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Causal boundary and singularities

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ABSTRACT

The relations between causality violation and singularities are investigated. We give a theorem that singularities necessarily occur when the boundary of causality violating set exists in a space-time under the physically suitable assumptions in the Hawking-Penrose singularity theorems except the global causality. The present theorem combined with the Hawking-Penrose theorem shows that mere existence of a trapped surface in a physical space-time implies the existence of singularities without referring to the causality condition.

1 Introduction

The space-time singularity has been discussed for a long time in general relativity. In 1970, Hawking and Penrose[1] showed that singularities, which mean causal geodesic incompleteness, could occur in a space-time under seemingly reasonable conditions in classical gravity. Their singularity theorem has an important implication that our universe has an initial singularity if we do not consider quantum effects. However, this theorem is physically unsatisfactory in the sense that the causality requirement everywhere in a space-time seems too restrictive. We can only know local events and there is no guarantee that the causality holds for the entire universe. For example, we cannot say by this theorem that there are singularities in Kerr type black holes, which have causality violating regions. Therefore, it is important to investigate the nature of causality violation.

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There are some works on a causality violation concerned with the occurrence of singularity. For a question whether one can manufacture closed timelike curves as a time-machine or not by the future technology, Tipler[2],[7] has shown that any attempt to evolve closed timelike curves from an initial regular cauchy data would cause singularities to form in a space-time in a classical theory. This work is also motivated by a question whether causality violation can prevent singularities or not. He answered to it by presenting his singularity theorem in which the global causality condition in the Hawking-Penrose theorem is replaced by the weaker one and adding the stronger energy condition. It has also been shown that causality violating set had incomplete null geodesics if the boundaries of causality violating set were compact (Kriele[3]). Kriele[4] also proved a generalization of the singularity theorem of Hawking-Penrose. Unfortunately his theorem is not very useful for predicting whether singularities exist or not, because it is too difficult to confirm that the conditions in the theorem are satisfied in a given space-time. In any case, these theorems are unsatisfactory because the conditions seem too restrictive.

Newman[5] found a black hole solution which had no singularities. This black hole solution was obtained by a suitable conformal transformation of the Gödel universe. His conclusion was that causality violating set could prevent singularities from occurring. However, this case is too special, because causality is violated in the whole space-time. On the other hand, there must be causality preserving regions in a physical space-time. Moreover, when we inspect exact solutions, for example Kerr black hole, Taub-NUT universe, which contain causality violating regions, it seems that the boundaries of causality preserving and violating regions just cause singularities.

In this paper we shall present a theorem that causality violating set necessarily makes singularities if only its boundary exists and energy conditions are satisfied in a given space-time. We also discuss the relation between our theorem and the Hawking-Penrose theorem.

In the next section, we briefly review Tipler's and Kriele's singularity theorems. In section 3, the definition and the lemmas for discussing causal structure and singularities are listed up. We present our singularity theorems for partially causality violating space-times in section 4. Conclusions are summarized in section 5.

2 Tipler and Kriele's theorems

We review Tipler and Kriele's theorems in this section. In addition, we discuss whether one can apply these theorems to various space-times in which causalities are violated.

Definition
A space-time (M,g) is said to be *asymptotically deterministic* if
(i) (M,g) contains a partial cauchy surface S such that
(ii) either $H(S) = H^+(S) \cup H^-(S)$ is empty, or if not, then

$$\lim_{s \to a}[\inf T_{ab}K^a K^b] > 0$$

on at least one of the null geodesic generators $\gamma(s)$ of $H(S)$, where a is the past limit of the affine parameter along γ if $\gamma \in H^+(S)$, and the future limit if $\gamma \in H^-(S)$. (K^a is the tangent vector to γ.)

Tipler's theorem(1977)
A space-time (M,g) cannot be null geodesically complete if
(1) $R_{ab}K^aK^b \geq 0$ for all null vectors K^a;
(2) there is a closed trapped surface in M;
(3) the space-time is asymptotically deterministic, and the Einstein equations hold;
(4) the partial Cauchy surface defined by (3) is non-compact.

The condition (3) of this theorem is too restrictive because of the energy condition (ii) in the definition of an asymptotically deterministic spacetime. The null geodesic γ has infinite conjugate points on $H^+(S)$ or $H^-(S)$ by this condition and therefore nobody knows whether this new energy condition is satisfied or not in a space-time under consideration.

