

Accelerating Berends-Giele recursion for gluons in arbitrary dimensions over finite fields

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Abstract

This work provides a proof of concept for the computation of pure gluonic amplitudes in quantum chromodynamics (QCD) on graphics processing units (GPUs). The implementation relies on the Berends-Giele recursion algorithm and, for the first time on a GPU, enables the numerical computation of amplitudes in an arbitrary number of space-time dimensions and over finite fields. This demonstrates the advantages of hardware acceleration, not only for the computation of tree-level amplitudes for real-radiation processes in four dimensions over complex numbers but also for generating loop integrands for virtual corrections in d dimensions over finite fields. The associated computer program is publicly available.

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1 Introduction

The precision phenomenology program at CERN’s Large Hadron Collider (LHC) requires very precise and accurate theoretical calculations in order to match the unprecedented precision achieved by experimental measurements [1]. Theoretical prediction, mainly obtained through Monte Carlo techniques, are very intensive computationally and their cost is expected to grow in the future as we move towards the high-luminosity phase of the LHC, when even more accurate predictions will be necessary [2–5]. In addition, this increasing demand for computational resources is affecting all aspects of theoretical high-energy physics, ranging from parton distribution function (PDF) determination, to loop calculations and Monte Carlo (MC) integration. To remedy this problem, in addition to developing custom solutions, the use of graphics processing units (GPUs) has been found to be a particularly attractive solution given the high parallelisability of many particle-physics problems.

In recent years, the rapid development of GPUs, driven primarily by the growing demand for artificial intelligence (AI) applications, has transformed them into general-purpose graphics-processing units (GPGPUs), extending their use beyond traditional graphics rendering to encompass high-performance scientific computing. While central processing units (CPUs) have traditionally been optimised for high-speed execution of single tasks, making them well-suited to sequential processing, GPUs are designed to handle large sets of similar tasks in parallel, prioritising massive throughput over per-task execution speed. Thus, the key difference between optimising code for GPUs and CPUs lies in the shift to maximising data throughput by exposing as much concurrency as possible. In turn, this can lead to trade-offs in performance when the calculation has to be executed sequentially. However, in today’s era of multi-core processors and large computing clusters, many of the optimisation techniques targeting hardware accelerators can similarly be advantageous on traditional CPUs. Currently, most GPGPU libraries focus on float or double precision types, which the devices can support natively. More complex numeric types require custom implementations, resulting in performance penalties and the inability to leverage general-purpose libraries. These limitations can affect development efficiency and flexibility.

Particle physics has not remained indifferent to these developments. While the most-used software in particle physics phenomenology are still CPU-based, the interest in GPGPU has translated into libraries for device-agnostic Monte Carlo integration [6–8], for PDF determination [9, 10], interpolation [11], parton shower [12], and amplitude generation and evaluation [13, 14]. Some of these efforts have culminated in the recent years on full device-agnostic event generators [15–17] targeting leading-order (LO) calculations. Comparative studies presented in the referred work show gains that can go up to a factor of two orders of magnitude for some exemplary cases. It is also worth mentioning efforts in MADGRAPH5_AMC@NLO to migrate to a fully GPU implementation, with preliminary results shown in Refs. [18–22].

If we restrict ourselves to fixed orders calculation, only LO generators are available, although a proof-of-concept calculation at next-to-leading order (NLO) QCD on a GPU was presented in Ref. [11] showing that these devices were capable of handling more complicated calculations beyond massive parallelisation. This last example, however, was obtained with hard-coded matrix elements with all the limitations that it implies. As of now, the method of choice to obtain fully general numerical amplitudes in the Standard Model and beyond is based on recursion techniques [23, 24]. These algorithms have led to the implementation of efficient numerical codes running on CPUs [25–27]. It is therefore of prime importance to design new computer programs able to handle these algorithms on GPUs. The first proof of concept in this direction was released more than 10 years ago [28, 29] but was restricted to massless QCD with only one public program limited to specific QCD processes [29, 30]. The recent efforts in the context of MADGRAPH5_AMC@NLO [15, 16, 18–22] and SHERPA [13, 17] points to a future in which GPUs will be regularly used in event generation. However, presently these efforts are limited to leading-order event generator and no matrix-element provider targeting high-accuracy calculations is available.

