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Algorithmic construction of O(3) chiral fields equations hierarchy and the Landau-Lifshitz equation hierarchy via polynomial bundle

by

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Abstract

We investigate new polynomial hierarchies of Lax pairs which contains the polynomial pairs for the system of O(3) chiral fields equations and Landau-Lifshitz equation introduced recently and give algorithmic construction of the corresponding hierarchies of soliton equations. We compare the Landau-Lifshitz equation hierarchy obtained via polynomial bundle with the hierarchy obtained via elliptic bundle.

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1 Introduction

It is well known that the so called inverse scattering method, see [1] allows one to apply different and fruitful approaches to the investigation of the class of nonlinear evolution equations, called soliton equations. Their characteristic property is that they can be expressed as compatibility condition of two linear operators L and M:

$$[L, M] = 0 \tag{1}$$

(This representation is called Lax representation and the couple L, M - Lax pair.) In a recent work [2] we have introduced new Lax pairs, polynomial in the spectral parameter, for two important physical systems[1]:

A) Landau-Lifshitz equation [3] (LL)

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \vec{S} \times R\vec{S}. \tag{2}$$

Here $\vec{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))$ is vector field depending on the spatial variable x and the time t, taking its values on the unit sphere $S^2 \subset \mathbf{R}^3$:

$$\vec{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1, \tag{3}$$

R is the diagonal matrix

$$R = diag(r_1, r_2, r_3), \quad r_i > 0$$

with nonnegative entries, $(R\vec{S})_i \equiv r_i S_i; \quad i = 1,2,3$. The LL equation describes perturbations propagating in a direction orthogonal to the anisotropy axis in a ferromagnet and the boundary conditions for it naturally arise from the physical background. These boundary conditions can be expressed as follows

$$\lim_{x \to \pm\infty} \vec{S} = (0, 0, 1). \tag{4}$$

Remark

It should be mentioned that LL equation is related to a number of other systems of the classical mechanics, see[4].

B) $O(3)$ - chiral fields equations (CF)

$$\begin{aligned} \vec{u}_t + \vec{u}_x + \vec{u} \times R\vec{v} = 0, \\ \vec{v}_t - \vec{v}_x + \vec{v} \times R\vec{u} = 0. \end{aligned} \tag{5}$$

Here $\vec{v}, \vec{u}$ are two vector fields depending on x, t taking values on the unit sphere S^2 and $\times$ is the vector product symbol.

[1] Below we shall obtain the pairs for these systems as a consequence of a general construction and for this reason we do not present them now.

The system of O(3) chiral fields describes dynamics in antiferromagnets, in liquid crystals, [5] and has application in the quantum field theory [6].

It is well known that the Lax pairs divide in a classes (hierarchies) and in every such hierarchy the first operators in the Lax pairs (those containing differentiation with respect to the spatial variable) coincide. Usually the pairs in the hierarchy have some natural ordering, for example one can order polynomial pairs by the maximal degree of the spectral parameter in the second operator of the corresponding Lax pair (those containing differentiation with respect to time). The first nonlinear equation in the hierarchy of equations corresponding to the hierarchy of Lax pairs usually gives the name to the the whole hierarchy of equations and to the hierarchy of Lax pairs itself. Thus one speaks about the Nonlinear Schrödinger equation hierarchy, Nonlinear Heisenberg equation hierarchy and so on.

We must stress that the pairs which were known up to now both for the LL and CF were elliptic in the spectral parameter, [7, 8]. On the contrary, as we have mentioned ours are polynomial in the spectral parameter.

We shall construct explicitly the polynomial hierarchy of Lax pairs for the O(3) CFS and for LL case as well as the corresponding hierarchy of soliton equations. As far as we know the hierarchy for the chiral fields equations was not constructed explicitely until now. As to the hierarchy related to the Landau-Lifshitz equation, it will be interesting to compare the hierarchy of equations obtained via polynomial bundle with the hierarchy obtained via elliptic bundle, see [9, 10, 11, 12].

2 Polynomial hierarchy of Lax Pairs related to the system of O(3) chiral fields equations.

First of all, in order to make the calculations easier we shall introduce some notations and shall take into account that most of our matrices lay in the algebra $so(4)$ - the algebra of 4×4 skew-symmetric matrices with complex entries. This algebra is semisimple, but not a simple one. Actually, $so(4)$ splits into a direct sum of two algebras, each of them isomorphic to $so(3)$. It can be verified that the splitting means that every element A of $so(4)$ have unique representation of the following form

$$A = \{\vec{u}\}_I + \{\vec{v}\}_{II}, \tag{6}$$

where

$$\{\vec{u}\}_I = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix}, \tag{7}$$

$$\{\vec{v}\}_{II} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} \tag{8}$$

With these notations one has the commutation relations:

$$
\begin{aligned}
[\{\vec{x}\}_I, \{\vec{y}\}_I] &= -2\{\vec{x}\times\vec{y}\}_I, \\
[\{\vec{x}\}_{II}, \{\vec{y}\}_{II}] &= -2\{\vec{x}\times\vec{y}\}_{II}, \\
[\{\vec{x}\}_I, \{\vec{y}\}_{II}] &= 0,
\end{aligned} \tag{9}
$$

which prove that $so(4)$ is a direct sum of two $so(3)$ algebras. There are however some more interesting properties of the above splitting. If J is the diagonal matrix

$$
J = \begin{pmatrix} -j_1 - j_2 + j_3 & 0 & 0 & 0 \\ 0 & -j_1 + j_2 - j_3 & 0 & 0 \\ 0 & 0 & j_1 - j_2 - j_3 & 0 \\ 0 & 0 & 0 & j_1 + j_2 + j_3 \end{pmatrix}, \tag{10}
$$

we have:

$$
\begin{aligned}
\{\vec{x}\}_I J\{\vec{y}\}_{II} - \{\vec{y}\}_{II} J\{\vec{x}\}_I &= 2\left(\{\vec{x}\times K\vec{y}\}_I + \{K\vec{x}\times\vec{y}\}_{II}\right), \\
\{\vec{x}\}_I J\{\vec{y}\}_I - \{\vec{y}\}_I J\{\vec{x}\}_I &= -2\{K(\vec{x}\times\vec{y})\}_{II}, \\
\{\vec{x}\}_{II} J\{\vec{y}\}_{II} - \{\vec{y}\}_{II} J\{\vec{x}\}_{II} &= -2\{K(\vec{x}\times\vec{y})\}_I.
\end{aligned} \tag{11}
$$

where $K = diag(j_1, j_2, j_3)$ and we use the notation $(K\vec{z})_i \equiv j_i z_i$.

Let us consider the hierarchy of Lax pairs having the following form:

$$
L \equiv \frac{\partial}{\partial x} - U, \quad M_N \equiv \frac{\partial}{\partial t} - V_N, \tag{12}
$$

$$
\begin{aligned}
U(\lambda) &= \tfrac{1}{2}A(\lambda + J), \\
V_N(\lambda) &= \tfrac{1}{2}(\lambda^N B_0 + \lambda^{N-1}B_1 + \ldots + B_N)(\lambda + J),
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
A &= \{\vec{u}\}_I + \{\vec{v}\}_{II}, \\
B_n &= \{\vec{b}_n\}_I + \{\vec{c}_n\}_{II}.
\end{aligned} \tag{14}
$$

Remark

One can see that these pairs are natural generalizations of the 4×4 pairs we have obtained in [2]. Strictly speaking from the beginning we obtained the pairs for LL equation and for the CFS in 6×6 form and then making use of the classical isomorphism between $so(3,3)$ and $sl(4)$ casted them into 4×4 form. In the present work we prefer the simpler 4×4 form.

