CRN 95-37

About the determination of critical exponents related to the fragmentation of finite systems

B. Elattari[1,2], J. Richert[1,3], P. Wagner[1]
and
A. Nourredine[2]

1) Centre de Recherches Nucléaires, IN2P3/CNRS and Université Louis Pasteur, BP28, 67037 Strasbourg Cedex 2, France

2) Université Bouchaïb Doukkali, Eljadida, Morocco

3) Laboratoire de Physique Théorique, 3 rue de l'Université, 67084 Strasbourg Cedex, France

Abstract. The outcome of percolation models and other approaches shows that the fragmentation of nuclei is possibly characterized by a phase transition whose effects are smoothed by the finite size of the systems. We review and discuss different methods aimed to extract critical exponents related to physical observables which get singular at the critical point in the corresponding infinite system.

PACS : 25.70 Pq , 05.70 Jk

1. Introduction

The success of percolation models [1-3] and recent phase space descriptions of fragmenting nuclei [4] raises the question of the existence of a phase transition which may govern the onset of the fragmentation regime induced by means of energetic nucleon-nucleus or nucleus-nucleus collisions. Whatever the type of phase transition which may be at work, its effects are of course smoothed by the small size of nuclei. Hence finite size effects may be strong and make it a priori difficult to extract associated critical exponents from the knowledge of physical observables related to fragment size distributions [1].

Here we present attempts to work out the critical exponents for percolation models characterized by a bond probability p. In practice, the methods may also be applied in the framework of other approaches like the phase space descriptions of ref. [4]. The paper is divided into two parts. We present and discuss first methods which from our experience are not reliable or even do not work. We do this in order to show the pitfalls and difficulties related to these methods. In a second step we present a reliable and successful method which allows to extract critical exponents characterizing the infinite case even if the considered systems are very small.

2. Unsuccessful methods

We tried to work out the value of the exponents β and γ [1] which characterize the behaviour of $A_{\mathrm{max}}(p) = A - m_1(p)\ \alpha\ (p - p_c)^{\beta}$ and $m_2(p)\ \alpha\ \mid p - p_c \mid^{-\gamma}$ in the infinite system close to the critical bond probability p_c [7]. Here m_1 and m_2 are the expressions of the first and second moment of the fragment size distribution [1], A is the total number of particles and A_{max} the average size of the largest fragment in events with probability p.

2a. Method proposed by Elliott et al. [5]

In order to get γ from the knowledge of the second moment $m_2(p)$ one chooses intervals of p on the left and right hand side of the maximum of $m_2(p)$ which corresponds to p_c in the infinite system. These intervals are chosen following the prescriptions of ref. [5] and correspond to values of p not too close to p_c where the curvature of m_2 for finite systems is no longer in agreement whith the singular behaviour of this quantity in the infinite system. By matching the right and left contributions one gets p_c and the slopes of ln m_2 as a function of ln $\mid p - p_c \mid$ allow in principle to fix γ.

We worked out p_c and γ for systems of different sizes corresponding to cubes of size A in 3-dimensional space. Typical results for different p-interval sizes $\Delta p_\ell = \mid p - p_c \mid_\ell$ on the left and $\Delta p_r = \mid p - p_c \mid_r$ on the right side of the maximum p_c are shown in Table 1. For large systems ($A = 63^3$) we obtained $p_c \sim 0.25 - 0.26$ and $\gamma \sim 1.85 - 2.0$. In the case of small systems ($A = 6^3$ f.i.) the dependence of p_c and γ on the choice of the intervals gets strong and we were unable to reproduce the results of ref. [5], the values of γ are systematically larger than those found there ($\gamma \simeq 1.75$).

In order to analyze experimental results, p should be replaced by the fragment multiplicity m [1]. Such an analysis has been performed recently. Incidentally it should be realised that there exists no one-to-one correspondence between p and m and second, the relationship between p and some average m is non linear, particularly in the vicinity of $p \sim p_c$. Hence $m_2(m) \sim \mid m - m_c \mid^{-\widetilde{\gamma}}$ for $m \simeq m_c$, maximum of m_2 and, in principle, $\widetilde{\gamma} \neq \gamma$. We tried to work out m_c and $\widetilde{\gamma}$. For $A = 63^3$ $m_c \simeq 0.2634$ and $\widetilde{\gamma} \sim 2.15$. Depending on the choice of the left and right hand side intervals in m, one observes changes in m_c and sizable variations in $\widetilde{\gamma}$ which can go down to 1.98. For $A = 6^3$ $\widetilde{\gamma}$ lies between 1.5 and 1.8 depending on the choice of the intervals.