In particular, while most of the effort has been focusing on the development of Monte Carlo aspects where the gains expected from GPU implementations are the most obvious, the advantages of using GPU for loop calculations can also be critical. Such an example has already been provided in Ref. [31, 6] where loop integrals are computed numerically using GPU-based integration routines. In addition, the calculation of multiloop amplitudes with numerical unitarity method [32] requires a large number of evaluations of tree amplitude along the cuts. The challenge is that these have to be computed for finite fields [33, 34] in d dimensions. This part of the calculation can therefore be particularly computationally intensive, and it could benefit from massive parallelisation on GPUs.

For these reasons, we have developed the first public program for the computation of tree-level amplitudes in arbitrary dimensions using both real numbers and finite fields running on both CPU and GPU. In this study we rely on integer arithmetic capabilities of GPUs. This proof of concept is for now restricted to purely gluonic amplitudes but it is foreseen to be extended to the full Standard Model in the future. The code is publicly available at

<https://github.com/Amps-GPU/BG-Trees>

and it can be readily used. We provide an [example jupyter notebook](#) within the repository.

The organisation of the article is as follows. Section 2 introduces the general motivation of the work, the formalism used as well as the conventions. In Section 3, the general strategy is presented along with the checks that have been performed and a benchmarking exercise. Section 4 contains a summary of the work as well as concluding remarks.

2 Use cases and method overview

In this section, we motivate the importance of tree-level amplitudes and their implementations in GPU. After this, the Berends-Giele recursion used in the present work is briefly introduced. Finally, the notations and conventions that we have adopted for our implementation in d dimension are presented.

2.1 Context

While LO accuracy is no longer the standard for particle-physics phenomenology, tree-level amplitudes continue to be of critical interest. For example, in Refs. [35, 36], it has been found that, for typical theoretical predictions, a fair share of the total amount of CPU time is spent in evaluating tree-level matrix elements. Also, for fixed-order predictions, tree-level amplitudes are needed for subtraction terms or real radiations and thus play a crucial role, in particular regarding to their numerical stability.

In additions to these well known use cases, the generalised unitarity method [37–40] in its numerical [41–43] and multi-loop [44–46] formulation, also requires the fast and reliable evaluation of tree amplitudes, usually through Berends-Giele recursion [23]. In the following, we summarise this method to compute multi-loop amplitudes, as implemented in the computer code CARAVEL [32].

In general, the non-colour part of an amplitude can be written as

$$\mathcal{A}^{(L)} = \sum_{\Gamma \in \Delta} \sum_{i \in M_{\Gamma}} c_{\Gamma,i} \mathcal{I}_{\Gamma,i}, \quad (1)$$

where the set Δ contains all propagator structures Γ while M_{Γ} is the full set of master integrals associated to Γ . The master integrals are thus $\mathcal{I}_{\Gamma,i}$ while the coefficients $c_{\Gamma,i}$ are functions of the kinematic invariants. The corresponding integrand, with ℓ_l the set of L loop momenta, can be further parametrised as

$$\mathcal{A}^{(L)}(\ell_l) = \sum_{\Gamma \in \Delta} \sum_{k \in Q_{\Gamma}} c_{\Gamma,k} \frac{m_{\Gamma,k}(\ell_l)}{\prod_{j \in P_{\Gamma}} \rho_j(\ell_l)}. \quad (2)$$

The set P_{Γ} contains all inverse propagators ρ_j for the diagram Γ (or equivalently the propagator structure) while the Q_{Γ} labels all the possible integrand $m_{\Gamma,k}$.