Compatibility condition between operators L and M_N gives the following matrix equation, which must be satisfied for arbitrary λ :

$$
U_t - (V_N)_x + [U, V_N] = 0. \tag{15}
$$

The left hand side of this equation is a polynomial in the spectral parameter λ and therefore all the coefficients of this polynomial must be equal to zero. This gives us the following relations :

$$
\begin{aligned}
&[A, B_0] = 0, \\
&[A, B_{n+1}] = 2(B_n)_x - (AJB_n - B_nJA), \quad n = 0, 1, \ldots, N-1, \\
&2A_t + AJB_N - B_NJA - 2(B_N)_x = 0.
\end{aligned} \tag{16}
$$

In order to obtain the evolution equation corresponding to the Lax pair $\{L, M_N\}$ one must be able to resolve recursively the first $N+1$ of these equations and to insert the result into the last equation. Making use of the particular form of the matrices A and B_n we readily arrive to the following chain of relations :

$$\begin{aligned} \vec{u} \times \vec{b}_0 = 0, \quad \vec{v} \times \vec{c}_0 = 0, \\ \vec{u} \times \vec{b}_{n+1} = -(\vec{b}_n)_x - K(\vec{v} \times \vec{c}_n) + \vec{u} \times K(\vec{c}_n) - \vec{b}_n \times K(\vec{v}), \\ \vec{v} \times \vec{c}_{n+1} = -(\vec{c}_n)_x - K(\vec{u} \times \vec{b}_n) + K(\vec{u}) \times \vec{c}_n - K(\vec{b}_n) \times \vec{v}, \\ n = 0, 1, \dots N-1. \end{aligned} \tag{17}$$

If we are looking for an infinite set of Lax pairs then we have an infinite system of equations. We shall call this system the O(3) CF chain system. Below, in order to simplify the solution of the chain system and also for the reason that the conditions below must hold for the O(3) chiral fields equations we shall assume that

$$\vec{u}^2 = 1, \quad \vec{v}^2 = 1. \tag{18}$$

Every solution of the O(3) CF chain system allows us to obtain the system of evolution equations

$$\begin{aligned} \vec{u}_t = (\vec{b}_N)_x + K(\vec{v} \times \vec{c}_N) - \vec{u} \times K(\vec{c}_N) + \vec{b}_N \times K(\vec{v}), \\ \vec{v}_t = (\vec{c}_N)_x + K(\vec{u} \times \vec{b}_N) - K(\vec{u}) \times \vec{c}_N + K(\vec{b}_N) \times \vec{v}, \\ N = 0, 1, 2, \dots . \end{aligned} \tag{19}$$

Using the next terms of the hierarchy one can also write down these equations into the form

$$\begin{aligned} \vec{u}_t = -\vec{u} \times \vec{b}_{N+1}, \\ \vec{v}_t = -\vec{v} \times \vec{c}_{N+1}, \\ N = 0, 1, 2, \dots . \end{aligned} \tag{20}$$

Then it is clear that the constraints $\vec{u}^2 = 1, \quad \vec{v}^2 = 1$ are compatible with the evolution.

The solution of the first equations in the chain is clear :

$$\vec{b}_0 = \epsilon \vec{u}, \quad \vec{c}_0 = \mu \vec{v}, \tag{21}$$

where ϵ, μ are arbitrary scalar functions. The corresponding evolution equations are

$$\begin{aligned} \vec{u}_t = \epsilon \vec{u}_x + \epsilon_x \vec{u} + (\epsilon - \mu)(\vec{u} \times K(\vec{v})), \\ \vec{v}_t = \mu \vec{v}_x + \mu_x \vec{v} - (\epsilon - \mu)(\vec{v} \times K(\vec{u})). \end{aligned} \tag{22}$$

However, in order to resolve the next equation of the chain system (or in order to obtain evolution equations compatible with the constraints) the left hand sides of this equations must be orthogonal to the vector fields $\vec{u}$ and $\vec{v}$ respectively. This readily gives that ϵ, μ must be some parameters that do not depend on x. For appropriate choice of the parameters ϵ, μ and $j_i, i = 1, 2, 3$ we obtain the O(3) CF.

At each step the situation is similar - the solution of the N - th equation is not unique and the freedom can be used to ensure that the next equation in the hierarchy can be resolved. However some freedom still exists, but it disappears if we can fix the values of the fields and their x - derivatives at some point (finite or not). We shall assume that

$$\begin{gathered}\lim_{x\to\pm\infty}\vec{u}=\vec{u}_0=const,\\ \lim_{x\to\pm\infty}\vec{v}=\vec{v}_0=const,\\ \lim_{x\to\pm\infty}\left(\frac{\partial}{\partial x}\right)^n\vec{u}=0,\\ \lim_{x\to\pm\infty}\left(\frac{\partial}{\partial x}\right)^n\vec{v}=0,\\ n=1,2,\ldots.\end{gathered}\tag{23}$$

These requirements are by no means indispensable in order to find the hierarchy of Lax pairs and the corresponding hierarchy of evolution equations. If we choose another ones we can obtain the corresponding hierarchy just in the same way as it will be done below.

As we shall need it let us briefly outline the properties of the linear operator $P_S:\mathbf{C}^3\to\mathbf{C}^3$ defined by

$$P_S(\vec{\xi})=\vec{S}\times\vec{\xi},\quad \vec{S}^2=1,\tag{24}$$

$\vec{S}$ being some fixed vector. It is readily seen that the linear space $\mathbf{C}^3$ splits into the following sum of linear subspaces :

$$\mathbf{C}^3=kerP_S\oplus imP_S.\tag{25}$$

This subspaces are orthogonal with respect to the scalar product

$$\langle\vec{\xi},\vec{\eta}\rangle=\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3,\tag{26}$$

and

$$kerP_S=\mathbf{C}\vec{S},\quad imP_S=\{\vec{\xi}:\langle\vec{\xi},\vec{S}\rangle=0\}.\tag{27}$$

Let us denote by an upper index "S" the projection of a given vector $\vec{b}$ onto the subspace imP_S, that is

$$\vec{b}^S=\vec{b}-\langle\vec{b},\vec{S}\rangle\vec{S}=-(P_S)^2\vec{b}.\tag{28}$$

If one has to solve for $\vec{x}$ the equation

$$\vec{S}\times\vec{x}=\vec{b}$$

then the problem has a solution if and only if the compatibility condition

$$\langle\vec{x},\vec{S}\rangle=0\Leftrightarrow(\vec{b}=\vec{b}^S)$$

is satisfied. In that case all the solutions of the above equation are given by the formula

$$\vec{x}=-\vec{S}\times\vec{b}+\mu\vec{S},\tag{29}$$

where μ is scalar parameter.