2b. The slope method

We used also the correlation between ln $A_{\max}(p)$ and ln m_2/m_1 where $A_{\max}(p)$ has been defined above [1]. The graph shows two branches with an extremum and the slope of the lower branch is given by $\lambda = 1 + \beta/\gamma$. We tried to extract β/γ for systems of different sizes $A = 63^3, 6^3, 5^3$ and 4^3. We obtained respectively $\lambda = 1.115, 1.39, 1.44$ and 1.53. For $A = 63^3$ λ is smaller then the slope expected for the infinite system, 1.25. In a second step we have extracted β from the behaviour of $A_{\max}(p)$ in the vicinity of p_c, see above. The same procedure allows to get $\widetilde{\beta}$ corresponding to $A_{\max}(m)$ α $\mid m - m_c \mid^{\widetilde{\beta}}$. It comes out that $\beta\,(\widetilde{\beta})$ depends very sensitively on the exact location of the maximum $p_c(m_c)$. For example, $0.38 \leq \beta \leq 0.45$ if $0.25 \leq p_c \leq 0.254$ and $A = 63^3$. The problem gets worse when A is small since $p_c(m_c)$ is more and more difficult to locate because of the shallow and smooth behaviour of moments m_k. We find for instance $0.256 \leq \widetilde{\beta} \leq 0.469$ when $0.208 \leq m_c \leq 0.277$ for $A = 6^3$. The results combined with the strong size dependence of λ makes it quite impossible to extract reliable values of $\gamma\,(\widetilde{\gamma})$. For $A = 6^3$ we find $0.656 \leq \widetilde{\gamma} \leq 1.228$.

2c. Padé approximants

We worked out a third method in order to determine γ. Starting from $\ln m_2 \; \alpha - \gamma \ln \mid p - p_c \mid$ one can formally develop the log expression for $p \sim p_c$. This leads to an infinite diverging series. Following an idea developped in ref. [7], one can construct the $[n, m]$ Padé approximants to this series, the ratio of polynomials of order n and m in the numerator and the denominator. We first checked that all approximants with $n, m \geq 2$ reproduce very well $\ln m_2$ for $p \sim p_c$. In a second step we worked out, by means of MAPLE, the algebraic expressions of the Padé approximants $[n, m]$ $\ln m_2(p) \simeq \sum_{i=0}^{n} b_i(C, p_c, \gamma) p^i / \sum_{i=0}^{m} c_i(C, p_c, \gamma) p^i$ corresponding to $m_2(p) \sim C \mid p - p_c \mid^{-\gamma}$. We then used numerical values of $m_2(p)$ obtained from percolation simulations in order to determine C, p_c and γ from a set of $10^2 - 10^3$ values of p by means of a least square fit procedure. In practice we used essentially the $[2, 2]$ approximant. Higher approximants give similar results. If we keep C, p_c and γ a priori unconstrained we find, for decreasing intervals of values of p on the left side of $p_c, 0.15 \leq p \leq 0.25$ to $0.24 \leq p \leq 0.25$ and $A = 63^3$, values of p_c which are too small and decreasing (from 0.2265 to 0.2218) and also decreasing values of γ (from 1.19 to 1.02). If one constrains arbitrarily γ to lie in a fixed interval, say $(1.5 - 2.5)$, p_c in $[0.2, 0.4]$ one finds, for $A = 63^3$ $p_c \sim 0.23 - 0.24$ and $\gamma \sim 1.5 - 1.6$. In all cases χ^2 tests lie between 1 and 10 and decrease with the size of the intervals Δp. We applied this method also to small systems, $A = 6^3$. For an unconstrained choice of parameters and decreasing intervals p_c decreases from unreasonable values $(0.4 - 0.5)$ for p in $[0.1 - 0.225]$ to $p_c \simeq 0.226$ for p in $[0.235 - 0.24]$ and γ decreases at the same time from 2.8 to 1.3. Typical results are shown in Table 2. These inconsistent results show that this third method is not working either. Other attempts to fix γ [8] failed in similar ways. Hence it seemed to us that an essential ingredient was missing in the procedures which were used up to now. An improved method which solves the problem is developped below.

3. Introduction of corrections and reliable approach to the determination of critical exponents

In the former section we tried to fix critical exponents from finite size systems using the expressions of physical observables which correspond to the description of the corresponding infinite system. In fact, doing this, we neglected the fact that we deal with finite systems and should take care of this fact on the physical observables. We tried to introduce these effects by postulating that [7] the fragment

size distribution $n_i(p)$ can be well reproduced for $p \sim p_c$ by an expression of the form

$$n_i(p) = i^{-\tau} f\left((p - p_c)\, i^{\sigma}\right) \tag{1}$$

where τ is the power law index at $p = p_c$, σ a further parameter and f a smooth function which verifies $f(0) = 1$. Using this expression one can write moments

$$\begin{aligned} m_k(p) &= \sum_{i=1}^{A_{\max}} i^k\, n_i(p) \\ &= \sum_{i=1}^{A_{\max}} i^{k-\tau} f_i((p - p_c)\, i^{\sigma}) \end{aligned} \tag{2}$$

In the sequel we shall use this expression in order to fix exponents γ and β for systems of all sizes.