Definition
•*focal point*
Let S be a locally spacelike surface (not necessary achronal surface) and let us consider a future directed null geodesic, $\beta(t)$, from S parameterized by t. If for any points $\beta(t)$ such that $t \geq t_1$, there exists an arbitrary close timelike curve from S to the point $\beta(t)$, then β is called a *focal point* to S.

•*Generalized future horismos of S*
Generalized future horismos of S, denoted by $e^+(S,M)$, is a closure of all future null geodesics β from S which have no focal points(the future end points of $e^+(S,M)$ correspond to the focal points).

•*cut locus*: $cl(S,M,+)$
The set of future end points of $e^+(S,M)$.

•*almost closed causal curve*
Choose an arbitrary Riemannian metric h of M. Let α be a curve and β be a reparametrization of α with $h(\beta',\beta') = 1$. Then α is called *almost closed*

if there exists an $X \in \beta'(t)$ such that for every neighbourhood U of X in the tangent bundle, TM, there exists a deformation γ of β in $\pi_{TM}(U)$ which yields a closed curve and satisfies $\gamma(t) \in \pi(U) \Rightarrow \gamma'(t) \in U$.

Kriele's theorem
Theorem 1(1990)
(M, g) is causal geodesically incomplete if:
(1) $R_{ab}K^aK^b \geq 0$ for every causal vector K and the generic condition is satisfied.
(2) (a) there exists a closed locally spacelike but not necessarily an achronal trapped surface S or (b) there exists a point r such that on every past (or every future) null geodesic from r the divergence θ of the null geodesics from r becomes negative or (c) there exists a compact achronal set S without edge.
(3) neither $cl(S, M, +)$ (respectively $cl(r, M, +)$) nor any $cl(D, M, -)$, where D is a compact topological submanifold (possibly with boundary) with $D \cap S \neq \emptyset$ (respectively $r \in D$), contains any almost closed causal curve that is a limit curve of a sequence of closed timelike curves.

Up to now, this theorem is the maximum generalization of Hawking-Penrose theorem in the sense that causality may be violated in the almost all regions except the cut locus.

However it is very difficult to verify whether the condition (3) is satisfied or not in a given space-time. In theorem 2 it is shown that there exist singularities when the causality violating region is compact even if there is no trapped surface .

Theorem 2(1989)
Let (M, g) be a space-time with chronology violating set V that satisfies:
(1) $R_{ab}K^aK^b \geq 0$ for every null vectors K and the generic condition is satisfied.
(2) V has a compact closure but $M - V \neq \emptyset$.
Then V is empty or V is generated by almost closed but incomplete null geodesics.

However, the condition (2) is not so reasonable. For example, in the Kerr space-time, this condition is not satisfied. Because the region of causality violation extends to infinity as the Kerr space-time has a timelike killing vector.

3 Preliminaries

We consider a space-time (M, g), where M is a four-dimensional connected differentiable manifold and g is a Lorentzian and suitably differentiable metric. In this section, we quote some definitions and useful lemmas from (HE)[1] for the discussion of causal structure and space-time singularities.

Definition (HE)
A point p is said to be a *limit point* of an infinite sequence of non-spacelike curves l_n if every neighbourfood of p intersects an infinite number of the l_n. A non-spacelike curve l is said to be a *limit curve* of the sequence l_n if there is a subsequence l'_n of l_n such that for every $p \in l$, l'_n converges to p.

Proposition 1 (HE 6.4.1)
The chronology violating set V of M is the disjoint union of the form $I^+(q) \cup I^-(q), q \in M$.

Lemma 1 (HE 6.2.1)
Let O be an open set and let l_n be an infinite sequence of non-spacelike curves in O which are future-inextendible in O. If $p \in O$ is a limit point of l_n, then through p there is a non-spacelike curve l which is future-inextendible in O and which is a limit curve of the l_n.

Proposition 2 (HE 4.5.10)
If p and q are joined by a non-spacelike curve $l(v)$ which is not a null geodesic they can also be joined by a timelike curve.

Proposition 3 (HE 4.4.5)
If $R_{ab}K^aK^b \geq 0$ everywhere and if at $p = \gamma(v_1)$, $K^cK^dK_{[a}R_{b]cd[e}K_{f]}$ is non-zero, there will be v_0 and v_2 such that $q = \gamma(v_0)$ and $r = \gamma(v_2)$ will be conjugate along $\gamma(v)$ provided $\gamma(v)$ can be extended to these values.

Proposition 4 (HE 4.5.12)
If there is a point r in (q, p) conjugate to q along $\gamma(t)$ then there will be a variation of $\gamma(t)$ which will give a timelike curve from q to p.