In order to compute the coefficients $c_{\Gamma,k}$ of Eq. (2), the properties of loop integrands for on-shell configurations ℓ_l^{Γ} of the loop momenta are exploited. These configurations are defined for a propagator structure P_{Γ} as

$$\rho(\ell_l^{\Gamma}) = 0 \quad \text{iff} \quad j \in P_{\Gamma}. \quad (3)$$

In this case, the leading pole of Eq. (2) reads

$$\sum_{\text{states}} \prod_{i \in T_{\Gamma}} \mathcal{A}_i^{\text{tree}}(\ell_l^{\Gamma}) = \sum_{\Gamma' \geq \Gamma, k \in Q_{\Gamma'}} \frac{c_{\Gamma',k} m_{\Gamma',k}(\ell_l^{\Gamma})}{\prod_{j \in (P_{\Gamma'}/P_{\Gamma})} \rho_j(\ell_l^{\Gamma})}. \quad (4)$$

In this case, T_{Γ} is the set of all tree-level amplitudes corresponding to the vertices of Γ while the state sum runs over all D_s -dimensional particle states appearing in the loop. The diagrams Γ' have an equal or larger number of propagators than Γ . Using this cut equation to determine

the coefficients $c_{\Gamma,k}$ therefore requires the multiple evaluation (in order to generate a set of linear equations) of the tree amplitudes $\mathcal{A}_i^{\text{tree}}$ at many phase-space points. This step represents a substantial part of the total computational time required for obtaining the loop amplitude. It means that having a program offering a fast evaluation of tree-level amplitudes in arbitrary dimensions is particularly valuable. This is one of the main motivations for the present work.

2.2 Berends-Giele recursion

In QCD, any gluon amplitude can be cast in the following form

$$\mathcal{M}_n(1, \dots, n) = \sum_{P(1, \dots, n-1)} \text{Tr}(T^{a_1} \dots T^{a_n}) \mathcal{C}(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}), \quad (5)$$

where \mathcal{C} is a function enclosing the information about the momenta and the helicities of the gluons. The indices a_i denote the colours of the n gluons and T^{a_i} are colour matrices in the fundamental representation.

The colour part of the amplitude can be obtained with analytical or numerical techniques. On the other hand, the non-colour part *i.e.* the function \mathcal{C} can be obtained with various numerical or analytical techniques. In the present work, we revert to the Berends-Giele (BG) recursion [23] which has been shown to be particularly efficient. The general idea is that the kinematic part of the amplitudes can be efficiently built out of building blocks of lower multiplicities called *currents*.

In particular, the matrix element for n gluons can be built out of the $n-1$ -gluon current by removing the propagator of the off-shell gluon, by contracting the current with the polarisation vector of the n^{th} gluon, and by enforcing momenta conservation. The currents are then recursively defined as

$$J^\mu(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}) = \frac{-i}{P_{1,n}^2} \left[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i}, P_{i+1,n}) J_\nu(k_1^{\lambda_1}, \dots, k_i^{\lambda_i}) J_\rho(k_{i+1}^{\lambda_{i+1}}, \dots, k_n^{\lambda_n}) \right. \\ \left. + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} V_4^{\mu\nu\rho\sigma} J_\nu(k_1^{\lambda_1}, \dots, k_i^{\lambda_i}) J_\rho(k_{i+1}^{\lambda_{i+1}}, \dots, k_j^{\lambda_j}) J_\sigma(k_{j+1}^{\lambda_{j+1}}, \dots, k_n^{\lambda_n}) \right], \quad (6)$$

where the first and second term account for the triple- and quartic-gluon vertices, respectively. Furthermore,

$$V_3^{\mu\nu\rho}(P, Q) = i(g^{\nu\rho}(P-Q)^\mu + 2g^{\rho\mu}Q^\nu - 2g^{\mu\nu}P^\rho), \\ V_4^{\mu\nu\rho\sigma} = i(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}), \\ P_{i,j} = \sum_{l=i}^j k_l. \quad (7)$$

2.3 Notations in d dimensions

Here the definitions and related conventions of the polarisation states are provided, taking the example of 6 dimensions. In the case the polarisation is built for a massless 6-dimensional vector $k = \{k^0, \dots, k^5\}$ (obeying therefore the relation $(k, k) = 0$) and a 4-dimensional massless vector χ embedded in six dimensions as follow

$$\chi = \{\chi^0, \dots, \chi^3, 0, 0\}, \quad \text{with} \quad (\chi, \chi) = 0. \quad (8)$$

The 4-dimensional part of the vector k vector is defined as

$$k_{4d} = \{k^0, \dots, k^3, 0, 0\}, \quad \text{with} \quad (k_{4d}, k_{4d}) = \mu^2 \neq 0. \quad (9)$$