3a. The method

We start from (1) and fix $p_{\max}$ as the maximum of $m_2(p)$ which is obtained from numerical calculations. From the expression of $n_i(p_{\max})$ we read off the value of τ, in the interval I of i where n_i follows an exact power law (recall that finite size effects distort this power law behaviour for large values of i). The exponent τ could also be obtained from the plot of $\ln m_3/m_1$ *vs.* $\ln m_2/m_1$, see ref. [1]. We then construct f_i, from the same interval I of values of i and varying σ over a given interval of values around $\sigma \simeq 0.5$ in such a way that the corresponding $f_i's$ coincide, i.e. f_i behaves like a universal function of $z = (p - p_{\max})\, i^{\sigma}$ for i in I. A more rigorous procedure would consist in fixing σ by means of a least square fit. We noticed that in every case we considered the fit procedure is not very sensitive to σ, see below. Once we know the "universal" function $f(z_i) = n_i(p)/n_i(p_{\max})$, $z_i = (p - p_{\max})\, i^{\sigma}$ we construct the moments from the analytical expressions

$$m_k(p) = \sum_{i=1}^{A-1} i^{k-\tau} f(z_i) \tag{3}$$

From the graph of $m_k(p)$ as a function of p we read p_c as the maximum of $m_k(p)$. This quantity differs from the numerically determined $p_{\max}$ above. It is smaller than the former quantity, getting closer to the p_c which corresponds to the location of the singularity of $m_k(p)\,(k \geq 2)$ in the infinite system. Finally we use the $m_k's$ given by (3) in order to determine γ (for $k = 2$) and β (for $k = 1$; $A_{\max} = A - m_1(p), m_1(p) \sim C\,(p - p_c)^{\beta}$). This is done by means of the method developped by Elliott et *al.* [5],

i.e. by choosing intervals of p on the left or (and) right side of p_c and fixing γ (resp. β) from the slope of $\ln m_2(p)$ (resp. $\ln A_{\max}$) represented as a function of $\ln \mid p - p_c \mid$. It comes out that this procedure leads to consistently and well defined values of γ and β for systems of any size. Contrary to the case presented in section 2a, the results are largely insensitive to the choice of the intervals, as long as p does not enter the region close to p_c which corresponds to a change of curvature of $m_k(p)$ due to the finite size of the system.

3b. Application of the method to the determination of γ and β

Here we present results which are obtained by implementing the method described above. We considered systems of different sizes. For all of them we obtained exponents which are reliable and close to the known values for the infinite system. Here we discuss the extreme case $L = 4^3$. As we saw in section 2, the smaller the system the more it gets difficult to extract the exponents. Following the method of sect. 3a we read $p_{\max}$ from the numerically determined $m_2(p), 0.303 \leq p_{\max} \leq 0.3135$. We did not work out σ from an explicit fit of $f(z_i)$ but chose it to lie in the interval $0.50 < \sigma < 0.55$. Such a procedure could of course easily be implemented and it would fix σ to a definite value. Here we left σ as an open parameter in order to test the sensitivity of $f(z_i)$ to variations of this parameter. As already mentioned, it comes out that $f(z_i)$ is not very sensitive to σ. In a further step we fixed τ as the slope of $\ln n_i(p)$ *vs.* $\ln i$ in the interval of mass where this function is linear. This is the case for clusters of intermediate size. We find $\tau \simeq 2.05$, somewhat lower than the value expected for the infinite system.

From the knowledge of σ and τ we determined the "universal" function $f(z_i)$ shown in Fig. 1. This function is easily fitted by a gaussian in the neighbourhood of its maximum. Then we constructed the analytical expression of $m_2(p)$ from (2) which is represented in Fig. 2. The actual maximum lies at $0.286 \leq p_c \leq 0.290$, somewhat lower than $p_{\max}$. Finally the prescription of Elliott et *al.* [5] is used in the interval of values of i where $\ln m_2(p)$ is a linear function of $\mid p_c - p \mid$, see Fig. 3. We obtained perfect straight lines with no fluctuations, on the left as well as on the right side of p_c. The slope of this curve gives γ and comes out to be the same on both sides of p_c. We found $\gamma = 1.74$ for $\sigma = 0.55$ and $\gamma = 1.796$ for $\sigma = 0.52$. Furthermore these results are of course insensitive to the choice of the intervals $\mid p - p_c \mid$, as long as these intervals are chosen in the region of the straight line behaviour shown in Fig. 3, $0.16 \leq p \leq 0.20$. Finally Fig. 4 shows the behaviour of $\ln A_{\max}(p)$ *vs.* $\ln(p - p_c)$ where p lies on the right hand side of p_c. Here too