Proposition 5 (HE 6.4.6)
If M is null geodesically complete, every inextendible null geodesic curve has pair conjugate points, and chronology condition holds on M, then the strong causality condition holds on M.

Proposition 6 (HE 6.4.7)
If the strong causality condition holds on a compact set φ, there can be no

past-inextendible non-spacelike curve totally or partially past imprisoned in φ.

Prop.5 physically means that chronology condition is equivalent to the strong causality condition if energy conditions are satisfied.

4 The theorem

A chronology violating set, V, can be seen in some exact solutions, for example, in the Kerr black hole and the Taub-NUT universe. These space-times have closed null geodesic curves of which one lap affine lengths are finite on their causal boundaries. The exact solutions suggest that the causal boundary generates a geodesically incompleteness. These motivate us to consider a possibility that more physically reasonable space-times also have closed null curves on their causal boundary and singularities. [3] More precisely, we consider the chronology violating space-times in which there exists at least one causal curve which is closed in a finite length through some points on the boundary of V.

Theorem 1
If a space-time (M, g) is causally complete, then the following three conditions cannot be all satisfied together:
(a) There exists a chronology violating region V which does not coincide with the whole space-time, i.e. $M - V \neq \emptyset$,
(b) every inextendible non-spacelike geodesic in (M, g) contains a pair of conjugate points,
(c) there exists at least one point p on the boundary of V such that each closed timelike curve through a point in the $V \cap \varepsilon$ can be contained in some compact set. (ε is an arbitrary small neighbourhood of p)

As mentioned above, if the condition (c) is satisfied, one can always pick out an infinite sequence such that the one lap length of each closed timelike curve does not diverge and their shape does not change abruptly when a point on each closed curve approaches to the boundary of V.
Here, we regard the one lap length of closed causal curve as the one measured

[3] When Hawking[6] discussed the chronology violations appearing in a bounded region of general space-time without curvature singularities, he introduced the notion of *the compactly generated cauchy horizon* defined as a cauchy horizon such that all the past directed null geodesic generators enter and remain within a compact set. This is analogous to the existence of closed null curves on the boundary of V. He asserted that one cannot make such a Cauchy horizon while the weak energy condition is satisfied. These also supports our expectations above.

by a generalized affine parameter[1].

Proof.
The chronology violating set V is an open set by *Prop 1*. If $V \neq \emptyset$, from the condition (a), we can find a boundary set $\dot{V}$ somewhere in $M - V$. Let us consider a sequence of points q_n in $V \cap \varepsilon$ which converges to p ($\lim_{n \to \infty} q_n = p$). By definition of V there is a closed timelike curve l_n through q_n. From the condition (c), there exists a compact set K such that each l_n is contained in $K \cap V$. Let l be a limit curve of the sequence l_n which passes through the limit point p. Choosing a suitable parameter of each l_n so that l_n is inextendible, the limit curve l is also non-spacelike inextendible curve in $K \cap \bar{V}$ by *Lemma 1*.

Let us consider the case that the limit point $p \in \dot{J}^+(q)$, $q \in V$ without loss of generality. This limit curve must also be contained in $K \cap \bar{V}$ because of the condition (c). Therefore l is totally past and future imprisoned in $K \cap \bar{V}$. If some point p' of l which is in the past of p is contained in V, there exists a closed non-spacelike curve but not null geodesic through p. Because one can connect the limit point p to some point c in V in the future of the p with some non-spacelike curve λ, one can always find a closed non-spacelike curve but not null geodesic such that $p \to c \to q \to p' \to p$ as depicted in Figure 1. This curve can be varied to a closed timelike curve through p by *Prop.2*. This contradicts with the achronality of the causal boundary $\dot{V}$ in which p is contained. If any point of l in the past of p is not contained in V, l is past imprisoned in $\dot{J}^+(q)$. On the other hand if each null geodesic generator of $\dot{J}^+(q)$ is null geodesically complete, then the strong causality condition holds on $\dot{J}^+(q)$ from the condition (b) because chronology condition is satisfied on $\dot{V}$ (see *Prop.5*). Therefore, from *Prop.6*, there can be no past-inextendible non-spacelike curve totally or partially past imprisoned in $\dot{J}^+(q)$. This contradicts with the fact that l is past imprisoned in $\dot{J}^+(q)$. □

Combining *Theorem 1* and *the Hawking-Penrose theorem*[1], we immediately get the following corollary.

