1/2)\, d^2/dk^2 (T_o k_r^2 I_1') - T_o I_1' + k^{-1}\, d/dk\,(T_o k^2 I_1') \right] \Big\}$$
$$+ (eV_o/\omega)\cos\varphi_o \Big\{ \dot{r}_o \left[(r_o - r_1) + \dot{r}_o(\varphi_o - \varphi_1) \right] T_o k^2 I_1' \tag{10.63}$$
$$- \left[\dot{r}_o^2 (\varphi_o - \varphi_1)^2 + \dot{r}_o (r_o - r_1)\, k\, d/dk\,(T_o k_r^2 I_1') \right] \Big\}$$

In all these formulae we suppressed the argument k in $T_o(k)$ and $k_r r_1$ in $I_n(k_r r_1)$ where

---

* one has to set $\varphi_1 = \varphi_o$ after the partial derivations of H_{TL}, see (23).

$$k_r^2 = (k^2 - k_o^2)^{1/2} \tag{10.64}$$

and I_n is the modified Bessel function of order n. T_o is the Transit Time Factor:[5)9)]

$$T_o(k) = \int_{-\infty}^{\infty} E_z(z,r=0,t)\cos(kz)\,dz \Big/ \int_{-\infty}^{\infty} E_z(z,0,t)\,dz = b(k)/I_o(k_r a) \tag{10.65}$$

Finally we specialize for $r_1 = 0$[*]:

$$H_{TL}^{(1)} = (eV_o/\omega)\sin\varphi_o\, T_o(k) \tag{10.66}$$

$$\begin{aligned} H_{TL}^{(2)} = {} & (eV_o/(2\omega))\sin\varphi_o \Big\{ \left[r_o + \dot{r}_o(\varphi_o - \varphi_1)\right]^2 /2\; T_o k_r^2 \\ & -(r_o k)^2 \left[T_o k^2 + k^{-1}\, d/dk(T_o k^2) + (1/2)\, d^2/dk^2(T_o k_r^2)\right] \Big\} \\ & + (eV_o/(2\omega))\cos\varphi_o \Big\{ \left[\dot{r}_o^2 + \dot{r}_o^2(\varphi_o - \varphi_1)\right] T_o k^2 \\ & - \left[\dot{r}_o^2(\varphi_o - \varphi_1)^2 + \dot{r}_o r_o\right] k\, d/dk\,(T_o k_r^2) \Big\} \end{aligned} \tag{10.67}$$

The shift of the representation point in phase space which corresponds to the integrated effect of a linac gap, is given by the following equations in "Hamiltonian" form (linear in the potentials and quadratic in the transversal quantities):

$$P_{\varphi+} - P_{\varphi-} = -(W_{n+1} - W_n)/\omega = -\partial H_{TL}/\partial\varphi_o \tag{10.68}$$

$$Q_{\varphi+} - Q_{\varphi-} = \varphi_{n+1} - \varphi_n = \partial H_{TL}/\partial P_\varphi^{(o)} = (k^3/(m\omega))\,\partial H_{TL}/\partial k \tag{10.69}$$

$$P_{r+} - P_{r-} = p_{r+} - p_{r-} = m(\dot{r}_{n+1} - \dot{r}_n) = -\,\partial H_{TL}/\partial Q_r^{(o)} = \partial H_{TL}/\partial r_o \tag{10.70}$$

$$Q_{r+} - Q_{r-} = \bar{r}_{n+1} - \bar{r}_n = \partial H_{TL}/\partial P_r^{(o)} = m^{-1}\,\partial H_{TL}/\partial\dot{r}_o \tag{10.71}$$

Note that the second column of (69) and (71) φ and $\bar{r}$ denote reduced quantities as in earlier ref.[3)5)8)9)]. We also used the

[*] Put $\varphi_1 = \varphi_o$ after the partial derivations of H_{TL}; see (23).

fact that D and B vanish for $L \rightarrow \infty$.

The above set of difference equations for a linac gap is more interesting from the theoretical point of view, while for practical beam dynamics calculations one prefers to employ $r' = dr/dz$ instead of p_r or P_r, since r and r' are quantities one can measure by slits [18]. There is no objection to such a procedure. One is completely at liberty to use any set of independent variables, even if they are not canonically conjugated, to describe the motion. By this reason we do not work out in more detail the expression (68) till (71). Instead we give at the end of this paper in Table I and II a set of diffence equations for the variables currently in use. Their entries differ from earlier results in as much as the earlier expressions for $r'_+ - r'_-$ contained an error recently detected by Promé [19]*. The necessary corrections are treated in appendix B.