Let us pass now to the solution of the O(3) CF chain system. Here we have two equations of the same type as considered above. Let us denote by the upper indices "u" and "v" the projections on the vector subspaces orthogonal to the vectors $\vec{u}$ and $\vec{v}$ respectively. Let us consider the equations

$$\begin{aligned} \vec{u} \times \vec{b}_{n+1} &= -(\vec{b}_n)_x - K(\vec{v} \times \vec{c}_n) + \vec{u} \times K(\vec{c}_n) - \vec{b}_n \times K(\vec{v}), \\ \vec{v} \times \vec{c}_{n+1} &= -(\vec{c}_n)_x - K(\vec{u} \times \vec{b}_n) + K(\vec{u}) \times \vec{c}_n - K(\vec{b}_n) \times \vec{v}. \end{aligned} \tag{30}$$

It is evident that from this system one can uniquely determine the projections of the vector fields - $\vec{b}^u_{n+1}$ and $\vec{c}^v_{n+1}$ and the projections over the subspaces $\mathbf{C}\vec{u}$ and $\mathbf{C}\vec{v}$ respectively remains indefinite. These projections are given by

$$\vec{u}\langle \vec{b}_{n+1}, \vec{u}\rangle, \vec{v}\langle \vec{c}_{n+1}, \vec{v}\rangle.$$

We shall find how these projections can be expressed thorough $\vec{b}^u_{n+1}, \vec{c}^v_{n+1}$. Let us consider the expression

$$\frac{\partial}{\partial x}\langle \vec{b}_{n+1}, \vec{u}\rangle = \langle \frac{\partial}{\partial x}\vec{b}_{n+1}, \vec{u}\rangle + \langle \vec{b}_{n+1}, \frac{\partial}{\partial x}\vec{u}\rangle.$$

Suppose now that the $n+2$ - th equation in the chain system can be resolved. Then we can write

$$(\vec{b}_{n+1})_x = -\vec{u} \times \vec{b}_{n+2} - K(\vec{v} \times \vec{c}_{n+1}) + \vec{u} \times K(\vec{c}_{n+1}) - \vec{b}_{n+1} \times K(\vec{v}).$$

Then evidently

$$\begin{aligned} \langle \vec{u}, (\vec{b}_{n+1})_x\rangle &= \langle \vec{u} \times K(\vec{v}), \vec{b}_{n+1}\rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}_{n+1}\rangle = \\ &= \langle \vec{u} \times K(\vec{v}), \vec{b}^u_{n+1}\rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}^v_{n+1}\rangle, \end{aligned}$$

and we have

$$\langle \vec{u}, (\vec{b}_{n+1})_x\rangle = \int_{\pm\infty}^{x} \Big(\langle \vec{b}^u_{n+1}, \vec{u}_x\rangle + \langle \vec{u} \times K(\vec{v}), \vec{b}^u_{n+1}\rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}^v_{n+1}\rangle \Big)\, dx. \tag{31}$$

Just in the same manner,

$$\langle \vec{v}, (\vec{c}_{n+1})_x\rangle = \int_{\pm\infty}^{x} \Big(\langle \vec{c}^v_{n+1}, \vec{v}_x\rangle + \langle \vec{u} \times K(\vec{v}), \vec{b}^u_{n+1}\rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}^v_{n+1}\rangle \Big)\, dx. \tag{32}$$

Inserting these expressions into the formulae

$$\vec{b}_{n+1} = \vec{b}^u_{n+1} + \vec{u}\langle \vec{u}, \vec{b}_{n+1}\rangle, \quad \vec{c}_{n+1} = \vec{b}^v_{n+1} + \vec{v}\langle \vec{v}, \vec{c}_{n+1}\rangle \tag{33}$$

we get $\vec{b}_{n+1}, \vec{c}_{n+1}$ expressed through the projections $\vec{b}^u_{n+1}, \vec{c}^v_{n+1}$. In what follows we shall assume that the projections over the subspaces $\mathbf{C}\vec{u}$ and $\mathbf{C}\vec{v}$ are given by the above formulae. Then the last thing that remains to be done is to express $\vec{b}^u_{n+1}, \vec{c}^v_{n+1}$ through $\vec{b}^u_n, \vec{c}^v_n$. We formulate the final answer in the following

Proposition 1.

The CF chain system has the following solution :

$$
\begin{gathered}
\vec{b}_0 = \epsilon\vec{u}, \quad \vec{c}_0 = \mu\vec{v}, \\
\vec{b}^u_{n+1} = \vec{u} \times (\vec{b}^u_n)_x + \langle \vec{u}, \vec{b}_n \rangle \vec{u} \times \vec{u}_x + [K(\vec{c}^v_n)]^u - (\langle \vec{u}, \vec{b}_n \rangle - \langle \vec{v}, \vec{c}_n \rangle)[K(v)]^u + \\
\vec{u} \times K(\vec{v} \times \vec{c}^v_n) + \langle \vec{u}, K(\vec{v}) \rangle \vec{b}^u_n, \\
\vec{c}^v_{n+1} = \vec{v} \times (\vec{c}^v_n)_x + \langle \vec{v}, \vec{c}_n \rangle \vec{v} \times \vec{v}_x + \left[K(\vec{b}^u_n)\right]^v + (\langle \vec{u}, \vec{b}_n \rangle - \langle \vec{v}, \vec{c}_n \rangle)[K(u)]^v + \\
\vec{v} \times K(\vec{u} \times \vec{b}^u_n) + \langle \vec{u}, K(\vec{v}) \rangle \vec{c}^v_n, \\
n = 0, 1, 2, \dots,
\end{gathered}
\tag{34}
$$

where ϵ, μ are arbitrary constants and

$$
\begin{aligned}
\langle \vec{u}, \vec{b}_n \rangle &= \int_{\pm\infty}^{x} \Big(\langle \vec{b}^u_n, \vec{u}_x \rangle + \langle \vec{u} \times K(\vec{v}), \vec{b}^u_n \rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}^v_n \rangle \Big)\, dx, \\
\langle \vec{v}, \vec{c}_n \rangle &= \int_{\pm\infty}^{x} \Big(\langle \vec{c}^v_n, \vec{v}_x \rangle + \langle \vec{u} \times K(\vec{v}), \vec{b}^u_n \rangle + \langle \vec{v} \times K(\vec{u}), \vec{c}^v_n \rangle \Big)\, dx.
\end{aligned}
\tag{35}
$$

One can see that the couple of functions $(\vec{b}^u_{n+1}, \vec{c}^v_{n+1})$ is expressed through the couple $(\vec{b}^u_n, \vec{c}^v_n)$ with the help of some integro - differential operator $\mathbf{A}_\pm(\vec{u}, \vec{v})$, depending on $\vec{u}(x), \vec{v}(x)$. The choice of the subscript "+" or "-" corresponds to the choice of $+\infty$ or $-\infty$ for the integration limit in the corresponding expressions.In other words :

$$
\begin{pmatrix} \vec{b}^u_{n+1} \\ \vec{c}^v_{n+1} \end{pmatrix} = \mathbf{A}_\pm(\vec{u}, \vec{v}) \begin{pmatrix} \vec{b}^u_n \\ \vec{c}^v_n \end{pmatrix}. \tag{36}
$$

We shall not write the explicit formula for $\mathbf{A}_\pm$ as it is too complicated and besides it can be easily derived from the above proposition. The operators that allows recursively to obtain the hierarchies of soliton equations are called recursion operators or generating operators. It turns out that their existence of is very important. For example they plays crucial role in describing the hierarchies of Hamiltonian structures for the soliton equations. Other important application of the generating operators is that their spectral decomposition allows to obtain the so called expansions over the squared (or adjoint) solutions which proved to be very useful tool for the investigation of the soliton equations, see for example [14], where the case of the generating operator for the hierarchy of Heisenberg ferromagnet is considered.