Corollary
If a space-time (M, g) is causally complete, then the following conditions cannot all hold:
(1) every inextendible non-spacelike geodesic contains a pair of conjugate points,
(2) the chronology condition holds everywhere on (M, g) or even if chronology condition is violated somewhere, such a region satisfies the condition (c),
(3) there exists a future-(or past-)trapped set S.

So far, we have considered the case that a chronology violating set sat-

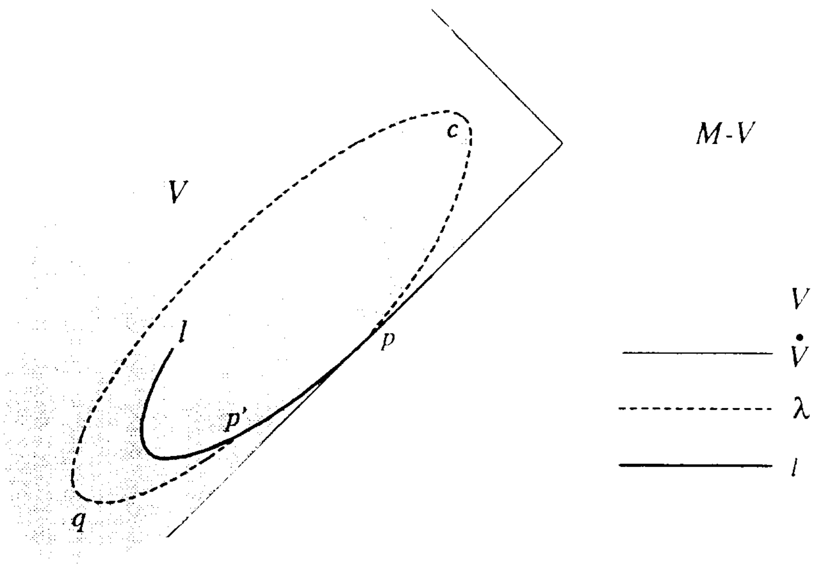

Figure 1: In the case that the past points of the limit curve l into the V, we can find a closed non-spacelike non null geodesic curve like a $p \to c \to q \to p' \to p$, which is the union of λ and a segment $p' \to p$.

isfies the condition (c). Though the reasonableness of the condition (c) is supported by the well-known exact solutions, no one can assure that the causality violating region appeared in a physical space-time always satisfies this condition. For instance, a space-time in which the condition (c) does not hold are illustrated in Figure 2. However, even in such a case, we could still have a statement concerning the existence of singularities if a given space-time satisfies the condition below.

condition (c)

Let each chronology violating set be V_i. Any V_i is causally separated from $V_{j\neq i}$ for each $q \in V_i$, $(\dot{J}^+(q) \cup \dot{J}^-(q)) \cap V_{j\neq i} = \emptyset$.

If a space-time (M, g) satisfies this condition (c') but not the condition (c), we can apply Kriele's theorem to a compact set $S := \dot{J}^+(q) \cap \dot{J}^-(q)$ (even if S is not compact, the quotient space $e^+(S)/S$ is compact. Then we can use Kriele's theorem). Hence, we obtain the following theorem.

Theorem 2

A space-time (M, g) which satisfies the conditions (a), (b), and either at least (c) or (c') is null or timelike geodesically incomplete.

proof.

We suppose that (M, g) is null or timelike geodesically complete. We only have to prove the case that the condition (c') hold but the condition (c) does