The author is very indebted to Dr. P.M. Lapostolle for the suggestion of this topic and for many helpful discussions. He also drew great profit from discussions with Prof. A. Sessler.

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* $(\varphi_+ - \varphi_-)_r$ and $(\bar{r}_+ - \bar{r}_-)_r$ in table II differ slightly from the same quantities given in table IV of ref. [5] as already noted in the two last sentences of Sect. 1

Appendix A.

Solution of the partial differential equation (10.6).

Inserting the expansion (10.9) into $L(X) = 0$, we find:

$$X_r \, p_r/(m\omega) - X_{p_r}\left[(e/\omega)\, D + (r - r_1)\cos(\varphi+\varphi_o)\right] + X_\varphi = 0 \qquad (A.1)$$

According to standard theory [15], in order to get the general solution of (1), we have to solve the system of ordinary differential equations belonging to tho characteristic curves:

$$\begin{aligned} dr/du &= p_r/(m\omega) && (A.2)\\ dp_r/du &= -(e/\omega)\left[D + (r - r_1)\,B\right]\cos(\varphi + \varphi_o) && (A.3)\\ d\varphi/du &= 1 && (A.4)\end{aligned}$$

a) $B = 0$.

The general solution of the system (2) to (4) is:

$$\begin{aligned} \varphi &= u + C_3 && (A.5)\\ p_r &= -(eD/\omega)\sin(\varphi_o + u + C_3) + C_2 && (A.6)\\ r &= eD/(m\omega^2)\cos(\varphi_o + u + C_3) + C_2 u/(m\omega) + C_1 && (A.7)\end{aligned}$$

From these we form the expressions:

$$X_1 = p_r + (eD/\omega)\sin(\varphi + \varphi_o) = C_2 \qquad (A.8)$$

$$X_2 = r - \varphi p_r/(m\omega) - eD/(m\omega^2)\cos(\varphi + \varphi_o) + \varphi \sin(\varphi + \varphi_o) = C_1 - C_2 C_3/(m\omega) \qquad (A.9)$$

X_1 and X_2 regarded as functions of r, p_r and φ are constant and this for arbitrary values of C_i, so they are constant along every characteristic curve and therefore a solution of (1) (with $B = 0$).

The general solution is an arbitrary function of X_1 and X_2.

b) $B \neq 0$.

From (2) and (3) we generate ($u_o = \varphi_o + C_3$, $\alpha = eB/(m\omega^2)$):

$$d^2r/du^2 + r(u) \cdot \alpha \cos(u + u_o) = (r_o B - D)e/(m\omega^2) \cos(u + u_o) \quad (A.10)$$

Substituting in the Mathieu [16] equation:

$$d^2y/dz^2 + (a - 2q \cos(2z))\, y(z) = 0 \quad (A.11)$$

$$2z = u + u_o + \pi \qquad r(u) = y\,((u+u_o+\pi)/2) \quad (A.12)$$

we find:

$$d^2r/du^2 + (a/4 + (2q/4)\cos(u+u_o))\, r(u) = 0 \quad (A.13)$$

The solutions of the homogeneous equation (10) are related to solutions of the Mathieu equation for the following special values of its parameters:

$$a = 0 \qquad q = 2\alpha \quad (A.14)$$

The difficulties of solving this equation are well known [16][17], so we do not try to get the general solution of (10) or (1). We may draw one conclusion for application in Sect. 10 b): The solutions of (13) are transcendental functions of q. Therefore we expect the solutions of (1) to be transcendental functions of B.

Appendix B

Corrections to

REVISED LINAC BEAM DYNAMICS EQUATIONS

by

B. Schnizer

1. Introduction

In a paper[5)] with the above title the author gave a set of difference equations for energy gain, phase change, change of radial position and radial velocity across a linac gap; these are generalisations of the so called Panofsky equations [6)] and have been introduced by P. Lapostolle and other authors[3)8)9)]. M. Promé has pointed out[19)] that all these derivations contain either an error in the equations for the radial velocity [5)], or that the approximation omits an appreciable term. The same fault is the source of difficulties which have arisen when comparing the relativistic and the non-relativistic energy gain in ref.[5)] It is the purpose of this note to correct these mistakes.

