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Einstein gravity from a matrix integral - Part I

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Abstract: We construct backreacted geometries dual to the supersymmetric mass deformation of the IKKT matrix model. They are Euclidean type IIB supergravity solutions given in terms of an electrostatic potential, having $SO(7) \times SO(3)$ isometry and 16 supersymmetries. Quantizing the fluxes, we find that the supergravity solutions are in one-to-one correspondence with fuzzy sphere vacua of the matrix model.

Contents

1 Introduction

The gauge/gravity duality is our best UV complete model of quantum gravity. It tells us that some quantum mechanical systems are well described by Einstein gravity in some regimes, usually involving many strongly coupled degrees of freedom. We would like to find out what is special about such systems.

In this paper, we study the simplest toy model that has an emergent Einstein gravity description, namely, a matrix integral. More precisely, we study the *polarized IKKT matrix model* [1, 2]. This is a supersymmetric mass deformation of the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model [3]. It has vacua[1] given by N-dimensional representations of $SU(2)$. These fuzzy sphere configurations are depicted on the left of Figure 2. For $N \times N$ matrices, the number of such vacua is $P(N)$, the number of integer partitions of N.

The IKKT matrix model was originally proposed as a non-perturbative definition of (type IIB) string theory. This is similar in spirit to the BFSS conjecture [4, 5], which proposes that the BFSS matrix quantum mechanics provides a non-perturbative definition of M-theory in flat space. While testing or verifying the BFSS conjecture remains challenging (see [6, 7] and references therein for recent discussions), the BFSS model was later revisited from the perspective of standard holography [8, 9], which helped to clarify its relation to M-theory and sharpen the original conjecture. In contrast, little work has been done on holography for the IKKT model, with the notable exception of studies of the decoupling limit of the D-instanton background [10–12]. This was partly due to the absence of observables that could be computed and compared on both sides. In this paper and a companion paper [13], we demonstrate a one-to-one correspondence between vacua of the polarized IKKT matrix integral and the dual backreacted geometries.

More precisely, we identify a family of geometries dual to the polarized IKKT matrix integral, that are solutions of Euclidean type IIB supergravity with vanishing 5-form flux, see (3.9). The metric takes the form of a warped product, with a 2-sphere and a 6-sphere fibered over a 2-dimensional plane parametrized by (ρ, z),

$$ds^2 = R_2(\rho, z)^2 d\Omega_2^2 + R_6(\rho, z)^2 d\Omega_6^2 + H(\rho, z)^2 (d\rho^2 + dz^2). \tag{1.1}$$

The solutions are given in terms of a single function $V(\rho, z)$ that solves the 4 dimensional axially symmetric Laplace equation,

$$V'' + \frac{2}{\rho}\dot{V} + \ddot{V} = 0 \tag{1.2}$$

where $V' \equiv \partial_z V$ and $\dot{V} \equiv \partial_\rho V$. We can think of ρ as the radial coordinate on $\mathbb{R}^3$ and z a coordinate along the orthogonal direction in $\mathbb{R}^4$. This allows us to use an electrostatic analogy, where V is an electrostatic potential. This is similar to the Lin-Maldacena geometries [14, 15] that describe the vacua of the BMN model [16].

Let us describe the electrostatic analogy in more detail. Boundary conditions are imposed by considering q conducting disks[2] of radius ρ_s, centered on the z-axis at different

[1] By vacuum we simply mean a local minimum of the action.

[2] We use the word *disks* by analogy with [14] but in our case we have 3D conducting balls in $\mathbb{R}^4$.

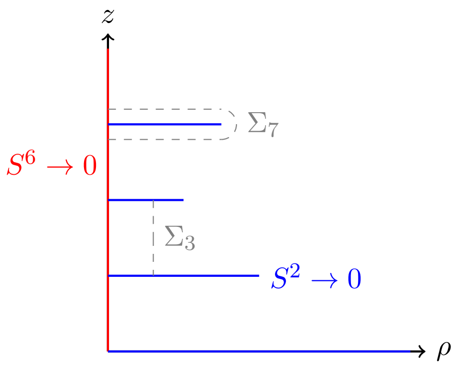

Figure 1: Picture of the geometry in the (ρ, z) plane. The 10d geometry is obtained by fibering an S^2 and an S^6. The blue lines are defined by $\dot{V} \equiv \partial_\rho V = 0$ and are the regions where S^2 shrinks. The red line is the z-axis $\rho = 0$ where the S^6 shrinks. We can construct 7-cycles Σ_7 as the product of the 6-sphere times a segment on the (ρ, z) plane whose endpoints have $\rho = 0$ (where S^6 shrinks). Similarly, there are 3-cycles Σ_3 given by the 2-sphere times a segment connecting points where S^2 shrinks.

positions z_s and of given charges Q_s, with $s = 1, ..., q$. In addition we add a background potential that grows at infinity, see (3.13), and an infinite grounded disk (or grounded plane) at $z = 0$. Those boundary conditions come from requiring that the supergravity solution is regular. This also determines the size of the disks $\rho_s(Q_s, z_s)$ since in general we expect the electric field to be infinite at the tip of a disk. However, since there is a background potential, we can choose the size of the disks so that the total electric field vanishes at the tip. This ensures that we don't have any singularity there.

These geometries are topologically non-trivial and they have various non-contractible cycles. This comes from the fact that the 6-sphere collapses on the z-axis and the 2-sphere collapses on the disks. We can then construct 3-cycles and 7-cycles by considering curves in the (ρ, z) plane ending on locations where the spheres shrink. The cycles are constructed by fibering a sphere over such curves, as shown in Figure 1.

The different supergravity fluxes can then be integrated over those cycles. Imposing the Dirac quantization conditions we find that the charges of the disks and their positions are quantized. Relating those quantum numbers to the dimensions and degeneracies of the $SU(2)$ irreducible representations, we find a one-to-one correspondence between the fuzzy sphere vacua and the supergravity solutions, as depicted in Figure 2.

We also study the system from the perspective of a probe brane. We consider a probe spherical $D1$-brane in a supergravity background that has a constant Ramond-Ramond flux, and compute its size by minimizing the DBI action. We get an exact correspondence between this radius and the radius of the fuzzy sphere vacua of the matrix model. Furthermore we show that this radius can also be computed in the backreacted geometry as the geodesic distance between the grounded conducting plane and a disk.

Structure of the paper. In section 2, we review the supersymmetric mass deformation of the IKKT matrix model. We study the saddle points of the matrix integral and show

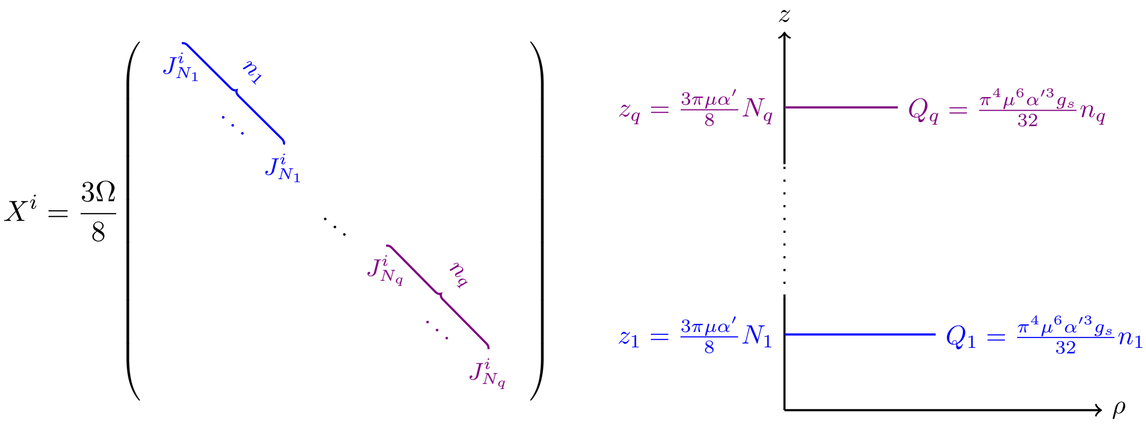

Figure 2: Correspondence between the fuzzy sphere vacua of the matrix model and the dual geometries. For each spin j_s $SU(2)$ irreducible representation of dimension $N_s = 2j_s+1$ we put a disk at position $z_s \sim N_s$. The number n_s of copies of this representation appearing determines the charge of that disk $Q_s \sim n_s$. Those integers are related to the $D1$ brane charge and $NS5$ brane charge by $N_{D1,s} = n_s$ and $N_{NS5,s} = N_s - N_{s-1}$.

that they are fuzzy spheres. In section 3, we construct Euclidean type IIB geometries with $SO(3) \times SO(7)$ symmetry and 16 supercharges. We conclude in section 4 with a discussion of our results. Our appendix A shows how to obtain the Euclidean IIB solution from the Lorentzian of [17]. In appendix B we show that the geometries we obtained are regular. In appendix C we review the solution to the electrostatic problem when the disks are widely separated.

2 The IKKT model and its mass deformation

The (Euclidean) IKKT integral is

$$Z_{\text{IKKT}} = \int \prod_{I=1}^{10} dX_I \prod_{\alpha=1}^{16} d\psi_\alpha \, e^{-S_{\text{IKKT}}}, \tag{2.1}$$

where[3]

$$S_{\text{IKKT}} = -\operatorname{Tr}\left[\frac{1}{4}\sum_{I,J=1}^{10}[X_I, X_J]^2 + \frac{i}{2}\sum_{\alpha,\beta=1}^{32}\sum_{I=1}^{10}\psi_\alpha(\mathcal{C}\Gamma^I)_{\alpha\beta}[X_I,\psi_\beta]\right], \tag{2.2}$$

[3]The action has the same convention as [18–20], which is natural from the perspective of dimensionally reducing 10D $\mathcal{N}=1$ SYM to a point. This convention is related to that in [2] through a simple redefinition of the charge conjugation matrix: $i\mathcal{C}_{\text{here}} = \mathcal{C}_{\text{there}}$. After Weyl projection to the suitable 16-component spinors, the actions are the same. The Weyl projection also explains why α only goes up to 16 in the fermion integral in (2.1). For more details, see [13].

where X_I are $N \times N$ traceless and hermitian bosonic matrices, ψ_α are $N \times N$ hermitian traceless matrices of 10D Euclidean Majorana-Weyl spinors, and Γ^I are the $SO(10)$ gamma matrices, with $\mathcal{C}$ the charge conjugation matrix.[4]

The result of the matrix integral (2.1) was conjectured in [18] and later computed in [19] up to a group-theoretical prefactor, which was calculated in [22] (see [23] for a review). The convergence of the partition function has been shown in [24, 25], where they also show that the correlation functions of matrix polynomials below degree 14 are finite (see [26] for some explicit correlation function examples). The gravitational dual to the IKKT model was studied in [10] by extending Maldacena's decoupling limit [8, 27] to D-instantons, and the result is flat space with non-vanishing, running dilaton and axion. In this paper we focus on Euclidean signature. For recent discussions on the Lorentzian IKKT model, see e.g. [28, 29].

The IKKT matrix model admits a mass deformation that preserves the sixteen dynamical[5] supersymmetries [1, App.A]. The action of the mass-deformed IKKT model reads

$$\begin{aligned} S_\Omega &= S_{\text{IKKT}} + S_{\text{def}} \\ S_{\text{def}} &= \operatorname{Tr}\left[\frac{3\Omega^2}{4^3}\sum_{i=1}^{3} X_i X_i + \frac{\Omega^2}{4^3}\sum_{p=4}^{10} X_p X_p + i\frac{\Omega}{3}\epsilon_{ijk}X_i X_j X_k - \frac{\Omega}{8}\psi_\alpha(\mathcal{C}\Gamma^{123})_{\alpha\beta}\psi_\beta\right], \end{aligned} \tag{2.3}$$

where the indices $i, j, k \in \{1, 2, 3\}$, $p \in \{4, 5, \ldots, 10\}$ and $\alpha, \beta \in \{1, 2, \ldots, 32\}$. Here $\Gamma^{123} = \Gamma^1\Gamma^2\Gamma^3$. Following [2], we will refer to this model as the *polarized IKKT matrix model*. We will also take X_I, ψ_α and Ω to be all dimensionless. In the limit $\Omega \to \infty$, S_Ω is dominated by the Gaussian mass terms and so the system is almost free. On the other hand, $\Omega \to 0$ corresponds to the strong coupling limit of the polarized IKKT model.

2.1 Symmetries

The mass deformation terms in (2.3) explicitly break the $SO(10)$ symmetry of the original IKKT model to $SO(3) \times SO(7)$. The matrices X^I and ψ transform in the adjoint representation of $SU(N)$. This is a global symmetry of the matrix model but we will often refer to it as the *gauge* symmetry by analogy with higher dimensional models. In addition, there are 16 supersymmetries acting as

$$\begin{cases} \delta X^I = -\psi_\alpha(\mathcal{C}\Gamma^I)_{\alpha\beta}\epsilon_\beta\,, \\ \delta\psi_\alpha = \frac{i}{2}[X_I, X_J]\Gamma^{IJ}_{\alpha\beta}\epsilon_\beta + \frac{3\Omega}{8}X_i(\Gamma^{123}\Gamma^i)_{\alpha\beta}\epsilon_\beta + \frac{\Omega}{8}X_p(\Gamma^{123}\Gamma^p)_{\alpha\beta}\epsilon_\beta\,, \end{cases} \tag{2.4}$$

where ϵ is deemed as Grassmann-even and δ is Grassman-odd susy generator.