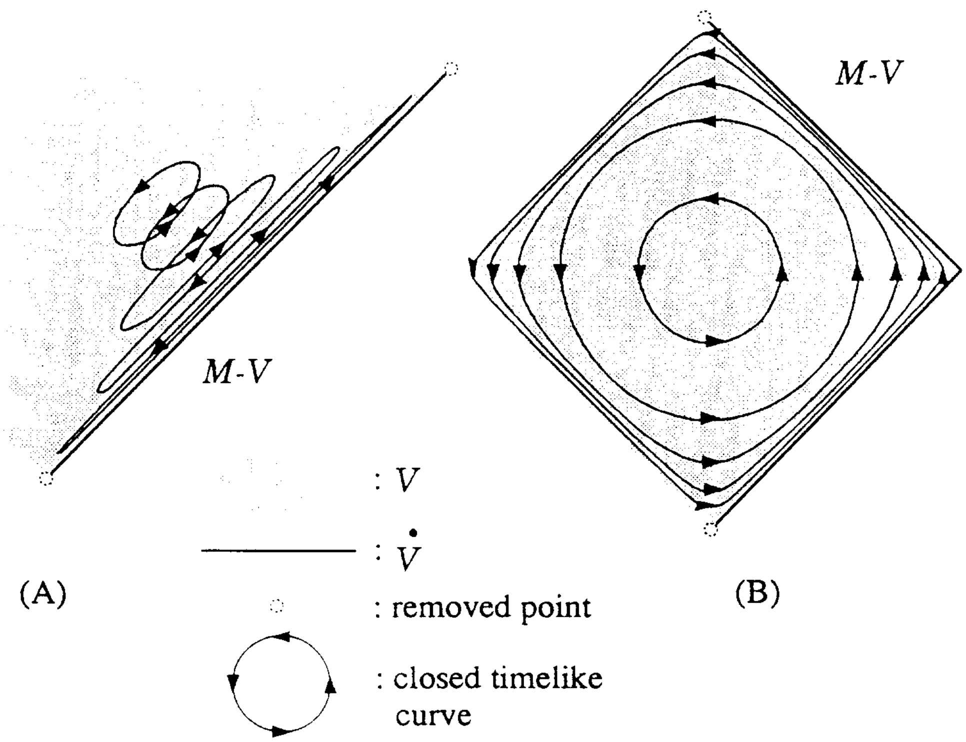

Figure 2: Two examples of space-times in which the limit curves of infinite sequences of closed timelike curves do not close are shown.

not. In such a space-time (M, g), every null geodesic generator on $\dot{V}$ is not closed.

Now we consider a non-closed null geodesic on $\dot{V}$. This null geodesic belongs to $\dot{J}^+(q)$ or $\dot{J}^-(q)$, as V is $I^+(q)\cap I^-(q)$ $(q \in V)$. Let this null geodesic belong to $\dot{J}^+(q)$ without losing generality. If this null geodesic has a past end point, it must be q. Let us take a point $p(\neq q)$ on this null geodesic and also let it be on the $\dot{V}$. From the fact that for $q \in V$ there is a closed timelike curve through q. This means that a timelike curve from q to p exists by *Prop.2*. Therefore, p belongs to $I^+(q)$. This contradicts $p \in \dot{V}$. If this null geodesic has no past end point, it is inextendible in the past. If the boundary of V contains this null geodesic entirely, from the condition (b), this boundary can be connected by timelike curves by *Prop.4*. This is also contradiction to the achronality of $\dot{V}$. Hence, let us consider the case that the boundary of V does not contain the whole segment of this null geodesic, that is, the null geodesic has an end point on the compact surface $S := \dot{J}^+(q) \cap \dot{J}^-(q)$. Extending this null geodesic beyond the future end point, we obtain an inextendible null geodesic lying on $\dot{J}^+(q)$ and call it outgoing. We also obtain an inextendible null geodesic belongs to $\dot{J}^-(q)$ and call it ingoing. The outgoing null geodesic has a pair conjugate points from the condition (b). One of the pair conjugate points is on the segment lying on the $\dot{V}$. The other is on the segment lying on the $\dot{J}^+(q) - \dot{V}$. The ingoing null geodesic on the $\dot{J}^-(q)$

also has a pair conjugate points in the same way as the outgoing case. Thus, S plays the same role as the trapped surface in the Kriele's theorem. From the condition (c'), the condition (3) of Kriele's theorem is satisfied in the cut locuses of S, intersections of outgoing and ingoing null geodesics. because the condition (c) is not satisfied (if the condition (c) is satisfied, there exists an almost closed causal curve.). Therefore we can show the existence of singularities from Kriele's theorem. □

5 Conclusions and discussion

We present the two theorems which state that singularities could occur whenever the boundary of causality violating set exists in a space-time under the physically suitable assumptions. We have shown that the existence of the boundaries of causality preserving and violating regions are closely related to singularities. that is, *the boundaries of causality violating set cause singularities in a physical space-time.*

We would like to emphasize that the theorem 1 supplements the Hawking-Penrose theorem in the sense that global causality is relaxed to some degree and instead the condition (c) is imposed in chronology violating regions, which may be supported by exact solutions. In other words, it is possible for a local observer to state about existence of singularities if *our space-time has a causality preserving region* which conforms with our experience.

Whether the quasi-global condition (c') is removable or not is still an open question.

As well as the Hawking-Penrose theorem, our theorem cannot predict where singularities exist and how strong they are. We think that how we predict these things is one of themes in the future.

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