It will be superfluous to discuss here the the hierarchies of Hamiltonian structures for the CF and so we shall leave this kind of questions for a future publication. We only want to remark that as far as we know the generating operator for the CF hierarchy has been calculated in [11] but on quite different background - using the fact that this operator gives the relation between compatible Poisson tensors defined via elliptic bundle. However in [11] the hierarchies of Lax pairs were not obtained.

At the end of this section let us write the first two systems of the CF hierarchy :

1. $N = 0$. First equation in the CF hierarchy :

$$\begin{aligned} \vec{u}_t &= \epsilon \vec{u}_x + (\epsilon - \mu)(\vec{u} \times K(\vec{v})), \\ \vec{v}_t &= \mu \vec{v}_x - (\epsilon - \mu)(\vec{v} \times K(\vec{u})). \end{aligned} \tag{37}$$

After the following choice of the parameters :

$\epsilon = -1$, $\mu = 1$, $R = 2K$ and changing $\vec{u}$ to $-\vec{u}$ we obtain the O(3) chiral fields equations system (5).

2. $N = 1$. Second equation in the CF hierarchy :

$$\begin{aligned} \vec{u}_t = \epsilon \vec{u} \times \vec{u}_{xx} + 2\epsilon(\langle \vec{u}, K(\vec{v}) \rangle - \langle \vec{u}_0, K(\vec{v}_0) \rangle)\vec{u}_x - \\ \epsilon K(\vec{v}_x) + \epsilon \langle \vec{u}, K(\vec{v}_x) \rangle \vec{u} - \mu \vec{u} \times K(\vec{v} \times \vec{v}_x) + \\ (\epsilon - \mu)(\langle \vec{u}, K(\vec{v}) \rangle - \langle \vec{u}_0, K(\vec{v}_0) \rangle)(\vec{u} \times K(\vec{v})) - (\epsilon - \mu)\vec{u} \times K^2(\vec{u}), \\ \vec{v}_t = \mu \vec{v} \times \vec{v}_{xx} + 2\mu(\langle \vec{v}, K(\vec{u}) \rangle - \langle \vec{v}_0, K(\vec{u}_0) \rangle)\vec{v}_x - \\ \mu K(\vec{u}_x) + \mu \langle \vec{v}, K(\vec{u}_x) \rangle \vec{v} - \epsilon \vec{v} \times K(\vec{u} \times \vec{u}_x) - \\ (\epsilon - \mu)(\langle \vec{v}, K(\vec{u}) \rangle - \langle \vec{v}_0, K(\vec{u}_0) \rangle)(\vec{v} \times K(\vec{u})) + (\epsilon - \mu)\vec{v} \times K^2(\vec{v}). \end{aligned} \tag{38}$$

An interesting special case of this system is obtained for $\mu = 0$. Then we have

$$\begin{aligned} \vec{u}_t = \epsilon \vec{u} \times \vec{u}_{xx} + 2\epsilon(\langle \vec{u}, K(\vec{v}) \rangle - \langle \vec{u}_0, K(\vec{v}_0) \rangle)\vec{u}_x - \\ \epsilon K(\vec{v}_x) + \epsilon \langle \vec{u}, K(\vec{v}_x) \rangle \vec{u} + \\ \epsilon(\langle \vec{u}, K(\vec{v}) \rangle - \langle \vec{u}_0, K(\vec{v}_0) \rangle)(\vec{u} \times K(\vec{v})) - \epsilon \vec{u} \times K^2(\vec{u}), \\ \vec{v}_t = -\epsilon \vec{v} \times K(\vec{u} \times \vec{u}_x) - \epsilon(\langle \vec{v}, K(\vec{u}) \rangle - \langle \vec{v}_0, K(\vec{u}_0) \rangle)(\vec{v} \times K(\vec{u})) + \\ \epsilon \vec{v} \times K^2(\vec{v}). \end{aligned} \tag{39}$$

Another reduction of the general system (38) is obtained if we assume that $\epsilon = \mu$. Then we have

$$\begin{aligned} \vec{u}_t = \epsilon \vec{u} \times \vec{u}_{xx} + 2\epsilon(\langle \vec{u}, K(\vec{v}) \rangle - \langle \vec{u}_0, K(\vec{v}_0) \rangle)\vec{u}_x - \\ \epsilon K(\vec{v}_x) + \epsilon \langle \vec{u}, K(\vec{v}_x) \rangle \vec{u} - \epsilon \vec{u} \times K(\vec{v} \times \vec{v}_x), \\ \vec{v}_t = \epsilon \vec{v} \times \vec{v}_{xx} + 2\epsilon(\langle \vec{v}, K(\vec{u}) \rangle - \langle \vec{v}_0, K(\vec{u}_0) \rangle)\vec{v}_x - \\ \epsilon K(\vec{u}_x) + \epsilon \langle \vec{v}, K(\vec{u}_x) \rangle \vec{v} - \epsilon \vec{v} \times K(\vec{u} \times \vec{u}_x). \end{aligned} \tag{40}$$

The next systems of the hierarchy can also be obtained without much difficulties but for them there is less hope for any physical applications.

3 Polynomial hierarchy of Lax Pairs related to the Landau-Lifshitz equation

The Landau-Lifshitz equation can be obtained within the general scheme described above if we impose instead of the constraint $\vec{v}^2 = 1$ the constraint $\vec{v} = 0$. Unfortunately, as the condition $\vec{v}^2 = 1$ was essential in all our constructions one cannot simply insert $\vec{v} = 0$ in the solution for the O(3) CF chain system in order to obtain the solution for the corresponding chain system for LL equation.

Remark

It is not difficult to check that if instead of the constraints $\vec{v} = 0, \vec{u}^2 = 1$ we choose the constraints $\vec{u} = 0, \vec{v}^2 = 1$ we shall obtain the same hierarchy of Lax pairs. Thus in all the constructions there exists a symmetry between the two so(3) subalgebras in so(4).

In order to obtain the LL equation in the same terms as it was introduced we shall change the notations and in what follows shall put $\vec{u} \equiv \vec{S}$. Then the chain relations reduce to

$$\begin{aligned} \vec{S} \times \vec{b}_0 &= 0, \\ \vec{S} \times \vec{b}_{n+1} &= -(\vec{b}_n)_x + \vec{S} \times K(\vec{c}_n), \\ (\vec{c}_n)_x &= -K(\vec{S} \times \vec{b}_n) + K(\vec{S}) \times \vec{c}_n, \\ n &= 0, 1, \ldots, N-1. \end{aligned} \tag{41}$$

We shall refer the above system of equations as the LL chain system.