[4] See appendix A of [21] for explicit expressions of the gamma matrices. For the charge conjugation matrix we choose $\mathcal{C} = \begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}$.

[5] The IKKT model of $U(N)$ matrices has sixteen dynamical and sixteen kinematical supersymmetries [3, 30], with the latter describing the center-of-mass degrees of freedom of the D-instantons [3, 18]. Since we choose to focus on traceless matrices and so the other sixteen kinematical supersymmetries of the IKKT model are irrelevant here.

These bosonic and fermionic symmetries are expected to give (a real form of) the exceptional Lie superalgebra F_4, which is also the superconformal algebra in five dimensions. We will discuss the structure of the supersymmetry algebra in more detail in [13].

2.2 Classical supersymmetric vacua

To find the classical vacuum, we set $\psi = 0$ and look for the minima of the bosonic potential

$$V_B = \mathrm{Tr}\left[-\frac{1}{4}[X_I, X_J]^2 + \frac{3\Omega^2}{4^3}X_iX_i + \frac{\Omega^2}{4^3}X_pX_p + i\frac{\Omega}{3}\epsilon_{ijk}X_iX_jX_k\right]. \tag{2.5}$$

The equations of motion (EOM) obtained from varying X_i and X_p are, respectively,

$$\frac{1}{2}[X_I, [X_I, X_i]] + \frac{3\Omega^2}{4^3}X_i + \frac{i\Omega}{4}\epsilon_{ijk}[X_j, X_k] = 0\,, \tag{2.6}$$

$$\frac{1}{2}[X_I, [X_I, X_p]] + \frac{\Omega^2}{4^3}X_p = 0\,, \tag{2.7}$$

where $I = 1, 2, \ldots, 10$, $i, j, k = 1, 2, 3$ and $p = 4, \ldots, 10$. The only solution for X_p is[6]

$$X_p = 0\,. \tag{2.8}$$

As for X_i, the solutions can be easily found using the ansatz

$$X_p = 0\,, \qquad X_i = \alpha\Omega J_i\,, \qquad [J_i, J_j] = i\epsilon_{ijk}J_k\,, \tag{2.9}$$

with J_i being the $N \times N$ matrix representations of $SU(2)$ Lie algebra (not necessarily irreducible). In this configuration the bosonic matrices form a 3-dimensional *fuzzy sphere* whose radius has the (dimensionless) length scale Ω. Plugging the ansatz into the equation of motion we get

$$\alpha^3 - \frac{1}{2}\alpha^2 + \frac{3}{64}\alpha = 0\,. \tag{2.10}$$

The roots of this polynomial are

$$\alpha = 0, \qquad \alpha = \frac{1}{8}, \qquad \alpha = \frac{3}{8}. \tag{2.11}$$

When $\alpha = 0$ we have the trivial vacuum with $X^I = 0$ and the on-shell action vanishes. At the non-trivial extrema the potential takes values

$$\begin{aligned} V_B &= \alpha^2\Omega^4\,\mathrm{Tr}\left[\frac{\alpha^2}{4}\epsilon_{ijk}\epsilon_{ijl}J_kJ_l + \frac{3}{4^3}J_iJ_i - \frac{\alpha}{6}\epsilon_{ijk}J_i\epsilon_{jkl}J_l\right] \\ &= \alpha^2\Omega^4\left(\frac{\alpha^2}{2} + \frac{3}{4^3} - \frac{\alpha}{3}\right)\mathrm{Tr}\,J_iJ_i \\ &= \begin{cases} \frac{5}{3\times 2^{13}}\Omega^4\,\mathrm{Tr}\,J_iJ_i & (\alpha = \frac{1}{8}) \\ -\frac{9}{2^{13}}\Omega^4\,\mathrm{Tr}\,J_iJ_i & (\alpha = \frac{3}{8}) \end{cases}. \end{aligned} \tag{2.12}$$

[6]See [2, Sec.3] for the derivation.

The solutions with $\alpha = \frac{3}{8}$ are dominant since $\mathrm{Tr}\, J_i J_i \geq 0$. If J_i is the spin-j irrep (and $N = 2j+1$), then

$$\mathrm{Tr}\, J_i J_i = N j(j+1) = N\frac{N^2-1}{4}\,. \tag{2.13}$$

For a general reducible representation, J_i is block diagonal with q types of blocks, each having multiplicity n_s and size $N_s \times N_s$. Then we have

$$\mathrm{Tr}\, J_i J_i = \sum_{s=1}^{q} n_s N_s j_s(j_s+1) = \sum_{s=1}^{q} n_s N_s \frac{N_s^2-1}{4}\,, \qquad N = \sum_{s=1}^{q} n_s N_s\,. \tag{2.14}$$

It is straightforward to check that $\mathrm{Tr}\, J_i J_i$ is maximized by the spin-$\frac{N-1}{2}$ irrep. Thus these solutions are not degenerate and the leading contribution comes from having a single irreducible representation.

Finally we want to see if the saddles preserve supersymmetry. In the background (2.9), the susy transformation of fermions in (2.4) becomes

$$\delta\psi = \frac{i}{2}(\alpha\Omega)^2 i\epsilon_{ijk}J_k\Gamma^{ij}\epsilon + \frac{\alpha 3\Omega^2}{8}J_i\Gamma^{123}\Gamma^i\epsilon = -\frac{\Omega^2}{2}\alpha\left(\alpha - \frac{3}{8}\right) i\epsilon_{ijk}J_k\Gamma^{ij}\epsilon\,, \tag{2.15}$$

where we have used

$$\Gamma^{123}\Gamma^k = \frac{1}{2}\epsilon_{ijk}\Gamma^{ij}\,. \tag{2.16}$$

Therefore, the supersymmetric saddles preserving $\delta\psi = 0$ are $\alpha = 0$, the trivial vacuum, and $\alpha = \frac{3}{8}$, corresponding to the minima of the action.

3 Backreacted geometries

In this section we study Euclidean solutions of type IIB supergravity that are smooth everywhere, have no horizons and have 16 supersymmetries together with $SO(7) \times SO(3)$ symmetry.

Let us review quickly the basics of type IIB supergravity. The bosonic matter content consists of two scalars, the dilaton ϕ and the axion χ, and two 3-form field strengths, the Ramond-Ramond (RR) F_3 and the Neveu-Schwarz-Neveu-Schwarz (NSNS) H_3. In addition, there is also a self-dual 5-form but we will set it to zero in this paper. The bosonic part of the action reads

$$S_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\phi}(H_3)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 - \frac{1}{12}e^{\phi}(F_3)^2\right). \tag{3.1}$$

This is the Lorentzian action. Since we will construct the Euclidean solution by analytic continuation of a Lorentzian solution, we do not have to use explicitly the Euclidean action. Our solution will solve the Lorentzian equations of motion but the metric will have Euclidean signature and the axion and RR 3-form will be purely imaginary. These are the appropriate reality conditions for Euclidean IIB supergravity [10, 11].

The field strengths are expressed in terms of 2-forms gauge potentials B_2 and C_2 as

$$H_3 = dB_2, \qquad F_3 = dC_2 - \chi H_3. \tag{3.2}$$

There are therefore 3 types of charge in this theory, corresponding to the couplings to the fields χ, B_2 and C_2. The equations of motion for these fields are

$$d(e^{2\phi} * d\chi) = -e^{\phi} H_3 \wedge *F_3, \qquad d(e^{\phi} * F_3) = 0, \qquad d(e^{-\phi} * H_3) = e^{\phi} d\chi \wedge *F_3. \tag{3.3}$$

Therefore, in addition to the closed 3-forms dC_2 and dB_2, we can also write a closed 9-form

$$\tilde{F}_9 = e^{2\phi} * d\chi + e^{\phi} B_2 \wedge *F_3, \tag{3.4}$$

and closed 7-forms

$$\tilde{H}_7 = * \left(e^{-\phi} H_3 - e^{\phi} \chi F_3 \right), \qquad \tilde{F}_7 = * e^{\phi} F_3. \tag{3.5}$$

The equations of motion then ensure that $d\tilde{F}_9 = d\tilde{H}_7 = d\tilde{F}_7 = 0$. This allows us to define conserved "electric" charges by integrating those closed forms ($\tilde{F}_9$, $\tilde{H}_7$, $\tilde{F}_7$) on 9-cycles or 7-cycles. Similarly, "magnetic" charges are defined by integrating the closed 3-forms (dC_2, dB_2) on 3-cycles [7].

Lorentzian type IIB supergravity also has internal $SL(2,\mathbb{R})$ symmetry acting on the scalars and the 3-forms. The latter simply transform as a doublet

$$\begin{pmatrix} H_3 \\ dC_2 \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_3 \\ dC_2 \end{pmatrix}, \qquad ad - bc = 1. \tag{3.6}$$

For the transformation of the axion and dilaton, it is more convenient to group them in the axi-dilaton $\tau = \chi + ie^{-\phi}$ that transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{3.7}$$

When we complexify the fields, $SL(2,\mathbb{R})$ gets promoted to $SL(2,\mathbb{C})$. However, in order to obey the reality conditions of Euclidean IIB supergravity, only a subset of this $SL(2,\mathbb{C})$ is allowed.

3.1 Euclidean solution of IIB supergravity with 16 supercharges

Lorentzian solutions of IIB supergravity with 16 supercharges and $SO(5,2) \times SO(3)$ symmetry were constructed in [17] and further analysed in [31–33]. They are dual to five-dimensional superconformal field theories with the exceptional F_4 superconformal symmetry.

We can obtain the Euclidean version with $SO(7) \times SO(3)$ isometry by performing an analytic continuation as we show in appendix A. The resulting Euclidean supergravity background has vanishing 5-form field strength and consists of the metric, the axion χ, the

[7] Our solution will not have non-contractible 1-cycles and therefore we do not discuss the "magnetic" charge of χ, which would measure D7-brane charge.

dilaton ϕ, the NSNS 3-form H_3 and the RR 3-form F_3. It can be written in term of a single function $V(\rho, z)$ that satisfies the equation

$$V'' + \ddot{V} + \frac{2}{\rho}\dot{V} = 0, \tag{3.8}$$

where $V' \equiv \partial_z V$ and $\dot{V} \equiv \partial_\rho V$. This is the Laplace equation in a four-dimensional axially symmetric system, where ρ is the radial coordinate and z is the vertical direction. The remaining two angular coordinates are not spacetime coordinates.

The Einstein frame solution reads[8]

$$\begin{aligned}
ds^2 &= \frac{8}{\mu^{\frac{5}{2}}}\left(\frac{1}{3^3}\frac{\Delta\dot{V}}{(-V'')}\right)^{1/4}\left[\frac{(-V'')}{\dot{V}}(d\rho^2 + dz^2) + 3\rho d\Omega_6^2 + \frac{\rho(-V'')\dot{V}}{\Delta}d\Omega_2^2\right],\\
e^\phi &= -\mu^3\frac{3\dot{V} + \rho V''}{\rho\sqrt{3\Delta\dot{V}(-V'')}},\\
\chi &= -\frac{i}{\mu^3}\frac{3\dot{V}(V' + \rho\dot{V}') + \rho V'V''}{3\dot{V} + \rho V''},\\
H_3 &= dB_2, \qquad B_2 = -\frac{8}{3\mu}\left(z - \frac{\rho\dot{V}\dot{V}'}{\Delta}\right)\wedge d\Omega_2\\
F_3 &= dC_2 - \chi H_3, \qquad C_2 = i\frac{8}{3\mu^4}\left(V - \rho\frac{\dot{V}}{\Delta}\left(V'\dot{V}' + 3\dot{V}(-V'')\right)\right)\wedge d\Omega_2,\\
\Delta &\equiv 3\dot{V}V'' + \rho(\dot{V}'^2 + V''^2).
\end{aligned} \tag{3.9}$$

The coordinates z, ρ have dimensions of length, μ is a mass parameter and V has dimensions of inverse length squared. This ansatz solves the equations of motion but we still have to impose regularity and positivity of the metric components. This leads to additional constraints on the function V. To identify boundaries in the (ρ, z) plane, we look for regions where the S^2 and/or the S^6 shrink. If we write their respective radii as R_2 and R_6 we note that

$$R_6^3 R_2 \propto \rho^2\dot{V}. \tag{3.10}$$

Looking back at the metric we see that $\rho \to 0$ corresponds to $S^6 \to 0$ and $\dot{V} \to 0$ corresponds to $S^2 \to 0$. To get a smooth solution we also want to impose boundary conditions so that the other components of the metric are finite when one of the spheres shrinks.

We start by looking at the coefficients of the metric when S^2 shrinks ($\dot{V} \to 0$ and $\rho \sim 1$). We need $V''\Delta^{1/3} \to 0$ for the coefficient of $d\rho^2 + dz^2$ to be finite, and $V''\Delta^{-1} \to 0$ for S^6 to be finite. The solution is then $V'' \to 0$ and $\Delta \sim 1$. By using the Laplace's equation we also get $\ddot{V} \to 0$.

Now we do a similar study in the region where S^6 shrinks ($\rho \to 0$). S^2 is finite if $\Delta \sim V''\dot{V}^{5/3}\rho^{4/3}$, and the 2-dimensional space is finite if $\Delta \sim \frac{\dot{V}^3}{V''^3}$. Taking the ratio to

[8] The Lorentzian solution was also written in terms of a 4d electrostatic potential in [33]. We thank Nikolay Bobev,Pieter Bomans and Fridrik Freyr Gautason for pointing this to us.

eliminate Δ we find that $\dot{V} \to 0$, which means, looking at the expression for Δ, that we also have $\Delta \to 0$.