Hence, a massless four-dimensional vector can then be constructed in the following way

$$k_{4d}^b = k_{4d} - \frac{\mu^2}{2(\xi, k_{4d})} \chi. \quad (10)$$

Square roots are cumbersome when working in finite fields \mathbb{F}_p , since they require a field extension 50% of the time. However, they can be avoided by multiplying ϵ by appropriate factors while dividing out from the ‘‘conjugate’’ state ϵ^* . In this case the two states are no longer complex conjugate, like for spinors with complex momenta. Nevertheless the completeness relation and orthonormality conditions are still satisfied. The polarisation vectors can be written as

$$\begin{aligned} \epsilon_1 &= \left\{ \frac{\langle k_{4d}^b | \gamma^\mu | \chi \rangle}{2 [k_{4d}^b, \chi]}, 0, 0 \right\}, & \epsilon_2 &= \left\{ \frac{[k_{4d}^b | \gamma^\mu | \chi]}{\langle k_{4d}^b, \chi \rangle}, 0, 0 \right\}, \\ \epsilon_3 &= \left\{ \frac{1}{\mu} k_{4d}^b - \frac{\mu}{2(\chi, k_{4d})} \chi, 0, 0 \right\}, & \epsilon_4 &= \frac{1}{\mu} \{0, 0, 0, 0, k_5, -k_4\}. \end{aligned} \quad (11)$$

2 By definition, the polarisation vectors are transverse $(k, \epsilon_i) = 0$ and fulfil the following properties

$$\begin{aligned} (\epsilon_1, \epsilon_2) &= -1, \\ (\epsilon_3, \epsilon_3) &= (\epsilon_4, \epsilon_4) = -1, \\ (\epsilon_1, \epsilon_1) &= (\epsilon_2, \epsilon_2) = (\epsilon_3, \epsilon_4) = (\epsilon_{1,2}, \epsilon_{3,4}) = 0. \end{aligned} \quad (12)$$

Note that we have adopted the conventions of Ref. [47] where the following expressions are provided in arbitrary dimensions. In particular, the completeness relation reads

$$\sum_{i=1}^4 \epsilon_i^\mu \epsilon_i^{*\nu} = -g_{6d}^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{(k, \eta)}, \quad (13)$$

where η is the 6-dimensional reference vector such that $(k, \eta) \neq 0$ and $g_{6d}^{\mu\nu}$ is the mostly negative metric in 6 dimensions. This relation can be further split into a 4-dimensional and two-dimensional part as

$$\sum_{i=1}^4 \epsilon_i^\mu \epsilon_i^{*\nu} = \left(-g_{4d}^{\mu\nu} + \frac{k_{4d}^\mu k_{4d}^\nu}{\mu^2} \right) - \left(g_{2d}^{\mu\nu} + \frac{k_{2d}^\mu k_{2d}^\nu}{\mu^2} \right), \quad (14)$$

by taking the particular choice

$$\eta^\mu = k_{4d}^\mu - k_{2d}^\mu. \quad (15)$$

This construction of the polarisation states can be generalised to $D > 6$ by introducing more polarisation states, where keeping the number of states equal to $D - 2$. We define the extra polarisation states beyond $D = 6$ as,

$$\epsilon_{5 \leq x \leq D-2}^\mu = \frac{1}{\mu_{x+1} \mu_{x+2}} \left\{ 0, 0, 0, 0, k_4 k_x, \dots, k_{x-1} k_x, -\mu_{x+1}^2, \vec{0} \right\}. \quad (16)$$

where we define μ_x as

$$\mu_x^2 = k_4^2 + \dots + k_{x-1}^2, \quad (17)$$

meaning that $\mu^2 = \mu_D^2$. The expression in Eq. (16) can be obtained by requiring that the completeness relation and orthogonality conditions are satisfied while solving for the polarisation components. It is also convenient to require that $\epsilon_x^{\mu > x+1} = 0$, so that lower dimensional states are trivially embedded in higher dimensions. Keeping the same convention to deal with the square roots, the polarisation vectors read