we obtain a perfect straight line with no fluctuations for $0.39 \leq p \leq 0.47$. For the example shown one obtains $\beta \sim 0.438$. It varies, for $0.303 \leq p_{\max} \leq 0.3135$ and $0.52 \leq \sigma \leq 0.55$ between $\beta = 0.385$ and $\beta = 0.455$. The precision with which γ and β are determined is of the order of ± 0.05 for the case $L = 4^3$.

The same type of calculations has been repeated for systems of size $L = 6^3$. The conclusions are the same if one uses the same procedure as above. Typical values of γ lie between 1.7 and 1.8 and $\beta's$ around 0.4. Results are stable with respect to the choice of intervals in which ln m_2 behaves linearly without any fluctuation, like it is seen for $L = 4^3$ in Figs. 3 and 4.

3c. Exponents related to multiplicities

We have also used the method of section 3b in order to determine the exponents $\widetilde{\gamma}$ and $\widetilde{\beta}$ which define ln $m_2(m)$ α ln $\mid m - m_c \mid^{-\widetilde{\gamma}}$ and ln $A_{\max}(m)$ α ln $\mid m - m_c \mid^{\widetilde{\beta}}$ at the critical multiplicity m_c from finite systems. It is interesting to work out these exponents since they can in principle be obtained from the experiment [6]. We constructed and obtained the "universal" function $\widetilde{f}((m - m_c)i^{\widetilde{\sigma}})$ in the same way as above, see Fig. 4. From the knowledge of $\widetilde{f}$ we constructed $m_2(m)$ and $m_1(m)$ in order to get $\widetilde{\gamma}$ and $\widetilde{\beta}$. Typical results for $L = 6^3$ give $\widetilde{\sigma} = 0.45$ $\widetilde{\gamma} \simeq \mid 1.95$ and $\widetilde{\beta} = 0.36$. Hence $\widetilde{\gamma}$ and $\widetilde{\beta}$ are somewhat different from γ and β calculated with p which is not surprising in view of the non linear relations between p and m, see above.

4. Conclusions

We presented and discussed several methods which allow in principle to extract critical exponents from observables related to finite systems in the framework of bond percolation. These exponents are interesting quantities which may be useful in order to characterize the criticality of nuclear fragmentation processes. From our experience this is not an easy task because nuclei are small systems. Several methods presented in section 2 are unreliable and lead to inconclusive results.

In section 3 we worked out a method which takes account of finite size effects in the vicinity of the critical point corresponding to the infinite system into account. It involves not only the exponents γ and β related to the second moment m_2 and $A_{\max}$ but also the exponent τ and the coefficient σ. This method can be further improved as far as the determination of the coefficients σ and τ are concerned. Numerical tests show that our procedure is able to give reliable exponents even if one works with

systems as small as 4^3. It works in the framework of bond percolation models, as well if one works with the bond probability p as with the experimentally observable fragment multiplicity m.

It is now tempting to use this method in order to investigate the outcome of the phase space approach to fragmentation worked out in ref. [4]. Since the behaviour of m_2 and A_{max} are very close if not identical to those obtained with bond percolation models we expect that the exponents will come out to be very close to those obtained in the percolation framework. Last but not least, the present procedure can be applied to the analysis of experimental fragment size distributions like those analyzed in ref. [6] and hopefully those from future experiments in order to fix the precise nature of a possibly existing phase transition in nuclear fragmentation at high energy.

References

[1] X. CAMPI : *J. Phys. A. : Math. Gen.* **19**, L917 (1986) ; *Phys. Lett.* **B208**, 351 (1988)

[2] U. LYNEN ET AL. : *Nucl. Phys.* **A545**, 329c (1992) ; see also preprint 05-94, GSI, Darmstadt

[3] P. DÉSESQUELLES ET AL. : *Phys. Rev.* **C48**, 1828 (1993)

[4] B. ELATTARI, J. RICHERT, P. WAGNER, Y.M. ZHENG : *Phys. Lett.* **B356**, 181 (1995) ; preprint CRN 95-20, to appear in *Nucl. Phys. A*

[5] J.B. ELLIOTT ET AL. : *Phys. Rev.* **C49**, 3185 (1994)

[6] M.L. GILKES ET AL. : *Phys. Rev. Lett.* **73**, 1590 (1994)

[7] D. STAUFFER AND A. AHARONY : Introduction to Percolation Theory, Taylor and Francis (1994)

[8] A. WOERNER : ALADIN-Collaboration, GSI Scientific Report 1994, p.49

Table 1 : Behaviour of p_c and γ as a function of the system size and the choice of intervals in p (see text). The lower and upper boundary of Δp_ℓ and Δp_r respectively are given in columns 2 and 3. The quantities χ^2_ℓ and χ^2_r which give an appreciation of the quality of the fit correspond to intervals on the left and right hand side of p_c respectively.