In summary we find that the metric is smooth if in those regions we satisfy

$$\begin{aligned} S^2 \text{ region}: &\quad V'' \to 0, \quad \Delta \sim 1, \quad \dot{V} \to 0, \\ S^6 \text{ region}: &\quad \dot{V} \to 0, \quad \Delta \to 0, \quad \rho \to 0. \end{aligned} \tag{3.11}$$

The S^6 region is just the z axis. For the S^2 region we need more information about the implicit curves defined by $\dot{V} = 0$. We can compute the slope of those curves as

$$m = \frac{\partial_z \dot{V}}{\partial_\rho \dot{V}} = \frac{\dot{V}'}{\ddot{V}}. \tag{3.12}$$

The denominator vanishes due to the the Laplace equation together with $\dot{V} \to 0$ and $V'' \to 0$. However the numerator is finite due to $\Delta \sim 1$. Therefore the slope in the (ρ, z) plane is zero and those curves are at constant z. This allows us to use an electrostatic analogy where $\partial_\rho V = 0$ is interpreted as the vanishing of the tangential component of the electric field on the boundary of a conductor. To respect the axial symmetry of the system, those conductors need to be 3 dimensional balls that are infinitely thin in the z direction. They are the higher dimensional analog of the 2d conducting disks from [14] in a 3d axially symmetric system. In the following we will also refer to the balls as "disks".

Since the spheres go to zero at different points in the (ρ, z) plane, we can construct cycles by connecting such points with a curve that passes through a region where the spheres are finite. In particular we can draw a segment between two disks over which we tensor the S^2, and that defines a 3-cycle Σ_3[9]. Similarly we draw a curve around a disk with endpoints on the z axis, and, tensoring the S^6, that defines a 7-cycle Σ_7. This construction is depicted on Figure 1.

This is not enough to guarantee that the metric and dilaton are positive everywhere. We also need boundary conditions at infinity that are set by including a background potential that grows at infinity. The simplest possibility is to choose a harmonic polynomial. As we show in the next section, the polynomial that is relevant for the polarized IKKT matrix model is

$$V_{bg}(\rho, z) = -\eta z \mu^3 + \frac{\mu^5}{2^7}(z\rho^2 - z^3)\,, \tag{3.13}$$

where the term proportional to the dimensionless parameter η has no effect on the metric but can affect the different matter fields. We will later see that η is related to a constant term that can be added to the action of the matrix model. The different positivity conditions are satisfied in the physical region $\rho \geq 0$ and $z \geq 0$, as we show in appendix B. Since the S^6 shrinks when $\rho = 0$, the geometry terminates smoothly there and there is no physical boundary. Similarly we would like the geometry to stop at $z = 0$ smoothly, which can be done by introducing a grounded plane (or infinite disk) at $z = 0$, making the S^2 shrink there.

[9] This is similar to a sphere, which is a 2-cycle, that can be defined by tensoring an S^1 over a segment, with the S^1 shrinking at the endpoints, that are then the north and south poles of the sphere.

This concludes the presentation of the supergravity solution. It is determined by choosing a configuration of conducting disks, of respective charges Q_s, centered on the z axis at positions z_s. Note that the radii of the disks are not free parameters for the following reason: We want the metric to be smooth, and therefore the electric field should not diverge at the tip. This requires the charge density to vanish at the tip, which is a constraint that is possible to satisfy in a background potential. Physically, assuming $Q_s \geq 0$, the electric field produced by the charge of the disk, pushing outward, and the electric field from the background potential, pushing inward, need to cancel at the tip of the disks. This allows to solve for the radii R_s in terms of the charges Q_s and positions z_s, which are then the only free parameters of the solution.

Once the configuration of disks is chosen, one can solve the four-dimensional Laplace equation to get the function $V(\rho, z)$, which in turns gives the full supergravity solution.

3.2 Asymptotic region

Here we study the solution in the asymptotic region $r \to \infty$, where $r^2 \equiv z^2 + \rho^2$. We use a multipole approximation to solve the electrostatic problem in that limit. Using the method of images we find that the leading contribution from the disks is a dipole. Together with the background potential this gives

$$V(\rho, z) = -\eta z \mu^3 + \frac{\mu^5}{2^7}(z\rho^2 - z^3) + \frac{1}{2\pi^2}\frac{zP}{(z^2+\rho^2)^2} + \frac{1}{4\pi^2}\sum_{l=1}^{\infty} q_{2l+1}\frac{U_{2l+1}(\cos\theta)}{(z^2+\rho^2)^{l+3/2}}, \tag{3.14}$$

where $P = 2\sum_{s=1}^{q} z_s Q_s$ is the dipole moment for a configuration of q conducting disks of charges Q_s and distances z_s from the grounded plane. The q_l for $l \geq 3$ are the higher multipole moments, U_l are the Chebyshev polynomials of the second kind and $\cos\theta = \frac{z}{\sqrt{z^2+\rho^2}}$. We only have odd terms in l because the electrostatic system is antisymmetric under $z \to -z$. Asymptotically, the solution reads

$$\begin{aligned} ds^2 &= dz^2 + d\rho^2 + z^2 d\Omega_2^2 + \rho^2 d\Omega_6^2 + \mathcal{O}(r^{-4}), \\ e^{\phi} &= \frac{2^{14}P}{\pi^2\mu^7 r^8} + \mathcal{O}(r^{-10}), \\ \chi &= i\frac{\pi^2\mu^7 r^8}{2^{14}P} + \mathcal{O}(r^6) \\ dC_2 &= -i\mu\, z^2 dz \wedge d\Omega_2 + \mathcal{O}(r^{-6}), \qquad H_3 = \mathcal{O}(r^{-6}) \end{aligned} \tag{3.15}$$

Remarkably, the space is asymptotically flat. This is the solution [10, 11] dual to the pure IKKT matrix model with the addition of dC_2 flux. This flux is in fact constant because we can use (x^1, x^2, x^3) as the Cartesian coordinates associated to the S^2 with radius z. Then

$$dC_2 = -i\mu\, dx^1 \wedge dx^2 \wedge dx^3 + \mathcal{O}(r^{-6}). \tag{3.16}$$

Let us make a few comments on the asymptotic form of the solution (3.15). First, setting the dipole moment P to zero makes the axion diverge. However, this does not pose a problem to our analysis since, as we will discuss in section 3.4, the dipole moment is identified with the size of the matrices which is always nonzero. Second, note that asymptotically we have $i\chi = -e^{-\phi}$. This relation is still obeyed by the first subleading terms in the asymptotic expansion, proportional to q_3, q_5 and q_7. We can measure the first deviation as

$$e^{-\phi} + i\chi = \frac{3z^2 + \rho^2}{64}\mu^2 - \eta + \mathcal{O}(r^{-2}), \tag{3.17}$$

where we recognize the bosonic mass term of the polarized IKKT model (2.3).

S-duality. Our solutions are in a frame where we only have RR flux and no NSNS flux in the asymptotic region. We can go to another frame by using $SL(2,\mathbb{C})$ transformations. In particular the S-duality transformation with $a = d = 0$ and $c = b = i$ will map $H_3 \to idC_2$, and $dC_2 \to iH_3$, preserving the reality conditions for Euclidean IIB supergravity. In this new frame, the NSNS flux is purely real, and the RR flux is purely imaginary. Setting $\eta = \frac{1}{2}$, we get the solution of the form

$$e^{\phi} = 1 - \frac{\mu^2}{32}(3z^2+\rho^2) + \mathcal{O}(r^{-2}), \qquad \chi = -ie^{-\phi} + \mathcal{O}(r^{-8}), \qquad H_3 = -\mu dx^1 \wedge dx^2 \wedge dx^3 + \mathcal{O}(r^{-6}). \tag{3.18}$$

In this frame, we can set all the multipoles to zero without encountering divergences. Doing so kills the subleading terms $\mathcal{O}(r^{-\#})$ in (3.18) and makes the solution identical to the recently found "cavity" solution [2]. On the other hand, in our original frame (3.15), setting multipoles to zero would make the axion diverge[10] everywhere. We will comment more on the comparison between the two frames in section 3.7. Note that taking the asymptotic limit does not commute with S-duality. One should first take the $SL(2,\mathbb{C})$ transformation in (3.9) and then expand the solution asymptotically.

3.3 12D uplift

In the analogous case of the BMN matrix model, the asymptotic region is a 11 dimensional pp-wave, which is not manifest here. However Euclidean IIB supergravity can also be formulated in 12D [34] using

$$ds_{12}^2 = ds_{10}^2 + M_{ij}dy^i dy^j, \qquad M \equiv e^{\phi}\begin{pmatrix} \chi^2 + e^{-2\phi} & -\chi \\ -\chi & 1 \end{pmatrix}, \tag{3.19}$$

where ds_{10}^2 is the 10 dimensional metric of the IIB solution and (y^1, y^2) are coordinates on a non-dynamical 2-torus. Using our asymptotic solution (3.15) we get[11]

$$ds_{12}^2 = ds_{10}^2 - 2idy_1dy_2 + \left(\frac{\mu^2}{32}(3z^2+\rho^2) - 2\eta\right)dy_1^2 + \frac{2^{14}P}{\pi^2\mu^7 r^8}dy_2^2 + \mathcal{O}(r^{-2}). \tag{3.20}$$

The dy_2^2 term that we displayed is the leading contribution for this metric component. Identifying a time component $t = -iy_1$ all the components of the metric are real. Let us

[10]The axion diverges also in the cavity solution (3.18) but only at the boundary of the ellipsoid $3z^2 + \rho^2 \leq \frac{32}{\mu^2}$.

[11]The coefficient of dy_1^2 comes from the first non zero term in $\chi^2 + e^{-2\phi}$, measured by (3.17). Due to high degree of cancellations we need to keep many terms in the asymptotic expansion. The final result is obtained by expanding e^{ϕ} up to terms of order r^{-16} and χ up to terms of order 1.

now discuss the matter fields. A 10d solution with vanishing 5-form can also be obtained from a 12d solution with only a 4-form F_4 such that

$$(F_4)_{y_1\mu\nu\rho} = (dC_2)_{\mu\nu\rho}, \qquad (F_4)_{y_2\mu\nu\rho} = (H_3)_{\mu\nu\rho}. \tag{3.21}$$

We then get

$$F_4 = -\mu z^2 i dy_1 \wedge dz \wedge d\Omega_2. \tag{3.22}$$

Hence, writing $y \equiv y_2$, we find that the asymptotic form of our solution is the 12 dimensional pp-wave background

$$\begin{aligned} ds_{12}^2 &= 2dtdy + \sum_{i=1}^{3} dx^i dx^i + \sum_{p=4}^{10} dx^p dx^p - 2dt^2\mu^2\left(\frac{3}{4^3}x^i x^i + \frac{1}{4^3}x^p x^p - \eta\right), \\ F_4 &= \mu dt \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{3.23}$$

where $z^2 = \sum_{i=1}^3 x^i x^i$, and $\rho^2 = \sum_{p=4}^{10} x^p x^p$. We checked that this indeed satisfies the 12D Einstein equations[12].

We can also write the 12D uplift in general and, using (3.9) and (3.19), we get

$$\begin{aligned} ds_{12}^2 &= \left(\frac{1}{3\rho^2\Delta\dot{V}(-V'')}\right)^{1/2}\left[-\mu^3(3\dot{V}+\rho V'')dy^2 - 2\tilde{\Delta}dtdy - \frac{1}{\mu^3}\frac{\tilde{\Delta} - 3\rho^2\dot{V}(-V'')\Delta}{3\dot{V}+\rho V''}dt^2\right], \\ &+ \frac{8}{\mu^{\frac{5}{2}}}\left(\frac{1}{3^3}\frac{\Delta\dot{V}}{(-V'')}\right)^{1/4}\left[\frac{(-V'')}{\dot{V}}(d\rho^2+dz^2) + 3\rho d\Omega_6^2 + \frac{\rho(-V'')\dot{V}}{\Delta}d\Omega_2^2\right] \\ F_4 &= -idC_2 \wedge dt + dB_2 \wedge dy, \\ B_2 &= -\frac{8}{3\mu}\left(z - \frac{\rho\dot{V}\dot{V}'}{\Delta}\right) \wedge d\Omega_2, \qquad C_2 = i\frac{8}{3\mu^4}\left(V - \rho\frac{\dot{V}}{\Delta}\left(V'\dot{V}' + 3\dot{V}(-V'')\right)\right) \wedge d\Omega_2, \\ \tilde{\Delta} &= 3\dot{V}(V' + \rho\dot{V}') + \rho V'V'', \qquad \Delta = 3\dot{V}V'' + \rho(\dot{V}'^2 + V''^2). \end{aligned} \tag{3.26}$$

3.4 Quantization of fluxes

We found geometries parametrized by continuous parameters, the charges of the disks Q_s and their positions z_s. On the other hand, as we saw in section 2.2, the polarised IKKT model has a discrete set of classical vacua. The way to connect the two descriptions is to impose the Dirac charge quantization condition on the fluxes present in the geometry [15]. We will see that in practice this will quantize the charges of the disks, as well as

[12] The normalization for p forms is

$$S = \frac{1}{16\pi G}\int d^d x\sqrt{g}\left(R - \frac{1}{2(p+1)!}F^{\mu_1\cdots\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}}\right), \tag{3.24}$$

giving the stress energy tensor

$$T_{\mu\nu} \equiv -\frac{16\pi G}{\sqrt{g}}\frac{\partial S_m}{\partial g^{\mu\nu}} = \frac{1}{12}\left(F_{\mu abc}F_\nu{}^{abc} - \frac{1}{8}g_{\mu\nu}F_{abcd}F^{abcd}\right). \tag{3.25}$$

In these conventions the field equations are $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$.

their respective distances, their respective difference of potentials and the leading dipole moment. At the end the classical supergravity approximation is valid when those quantum numbers are large. Hence those quantization conditions effectively become invisible, as we will see explicitly below. Nevertheless we learn how those charges are mapped to the representations of $SU(2)$ characterizing the classical vacua (2.9) corresponding to saddle points of the matrix integral.