$$\begin{aligned} \epsilon_1 = \tilde{\epsilon}_2 &= \left\{ \frac{\langle k_{4d}^b | \gamma^\mu | \chi \rangle}{2 [k_{4d}^b, \chi]}, 0, 0 \right\}, & \tilde{\epsilon}_1 = \epsilon_2 &= \left\{ \frac{[k_{4d}^b | \gamma^\mu | \chi]}{\langle k_{4d}^b, \chi \rangle}, 0, 0 \right\}, \\ \epsilon_3 &= \left\{ k_{4d}^b - \frac{\mu^2}{2(\chi, k_{4d})} \chi, 0, 0 \right\}, & \tilde{\epsilon}_3 &= \frac{\epsilon_3}{\mu^2}, \\ \epsilon_4 &= \{0, 0, 0, 0, k_5, -k_4\}, & \tilde{\epsilon}_4 &= \frac{\epsilon_4}{\mu^2}. \end{aligned} \quad (18)$$

3 GPU acceleration

In this section we describe the specific GPU implementation. In this work we achieve the GPU acceleration of d -dimensional amplitudes using finite fields through two strategies. First, our focus is on future flexibility rather than solely performance. To that end, we utilise the TensorFlow library [48]. TensorFlow is a well-maintained library with a focus on Artificial Intelligence. It provides many kernels for mathematical operations capable of running in GPUs (of different vendors) as well as CPUs. Most importantly, it provides an easy interface for custom types, avoiding the need for specific implementations for each operation (at the cost of some overhead). Note that while our focus is on finite fields the library can operate with arbitrary precision objects (through the `mpmath` library [49]) as well as standard (`numpy` [50] or `tensorflow` [48]) float or complex numbers.

While it is possible to create every necessary operation required by the BG recursion using finite fields by composing existing TensorFlow primitives, this introduces a significant overhead relative to the actual complexity of the operations. To mitigate this, we have developed custom kernels (for CPU and GPU) for the subset of operations where this overhead is greater. The operations for which specific kernels have been developed are the many-dimensional tensor batch contractions and the multiplicative inverse computation in finite fields.

Since the parallelisation is performed strictly along the event dimension, tasks that do not depend on the number of events, or that are not performance-critical, are left to the host code. For example, the recursion algorithm is prepared entirely on the host. Likewise, the input momenta and polarisation states are also prepared on the host and rely on CPU dependencies `lips`, `syngular` [51], and `Singular` [52]. While in principle it might be beneficial to offload some other stages to the hardware accelerator, the added complexity does not justify the marginal performance gain at this stage.

The choice of CUDA [53] for the kernels and TensorFlow and python for the overall framework has been driven primarily by the availability of these tools (and compilers) on the computer resources we have access to. We have deliberately minimised reliance on platform-specific features in order to keep this proof-of-concept project flexible enough for future extensions. This ensures that kernels can be ported to alternative platforms (*e.g.* OpenCL) and that the wrapper code can, if necessary, be adapted to other frameworks such as PyTorch [54] or C++.

Our approach prioritises flexibility and modularity over performance, with the aim of creating a library easy to use, develop, and extend. True performance bottlenecks will only emerge once the library is integrated into larger frameworks, and excessive early optimisation might hinder future development. Another crucial aspect of GPU computing is memory management, but as will be shown in Section 3.2, our implementation scales well without the need for extensive optimisation in this area. We have thus decided to leave this for future development, as this is not currently a bottleneck for our project.

While the main objective of this work is computing scattering amplitudes on GPUs, the library developed in the context of this work is written such that it can be used standalone for calculations involving finite fields. The use of python and TensorFlow as the glue language for this project makes the library specially well-suited for machine learning applications or scenarios requiring differentiability [55]. In such cases, however, further extensions of the library should be developed in order to include GPU kernels for the gradient computations whenever necessary.