L	Δp_ℓ	Δp_r	p_c	γ	χ^2_ℓ	χ^2_r
63	$0.01 - 0.04$	$0.01 - 0.09$	0.255	1.97	5	12.8
63	$0.015 - 0.03$	$0.015 - 0.06$	0.254	1.93	1.6	1.5
63	$0.015 - 0.05$	$0.015 - 0.035$	0.254	1.865	4.2	1.3
6	$0.068 - 0.135$	$0.068 - 0.127$	0.346	1.935	1.82	1.06
6	$0.078 - 0.145$	$0.078 - 0.137$	0.344	2.025	1.52	1.03
6	$0.058 - 0.125$	$0.058 - 0.117$	0.352	1.825	2.95	1.07

Table 2 : Determination of C, p_c and γ as a function of the system size and the choice of intervals in p (see text). Δp is the size of the interval in p. χ^2 gives an appreciation of the quality of the fits.

	L	Δp	C	p_c	γ	χ^2
a	63	0.08	0.51×10^5	0.2256	1.16	2713/898
	63	0.05	0.75×10^5	0.2218	1.017	105/98
b	63	0.04	0.19×10^5	0.232	1.5	2970/398
	63	0.02	0.187×10^5	0.241	1.6	303/198
c	6	0.08	39	0.4835	2.31	36/38
	6	0.04	56.44	0.4	1.6	15/18

a) unconstrained calculations (see text)

b) constrained calculations (see text)

c) unconstrained calculations (see text)

Figure caption

Fig. 1 : The "universal" function $f(z_i)$ obtained from $n_i(p)$ for values of i in which $\ln n_i(p)$ shows a straight line behaviour as a function of $\ln i$ (fragments of intermediate size). The full line corresponds to a gaussian fit of $f(z_i)$. The system size is $L = 4^3$. See text.

Fig. 2 : Representation of the analytical expression of $m_2(p)$ given by (2) for $L = 4^3$. The exponent $\tau = 2.05$, the parameter $\sigma = 0.52$ and $f(z_i)$ have been determined as explained in the text. The arrow shows the location of $p_c = 0.285$.

Fig. 3 : $\ln m_2(p)$ as a function of $\ln \mid p_c - p \mid$ for $L = 4^3$. The slope gives $-\gamma$. See explanations in the text.

Fig. 4 : $\ln A_{\max}(p)$ as a function of $\ln(p - p_c)$, $p > p_c$ for $L = 4^3$. The slope gives β. See explanations in the text.

Fig. 5 : The "universal" function $f(z_i)$ obtained from $n_i(m)$ with the same procedure as in Fig. 1. Here the system size is $L = 6^3$.