To quantize the fluxes we study the equations of motion and Bianchi identities to identify conserved charges, as shown at the beginning of this section. We have closed p-forms with $p = 3, 7, 9$. As we explained, the geometries have 3-cycles and 7-cycles. We can then compute the fluxes of closed 3-forms on Σ_3 and of closed 7-forms on Σ_7. The flux of the closed 9-form can be computed on a large S^9 in the flat space asymptotic region. We start with the latter that computes the instanton charge. We recall

$$\tilde{F}_9 = e^{2\phi} * d\chi + e^{\phi} B_2 \wedge *F_3 \tag{3.27}$$

Much like other fundamental Dp-branes, the fundamental instanton has unit charge. Thus, the integral of $\tilde{F}_9$ counts the number of instantons. Evaluating the integral on S^9 at infinity, we find[13]

$$N_{(-1)} = \frac{1}{(2\pi)^8 \alpha'^4 g_s} \int_{S^9} \tilde{F}_9 = \frac{2^7}{3\pi^5 \mu^7 \alpha'^4 g_s} P. \tag{3.28}$$

The dipole moment is therefore quantized. Note that this value of $N_{(-1)}$ can also be checked by matching the coefficient of the dilaton in the D-instanton solution [8].

We now compute the other fluxes. The closed 7-forms are

$$\tilde{H}_7 = * \left(e^{-\phi} H_3 - e^{\phi} \chi F_3 \right), \qquad \tilde{F}_7 = * e^{\phi} F_3, \tag{3.29}$$

and the closed 3-forms are H_3 and dC_2. Integrating those fluxes counts the number of $D1$ and $D5$ branes, as well as the number of $F1$ fundamental strings and of $NS5$ fivebranes, as we see from the quantization conditions

$$\begin{aligned} \frac{1}{(2\pi)^2 \alpha'} \int_{\Sigma_3} H_3 &= N_{NS5}, \qquad & \frac{1}{(2\pi)^6 \alpha'^3 g_s} \int_{\Sigma_7} \tilde{F}_7 &= N_{D_1}, \\ \frac{1}{(2\pi)^2 \alpha' g_s} \int_{\Sigma_3} dC_2 &= N_{D5}, \qquad & \frac{1}{(2\pi)^6 \alpha'^3} \int_{\Sigma_7} \tilde{H}_7 &= N_{F_1}, \end{aligned} \tag{3.30}$$

where the labels of the integers N_{label} refer to the type of brane carrying the charges. The integral on Σ_3 can be done close to the axis at $\rho = 0$. There we expand

$$\rho \to 0: \qquad V(\rho, z) = f(z) - \frac{f''(z)}{6} \rho^2 + \frac{1}{120} f^{(4)}(z) \rho^4 + ..., \tag{3.31}$$

where the terms in the Taylor expansion are set by using Laplace's equation. If we integrate H_3 we get

$$4\pi^2 \alpha' N_{NS5} = \int_{\Sigma_3} i H_3 = \frac{8}{3\mu} \int_{z_{s-1}}^{z_s} d\Omega_2 dz = \Omega_2 \frac{8}{3\mu} d_s, \tag{3.32}$$

[13]The integral on S^9 of radius R gives $\frac{1}{(2\pi)^8 \alpha'^4 g_s} \int_{S^9} \tilde{F}_9 = \frac{2^7}{3\pi^5 \mu^7 \alpha'^4 g_s} P + \mathcal{O}(R^{-6})$. Sending $R \to \infty$ gives (3.28).

where $d_s \equiv z_s - z_{s-1}$ is the distance between the disks, that is therefore quantized. To do the integrals on the 7-cycle Σ_7 we can pick a contour close to the disk and expand

$$z \to z_s : \qquad V(\rho, z) = V_s + g(\rho)(z - z_s) - \frac{2\dot{g}(\rho) + \rho\ddot{g}(\rho)}{6\rho}(z - z_s)^3 + \ldots \tag{3.33}$$

Integrating $\tilde{F}_7$ we get

$$(2\pi)^6 \alpha'^3 g_s N_{D1} = \int_{\Sigma_7} i\tilde{F}_7 = \frac{2560}{\mu^6} \int_0^{\rho_s} d\Omega_6 d\rho\, \rho^3 \partial_\rho \left(\partial_z V(z^+) - \partial_z V(z^-)\right) = \frac{2560}{\mu^6} \Omega_6 \frac{Q_s}{\Omega_2}, \tag{3.34}$$

and therefore the charges of the disks are quantized. Here by z^+ we mean $z + \epsilon$ with $\epsilon > 0$ and similarly $z^- \equiv z - \epsilon$. To summarize, we get the following quantization conditions [14]

$$N_{(-1)} = \frac{2^7}{3\pi^5 \mu^7 \alpha'^4 g_s} P\,, \qquad N_{NS5,s} = \frac{8}{3\pi\mu\alpha'} d_s\,, \qquad N_{D1,s} = \frac{2^5}{\pi^4 \mu^6 \alpha'^3 g_s} Q_s\,. \tag{3.35}$$

Now using $P = 2\sum_s Q_s \sum_{t\leq s} d_t$, we get that all numerical factors cancel and

$$N_{(-1)} = \sum_s N_{D1,s} \sum_{t\leq s} N_{NS5,t}, \tag{3.36}$$

which expresses the partitions of $N_{(-1)}$ in the integers $\sum_{t\leq s} N_{NS5,t}$. We can compare this with expectations from the polarized IKKT matrix model. The vacua are fuzzy spheres with 3 matrices in the adjoint of $SU(2)$. Such a matrix can generically be written as

$$J^i = \bigoplus_{s=1}^{q} \mathbb{1}_{n_s} \otimes J^i_{(N_s)}\,, \qquad i = 1, 2, 3\,, \qquad N = \sum_{s=1}^{q} n_s N_s\,, \tag{3.37}$$

where $J^i_{(N_s)}$ is the $N_s = 2j_s + 1$ dimensional (spin-j_s) irrep, and n_s are the multiplicities of those irreps. There are $P(N)$ such configurations. It is then natural to identify

$$N \leftrightarrow N_{(-1)}\,, \qquad n_s \leftrightarrow N_{D1,s}\,, \qquad N_s \leftrightarrow \sum_{t\leq s} N_{NS5,t}\,. \tag{3.38}$$

Therefore the gravity solution dual to this vacuum corresponds to a configuration with q disks at positions $z_s = \sum_{t\leq s} d_t$ with respective charge Q_s, see Figure 2.

Let us now discuss the two quantization conditions that we omitted. The $\tilde{H}_7$ integral yields

$$(2\pi)^6 \alpha'^3 N_{F1} = \int_{\Sigma_7} \tilde{H}_7 = \frac{2^9}{\mu^9} \Omega_6 \int d\rho\, \rho^3 \left(3\rho(\partial_z\partial_\rho V)^2 - \partial_\rho(\partial_z V)^2\right)\Big|_{z=z_s-\epsilon}^{z=z_s+\epsilon} = 0, \tag{3.39}$$

and vanishes because the integrand takes identical values above and below the disk. On the other hand the integral of dC_2 gives

$$4\pi^2 \alpha' g_s N_{D5} = \int_{\Sigma_3} dC_2 = \frac{8}{3\mu^4} \int_{z_{s-1}}^{z_s} d\Omega_2 dz \partial_z V(z, 0) = \Omega_2 \frac{8}{3\mu^4}(V_s - V_{s-1}). \tag{3.40}$$

[14] The "s" in g_s is not a label for the disks, but the standard name for the string coupling constant.

where V_s gives the (constant) potential of the disk s. This shows that the difference of potentials between neighboring disks must be quantized in units proportional to $g_s N_{D5}$. This raises an apparent puzzle since the electrostatic problem is already fully determined and it is unclear that we could impose this additional constraint. A resolution to this puzzle comes from the fact that the supergravity approximation is valid when $g_s \to 0$ and the quantum number N_{D5} is large. Thus we can choose the product $g_s N_{D5}$ so that $V_s - V_{s-1}$ is arbitrarily close to any real number, which is determined by solving the electrostatic problem. Note that the same comments apply also to quantizations imposed by N_{-1} and N_{D1}, both of which come with a factor of g_s: strictly speaking, quantizations of these fluxes do not constrain the parameter space of the solutions in the supergravity limit.

3.5 Validity of the supergravity approximation

To trust the supergravity approximation we need small curvature of the string frame metric $ds^2_{(s)} \equiv e^{\phi/2} ds^2$ to avoid stringy higher curvature corrections, and small string coupling $g_s e^{\phi}$ to avoid string loop corrections. To probe the curvature we use the Ricci scalar R of the string metric in string units, i.e. we study $\alpha' R$.

Let us first study those quantities in the asymptotic region where we use the expansion (3.14). We get

$$r \to \infty : \qquad R\alpha' \sim \frac{r^2}{\sqrt{g_s N}\alpha'} \sim \frac{r^2}{\alpha'^2 \mu^2 \sqrt{\lambda}}, \qquad g_s e^{\phi} \sim \frac{g_s^2 N \alpha'^4}{r^8} \sim \frac{\mu^8 \alpha'^8 \lambda^2}{N r^8}, \tag{3.41}$$

where $\lambda \equiv N g_{YM}^2/\mu^4$ is the (dimensionless) 't Hooft coupling. The dilaton decays at infinity but is finite close to the axis and on the disks. We will estimate the value of the dilaton in those regions. We also estimate the curvature since the disks should be inside the trustworthy region. Since we trust the approximation (3.41) from infinity down to $r \sim \max(\rho_s, z_s)$, we also need to check that the conditions coming from the approximation near the disks are not weaker than the condition coming from the asymptotics at that point.

At finite distance, we study the case of a single disk (and its image) and write the potential

$$V(\rho, z) = V_{bg}(\rho, z) + \frac{1}{4\pi^2 \rho} \int_0^{\rho_s} dr f_s(r) \left(\frac{\rho - r}{(\rho - r)^2 + (z - z_s)^2} - \frac{\rho - r}{(\rho - r)^2 + (z + z_s)^2} \right), \tag{3.42}$$

where $f_s'(r) = -2\pi r \sigma_s(r)$ with $\sigma_s(r)$ the charge density of the s disk. This expression for $V(\rho, z)$ can be obtained from (C.2) by integrating by parts. Then, we plug this expression in the equations for the dilaton and the curvature and expand around the regions of interest. The main input to specify here is the function $f_s(r)$. Since only the integral of $f_s(r)$ is needed, we assume an even distribution $f_s(r) \sim Q_s/\rho_s$ and this suffices for our estimations.[15]

Now we analyze the curvature and the dilaton for different values of the dimensionless parameter ρ_s/z_s. In fact, to connect with our matrix model discussion in [13], it will be

[15] More refined analysis is given in [13], where a similar assumption on matrix model side is also discussed.

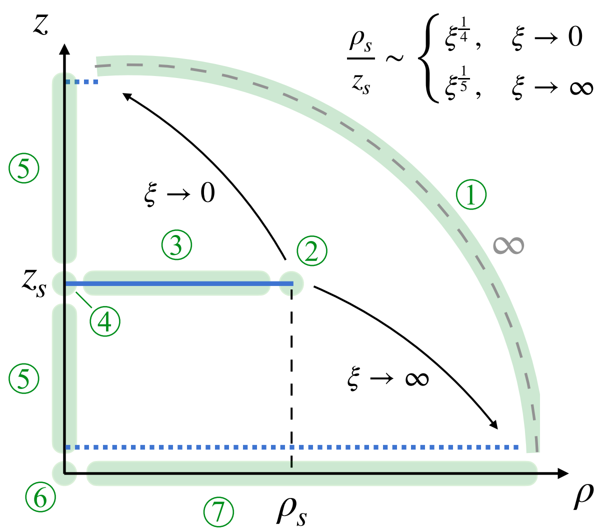

Figure 3: Summary of our analysis of the supergravity regime. We study the scaling of the string coupling $g_s e^{\phi}$ and the curvature probed by the Ricci scalar in regions ① to ⑦. The scaling behaviors are given in table 1. We find that the strongest conditions come from infinity and the tip of the disk (region ① and ②).

more convenient to use the parameter

$$\xi \equiv \frac{3^5}{2^{20}\pi^2} \frac{Q_s}{z_s^5 \mu^5} = \frac{n_s}{\Omega^4 N_s^5}, \tag{3.43}$$

which is related to ρ_s/z_s. As we show in [13], we can write $\frac{\rho_s}{z_s} = w(\xi)$, where $w(\xi) \sim \xi^{1/5}$ when $\rho_s/z_s \gg 1$ and $w(\xi) \sim \xi^{1/4}$ when $\rho_s/z_s \ll 1$. We can now study the dilaton and curvature for different regimes of ξ, in regions close to the axis, the disk, and the conducting plane. In all these regions, we get

$$R\alpha' \sim \frac{\alpha' \mu}{\rho_s} \tilde{q}_R(\xi) \sim \frac{q_R(\xi)}{\lambda^{1/6}}, \qquad g_s e^{\phi} \sim \frac{g_s \tilde{q}_\phi(\xi)}{\mu^2 \rho_s^2} \sim \frac{\lambda^{2/3}}{N} q_\phi(\xi), \tag{3.44}$$

where $\tilde{q}_R = \xi^{1/6} q_R / w(\xi)$ (and similarly for $\tilde{q}_\phi$), and the functions q_R and q_ϕ have different asymptotics in different regions, see Table 1. We get the stronger conditions from the tip of the disks where

$$(\rho, z) \to (\rho_s, z_s): \qquad q_R(\xi) \approx \begin{cases} \xi^{-2/15} & \text{if } \ \xi \gg 1 \\ \xi^{-1/12} & \text{if } \ \xi \ll 1 \end{cases}, \qquad q_\phi(\xi) \approx \begin{cases} \xi^{-4/15} & \text{if } \ \xi \gg 1 \\ \xi^{1/12} & \text{if } \ \xi \ll 1 \end{cases}. \tag{3.45}$$