The documentation for this library can be found in the following URL:

<https://amps-gpu.github.io/BG-Trees/>

3.1 Validations

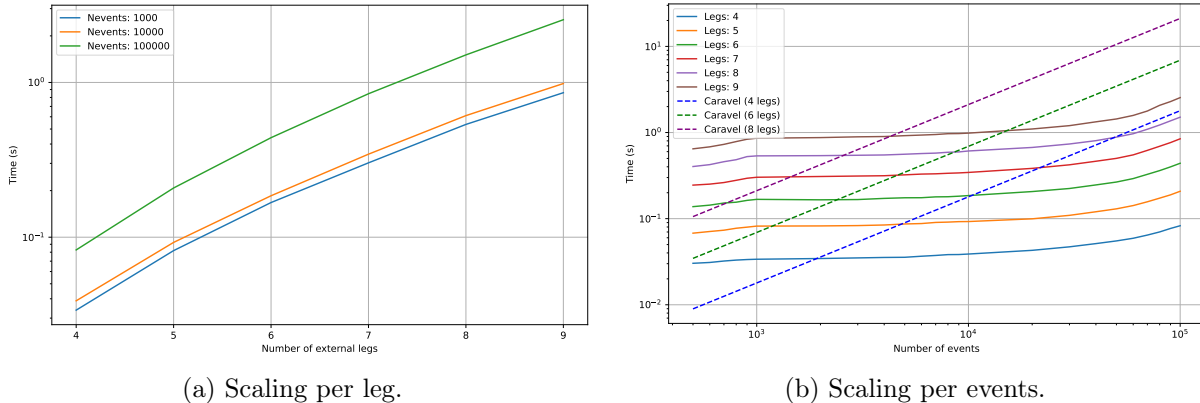
To validate the implementation, we have performed several checks. Most of them are implemented as continuous integration tests in the `tests` directory.

Starting from internal consistency checks, we verify the implementation of custom GPU kernels for finite field operations (`test_finite_gpufields.py`). The metric and vertices are verified to yield the same results on CPU and GPU (`test_metric_and_vertices.py`). For the states (a.k.a. polarisation vectors), we verify the completeness relations of Eqs. (13) and (14) in $d = \{4, 5, 6, 7, 8, 9, 10\}$ (`test_states.py`). For the d -dimensional phase space point generation, we verify on-shell relations and momentum conservation (`tests/test_phase_space.py`).

At the level of the amplitudes, we verify that the Ward identity is satisfied, that results in four dimensions reproduce known analytic expressions, and that d -dimensional results match against an external library (`test_currents.py`). The Maximally Helicity Violating (MHV) amplitudes are checked against Parke-Taylor formula [56]

$$\mathcal{M}_n(1^-, 2^-, 3^+, \dots, n^+) = ig^{n-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (19)$$

for a variety of multiplicities n (implemented in the CI as $n = 5$). We also verify NMHV amplitudes against known analytic expressions at $n = 6$ and $n = 7$, of which we retain the alternating NMHV amplitude at six-point as a representative. Lastly, our test suite shows how to reproduce the output of CARAVEL [32] when the momenta live beyond four dimensions. In order to perform this validation and to allow for a one-to-one comparison, we have allowed for a modification of the metric in our code to match CARAVEL’s alternating metric, $\text{diag}(1, -1, 1, -1, 1, -1, \dots)$. We generate d -dimensional phase space points (incorporating the metric change) and then pass the resulting d -momenta to CARAVEL. By using its `Forest` debug feature, we can obtain values for the states in CARAVEL’s convention which then can be used as the input of our BG recursion. This allowed us to match the result at the level of the color-ordered amplitude, up to a power of 2, which is easily accounted for by \mathbb{Q} -reconstruction from the \mathbb{F}_p value.



(a) Scaling per leg.

(b) Scaling per events.

Figure 1: Total time it takes to compute the amplitude for the given number of phase space points in 8 dimensions for amplitudes with increasing number of external legs for one helicity configuration. All runs are done in GPU. For comparison, the time taken by Caravel running in CPU for 4, 6 and 8 legs (and 8 dimensions) is also shown.

3.2 Benchmarks

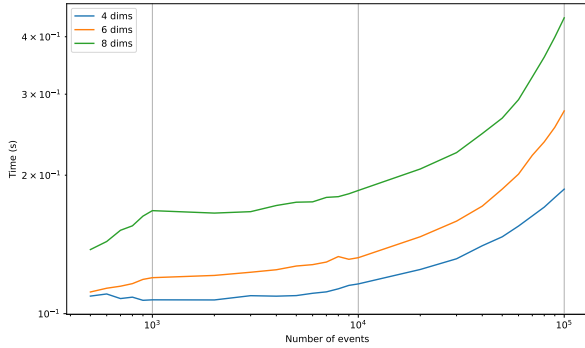
We end this section with a benchmarking of the library presented in this article by running it on different hardware platforms.