The corresponding hierarchy of evolution equations is then

$$\vec{S}_t = (\vec{b}_N)_x - \vec{S} \times K(\vec{c}_N), \quad N = 0, 1, 2, \ldots, \tag{42}$$

or using the next term in the hierarchy we can write

$$\vec{S}_t = -\vec{S} \times \vec{b}_{N+1}, \quad N = 0, 1, 2, \ldots. \tag{43}$$

Thus as it was for the case of the O(3) CF the N - th evolution equation respects the constraint $\vec{S}^2 = 1$ if the (N+1) relation in the chain can be resolved. Therefore if we can show that there exist solution of the infinite system defined above all the evolution equations will respect automatically the constraint. To begin with one must make a choice for the first terms. We shall consider the case

$$\vec{b}_0 = \vec{S}, \quad \vec{c}_0 = 0 \tag{44}$$

as it leads directly to the LL equation. The general case $\vec{b}_0 = f\vec{S}$, f being some scalar function and $\vec{c}_0$ being a solution of the equation

$$(\vec{c}_0)_x = K(\vec{S}) \times \vec{c}_0 \tag{45}$$

seems more complicated, but in fact we must remind that in order to obtain evolution equation having the form

$$\vec{S}_t = \vec{F}(\vec{S}, \vec{S}_x, \ldots)$$

the function f and the solution $\vec{c}_0$ must depend on x, t only through $\vec{S}(x,t)$ and its derivatives. A brief analysis then shows that $\vec{c}_0 = 0$ is the only appropriate choice and the general case can be treated within the same lines.

The hierarchy of equations can be described explicitly if at each step we can present the solution of the equation

$$(\vec{c}_n)_x = -K(\vec{S} \times \vec{b}_n) + K(\vec{S}) \times \vec{c}_n. \tag{46}$$

In order to do it we shall need some preparations. Let us introduce the sequence of diagonal matrices $K^{(n)}, n = 1, 2, \ldots$, satisfying the relations :

$$K^{(1)} \equiv K,$$
$$K^{(1)}(\vec{a}) \times K^{(1)}(\vec{b}) = K^{(2)}(\vec{a} \times \vec{b}),$$
$$K^{(1)}(\vec{a} \times K^{(1)}K^{(1)}(\vec{b})) + K^{(1)}(\vec{a}) \times K^{(2)}(\vec{b}) = K^{(3)}(\vec{a} \times \vec{b}),$$
$$K^{(1)}(\vec{a} \times K^{(1)}K^{(2)}(\vec{b})) + K^{(2)}(\vec{a} \times K^{(1)}K^{(1)}(\vec{b})) + K^{(1)}(\vec{a}) \times K^{(3)}(\vec{b}) = K^{(4)}(\vec{a} \times \vec{b}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\sum_{i=1}^{n-2} K^{(i)}(\vec{a} \times K^{(1)}K^{(n-i-1)}(\vec{b})) + K^{(1)}(\vec{a}) \times K^{(n-1)}(\vec{b}) = K^{(n)}(\vec{a} \times \vec{b}),$$
$$n = 3, 4, \ldots, \tag{47}$$

for arbitrary choice of the vectors $\vec{a}, \vec{b}$.

Lemma

The sequence of diagonal matrices $K^{(n)}$

$$K^{(n)} = diag(K_1^{(n)}, K_2^{(n)}, K_3^{(n)}) \tag{48}$$

is well defined and the entries $K_i^{(n)}, i = 1, 2, 3$ of $K^{(n)}$ are homogeneous polynomials of degree n with respect to the variables j_1, j_2, j_3.

Proof

Let us calculate the first terms of the sequence. One readily obtains :

$$\begin{array}{llll} K_1^{(1)} = j_1, & K_1^{(2)} = j_2 j_3, & K_1^{(3)} = j_1(j_2^2 + j_3^2), & K_1^{(4)} = j_2 j_3(2j_1^2 + j_2^2 + j_3^2) \\ K_2^{(1)} = j_2, & K_2^{(2)} = j_1 j_3, & K_2^{(3)} = j_2(j_2^1 + j_3^2), & K_2^{(4)} = j_1 j_3(j_1^2 + 2j_2^2 + j_3^2) \\ K_3^{(1)} = j_3, & K_3^{(2)} = j_1 j_2, & K_3^{(3)} = j_3(j_1^2 + j_2^2), & K_3^{(4)} = j_1 j_2(j_1^2 + j_2^2 + 2j_3^2) \end{array} \tag{49}$$

The statement of the Lemma being true for $n = 1, 2, 3, 4$ one can try to prove the Lemma by induction. Suppose that the sequence $K^{(n)}$ has the needed properties for $n = 1, 2, \ldots ; N \geq 4$. Then we shall prove that there exists unique diagonal matrix $K^{(N+1)}$ such that

$$\sum_{i=1}^{N-1} K^{(i)}(\vec{a} \times K^{(1)}K^{(s-i)}(\vec{b})) + K^{(1)}(\vec{a}) \times K^{(N)}(\vec{b}) = K^{(N+1)}(\vec{a} \times \vec{b})$$

for arbitrary choice of the vectors $\vec{a}, \vec{b}$. Let us calculate the first component of the left hand side of this vector equality. We get :

$$\left(\sum_{i=1}^{N-1} K_1^{(i)} K_3^{(1)} K_3^{(N-i)} + K_2^{(1)} K_3^{(N)} \right) a_2 b_3 - \left(\sum_{i=1}^{N-1} K_1^{(i)} K_2^{(1)} K_2^{(N-i)} + K_3^{(1)} K_2^{(N)} \right) a_3 b_2.$$

In order to write this expression into the form

$$K_1^{(N+1)}(a_2 b_3 - a_3 b_2)$$

with some coefficient $K_1^{(N+1)}$ which does not depend on $a_i, b_i; i = 1,2,3$ it is necessary and sufficient to have

$$W \equiv \sum_{i=1}^{N-1} K_1^{(i)}(K_3^{(1)}K_3^{(N-i)} - K_2^{(1)}K_2^{(N-i)}) + K_2^{(1)}K_3^{(N)} - K_3^{(1)}K_2^{(N)} = 0.$$

We remind that by inductive assumption for all $2 \le s \le N-1$ we have

$$K_1^{(s+1)} = K_2^{(1)}K_3^{(s)} + K_3^{(1)}\sum_{i=1}^{s-1} K_1^{(i)}K_3^{(s-i)},$$

$$K_1^{(s+1)} = K_3^{(1)}K_2^{(s)} + K_2^{(1)}\sum_{i=1}^{s-1} K_1^{(i)}K_2^{(s-i)}.$$

and also four other relations which can be obtained from the above ones with cyclic permutation of the indices. They correspond to the other two components of the vector relations from the Lemma. Then we can write

$$W = K_2^{(1)}\left(K_2^{(1)}K_1^{(N-1)} + \sum_{l=1}^{N-2} K_3^{(l)}K_1^{(N-l-1)}K_1^{(1)}\right) -$$

$$K_3^{(1)}\left(K_3^{(1)}K_1^{(N-1)} + \sum_{l=1}^{N-2} K_2^{(l)}K_1^{(N-l-1)}K_1^{(1)}\right) +$$

$$\sum_{i=1}^{N-1}\left(K_1^{(i)}K_3^{(N-i)}K_3^{(1)} - K_1^{(i)}K_2^{(N-i)}K_2^{(1)}\right) =$$

$$K_3^{(1)}\left(\sum_{l=1}^{N-3} K_1^{(l)}(K_3^{(N-l)} - K_1^{(1)}K_2^{(N-l-1)})\right) + K_3^{(1)}K_1^{(N-2)}(K_3^{(2)} - K_1^{(1)}K_2^{(1)}) -$$

$$K_2^{(1)}\left(\sum_{l=1}^{N-3} K_1^{(l)}(K_2^{(N-l)} - K_1^{(1)}K_3^{(N-l-1)})\right) - K_2^{(1)}K_1^{(N-2)}(K_2^{(2)} - K_1^{(1)}K_3^{(1)}).$$