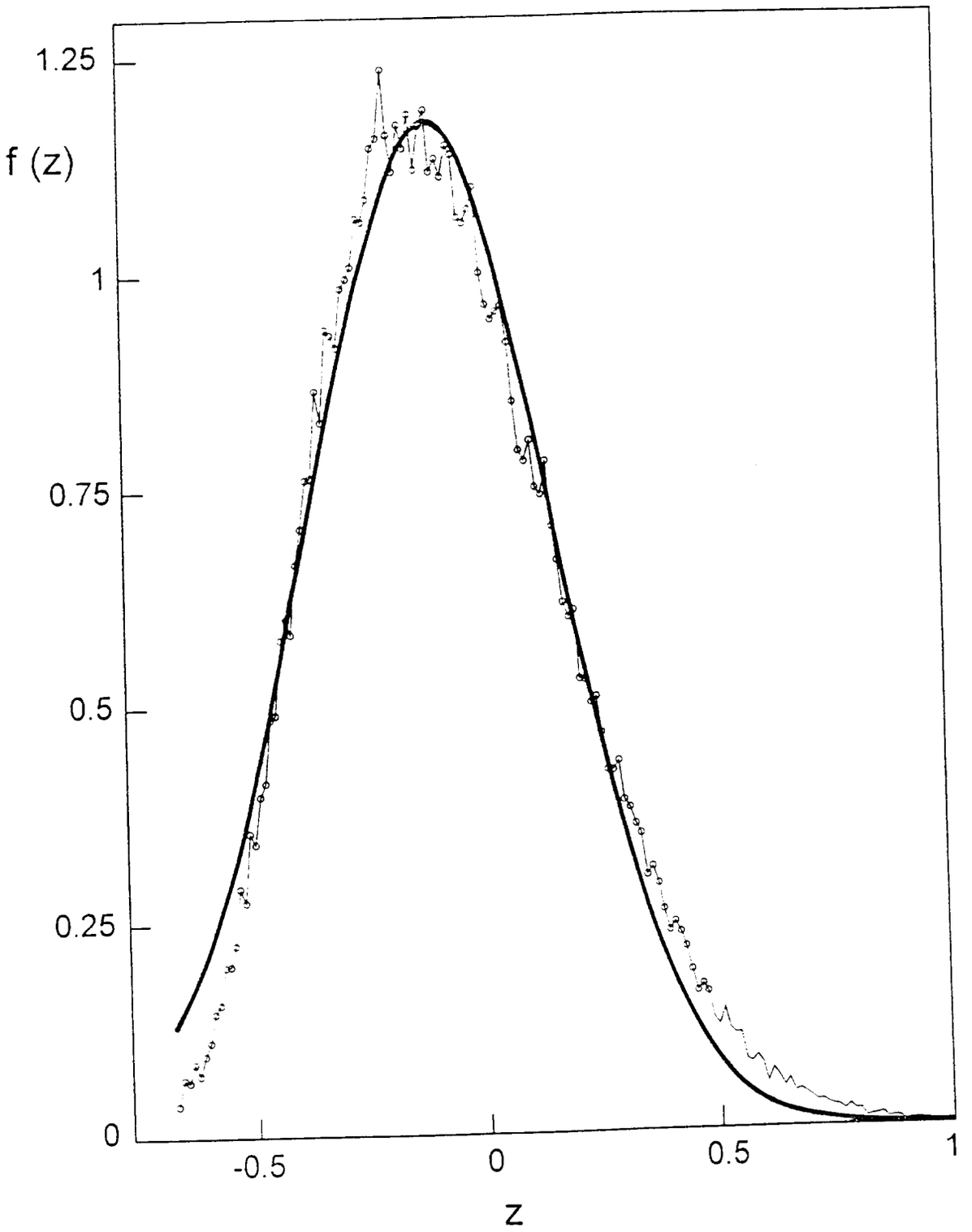

Fig. 4