In fact when $\xi \gg 1$ we get the same conditions in the other regions as well. However when $\xi \ll 1$ we get larger powers in the other regions. For example on the axis we have $q_\phi \sim \xi^{4/3}$ which is much less than $\xi^{1/12}$ when ξ is small. We also find that those conditions are stronger than imposing small string coupling in (3.41) at $r \sim \max(\rho_s, z_s)$. However the

Regions in Fig.3	$q_R(\xi)$		$q_\phi(\xi)$	
	$\xi \gg 1$	$\xi \ll 1$	$\xi \gg 1$	$\xi \ll 1$
①	$\xi^{1/15}$	$\xi^{-1/3}$	$\xi^{-4/15}$	$\xi^{4/3}$
②,④	$\xi^{-2/15}$	$\xi^{-1/12}$	$\xi^{-4/15}$	$\xi^{1/12}$
③	$\xi^{-2/15}$	$\xi^{-1/12}$	$\xi^{-4/15}$	$\xi^{1/6}$
⑤,⑥,⑦	$\xi^{-2/15}$	$\xi^{2/3}$	$\xi^{-4/15}$	$\xi^{4/3}$

Table 1: Asymptotic behavior of the scalar curvature and the dilation in different regions in figure 3. For region ① we have used (3.41) with $r^2 = \rho_s^2 + z_s^2$. The relevant entries that lead to the final result (3.46) are from regions ① and ②.

asymptotic curvature gives a stronger condition, where we find $q_R \sim \xi^{1/15}$ when $\xi \gg 1$ and $q_R \sim \xi^{-1/3}$ when $\xi \ll 1$. For the former, it turns our that the condition is automatic when $\xi \gg 1$. However for the latter we get an extra constraint.

In terms of the parameters $(x^{(s)}, n_s, N_s)$, where $x^{(s)} = \frac{1}{2\pi\mu\alpha'}\rho_s$ [13], the supergravity approximation is valid for

$$\begin{cases} n_s \gg N_s (x^{(s)})^2 & \text{if } \; x^{(s)} \gg N_s \\ n_s \gg (x^{(s)})^3 \gg N_s x^{(s)} & \text{if } \; N_s \gg x^{(s)} \text{ or } N_s \sim x^{(s)} \end{cases} . \tag{3.46}$$

In particular, the irreducible vacuum with $n_s = 1$ is not well described by supergravity.

It is interesting to compare these conditions with the ones obtained in [13] for the derivation of the electrostatic problem from the exact localization computation. There, we needed

$$n_s \gg (x^{(s)})^2 \log x^{(s)} \gg 1 \,, \tag{3.47}$$

which is a weaker condition than the second line of (3.46) but can be slightly stronger than the first line [16]. Perhaps, a more careful 1-loop computation in supergravity could lead to the extra $\log x^{(s)}$ needed to strengthen the conditions for validity of the classical approximation in the first line of (3.46).

3.6 Free energy

With the geometries at hand, we want to compute the on-shell action associated to each solution. This is expected to match the free energy of the matrix model in the large N limit.

As shown in [35] (see also [36] for a similar computation), in type IIB supergravity with vanishing 5-form flux the on-shell action is the boundary term

$$S_E = \frac{1}{2\kappa^2} \int d\left(-\frac{1}{4} M_{ij} \mathcal{C}_2^i \wedge * \mathcal{F}_3^j \right) , \tag{3.48}$$

[16] In fact it is stronger when $\log x^{(s)} > N_s$

where

$$M_{ij} = e^{\phi} \begin{pmatrix} \chi^2 + e^{-2\phi} & -\chi \\ -\chi & 1 \end{pmatrix}, \qquad \mathcal{C}_2 = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \qquad \mathcal{F}_3 = d\mathcal{C}_2. \tag{3.49}$$

This integral gets contributions from the different cycles in the geometry. This can be understood as follows. Pick the contour going around the disks and along the z axis, as shown on Figure 4. The interior is a patch covering the entire spacetime minus some part of measure zero. However all the cycles in that patch are contractible. We can therefore integrate over that patch and use Stokes theorem to evaluate the on-shell action. We find that the integrand vanishes on the z axis. Therefore we are left with the contributions from going around the disks, which give

$$2\kappa^2 S_E = -\frac{5120}{3\mu^{10}} \Omega_6 \Omega_2 \sum_{\text{disks } s} \int d\rho \left(\rho^3 V_s \dot{V}' - \frac{1}{2} z_s \rho^3 \partial_\rho (V'^2) \right) \tag{3.50}$$

where the domain of integration is above and below the disk. The second term vanishes since V'^2 is continuous across the disk. Evaluating and using $2\kappa^2 \equiv 16\pi G_N^{(10)} = (2\pi)^7 \alpha'^4 g_s^2$ we get the answer

$$S_E = \frac{2^7}{3\pi^4 g_s^2 \alpha'^4 \mu^{10}} \sum_{s=1}^{q} Q_s V_s. \tag{3.51}$$

We then find that the free energy of the supergravity solution corresponds to the electrostatic energy of the disk configuration[17].

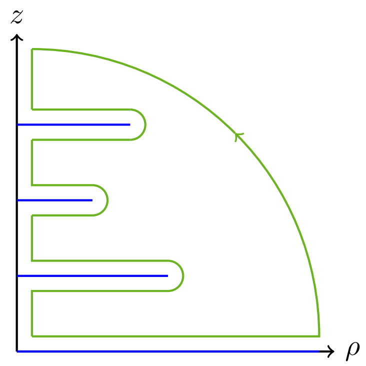

Figure 4: Integration contour for the on-shell action. The interior is a patch covering the entire spacetime minus regions of measure zero. All cycles in that patch are contractible.

The explicit expression of the gravitational on-shell action requires computing the potentials V_s by solving an electrostatic problem. Let us focus on the particular limit when $d_s \gg 1$. In this case we can consider only one disk at a time in the background potential. Its charge density can be computed explicitly as we show in appendix C. Using the constraint that it vanishes at the tip we find

$$V_s = -\mu^5 \frac{z_s^3}{2^7} + \mathcal{O}(\sqrt{Q_s z_s}), \tag{3.52}$$

[17] This is not entirely exact because of the presence of the background electric field.

which is just the contribution from the background potential evaluated at the center of the disk. Inserting all the factors, together with $g_{YM}^2 = \frac{g_s}{(2\pi)^3\alpha'^2}$ the gravitational on-shell action gives

$$S_E = -\frac{\mu^4}{g_{YM}^2}\frac{9}{2^{17}}\sum_s n_s N_s^3 + \mathcal{O}(\sqrt{n_s N_s})\,. \tag{3.53}$$

It is enlightening to compare this result to expectations from the matrix model. Because of the identification (3.38), the limit $d_s \gg 1$ corresponds to the scenario in the matrix model side where the saddle points (3.37) consist of $SU(2)$ irreducible representations with large matrix dimension and small degeneracy. This is the limit where the fuzzy spheres become almost classical and the quantum fluctuations are expected to be negligible.[18] Taking only the contributions from the saddles, we find that the value of the action is

$$S_\Omega^{\text{saddle}} = -\Omega^4\frac{9}{2^{15}}\sum_{s=1}^{q} n_s N_s(N_s^2-1), \tag{3.54}$$

where n_s is the quantized charge Q_s and N_s is the quantized height z_s. We see that the functional forms of (3.53) and (3.54) are identical (at large N_s).

There is still a leftover piece in the evaluation of the on-shell action, namely the contribution from the asymptotic boundary of spacetime. This gives both a divergent term and a finite term. The finite term comes from the octopole of the electrostatic configuration q_3. To see this, we expand

$$V(\rho,z) = -\eta\mu^3 z + \frac{\mu^5}{2^7}(z\rho^2 - z^3) + \frac{P}{2\pi^2}\frac{z}{(z^2+\rho^2)^2} + \frac{q_3}{\pi^2}\frac{z(z^2-\rho^2)}{(z^2+\rho^2)^4} + \dots \tag{3.55}$$

and evaluate the contribution from the arc at $\rho^2 + z^2 = R^2$:

$$2\kappa^2 S_E^{(\infty)} = \Omega_2\Omega_6\frac{2^4}{\mu^5}\left(\frac{PR^2}{\pi} + \frac{5q_3}{3\pi}\right) + \mathcal{O}(R^{-2}). \tag{3.56}$$

Since we are considering this spacetime boundary we should also add the Gibbons-Hawking term, for which we get

$$\kappa^2 S^{(GH)} = -\int_{\partial M}\sqrt{h}K = \Omega_2\Omega_6\left(\frac{45\pi R^8}{256} - \frac{36PR^2}{\pi\mu^5} - 4\frac{q_3}{\pi\mu^5}\right) + \mathcal{O}(R^{-2}). \tag{3.57}$$

As can be seen from these expressions, the Gibbons-Hawking term alone cannot remove the diverging terms. This indicates that one needs to supplement it with suitable counter terms. This procedure will in principle change the finite term. We leave for the future the determination of the counter terms and the precise match with the matrix model side.

For now, note that in the limit $d_s \gg 1$, we can compute the octopole q_3 as shown in appendix C, where we find

$$q_3 = 2\sum_s Q_s d_s^3 + \mathcal{O}\left(\sqrt{N_s n_s^3}\right). \tag{3.58}$$

[18] In the companion paper [13] we show that, using supersymmetry localization, the *full* free energy of the matrix integral has exactly the same expression as (3.54) in the same limit, despite the computation being highly non-trivial.

The finite pieces of the contributions (3.56) and (3.57) are then

$$S^{(\infty)}_{\text{finite}} = \frac{\mu^4}{g^2_{YM}}\frac{3}{2^{13}}\sum_s n_s N_s^3 + \mathcal{O}\left(\sqrt{N_s n_s^3}\right), \qquad S^{(GH)}_{\text{finite}} = -\frac{\mu^4}{g^2_{YM}}\frac{9}{5\cdot 2^{14}}\sum_s n_s N_s^3 + \mathcal{O}\left(\sqrt{N_s n_s^3}\right), \tag{3.59}$$

and we see that they again take the expected functional form in terms of n_i and N_i. Adding those contributions together with the bulk term (3.53) does not reproduce the numerical prefactor of (3.54) under the identification of parameters (3.77) that we discuss in the next section, suggesting that the counterterms modify the finite contributions.

Scaling of the free energy. The on-shell action depends on 3 parameters: the positions of the disks z_s, their charges Q_s and the parameter μ. To consider dimensionless quantities we write $S = S(Q_s, \mu z_s, \mu\sqrt{\alpha'})$. We will use two symmetries. Firstly, notice that

$$S(Q_s, \mu z_s, \mu\sqrt{\alpha'}) = \lambda_1^{-8} S(Q_s, \mu z_s, \lambda_1^{-1}\mu\sqrt{\alpha'}) \,. \tag{3.60}$$

This follows from the transformation

$$(\rho, z) \to (\lambda_1\rho, \lambda_1 z), \qquad \mu \to \lambda_1^{-1}\,\mu, \tag{3.61}$$

which implies $V \to \lambda_1^{-2}\,V$ and Q_s remains fixed. Secondly, notice that if we rescale $\mu \to \lambda_2^{1/5}\mu$ and $Q_s \to \lambda_2 Q_s$ keeping z_s fixed, then $V \to \lambda_2 V$. This leads to

$$S(\lambda_2 Q_s, \lambda_2^{1/5}\mu z_s, \lambda_2^{1/5}\mu\sqrt{\alpha'}) = S(Q_s, \mu z_s, \mu\sqrt{\alpha'}) \,. \tag{3.62}$$

Note that both transformations leave invariant the quantity[19]

$$\xi \equiv \frac{3^5}{2^{20}\pi^2}\frac{Q_s}{z_s^5\mu^5} = \frac{n_s}{\Omega^4 N_s^5}. \tag{3.63}$$

Using the symmetries above we have the scaling relation

$$S(Q_s, \mu z_s, \mu\sqrt{\alpha'}) = \lambda_1^{-8} S(Q_s, \mu z_s, \lambda_1^{-1}\mu\sqrt{\alpha'}) = \lambda_1^{-8} S\left(\lambda_2 Q_s, \lambda_2^{1/5}\mu z_s, \lambda_2^{1/5}\lambda_1^{-1}\mu\sqrt{\alpha'}\right). \tag{3.64}$$

Using $\lambda_1 = \sqrt{\alpha'}\mu Q_s^{-1/5}$ and $\lambda_2 = Q_s^{-1}$ we get

$$S(Q_s, \mu z_s, \mu\sqrt{\alpha'}) = \frac{Q_s^{8/5}}{\alpha'^4\mu^8} S\left(1, z_s\mu Q_s^{-1/5}, 1\right) = \frac{Q_s^{8/5}}{\alpha'^4\mu^8} S\left(1, \xi^{-1/5}, 1\right). \tag{3.65}$$

Using the relations (3.35) we get

$$\frac{Q_s^{8/5}}{\alpha'^4\mu^8} \propto \xi^{4/15} N^{4/3}\Omega^{8/3}, \tag{3.66}$$

such that

$$S(Q_s, \mu z_s, \mu\sqrt{\alpha'}) = \frac{N^2}{\lambda^{2/3}} H(\xi), \tag{3.67}$$

where $\lambda \equiv N/\Omega^4 = N g^2_{YM}/\mu^4$ is the dimensionless 't Hooft coupling. This is the predicted scaling that has been discussed from scaling similarity [37–39].