From the explicit formulae for $K^{(2)}$ it follows that the terms which are not under the summation are zero. As to the rest of the expression W it vanishes due to the the relations

$$K_3^{(N-l)} - K_1^{(1)}K_2^{(N-l-1)} = \sum_{j=1}^{N-l-2} K_2^{(1)}K_3^{(j)}K_2^{(N-l-j-1)},$$

$$K_2^{(N-l)} - K_1^{(1)}K_3^{(N-l-1)} = \sum_{j=1}^{N-l-2} K_3^{(1)}K_3^{(j)}K_2^{(N-l-j-1)}.$$

Absolutely the same procedure can be applied for the other two components. The statement that the entries of $K^{(n)}$ are homogeneous polynomials of degree n readily follows from the proof.

Proposition 2.
Suppose $\vec{b}_m, \vec{c}_m$*:* $m = 0, 1, \ldots, n-1 \geq 0$ *are solutions of the chain system :*

$$\begin{aligned} \vec{b}_0 &= \vec{S}, \vec{c}_0 = 0, \\ \vec{S} \times \vec{b}_{m+1} &= -(\vec{b}_m)_x + \vec{S} \times K(\vec{c}_m), \\ (\vec{c}_m)_x &= -K(\vec{S} \times \vec{b}_m) + K(\vec{S}) \times \vec{c}_m, \\ m &= 1, 2, \ldots, n-1. \end{aligned} \tag{50}$$

Then

$$\vec{c}_n = \sum_{q=1}^{n} (-1)^{q-1} K^{(q)}(\vec{b}_{n-q}) \tag{51}$$

is a solution of the equation

$$(\vec{c}_n)_x = -K(\vec{S} \times \vec{b}_n) + K(\vec{S}) \times \vec{c}_n.$$

Proof
We shall prove this proposition by induction. For $n = 1$ we have

$$\vec{S} \times \vec{b}_1 = (\vec{b}_0)_x = (\vec{S})_x.$$

As $\vec{S}^2 = 1$, the vector $\vec{S}_x$ is orthogonal to $\vec{S}$ and one gets $\vec{b}_1 = \vec{S} \times \vec{S}_x + \alpha \vec{S}$, where α is scalar parameter.Then it is readily seen that $\vec{c}_1 = K(\vec{S})$ solves the equation

$$(\vec{c}_1)_x = -K(\vec{S} \times \vec{b}_1) + K(\vec{S}) \times \vec{c}_1.$$

We shall assume now that the proposition is true for all $n = 1, 2, \ldots, N-1$ and shall prove it for $n = N$. In order to do it let us calculate

$$(\vec{c}_N)_x = \sum_{q=1}^{N} (-1)^{q-1} K^{(q)}((\vec{b}_{N-q})_x).$$

Taking into account that $\vec{b}_i$ are solutions of the chain system we get

$$(\vec{c}_N)_x = -\sum_{q=1}^{N} (-1)^{q-1} K^{(q)}(\vec{S} \times \vec{b}_{N-q}) + \sum_{q=1}^{N-1} (-1)^{q-1} K^{(q)}(\vec{S} \times K^{(1)}(\vec{c}_{N-q})),$$

where we have used that $\vec{c}_0 = 0$. Inserting into this equation the expressions for $\vec{c}_{N-q}$ we obtain

$$(\vec{c}_N)_x = -\sum_{q=1}^{N} (-1)^{q-1} K^{(q)}(\vec{S} \times \vec{b}_{N-q}) +$$

$$\sum_{q=1}^{N-1} (-1)^{q-1} K^{(q)}(\vec{S} \times K^{(1)}(\sum_{r=1}^{N-q} (-1)^{r-1} K^{(r)}(\vec{b}_{N-q-r}))).$$

Then

$$(\vec{c}_N)_x + K(\vec{S}\times\vec{b}_N) - K(\vec{S})\times\vec{c}_N =$$

$$\sum_{q=1}^{N-1}(-1)^{q-1}K^{(q)}(\vec{S}\times K^{(1)}(\sum_{r=1}^{N-q}(-1)^{r-1}K^{(r)}(\vec{b}_{N-q-r})))-$$

$$\sum_{q=1}^{N}(-1)^{q-1}K^{(q)}(\vec{S}\times\vec{b}_{N-q}) + K^{(1)}(\vec{S}\times\vec{b}_N) - K^{(1)}(\vec{S})\times\sum_{q=1}^{N}(-1)^{q-1}K^{(q)}(\vec{b}_{N-q}) =$$

$$\sum_{q=1}^{N-1}(-1)^{q-1}K^{(q)}(\vec{S}\times K^{(1)}(\sum_{r=1}^{N-q}(-1)^{r-1}K^{(r)}(\vec{b}_{N-q-r})))-$$

$$\sum_{q=2}^{N}(-1)^{q-1}K^{(q)}(\vec{S}\times\vec{b}_{N-q}) - K^{(1)}(\vec{S})\times\sum_{q=1}^{N}(-1)^{q-1}K^{(q)}(\vec{b}_{N-q}) =$$

$$\sum_{q=2}^{N-1}(-1)^{q+1}\left(K^{(q+1)}(\vec{S}\times\vec{b}_{N-q}) - K^{(1)}(\vec{S})\times K^{(q)}(\vec{b}_{N-q})\right) -$$

$$\sum_{q=2}^{N-1}(-1)^{q+1}\left(\sum_{l+r=q}K^{(l)}(\vec{S}\times K^{(1)}K^{(r)}(\vec{b}_{N-q})\right) +$$

$$(-1)^N\left(K^{(1)}(\vec{S})\times K^{(q)}(\vec{b}_0) + \sum_{q+r=N}K^{(q)}(\vec{S}\times K^{(1)}K^{(r)}(\vec{b}_0)\right) +$$

$$K^{(2)}(\vec{S}\times\vec{b}_{N-1}) - K^{(1)}(\vec{S})\times K^{(1)}(\vec{b}_{N-1}).$$

From the definition of the matrices $K^{(q)}$ it follows that the above expression is equal to

$$(-1)^N K^{(N+1)}(\vec{S}\times\vec{b}_0) = 0.$$

Thus $\vec{c}_N$ is a solution of the equation

$$(\vec{c}_N)_x = -K(\vec{S}\times\vec{b}_N) + K(\vec{S})\times\vec{c}_N$$

and the Proposition is proved.