In Fig. 1, we show the scaling of the calculation as a function of the number of external legs (Fig. 1a) and number of events (Fig. 1b). While running on graphics cards speeds up the calculation considerably, we can see in Fig. 1a that the exponential scaling in the number of legs is maintained and independent of the number of events being run. This can be easily understood, as the change of hardware has no effect on the complexity of the calculation itself. Instead, the benefits of running on GPU can be immediately seen when we study how the computational cost scales with the number of events. In Fig. 1b, the time it takes to finish the computation is shown as a function of the number of events. This time is kept approximately constant as we are filling up the computational capabilities of the GPU up to a threshold. From that threshold onwards the scaling becomes linear. Note that the threshold does not depend on the number of legs but only on the size of the input array of phase space points. Increasing the complexity of the calculation shifts the curve, but it does not affect the general behavior.

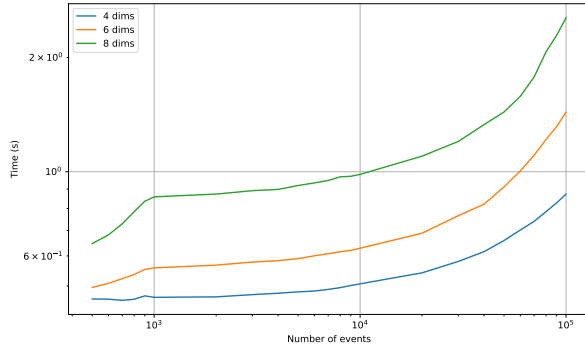
In Fig. 2, instead we benchmark the framework for a different number of dimensions for two choices of the number of external particles (6 and 9). While the computational cost grows with the number of dimensions, the growth is slower compared to the growth with the number of external particles, meaning that extending the computation to an arbitrary number of dimensions can be achieved.

In the last of these comparisons, in Fig. 3, we show the behaviour of the code on different devices. We choose a representative set of graphics cards that can be found both on laptops and home desktop computers as well as enterprise-grade GPUs. Note that this cannot be considered a fair comparison between the devices since the rest of the hardware in which they are running should also be considered and (for practical reasons) is wildly different in each case. On the other hand, the similarity in runtime across different hardware is very good news, as it means the choice of running on GPU does not necessarily require costly CPU resources to use it adequately.

In all plots in Figs. 1, 2, and 3 we observe also a notable offset even for a small number of events. This is partly due to the prioritisation of flexibility over performance. However, in



(a) 6 external particles



(b) 9 external particles

Figure 2: Total time it takes to compute the amplitude for the given number of phase space points for different dimensions for one helicity configuration. We show the 6 (left) and 9 (right) particles cases.

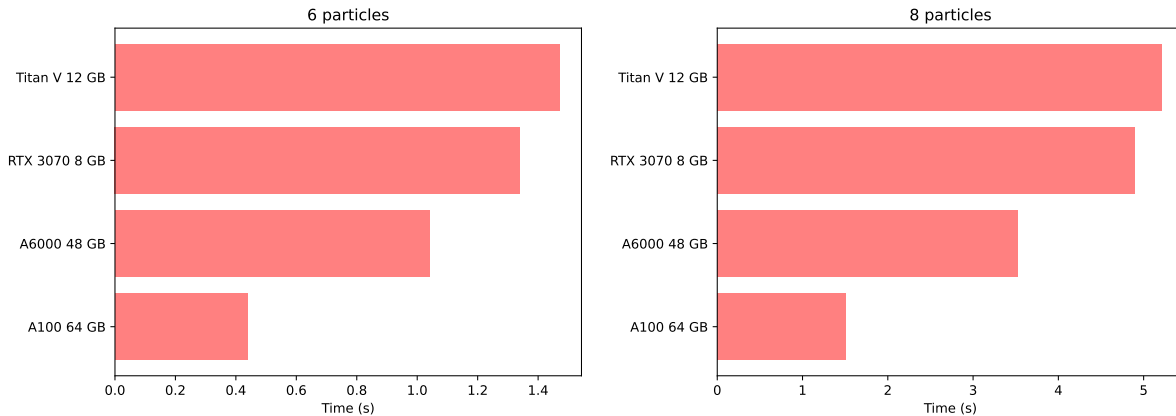


Figure 3: Total time it takes to compute the amplitude for 10^5 events, 6 and 8 external particles, 4 dimensions for different devices. Note that each device is running in a computer with different capabilities, however we observe a general (expected) positive correlation between the memory of the device and the speed, regardless of the number of external particles being considered.