2. The radial velocity.

Using the notations of ref[5)] (a grave accent ` denotes a derivation w.r.t. $\varphi = \omega t$, a prime ′ one w.r.t. $z$), one may describe the error in the following way: At first the radial velocity has been evaluated as function of $\varphi$: $r^{`}(\varphi) = dr/d\varphi$. But we want to use $r' = dr/dz$. To evaluate the change of this quantity across a gap in a correct way, we have to form:

$$(r'_+ - r'_-) = \left(\frac{dr}{dz}\right)_+ - \left(\frac{dr}{dz}\right)_- = \left(\frac{dr}{d\varphi}\right)_+\left(\frac{d\varphi}{dz}\right)_+ - \left(\frac{dr}{d\varphi}\right)_- \left(\frac{d\varphi}{dz}\right)_- = \frac{r`(kL)}{z`(kL)} - \frac{r`(-kL)}{z`(-kL)} \quad (B.1)$$

In [5] z`(kL) and z`(-kL) had been erroneously replaced by $z`_o = z`(o) = 1/k \neq z`(kL) \neq z`(-kL)$.

That means the change of z` across the gap has been neglected in the expressions of $r'_+ - r'_-$ of Table I and IV of ref [5].

To deal with equ. (1) in a correct way, we employ:

$$z(\varphi) = kz + \bar{E}\, z^{(1)}(\varphi) + \bar{E}^2 \ldots \qquad r`(\varphi) = r`_o + \bar{E}\, r`^{(1)}(\varphi) + \bar{E}^2 \ldots \quad (B.2)$$

$$(z(\varphi))^{-1} = k - \bar{E}\, k^2 z`^{(1)}(\varphi) - \bar{E}^2 \ldots$$

where the last equation has been derived from the first by expanding w.r.t. $\bar{E} = eE_1/(mz\omega^2)$.

The right expression of the change of the radial velocity then reads :

$$r'_+ - r'_- = \bar{E}\left[k\dot{r}^{(1)}(kL) - k\dot{r}^{(1)}(-kL) - k\dot{r}_o\left(k\dot{z}^{(1)}(kL) - k\dot{z}^{(1)}(-kL)\right)\right] \quad (B.3)$$

In the <u>non-relativistic case</u> (Table I of ref.[5]) this can be written:

$$r'_+ - r'_- = (r'_+ - r'_-)_{\ell o} - r'(W_+ - W_-)/(2W) + \bar{E}^2 \ldots \quad (B.4)$$

where $(r'_+ - r'_-)_{\ell o}$, $W_+ - W_-$, r' and W are the quantities as denoted in the tables quoted ($L \to \infty$), except terms proportional to r'^2 have to be dropped in the second term of (4). The explicit expression is given in Table I.

For the relativistic case (Table IV) we need:

$$k\grave{z}^{(1)}(kL)_{L\longrightarrow\infty} - k\grave{z}^{(1)}(-kL) = (1-\beta_o^2)^{1/2}\left[1-(E_c/\bar{E}).(k_o/k)\right]\int_{-kL}^{kL} \cos(\varphi+\varphi_o)d\varphi$$

$$\times \; \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z \; b(kz) \; e^{ikz(o)} J_o(\gamma r^{(o)})/J_o(\gamma a) \qquad (B.5)$$

$$= (1-\beta_o^2)^{1/2}(1-k_o^2/k^2)\cos\varphi_o V_o k \, T_o(k) \, I_o(k_r r_o)/E_1 + r_o' \ldots$$

Inserting this in (3) yields the correct relativistic expression of the change of radial velocity $(r_+' - r_-')_r$ indicated below.

3. Relativistic and Non-Relativistic Energy Gain.

In the non-relativistic case (Table I of ref. 5)) $W_+ - W_-$ solely denotes the gain in longitudinal kinetic energy. The second definition different from the first one has been adopted, because we thought it is not admissible to take the sole longitudinal kinetic energy for a relativistic problom, since their longitudinal and radial energy are coupled. But we now believe this separation is permissible to the degree of approximations introduced.

The non-relativistic change of total kinetic energy $(W_+ - W_-)_{tot}$, is:

$$(W_+ - W_-)_{tot} = W_+ - W_- + (m/2)(dr/dt)_+^2 - (m/2)(dr/dt)_-^2 \qquad (B.7)$$

$$(m/2)(dr/dt)_+^2 - (m/2)(dr/dt)_-^2 = m\omega^2(\grave{r}_+^2 - \grave{r}_+^2)/2 =$$

$$m\omega^2 \bar{E} \, \grave{r}_o\left(\grave{r}_o(kL) - \grave{r}(-kL)\right) = eV_oT_o(k)\frac{k}{k_r}I_1(k_r r_o) \, r_o' \sin\varphi + \ldots \qquad (B.8)$$

This is just the extra term by which $(W_+ - W_-)_r$ differs from $W_+ - W_-$. Thus the relativistic and the non-relativistic gain of longitudinal or total kinetic energy are equal to that degree of approximation.