Now we know how to solve recursively the second part of the equations in the chain system. The first part of these equations runs as follows :

$$\vec{S}\times\vec{b}_{n+1} = -(\vec{b}_n)_x + \vec{S}\times K(\vec{c}_n).$$

We have considered similar equations dealing with the CF chain system. As it was outlined in the previous section in order to solve for $\vec{b}_{n+1}$ the compatibility condition

$$\langle(\vec{b}_n)_x, \vec{S}\rangle = 0 \tag{52}$$

must be satisfied and to ensure it we must use the freedom in the determination of the solution for the previous equation. Let us consider the following decomposition of $\vec{b}_n, n \geq 1$

$$\vec{b}_n = \vec{b}_n^S + \vec{S}\langle\vec{b}_n, \vec{S}\rangle. \tag{53}$$

As before when one solves the chain relations one recovers uniquely $\vec{b}_n^S$ and all the nonuniqueness appears in the determination of $\langle\vec{b}_n, \vec{S}\rangle$. They are recovered from $\vec{b}_n^S$ if in addition one can fix the values of the field $\vec{S}$ and its x-derivatives at some point. Then the calculations are exactly the same as in the case of CF chain system and we shall omit them. Before presenting the final result however there is one point we want to discuss. As already mentioned in order to obtain unique solution on the each step one must fix the values of the vector field $\vec{S}$ and its x-derivatives at some point of $\mathbf{R}^3$ (finite or not). We shall assume that the function $\vec{S}$ has the following property

$$\begin{aligned} \lim_{x\to\pm\infty} \vec{S} = \vec{S}_0 = const, \\ \lim_{x\to\pm\infty} \left(\tfrac{\partial}{\partial x}\right) \vec{S} = 0, \\ n = 1, 2, \ldots . \end{aligned} \tag{54}$$

Usually, for the LL equation the condition

$$\lim_{x\to\pm\infty} \vec{S} = (0, 0, 1)$$

is imposed. We consider the above condition in order to obtain more symmetrical expressions with respect to a cyclic permutation of the indices 1, 2, 3.

Remark

The above requirements seem quite natural for the Landau-Lifshitz equation, but, of course , if one is looking only for the hierarchy of equations they are not absolutely necessary.

Thus we arrive to the following solution of the LL chain system

$$\vec{b}_{n+1}^S = \vec{S} \times \frac{\partial}{\partial x}(\vec{b}_n^S) + (\vec{S} \times \vec{S}_x) \int_{\pm\infty}^{x} \langle\vec{b}_n^S, \vec{S}_x\rangle dx + (K(\vec{c}_n))^S. \tag{55}$$

The problem is solved, but in order to put it into more convenient form let us introduce the operator :

$$\Lambda_{\pm}(X(x)) \equiv \vec{S} \times \frac{\partial}{\partial x}\vec{X}(x) + (\vec{S} \times \vec{S}_x) \int_{\pm\infty}^{x} \langle\vec{X}(x), \vec{S}_x\rangle dx, \tag{56}$$

$\vec{X}(x)$ being vector field. Then we can formulate our results in the following

Proposition 3.

The LL chain system has the following solution

$$\begin{aligned}
\vec{b}_0 &= \vec{S}, \quad \vec{c}_0 = 0, \\
\vec{b}_1 &= \vec{S} \times \vec{S}_x, \quad \vec{c}_1 = K(\vec{S}), \\
\vec{b}_{n+1} &= \vec{b}^S_{n+1} + \vec{S} \int_{\pm\infty}^{x} \langle \vec{b}^S_{n+1}, \vec{S}_x \rangle dx, \\
\vec{b}^S_{n+1} &= \Lambda_{\pm}(\vec{b}^S_n) + (K(\vec{c}_n))^S, \\
\vec{c}_n &= \sum_{q=1}^{n} (-1)^{q-1} K^{(q)}(\vec{b}_{n-q}), \\
& n = 1, 2, \ldots .
\end{aligned} \tag{57}$$

The operator $\Lambda_{\pm}$ is the so called recursion or generating operator for the Heisenberg ferromagnet equation hierarchy of soliton equations. It was calculated for the first time in [13], see also [14], using the gauge equivalence between Heisenberg Ferromagnet equation and Nonlinear Schrödinger equation [16]. There are at least two other possibilities to arrive to this operator - geometrical, see for example [15], or solving the corresponding chain system for the Heisenberg Ferromagnet equation hierarchy, see [10]. The Heisenberg Ferromagnet (HF) equation is called the following system :

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}. \tag{58}$$

Here $\vec{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))$ is vector field depending on the spatial variable x and the time t, taking its values on the unit sphere $S^2 \subset \mathbf{R}^3$. The boundary conditions for this equation are similar to those for the LL equation :

$$\lim_{x \to \pm\infty} \vec{S} = (0, 0, 1). \tag{59}$$

Formally the HF equation is obtained from the LL equation if $r_i = 0$. Therefore it is natural to expect that when certain parameters (in our case j_i) tend to zero one can obtain from the recursion scheme in LL case the recursion scheme of HF. As it is seen from the above proposition this is indeed the case in our approach. Surprisingly, for the elliptic bundle when the parameters r_i tend to zero one obtains not Λ, but Λ^2, see [9]. However, as we shall see below the hierarchies of equations obtained via elliptic and via polynomial bundle seem to be equivalent in a sense to be described below.

Finally, let us write the first evolution equations from the hierarchy corresponding to the chain system solution that was given in Proposition 3. These equations as mentioned are written into the form

$$\vec{S}_t = -\vec{S} \times b_{N+1}, \quad N = 0, 1, 2, \ldots . \tag{60}$$

We have the following equations :

1. $N = 0$. The first equation in the hierarchy as it often happens is linear.

$$\vec{S}_t = \vec{S}_x. \tag{61}$$

2. $N = 1$. Second equation of the hierarchy.

$$\vec{S}_t = (\vec{b}_1)_x - \vec{S} \times \vec{c}_1 = \vec{S} \times \vec{S}_{xx} - \vec{S} \times K^2(\vec{S}). \tag{62}$$

If we choose $j_i^2 = -r_i; i = 1, 2, 3$ we obtain the Landau-Lifshitz equation.

3. $N = 2$. Third equation of the hierarchy.

$$\vec{b}_2^S = \vec{S} \times (\vec{S} \times \vec{S}_{xx}) - \vec{S} \times (\vec{S} \times K^2\vec{S}) = (-\vec{S}_{xx} + K^2(\vec{S}))^S.$$

$$\vec{b}_2 = (-\vec{S}_{xx} + K^2(\vec{S}))^S + \vec{S}\int_{-\infty}^{x} \langle -\vec{S}_{xx} + K^2(\vec{S}), \vec{S}_x\rangle dx =$$

$$(-\vec{S}_{xx} + K^2(\vec{S}))^S + \frac{1}{2}\vec{S}(\langle K^2(\vec{S}), \vec{S}\rangle - \langle K^2(\vec{S}_0), \vec{S}_0\rangle) - \frac{1}{2}\vec{S}\langle \vec{S}_x, \vec{S}_x\rangle =$$

$$-\vec{S}_{xx} + K^2(\vec{S}) + \vec{S}\left(\langle \vec{S}_{xx}, \vec{S}\rangle - \frac{1}{2}(\langle K^2(\vec{S}), \vec{S}\rangle - \langle K^2(\vec{S}_0), \vec{S}_0\rangle)\right) =$$

$$-\vec{S}_{xx} + K^2(\vec{S}) - \vec{S}\left(\frac{3}{2}\langle \vec{S}_x, \vec{S}_x\rangle + \frac{1}{2}(K^2(\vec{S}), \vec{S}\rangle - K^2(\vec{S}_0), \vec{S}_0\rangle)\right).$$