[19] In principle we should have a different ξ for each disk, i.e. $\xi = \xi_s$, but the scaling argument follows through.

3.7 Polarized probe $D1$ brane

Here we study the system from a probe brane perspective. Starting from a stack of N Dp branes, its non-abelian DBI action contains couplings to higher form gauge fields with respect to which a single Dp brane is neutral. In the particular case of $p = 0$, a 4-form RR flux polarizes the $D0$ branes into a $D2$ brane, which is known as the Myers effect [40]. See [41] for an analysis of this effect in the BMN model. Here we study a similar system where N D-instantons polarize into a $D1$ brane under the effect of an external constant RR flux. A similar analysis with an external NSNS flux was conducted in [2].

The DBI action for a single $D1$ brane reads

$$\begin{aligned} S_{D1} = &-\frac{1}{2\pi\alpha' g_s}\int d^2\sigma e^{-\phi}\sqrt{-\det\left(g^{(s)}_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)} \\ &+ \frac{1}{2\pi\alpha' g_s}\int\left(\chi\left(B_2 + 2\pi\alpha' F_2\right) - C_2\right), \end{aligned} \tag{3.68}$$

where g, B_2, χ and C_2 are respectively the pullbacks of the string frame metric $g^{(s)} \equiv e^{\phi/2} g$, NSNS potential, axion and RR potential, while F_2 is the worldvolume $U(1)$ field strength. For a bound state of N D-instantons it is given by $F_2 = \frac{N}{2} d\Omega_2$ [2, 40]. This is the Lorentzian action evaluated with purely imaginary RR fields and a metric with Euclidean signature. We get a real Euclidean action by sending $S \to iS$.

Following our physical picture, we would like to start with D-instantons in flat space, and add a flux, which would define the background fields. However, flat space and constant flux is not a supergravity solution unless we also add a non-trivial axi-dilaton. We can study what this solution can be by using our backreacted geometries and take a *probe* limit where the backreaction goes to zero. Therefore we expand

$$V(\rho, z) = -\eta z\mu^3 + \frac{\mu^5}{2^7}(z\rho^2 - z^3) + \epsilon f(\rho, z), \tag{3.69}$$

and we take the limit $\epsilon \to 0$. We find

$$\begin{aligned} ds^2 &= dz^2 + d\rho^2 + z^2 d\Omega_2^2 + \rho^2 d\Omega_6^2 + \mathcal{O}(\epsilon), \\ C_2 &= -i\frac{\mu}{3} z^3 \wedge d\Omega_2 + \mathcal{O}(\epsilon), \qquad B_2 = \mathcal{O}(\epsilon) \\ i\bar{\tau} &\equiv e^{-\phi} + i\chi = \frac{\mu^2}{64}(3z^2 + \rho^2) - \eta + \mathcal{O}(\epsilon), \end{aligned} \tag{3.70}$$

where we only displayed a particular combination of the axion and dilaton that we will shortly justify. Individually we find that the e^{ϕ} goes to zero and the axion diverges, leaving this configuration finite, whereas the other combination $i\tau \equiv e^{-\phi} - i\chi \sim \frac{1}{\epsilon}$ diverges. We also note again that $3z^2 + \rho^2$ is exactly the bosonic part of the mass-deformed IKKT matrix model.

Let us now go back to the DBI action. We will evaluate it in the constant RR flux background (3.70) first without specifying the axi-dilaton. The embedding of the $D1$ is chosen to be localized in the S^2, at $\rho = 0$ and spherically symmetric with radius r. Since everything is spherically symmetric we can integrate over the angles and we get

$$S_{D1} = \frac{2}{g_s\alpha'} e^{-\phi}\sqrt{\pi^2\alpha'^2 N^2 + e^{\phi} r^4} + \frac{2\pi N}{g_s}(i\chi) + \frac{2r^3\mu}{3\alpha' g_s}. \tag{3.71}$$

At large N we get

$$S_{D1} = \frac{2\pi N}{g_s}\left(e^{-\phi} + i\chi\right) + \frac{r^4}{g_s \pi \alpha'^2 N} - \frac{2r^3\mu}{3\alpha' g_s} + \mathcal{O}(N^{-2}), \tag{3.72}$$

where we see the relevant axi-dilaton appearing. We can now plug in our background (3.70) and we get

$$S_{D1} = \frac{r^4}{N g_s \pi \alpha'^2} - \frac{2r^3\mu}{3\alpha' g_s} + \frac{3N\pi r^2\mu^2}{32 g_s} - \frac{2\pi N\eta}{g_s} + \mathcal{O}(N^{-2}). \tag{3.73}$$

The allowed configurations of the $D1$ are found by looking at extrema of the action $S'_{D1}(r) = 0$. We get the solutions

$$r = 0, \qquad r = \frac{1}{8}N\pi\mu\alpha', \qquad r = \frac{3}{8}N\pi\mu\alpha', \tag{3.74}$$

which are the same roots as the ones for the fuzzy sphere saddles of the polarized IKKT integral (2.11). The global minimum occurs for the third solution, and the on-shell action reads

$$S_{D1} = -\frac{2\pi N\eta}{g_s} - \frac{9}{2^{15}}N^3\frac{\mu^4}{g_{YM}^2} = \frac{2\pi i N\bar{\tau}(r=0)}{g_s} - \frac{9}{2^{15}}N^3\frac{\mu^4}{g_{YM}^2}. \tag{3.75}$$

In the second equality, we used the relation $i\bar{\tau}(r=0) = i\chi + e^{-\phi}|_{r=0} = -\eta$, obtained from (3.70).

The first term in (3.75) can be identified with the on-shell action of N D-instantons at $r = 0$ (cf. [42]). As discussed in [2, Sec.4.3], this can be reproduced on the matrix model side by adding a constant term $\frac{2\pi\bar{\tau}(r=0)}{g_s}\mathrm{Tr}(\mathbb{1})$ to the action. On the other hand, the second term is a contribution from polarizing D-instantons into $D1$ and is dominant when $N \gg 1$. This second term should be compared with the (irrep.) fuzzy sphere on-shell action on the matrix model

$$S_\Omega^{\text{fuzzy sphere}} = -\frac{9}{2^{15}}N^3\Omega^4. \tag{3.76}$$

The comparison leads to a matching of the parameters on both sides,

$$\Omega = \frac{\mu}{\sqrt{g_{YM}}}. \tag{3.77}$$

To further test the correspondence between the matrix model side and the gravity side, we consider the size of the (irrep.) fuzzy spheres on the matrix side using

$$r^2_{\text{fuzzy sphere}} = (2\pi\alpha')^2\frac{g_{YM}}{N}\,\mathrm{Tr}\,X^2 = (2\pi\alpha')^2 g_{YM}\frac{3^2}{8^2}\Omega^2\frac{N^2-1}{4} \simeq \left(\frac{3}{8}\pi\alpha'\sqrt{g_{YM}}\Omega N\right)^2. \tag{3.78}$$

This exactly matches the radius of the probe $D1$ brane

$$r_{D1} = \frac{3}{8}N\pi\mu\alpha', \tag{3.79}$$

if we again use the matching conditions (3.77). We can also look for an analog in the backreacted geometries. There we can compute the height of the disk using the asymptotic flat metric. That gives

$$r_{\text{disk}} = \int_0^d \sqrt{g_{zz}}dz = d = \frac{3}{8}N\pi\mu\alpha', \tag{3.80}$$

again matching the previous results. We thus find a perfect match among the three computations,

$$r_{\text{fuzzy sphere}} = r_{D1} = r_{\text{disk}} \,. \tag{3.81}$$

Comparison with the probe brane analysis in [2]. Our identification of parameters (3.77) differs from the one obtained in [2] by a factor of 2. However the physical systems we analysed are different, hence it is not a direct contradiction. As we explained in 3.2, our background (3.70) is related to their background (3.18) by S-duality. However we are both studying a $D1$ brane in our respective backgrounds, whose S-dual is an $F1$ string. Hence the two systems are not equivalent and correspond to different physics.

Let us argue for our choice. We will give two reasons. First, studying the polarization of Dp branes due to an external RR flux (rather than NSNS) seemed more natural in view of previous works [40, 41]. Second, and more importantly, in our backreacted geometries we could measure the number of $F1$ strings (3.39) and found zero. Therefore, even if it is perfectly fine to study the dynamics of a probe $F1$ on this background, we cannot argue that the backreacted geometry corresponds to the backreaction of this $F1$ string. If we S-dualize (3.9) so that the asymptotic background (3.18) is identical to the one considered in [2], we find that the number of $D1$ branes is zero. Thus it is more natural to consider a probe $F1$ string as opposed to a probe $D1$ brane as was done in [2].

4 Discussion

Summary and main results. In this paper we studied Euclidean type IIB geometries dual to the mass-deformed IKKT matrix model, which are closely related to Lin-Maldacena geometries. Indeed they are smooth and without horizons, and can be formulated in terms of a four-dimensional electrostatic potential $V(\rho, z)$. Asymptotically the metric describes flat Euclidean space (in the Einstein frame) with an axi-dilaton corresponding to the D-instanton solution [10, 11]. In addition there is a constant 3-form Ramond-Ramond flux. This is consistent with the picture that those geometries correspond to the backreaction of N D-instantons polarizing into a $D1$ brane [2]. Those geometries are in one-to-one correspondence with the fuzzy sphere vacua of the polarized IKKT matrix model, as we showed explicitly by quantizing the different fluxes and relating their quantum numbers to the dimensions and degeneracies of the $SU(2)$ irreducible representations.

In the companion paper [13] we compute the partition function and some correlation functions of the polarized IKKT model using supersymmetric localization. Our preliminary results suggest that, in the large N limit, the localization equations are identical to the electrostatic problem that determines the supergravity solution, similar to the case for the BMN model [43]. In particular we can write the partition function in terms of the electrostatic variables, where it also receives two contributions, one from the electrostatic energy $Q_s V_s$, and one from the octopole q_3, matching respectively the bulk term and the boundary term of the supergravity on-shell action. The detailed comparison with the supergravity answer is in progress and we hope to report it soon.

Future directions.

- In this paper, we constructed backreacted geometries but did not evaluate their on-shell actions. As discussed in section 3.6, doing so requires determining the boundary counterterms. This can be determined by requiring supersymmetry to be restored, see e.g. [44].

- As mentioned above, the supersymmetric localization allows us to compute certain correlation functions as well. As in the usual holography, operators in the matrix model should be dual to some asymptotic behaviour of the supergravity fields. It would be enlightening to derive the dictionary carefully and perform quantitative comparison of correlation functions on both sides.

- In this paper, we focused on solutions without D7-brane charges. It would be interesting to generalize our construction to include D7-branes (i.e. nontrivial SL(2,$\mathbb{C}$) monodromies). Similar solutions dual to five-dimensional superconformal field theories were constructed in [45] and it should be possible to perform suitable analytic continuation to obtain geometries with $SO(7) \times SO(3)$ isometry.

- A closely related question would be to identify the matrix-model counterparts of geometries with D7-branes. One possibility may be to look for mass deformation of the $D(-1)/D7$ matrix models discussed recently in [46–48] that preserve exceptional F_4 superalgebra.

- In the limit $\Omega \to 0$, one could derive an effective action for the diagonal modes by explicitly integrating out the off-diagonal modes. This has been done for the IKKT model in [49]. For the BMN model, the effective Hamiltonian resulting from similar computation reduces to decoupled supersymmetric harmonic oscillators and its spectrum reproduces that of 11D (linearized) supergravity on the pp-wave background to the leading order [7]. However, it turns out to be significantly more difficult when one tries to compute subleading corrections to the BMN effective Hamiltonian, which is a key for understanding how backreacted geometry forms through interactions of graviton gas. With the simpler matrix model one can perhaps make more progress.

- In the case of the BFSS and BMN models, studying backreacted geometries [8, 15] was important for understanding the precise relationship to M-theory and sharpening the original BFSS conjecture [4, 5]. Therefore it would be interesting to revisit the IKKT conjecture in the light of our analysis, understand the relation to holography discussed in this paper, and possibly make the conjecture more precise. It would also be interesting to make contact with recent discussions on the (nonrelativistic) decoupling limits of string theory e.g. [50–53]. More generally it would be interesting if we can use matrix integrals to get a microscopic description of F-theory [54] and extract non-perturbative observables.