cases where performance is critical, while the overall scaling is expected to remain unchanged, the computing time can be reduced by ahead-of-time compilation (*e.g.* CUDA kernels for a set number of legs, dimension, etc). The overhead caused by the choice of python as the language to glue the components together can also be trivially eliminated by changing those components to a compiled language such as C, Rust or Fortran.

We conclude with a comparison to another GPU implementation of BG recursion named **BlockGen** and used in the code **Pepper** [13, 17]. While the scopes of this work and **BlockGen** [13] are quite different, it is illustrative to benchmark both methods. The target of **Pepper** is the calculation of physical observables in a way similar to that of other event generators like Sherpa [57], to achieve this, it uses highly optimised CUDA kernels for double-precision and a four-dimensional momenta phase space. Our focus instead is on computing amplitudes in an arbitrary number of dimensions using finite fields, with a very flexible code base which can serve other purposes, and so only the most critical components are compiled ahead of time. With these differences in

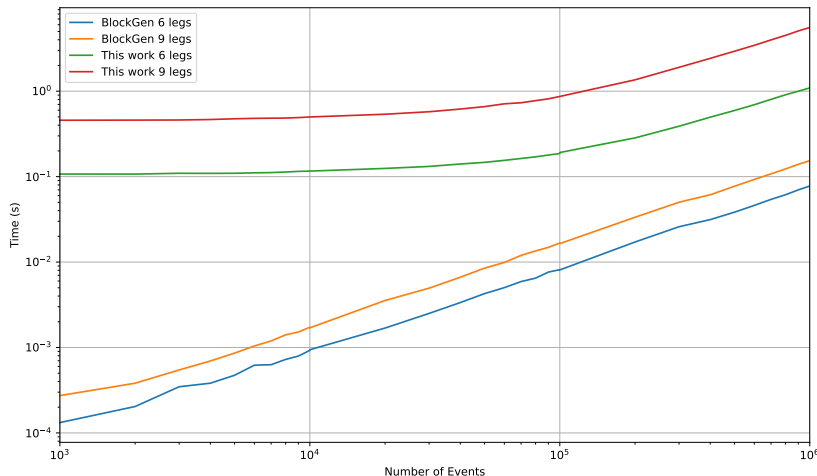


Figure 4: Comparison between the framework introduced in this work and `BlockGen` [13, 17] for 6 and 9 external particles. For this benchmark both calculations are run in 4 dimensions.

mind, a benchmark for 6 and 9 particles in the final state is presented in Fig. 4. As expected, the just-in-time compilation nature of our framework introduces an overhead that requires a large number of events to overcome. Eventually, the difference in performance between both approaches stabilises with `BlockGen` being about one order of magnitude faster. In this benchmark both codes are running with the same number of dimensions (4) and using the leading-color approximation for the calculation. Note however that they use different numerical settings: a custom type, finite fields, vs. double precision floats.

4 Conclusion

In this work we have developed a tree-level matrix element generator and evaluator based on the Berends-Giele recursion algorithm and optimised for execution on GPUs. Our primary objective was to demonstrate the advantages of hardware acceleration for computations extending beyond four dimensions and employing number types beyond traditional floating-point arithmetic, specifically finite fields. Our results show that hardware acceleration, which has so far been developed primarily for Monte Carlo integration, can indeed be extended to support higher-order calculations, where products of tree amplitudes in higher dimensions yield loop-level integrands.

Given the proof of concept nature of this work, we have explicitly prioritised flexibility over performance. So while we have developed dedicated kernels for the most costly operations, the more complex stages of the algorithm are performed in higher-level languages so that they can be easily modified and possibly improved.

A natural extension of this work is to cover the full Standard Model by handling also quarks and the electroweak part. This would enable the integration with existing tools, such as `CARAVEL` or Monte Carlo generators.

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