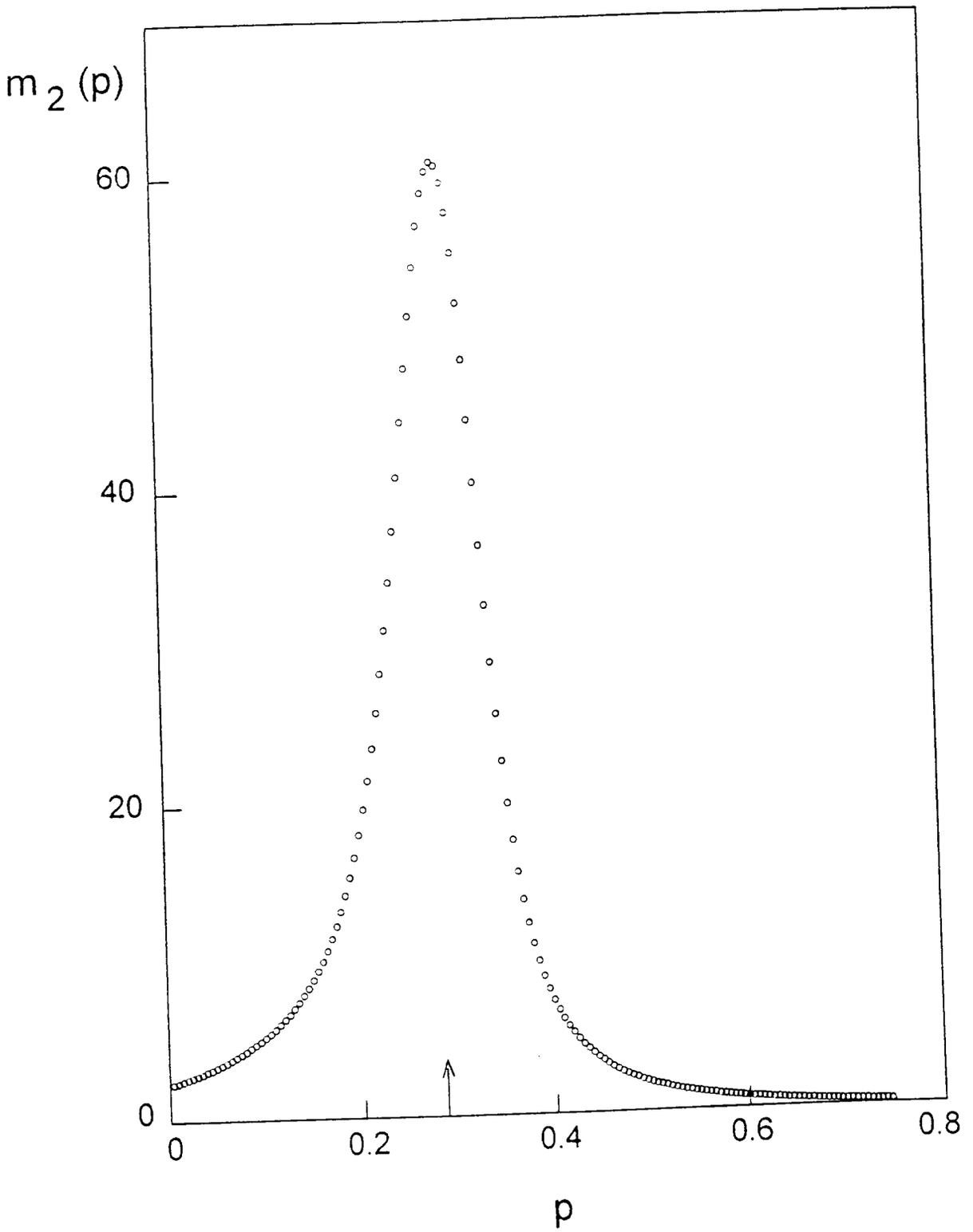

Fig. 2

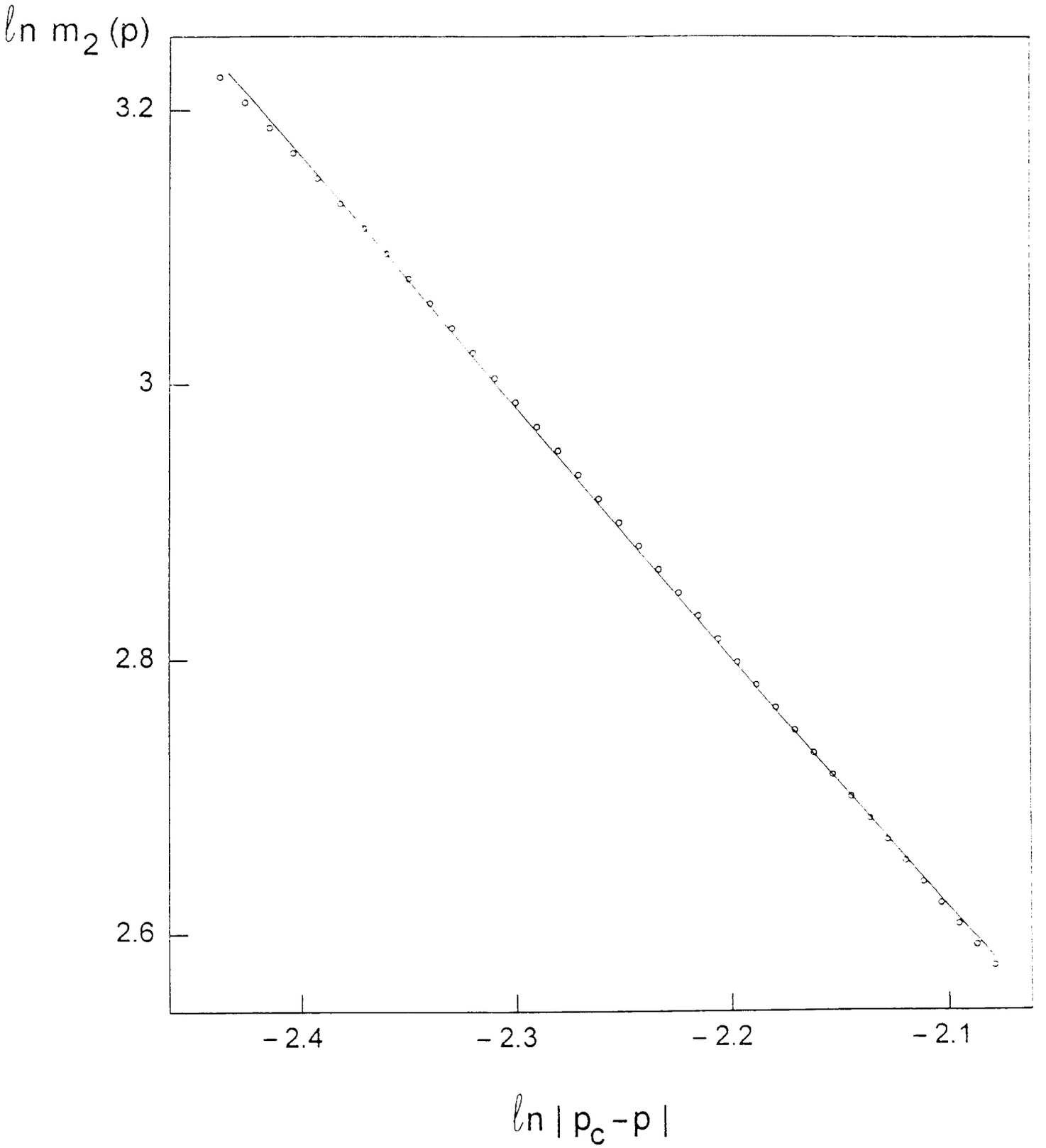