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A Analytic continuation of the IIB Lorentzian solution

In this appendix, we explain how to find the geometry (3.9) through analytic continuation. Our starting point is the Lorentzian solution constructed in [17], which takes the form of a warped product $AdS_6 \times S_2 \times \Sigma_2$. The 5-form field strength vanishes, while the axion χ, dilaton ϕ, NSNS flux $H_3 = dB_2$ and RR flux $F_3 = dC_2 - \chi B_2$ are written in terms of a complex scalar B and complex 3-form $\mathcal{F}_3$ as

$$B = \frac{1+i\tau}{1-i\tau}, \qquad \tau = \chi + ie^{-\phi}, \qquad \mathcal{F}_3 = H_3 + idC_2. \tag{A.1}$$

The full supergravity solution depends on only two holomorphic functions $\mathcal{A}_\pm(w)$ and reads

$$\begin{aligned}
&ds^2 = f_6^2(w,\bar{w})ds^2_{AdS_6} + f_2^2(w,\bar{w})d\Omega_2^2 + H^2(w,\bar{w})dwd\bar{w},\\
&\mathcal{F}_3 = d\mathcal{C}\wedge \mathrm{vol}_{S^2}, \qquad \mathcal{C} = \frac{4i}{9}\left(\frac{\bar\partial\bar{\mathcal{A}}_-\partial\mathcal{G}}{\kappa^2} - 2R\frac{\bar\partial\bar{\mathcal{A}}_-\partial\mathcal{G} + \partial\mathcal{A}_+\bar\partial\mathcal{G}}{(R+1)\kappa^2} - \bar{\mathcal{A}}_- - 2\mathcal{A}_+\right),\\
&B = \frac{\partial\mathcal{A}_+\bar\partial\mathcal{G} - R\bar\partial\bar{\mathcal{A}}_-\partial\mathcal{G}}{R\bar\partial\bar{\mathcal{A}}_+\partial\mathcal{G} - \partial\mathcal{A}_-\bar\partial\mathcal{G}},
\end{aligned} \tag{A.2}$$

where the metric functions are

$$f_6^2 = \sqrt{6\mathcal{G}f_R}, \qquad f_2^2 = \frac{1}{9}\sqrt{\frac{6\mathcal{G}}{f_R^3}}, \qquad H^2 = 4\kappa^2\sqrt{\frac{f_R}{6\mathcal{G}}}, \qquad f_R \equiv \frac{1+R}{1-R}, \tag{A.3}$$

where

$$\mathcal{G} \equiv |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}}, \qquad \kappa^2 \equiv -|\partial\mathcal{A}_+|^2 + |\partial\mathcal{A}_-|^2, \qquad f_R^2 = 1 + \frac{2}{3}\frac{|\partial\mathcal{G}|^2}{(-\partial\bar\partial\mathcal{G})\mathcal{G}}, \tag{A.4}$$

with the function $\mathcal{B}$ defined up to a constant through

$$\partial\mathcal{B} \equiv \mathcal{A}_+\partial\mathcal{A}_- - \mathcal{A}_-\partial\mathcal{A}_+. \tag{A.5}$$

This solution can be analytic continued to a Euclidean solution by the replacement[20]

$$ds^2_{AdS_6} \to -d\Omega_6^2. \tag{A.7}$$

We get a Euclidean solution but the metric has imaginary components. To make them real we define new coordinates $(w', \bar{w}')$ and new functions $\mathcal{G}'$, κ'^2 and f'_R as

$$w = e^{-i\alpha}w', \qquad \bar{w} = e^{-i\alpha}\bar{w}', \qquad \mathcal{G} = e^{i\alpha_\mathcal{G}}\mathcal{G}', \qquad \kappa = e^{i\alpha_\kappa}\kappa', \qquad f_R = e^{i\alpha_R}f'_R, \tag{A.8}$$

[20] This means that the metric

$$ds^2 = -f_6^2(w,\bar{w})d\Omega_6^2 + f_2^2(w,\bar{w})d\Omega_2^2 + H^2(w,\bar{w})dwd\bar{w} \tag{A.6}$$

solves the equations of motion. Indeed one can check that the corresponding changes of signs in the Ricci tensor when AdS_6 is replaced by S^6 are compensated by changes of signs of the time component when we go to Euclidean signature. This ensures that the Einstein equation is still satisfied. Then one can check that all the other equations do not depend on the sign of f_6^2 and are then invariant. See [32] for an explicit form of all equations of motion.

and we solve for $\alpha, \alpha_G, \alpha_\kappa$ and α_R so that each component of the metric is real when G', κ' and f'_R are real. We find

$$\alpha = \frac{\pi}{2}, \qquad \alpha_G = \frac{3\pi}{2}, \qquad \alpha_\kappa = \frac{3\pi}{4}, \qquad \alpha_R = \frac{\pi}{2}. \tag{A.9}$$

This solution manages to change the sign of f_6^2 by going to the second sheet of the square root in (A.3). We therefore have the solution

$$ds^2 = R_6^2 d\Omega_6^2 + R_2^2 d\Omega_2^2 + H^2 dw' d\bar{w}', \tag{A.10}$$

where

$$R_6^2 = \sqrt{6\mathcal{G}' f'_R}, \qquad R_2^2 = \frac{1}{9}\sqrt{\frac{6\mathcal{G}'}{f_R'^3}}, \qquad H^2 = 4\kappa'^2 \sqrt{\frac{f'_R}{6\mathcal{G}'}}. \tag{A.11}$$

We write the w'-holomorphic functions $g_\pm(w') \equiv \mathcal{A}_\pm(w(w'))$ and $h(w') \equiv \mathcal{B}(w(w'))$. Note that $\mathcal{A}_\pm(w(w'))$ and $\bar{\mathcal{A}}_\pm(\bar{w}(\bar{w}'))$ are no longer complex conjugate of each other (and similarly for $\mathcal{B}$ and $\bar{\mathcal{B}}$). Instead, we have $\bar{\mathcal{A}}_\pm(\bar{w}(\bar{w}')) = (\mathcal{A}_\pm(-w(w')))^*$[21]. However, for $g_\pm(w')$ and $h(w')$ we want to keep the bar to mean complex conjugation. To this end we identify

$$g_\pm(w') \equiv \mathcal{A}_\pm(w(w'))\,, \qquad \bar{g}_\pm(w') \equiv \bar{\mathcal{A}}_\pm(-\bar{w}(\bar{w}'))\,, \qquad \bar{g}_\pm(\bar{w}') = (g_\pm(w'))^*\,, \tag{A.13}$$

and similarly for $h(w')$. Now in terms of these holomorphic functions, the metric functions are

$$\begin{aligned} \kappa'^2 &= -i\left[-\partial_{w'} g_+(w')\partial_{\bar{w}'}\bar{g}_+(-\bar{w}') + \partial_{w'} g_-(w')\bar{\partial}_{\bar{w}'}\bar{g}_-(-\bar{w}')\right] \\ \mathcal{G}' &= i\left[g_+(w')\bar{g}_+(-\bar{w}') - g_-(w')\bar{g}_-(-\bar{w}') + h(w') + \bar{h}(-\bar{w}')\right] \\ \partial_{w'} h(w') &\equiv g_+(w')\partial_{w'} g_-(w') - g_-(w')\partial_{w'} g_+(w'), \end{aligned} \tag{A.14}$$

and the defining equation for f'_R is

$$f_R'^2 + 1 = \frac{2}{3}\frac{\partial\mathcal{G}'\bar{\partial}\mathcal{G}'}{\kappa'^2\mathcal{G}'}\,. \tag{A.15}$$

Also note that $\partial\bar{\partial}\mathcal{G}' = \kappa'^2$, so the reality condition for the metric boils down to the construction of the real function $\mathcal{G}'$ given in (A.14).

The simplest solution is to have

$$e^{-i\beta} g_-(-w') = g_+(w') \equiv g(w')\,, \tag{A.16}$$

with β a real constant, such that

$$(g_+(w')\bar{g}_+(-\bar{w}'))^* = \bar{g}_+(\bar{w}')g_+(-w') \stackrel{!}{=} g_-(w')\bar{g}_-(-\bar{w}')\,. \tag{A.17}$$

[21] We can see this by looking at complex conjugation of the Laurent series

$$\bar{\mathcal{A}}(\bar{w}) = \sum_n a_n^*(\bar{w} - \bar{w}_0)^n = \sum_n a_n^*(-i\bar{w}' - \bar{w}_0)^n = \left(\sum_n a_n(-(-iw') - w_0)^n\right)^* = (\mathcal{A}(-w))^* \tag{A.12}$$

From (A.16) we get automatically that $h(w') + \bar{h}(-\bar{w}')$ is imaginary (up to an inessential constant): With this identification $\partial_{w'} h$ is an even function in w'

$$\begin{aligned}(\partial_{w'} h)(w') =& e^{i\beta}\left[g_+(w')\partial_{w'}(g_+(-w')) - g_+(-w')\partial_{w'}(g_+(w'))\right]\\ =& e^{i\beta}\left[-g_+(w')(\partial_{w'} g_+)(-w') - g_+(-w')(\partial_{w'} g_+)(w')\right] = (\partial_{w'} h)(-w')\,,\end{aligned} \tag{A.18}$$

and thus $h(w') = -h(-w') + \text{const.}$, and similarly for $\bar{h}(\bar{w}')$. Therefore,

$$h(w') + \bar{h}(-\bar{w}') = h(w') - \bar{h}(\bar{w}') + \text{const.} = 2i\text{Im}h(w') + \text{const.} \tag{A.19}$$

Then, renaming $h \to he^{i\beta}$, we get

$$\begin{aligned}\kappa'^2 &= -2\,\text{Im}\,\partial_{w'} g(w')\partial_{\bar{w}'}\bar{g}(-\bar{w}')\\ \mathcal{G}' &= -2\,\text{Im}\,g(w')\bar{g}(-\bar{w}') - 2\,\text{Im}\,e^{i\beta}h(w') + \text{const.}\\ \partial_{w'} h(w') &= g(w')\partial_{w'} g(-w') - g(-w')\partial_{w'} g(w')\,.\end{aligned} \tag{A.20}$$

We can furthermore redefine $g \to e^{-i\beta/2} g$ and the β dependence above disappears.

To make the expressions dependent only on $\{w', \bar{w}'\}$ but not $\{-w', -\bar{w}'\}$, we write $g(w') = g_S(w') + g_A(w')$ where g_S and g_A are respectively symmetric and antisymmetric[22]. Then we get

$$\begin{aligned}&\kappa'^2 = 4\,\text{Im}\,\partial_{w'} g_S\,\partial_{\bar{w}'}\bar{g}_A\\ &\mathcal{G}' = 4\,\text{Im}\left[g_S\bar{g}_A - \frac{1}{2}h\right] + \text{const.}, \qquad \partial_{w'} h = 2g_A\partial_{w'} g_S - 2g_S\partial_{w'} g_A.\end{aligned} \tag{A.21}$$

Since nothing depends explicitly on the coordinates w', we can make a holomorphic transformation and treat g_A as the new coordinate. We note that all dependence on $\partial_{w'} g_A$ disappears in the chain rule. Let us see this in more detail. First we have

$$\begin{aligned}\kappa'^2 &= 4\,\text{Im}\,\partial_{w'} g_S\,\partial_{\bar{w}'}\bar{g}_A\frac{\partial_{w'} g_A}{\partial_{w'} g_A} = 4|\partial_{w'} g_A|^2\,\text{Im}\,\frac{\partial_{w'} g_S}{\partial_{w'} g_A},\\ \mathcal{G}' &= 4\,\text{Im}(g_S\bar{g}_A - \frac{h}{2}) + \text{const.}, \qquad \frac{\partial_{w'} h}{\partial_{w'} g_A} = 2g_A\frac{\partial_{w'} g_S}{\partial_{w'} g_A} - 2g_S.\end{aligned} \tag{A.22}$$

Next using the chain rule we have

$$\frac{\partial_{w'} g_S}{\partial_{w'} g_A} = \frac{\partial g_S}{\partial g_A} \equiv \partial_A g_S, \qquad \frac{\partial_{w'} h}{\partial_{w'} g_A} = \frac{\partial h}{\partial g_A} \equiv \partial_A h. \tag{A.23}$$

There is still the dependence on $|\partial_{w'} g_A|$ in κ'^2. However, the only appearances of κ'^2 are in f'_R as $\partial\mathcal{G}'\bar{\partial}\mathcal{G}'/\kappa'^2$ and the factors cancel using the chain rule, and in the metric as

$$ds^2_\Sigma \sim \kappa'^2 dw' d\bar{w}' \sim \partial_{w'} g_A dw' \partial_{\bar{w}'}\bar{g}_A d\bar{w}' = dg_A d\bar{g}_A \tag{A.24}$$

and they also cancel. Therefore, the solution does not depend on $\partial_{w'} g_A$. We will now simplify the notation as

$$g_A \equiv z + i\rho, \qquad g_S(g_A) \to f(z + i\rho), \qquad \partial_A \to \partial, \qquad |\partial g_A|^2\kappa'^2 \to \kappa'^2 \tag{A.25}$$

[22] This is always possible since $g(w') = \frac{1}{2}(g(w') + g(-w')) + \frac{1}{2}(g(w') - g(-w'))$.

and similarly for the barred quantities. Now we can write h more explicitly in terms of the primitive of f. We have

$$h = \int dg_A \, \partial_A h = 2 \int dg_A \, \left(g_A \partial_A f - f \right) = 2 g_A f - 4 \int f. \tag{A.26}$$

Plugging this result in $\mathcal{G}'$ we get

$$\kappa'^2 = 4 \, \mathrm{Im} \, \partial f = \partial \bar{\partial} \mathcal{G}', \qquad \mathcal{G}' = -8 \left(\rho \, \mathrm{Re} \, f - \mathrm{Im} \int f \right), \tag{A.27}$$

where the constant in $\mathcal{G}'$ is absorbed in the integral.