The corresponding evolution equation is

$$\vec{S}_t = (\vec{b}_2)_x - \vec{S} \times K(\vec{c}_2) = (\vec{b}_2)_x - \vec{S} \times K(K(\vec{b}_1) + K^{(2)}(\vec{S})).$$

But, $KK^{(2)} = j_1 j_2 j_3 \mathbf{1}_3$ and therefore for the equation we have

$$\vec{S}_t = (\vec{b}_2)_x - \vec{S} \times K^2(\vec{S} \times \vec{S}_x).$$

We get

$$\begin{gathered}\vec{S}_t = -\vec{S}_{xxx} - \vec{S} \times K^2(\vec{S} \times \vec{S}_x) + K^2(\vec{S}_x) - \\ \vec{S}_x\left(\tfrac{3}{2}\langle \vec{S}_x, \vec{S}_x\rangle + \tfrac{1}{2}(\langle K^2(\vec{S}), \vec{S}\rangle - \langle K^2(\vec{S}_0), \vec{S}_0\rangle)\right) - \\ \vec{S}\left(3\langle \vec{S}_x, \vec{S}_{xx}\rangle + \langle K^2(\vec{S}), \vec{S}_x\rangle\right),\end{gathered} \tag{63}$$

which after a brief calculation can be put into the final form

$$\begin{gathered}\vec{S}_t = -\vec{S}_{xxx} - 3\vec{S}\langle \vec{S}_x, \vec{S}_{xx}\rangle - \\ \tfrac{3}{2}\vec{S}_x\left(\langle \vec{S}_x, \vec{S}_x\rangle + \langle K^2(\vec{S}), \vec{S}\rangle - \tfrac{2}{3}trK^2 - \tfrac{1}{3}\langle K^2(\vec{S}_0), \vec{S}_0\rangle\right).\end{gathered} \tag{64}$$

If we accept that $\vec{S}_0 = (0, 0, 1)$ we get

$$\begin{gathered}\vec{S}_t = -\vec{S}_{xxx} - 3\vec{S}\langle \vec{S}_x, \vec{S}_{xx}\rangle - \\ \tfrac{3}{2}\vec{S}_x\left(\langle \vec{S}_x, \vec{S}_x\rangle + \langle K^2(\vec{S}), \vec{S}\rangle - \tfrac{1}{3}(2j_1^2 + 2j_2^2 + 3j_3^2)\right).\end{gathered} \tag{65}$$

The above equation differs from the next equation in the LL hierarchy found via the elliptic pairs, see for example [9]. Using our notations this equation can be written as follows :

$$\vec{S}_t = \vec{S}_{xxx} + 3\vec{S}\langle \vec{S}_x, \vec{S}_{xx}\rangle + \tfrac{3}{2}\vec{S}_x\left(\langle \vec{S}_x, \vec{S}_x\rangle + \langle K^2(\vec{S}), \vec{S}\rangle - j_3^2\right). \tag{66}$$

This equation was obtained by Date, Jimbo, Kashivara and Miwa, [17].

4 Discussion

As far as we know the set of polynomial Lax pairs for the CF hierarchy was not presented until now. Almost the same is true for the corresponding hierarchy of equations, because from the results of [11] it is very difficult to obtain the corresponding hierarchy of soliton equations. Therefore there is little possibility to compare our result with other ones. For the LL case however the corresponding hierarchy of soliton equations obtained via elliptic bundle exists, [9]. Let us briefly describe the situation about the LL hierarchies obtained via elliptic and via polynomial bundle. The first nonlinear evolution equation in both hierarchies coincide, it is the LL equation. The second nonlinear equations however are different. Nevertheless one can say that up to the third equation both hierarchies are equivalent. Indeed, the equations in the hierarchies have the form

$$\begin{aligned} \vec{S}_t &= \vec{X}_n(\vec{S}, \vec{S}_x, ...), \\ \vec{S}_t &= \vec{Y}_n(\vec{S}, \vec{S}_x, ...), \\ n &= 1, 2, \ldots, \end{aligned} \tag{67}$$

the right hand sides of them being vector fields on the infinite dimensional manifold of "potentials", i. e the set of functions $\vec{S}(x)$. The hierarchies would be equivalent not only if $\vec{X}_n = \vec{Y}_n, \quad n = 1, 2, \ldots$, but also in the case when every $\vec{X}_n$ is finite linear combination with constant coefficients of the fields $\vec{Y}_n$. For example if we denote by $\vec{Y}_n$ the fields obtained via polynomial bundle then the field corresponding to the equation of Date, Jimbo, Kashivara and Miwa can be written as

$$-\vec{Y}_3 + (j_1^2 + j_2^2)\vec{Y}_1. \tag{68}$$

We believe that both hierarchies are equivalent in the sense mentioned above but of course the question about the equivalence of both hierarchies remains open.

We also leave for the future the questions about the Hamiltonian structures of the equations in LL and CF hierarchies and about the commutativity of the corresponding flows. These results will be published elsewhere.

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References

[1] Faddeev L. D. . Takhtadjan (1987) Hamiltonian Method in the Theory of Solitons. (Berlin : Springer Verlag).

[2] L. A. Bordag. A. B. Yanovski (1995)*J. Phys.* **A**, Math. Gen. , **28**, p. 4007-4013.

[3] Landau L. D. and Lifshitz E. M. (1935)In : Collected papers of L. D. Landau. Ed. D. ter Haar, p. 101, New York, Pergamon, Gordon and Breach (1965)

[4] Vesselov A. P. (1983) *DAN SSSR* **270** p. 1094-1097.

[5] Vesselov A. P. (1984) *DAN SSSR* **276** p. 590-593

[6] Borovik A. E. (1978) *Lett JETP* **28**, N10 p. 629-632; **35**, N7 p. 3687-3723

[7] Sklianin A. E. (1979) *Steklov Math. Institute LOMI Preprint E-3-79.*

[8] Cherednik I. V. (1981) *Journal of Nucl. Phys.* **33** 278-282

[9] Barouch E. , Fokas A. S. and Pageorgiou V. G. (1987) Clarkson University preprint, Potsdam, NY 13676

[10] Barouch E. , Fokas A. S. and Papageorgiou V. G. (1988) *J. Math. Phys.* **29**, N12, p. 2628-2633

[11] Sidorenko U. N. (1987) *Zap. LOMI* **161**, p. 76-87

[12] Fuchssteiner (1984) *Physica* **13 D**, p. 387-394

[13] Gerdjikov V. S. and Yanovski A. B. (1984) *Phys. Lett* **103 A**, p. 232-236

[14] Gerdjikov V. S. and Yanovski A. B. (1984) *Comm. Math. Phys.* **103**, N4, p. 549-568

[15] Magri F. , Morosi C. , Ragnisco O. (1985) *Comm. Math. Phys.* , **99**, p. 115-140.

[16] Zakharov V. E. and Takhtadjan (1979) *Teor. i Mat. Fis.* **38**, p. 26-37

[17] Date E. , Jimbo M. , Kashiwara M. and Miwa T. (1983) *J. Phys.* **A16**, p. 221-236