It is now a matter of personal taste which kind of energy gain one likes to employ. If one wants to use the total kinetic energy then in both cases the expression $(W_+ - W_-)_r$ of Table IV of ref. [5] is the right one ; for the bare longitudinal kinetic energy take $W_+ - W_-$ of Table I.

TABLE I :

Nonrelativistic change of longitudinal kinetic energy, phase, transversal velocity and reduced radial position across a linac gap.

$$W_+ - W_- = eV_o \; T_o I_o \; \cos\varphi \; + \; eV_o \; d/dk(T_o k_r I_1) \; r' \sin\varphi$$

$$\varphi_+ - \varphi_- = \lim_{L \to \infty} \left[\varphi(kL) - \varphi(-kL) - L(z'(kL))^{-1} + L(z'(-kL))^{-1} \right]$$

$$= \alpha k \; d/dk(T_o I_o) \sin\varphi - \alpha k \; d^2/dk^2(T_o k_r I_1) \; r' \cos\varphi$$

$$r'_+ - r'_- = (dr/dz)_+ - (dr/dz)_-$$

$$= -\alpha \; T_o k \; I_1/k_r \sin\varphi + \alpha \left[d/dk(T_o k \; I_1') - T_o I_o \right] r' \cos\varphi$$

$$\bar{r}_+ - \bar{r}_- = \lim_{L \to \infty} \left[r(kL) - r(-kL) - kL(r'(kL) + r'(-kL)) \right]$$

$$= -\alpha \; d/dk(T_o k \; I_1/k_r) \cos\varphi - \alpha \; d^2/dk^2(T_o k \; I_1') \; r' \sin\varphi$$

TABLE II :

The change of the same quantities in the relativistic case:

$$(W_+ - W_-)_r = \; = W_+ - W_- \; \left[- eV_o \; T_o k \; I_1/k_r \; r' \sin\varphi \right]^{*}$$

* By adding the term in the square brackets one gets in both cases the gain in total energy. (Cf. (B.7), (B.8))

$$(\varphi_+ - \varphi_-)_r (1-\beta_o^2)^{-1/2} = (\varphi_+ - \varphi_-)(1-k_o^2/k^2) - \alpha k\, k_o^2/k^2\, T_o\, I_1\, k_r\, r' \cos\varphi$$

$$(r'_+ - r'_-)_r (1-\beta_o^2)^{-1/2} = (r'_+ - r'_-)(1-k_o^2/k^2) + \alpha\, k_o^2/k^2\, T_o (I'_1 - I_o)\, r' \cos\varphi$$

$$(\bar{r}'_+ - \bar{r}'_-)_r (1-\beta_o^2)^{-1/2} = (\bar{r}_+ - \bar{r}_-)(1-k_o^2/k^2) - \alpha\, k_o^2/k^2\, d/dk(2T_o I'_1 - T_o I_o) r' \sin\varphi$$

$$- \alpha\, k_o^2/k^2\, T_o I_1/k_r \cos\varphi$$

Common to both tables:

$$\alpha = eV_o/(2W) \quad W = m/2\ (dz/dt)_o^2 \quad m = \text{rest mass}$$

$$k_o^2/k^2 = (dz/dt)_o^2/c^2 \approx \beta_o^2 \qquad 1 - k_o^2/k^2 \approx k_r^2/k^2 \approx 1 - \beta_o^2$$

The arguments $k = \omega/\dot{z}_o$ of $T_o(k)$, $k_r(k)r_o$ of $I_n(k_r r_o)$ and the subscript $_o$ of φ_o and r'_o have been dropped.

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13. $E_o \cos(\omega t+\varphi_o)$ is not a solution of Maxwells Equations. But if one applies to the plates of a plane condenser a voltage

$V_o \cos(\omega t+\varphi_o)$, one might hope that the field in tho interior is approximately described by the expression above, as long as the wave length is great compared with the width of the condenser. Anyway, as far as dynamics is concerned, one may always assume such a field of force.

14. e.g. ref. 9., equ. (26)

15. ref. 1) chp. I, § 5

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19. M. Promé, private communication to P.M. Lapostolle, (April 1967).