Let us pause and summarize our chain of redefinitions and record the relation between the original functions $\mathcal{A}_\pm$ and the new ones. We have[23]

$$\begin{aligned}
\mathcal{A}_+(w) &= g(w') = g_S(w') + g_A(w') = f(z,\rho) + z + i\rho, \\
\bar{\mathcal{A}}_+(\bar{w}) &= \bar{g}(-\bar{w}') = \bar{g}_S(\bar{w}') - \bar{g}_A(\bar{w}') = \bar{f}(z,\rho) - z + i\rho, \\
\mathcal{A}_-(w) &= g(-w') = g_S(w') - g_A(w') = f(z,\rho) - z - i\rho, \\
\bar{\mathcal{A}}_-(\bar{w}) &= \bar{g}(\bar{w}') = \bar{g}_S(\bar{w}') + \bar{g}_A(\bar{w}') = \bar{f}(z,\rho) + z - i\rho.
\end{aligned} \tag{A.28}$$

Combining this with (A.4) and (A.5) quickly leads to (A.27). Now the metric reads

$$ds^2 = \sqrt{6\mathcal{G}'} \left(f_R'^{1/2} d\Omega_6^2 + \frac{1}{9 f_R'^{3/2}} d\Omega_2^2 \right) + 4\kappa'^2 \sqrt{\frac{f_R'}{6\mathcal{G}'}} (dz^2 + d\rho^2). \tag{A.29}$$

Now the geometry can be compactly described in the two-dimensional (ρ, z) plane. We want to identify the boundaries in this plane where the spheres S^2 and S^6 shrink to zero.[24] For this notice that

$$R_6^3 R_2 \propto \mathcal{G}'^2 \tag{A.30}$$

and therefore we identify the boundary as the region where $\mathcal{G}' = 0$. Writing the holomorphic function $\int f = U + iW$, we have that W is a harmonic function and

$$\frac{1}{8} \mathcal{G}' = W - \rho \partial_\rho W. \tag{A.31}$$

We can now introduce V through $W = -\rho V$. The harmonic condition becomes

$$\nabla^2 W = 0 \implies \partial_z^2 V + \frac{2}{\rho} \partial_\rho V + \partial_\rho^2 V = 0, \tag{A.32}$$

and therefore V is a harmonic function of a four-dimensional axially symmetric system, where ρ is the radial variable and z the height of the cylinder. The boundary condition becomes

$$\frac{1}{8} \mathcal{G}' = \rho^2 \partial_\rho V = 0, \tag{A.33}$$

[23] Here we set $\beta = 0$ since we saw that it can be eliminated by further redefinitions.

[24] The boundaries in (ρ, z) plane do not correspond to actual boundaries of physical spacetime. For example, consider a solid cylinder described by coordinates $(r\cos\theta, r\sin\theta, z)$. The axis $r = 0$ is the boundary in (r, z) plane where the S^1 shrinks but it is not the boundary of the cylinder.

and has a form suitable for the electrostatic analogy, as we derive carefully in the main text. Expressing all the supergravity fields in terms of V, changing coordinates and redefining V according to

$$(\rho, z) \to \frac{2}{\mu}(\rho, z), \qquad V \to \frac{2}{\mu^4} V\,, \tag{A.34}$$

we get a solution similar to (3.9), although with the wrong reality conditions. Namely the RR flux is real and the NSNS flux is imaginary. Also, following the logic of appendix B, we find that the dilaton needs to be everywhere negative. Thus the final step leading to (3.9) is to use the $SL(2,\mathbb{C})$ transformation on the axi-dilaton τ in (A.1) and the 3-forms

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} H_3 \\ dC_2 \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_3 \\ dC_2 \end{pmatrix}, \tag{A.35}$$

with $a = -d = i$ and $b = c = 0$ to get the correct reality conditions and change the sign of the dilaton.

B Regularity of the geometries

In this appendix we study in detail the regularity conditions for the solutions (3.9). We want to show that the metric and the exponential of the dilaton are positive and smooth everywhere, provided that we add a suitable background potential.

The metric is positive definite if

$$\dot{V} > 0, \qquad V'' < 0, \qquad \Delta > 0, \tag{B.1}$$

and e^{ϕ} is positive if in addition

$$3\dot{V} + \rho V'' < 0\,, \tag{B.2}$$

is satisfied everywhere.

To show these conditions we use properties of harmonic functions. We have that V is harmonic in 4D

$$V'' + \ddot{V} + \frac{2}{\rho}\dot{V} = 0, \tag{B.3}$$

and asymptotes to

$$V(\rho, z) \underset{r\to\infty}{\simeq} -\eta z + \frac{\mu^5}{2^7}(z\rho^2 - z^3). \tag{B.4}$$

We start with $-V''$, which is also harmonic in 4D. We recall the expansion near the disks and near the z axis

$$\begin{aligned} \rho \to 0: &\qquad V(\rho, z) = f(z) - \frac{f''(z)}{6}\rho^2 + \frac{1}{120} f^{(4)}(z)\rho^4 + \ldots, \\ z \to z_s: &\qquad V(\rho, z) = V_s + g(\rho)(z - z_s) - \frac{2\dot{g}(\rho) + \rho\ddot{g}(\rho)}{6\rho}(z - z_s)^3 + \ldots \end{aligned} \tag{B.5}$$

We find

$$-V'' \underset{r\to\infty}{\simeq} \frac{3z}{64}\mu^5, \qquad -V'' \underset{\rho\to 0}{\simeq} -f''(z), \qquad -V'' \underset{z\to z_s}{\simeq} 0 \tag{B.6}$$

It is zero on the disks, regular on the z axis, and is positive at infinity. Since no local minima can exist between the disks and infinity, it needs to be positive everywhere.

We can study $\dot{V}$ similarly. Since $\rho > 0$ it is equivalent to study the positivity of $\dot{V}/\rho$, which is harmonic in 5D. We get

$$\frac{\dot{V}}{\rho} \underset{r\to\infty}{\simeq} \frac{z}{64}\mu^5, \qquad \frac{\dot{V}}{\rho} \underset{\rho\to 0}{\simeq} -\frac{7}{6}f''(z), \qquad \frac{\dot{V}}{\rho} \underset{z\to z_s}{\simeq} 0 \tag{B.7}$$

Again we find that the function is positive at infinity, vanishes on the disks and is regular on the z-axis. In fact on the z axis we have $\dot{V}/\rho \sim -V'' > 0$. Therefore we find that the function is positive everywhere.

To conclude with the positivity of the metric we want to check that $\Delta > 0$. Using Laplace's equation we can write

$$\Delta = (\rho\ddot{V} - \dot{V})(-V'') + \rho\dot{V}'^2. \tag{B.8}$$

Therefore it is sufficient to show $\rho\ddot{V} - \dot{V} = \rho^2\partial_\rho(\dot{V}/\rho) > 0$. Since the function $\dot{V}/\rho$ is harmonic and growing at infinity, its derivative is positive. Therefore $\Delta > 0$. We conclude that the metric is positive everywhere in the region $z \geq 0$ and $\rho \geq 0$.

For the dilaton, we can use the Laplace's equation to write

$$-\rho V'' - 3\dot{V} = \rho\ddot{V} - \dot{V}, \tag{B.9}$$

which is the same function that we studied in the previous paragraph. Therefore we conclude that $e^\phi > 0$.

Having determined that the metric and dilaton are positive everywhere, we still have to study in detail the smoothness of the solution. The dangerous regions are the $z-$axis and the disks, where again we can expand with (B.5) to look at the expressions of all supergravity fields. Near the z-axis we find, up to terms of order ρ or higher,

$$\begin{aligned}
ds^2 &\underset{\rho\to 0}{\simeq} \left(\frac{2^{12}}{45\mu^{10}(5f'''^2 - 3f''f^{(4)})^3}\right)^{1/4} \left[(5f'''^3 - 3f''f^{(4)})(dz^2 + d\rho^2) + 5f''^2\rho d\Omega_2^2\right], \\
e^\phi &\underset{\rho\to 0}{\simeq} \frac{\mu^3}{\sqrt{5}} \frac{f^{(4)}}{f''\left(5f'''^2 - 3f''f^{(4)}\right)^{1/2}}, \qquad \chi \underset{\rho\to 0}{\simeq} \frac{i}{\mu^3}\frac{f''\left(5f'''^2 - 3f''f^{(4)}\right)^{1/2}}{f^{(4)}} \\
B_2 &\underset{\rho\to 0}{\simeq} -\frac{8}{3\mu}\left(z + \frac{5f''f'''}{3f''f^{(4)} - 5f'''^3}\right) d\Omega_2, \qquad C_2 \underset{\rho\to 0}{\simeq} \frac{8i}{3\mu^4}\left(f + \frac{5(f'f''f''' - 3f''^3)}{3f''f^{(4)} - 5f'''^3}\right) d\Omega_2,
\end{aligned} \tag{B.10}$$

and we see that nothing diverges. Similarly near the disks we find at the leading order in $(z - z_s)$

$$\begin{aligned}
ds^2 &\underset{z\to z_s}{\simeq} \left(\frac{2^{12}\rho^2\dot{g}^3}{3^3\mu^{10}(2\dot{g} + \rho\ddot{g})}\right)^{1/4} \left[\frac{2\dot{g} + \rho\ddot{g}}{\rho f'}(dz^2 + d\rho^2) + 3\rho d\Omega_6^2\right], \\
e^\phi &\underset{z\to z_s}{\simeq} -\mu^3 \frac{\dot{g} - \rho\ddot{g}}{\sqrt{3}\rho\left(2\dot{g}^4 + \rho\dot{g}^3\ddot{g}\right)^{1/2}}, \qquad \chi \underset{z\to z_s}{\simeq} -\frac{i}{\mu^3}\left(g + \frac{3\rho\dot{g}^2}{\dot{g} - \rho\ddot{g}}\right) \\
B_2 &\underset{z\to z_s}{\simeq} -\frac{8}{3\mu}z_s d\Omega_2, \qquad C_2 \underset{z\to z_s}{\simeq} \frac{8iV_s}{3\mu^4}d\Omega_2,
\end{aligned} \tag{B.11}$$

and again everything stays finite when $z \to z_s$. Therefore we determined that the solution is regular everywhere, even near $\rho = 0$ and the disks.

C Electrostatics with a high disk

In this appendix we solve the electrostatic problem in the limit where we have only one conducting disk at $z_s = d \gg 1$[25]. Our goal is to compute the constant potential on that disk and the octopole of the asymptotic electrostatic potential.

The potential of a ball of radius a and charge density $\sigma(\rho)$ is

$$V(\rho, z) = \frac{1}{4\pi^2} \int d\Omega_2 \int_0^a u^2 du \frac{\sigma(u)}{(z-d)^2 + \rho^2 + u^2 - 2u\rho\cos\theta}. \tag{C.1}$$

Integrating over the angles we get

$$V(\rho, z) = \frac{1}{4\pi\rho} \int_0^a du\ u\sigma(u) \log\left(1 + \frac{4\rho u}{(\rho-u)^2 + (z-d)^2}\right). \tag{C.2}$$

In particular

$$V(\rho, d) = \frac{1}{2\pi\rho} \int_0^a du\ u\sigma(u) \log\left|\frac{\rho+u}{\rho-u}\right|. \tag{C.3}$$

So we want to solve for V_0 in

$$\frac{1}{2\pi\rho} \int_0^a du\ u\sigma(u) \log\left|\frac{\rho+u}{\rho-u}\right| = V_0 - V_{bg} - V_{\text{other disks}} \equiv \frac{1}{2\pi\rho} f(\rho), \qquad r < a, \tag{C.4}$$

where $V_{bg}(\rho) = \alpha(\rho^2 d - d^3)$ is the background potential at $z = d$. The solution to that integral equation is [56]

$$\rho\sigma(\rho) = \frac{1}{\pi^2} \text{p.v.} \int_{-a}^a du \sqrt{\frac{a^2-u^2}{a^2-\rho^2}} \frac{f'(u)}{\rho-u}. \tag{C.5}$$

In the limit $d \to \infty$ we can neglect the contributions from the image disk since it decays as d^{-2}. Requiring that $\sigma(a) = 0$ we get the radius

$$a = \sqrt{\frac{2(\alpha d^3 + V_0)}{3\alpha d}}. \tag{C.6}$$

We still have to determine V_0 which can be done by computing $Q = 4\pi \int_0^a \rho^2 \sigma(\rho)$. Using the result for a we get

$$Q = 2\pi^2 \frac{(\alpha d^3 + V_0)^2}{3d\alpha}, \tag{C.7}$$

which allows to solve for V_0 simply by inverting

$$V_0 = -\alpha d^3 + \frac{1}{\pi}\sqrt{\frac{3}{2} Q d\alpha}. \tag{C.8}$$

[25] See [55] for a solution in usual $D = 3$ electrostatics.

We also compute the first terms in the multipole expansion, taking into account the image disk. The distance between the two disks is $2d$. With the asymptotic expansion (3.14) we get the first terms

$$
\begin{aligned}
V(\rho, z) &= \frac{P}{2\pi^2} \frac{z}{(z^2+\rho^2)^2} + \frac{q_3}{\pi^2} \frac{z(z^2-\rho^2)}{(z^2+\rho^2)^4} + \dots \\
P = 2Qd, &\qquad q_3 = 2Qd^3 - 8\pi d \int_0^a du\; u^4 \sigma(u).
\end{aligned} \tag{C.9}
$$

Using our result for $\sigma(\rho)$ in the limit $d \gg 1$, we can then compute

$$
q_3 = 2Qd^3 + \sqrt{\frac{2dQ^3}{3\alpha}} + \dots \tag{C.10}
$$

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