Fig. 3

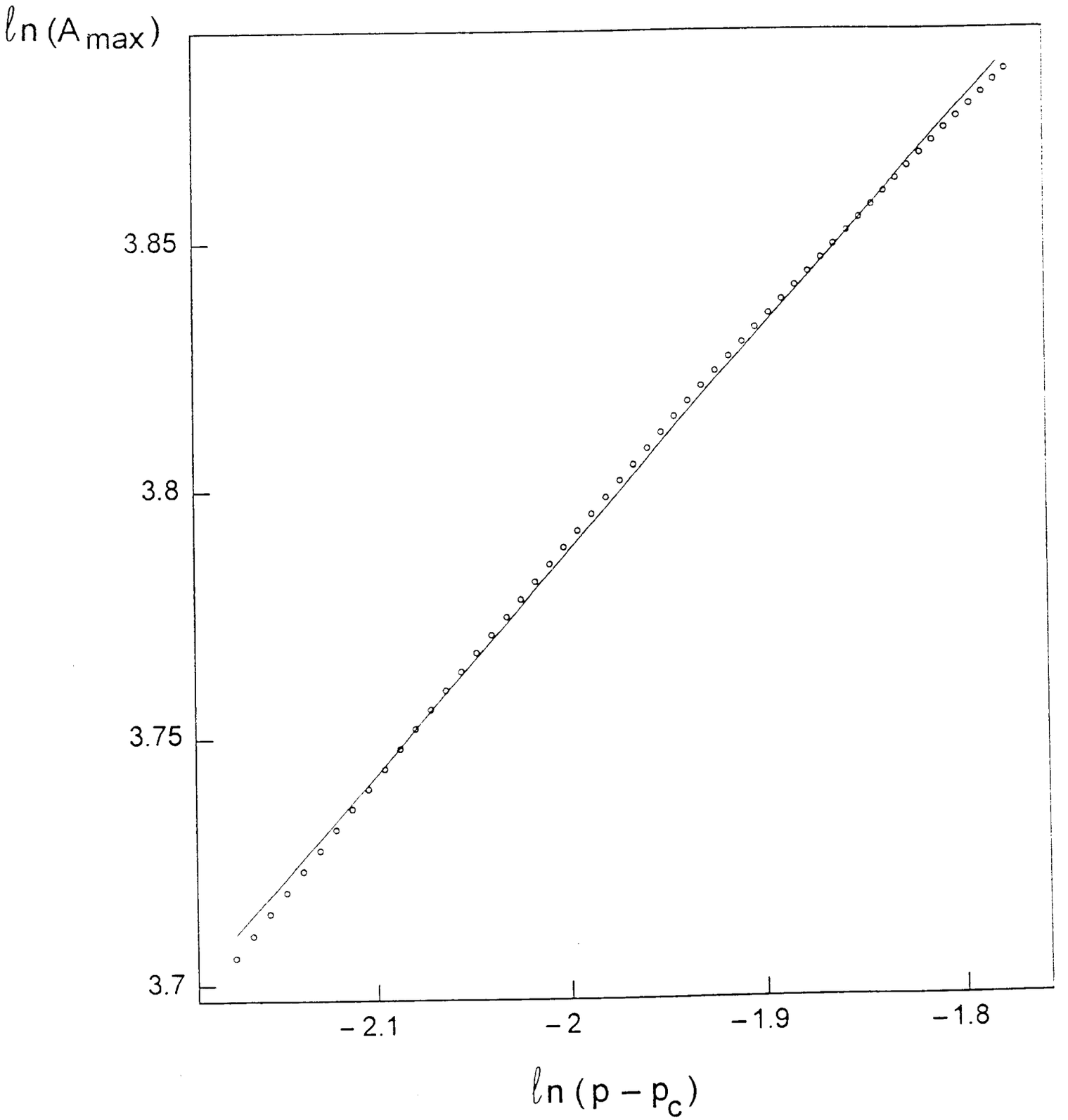

Fig. 4

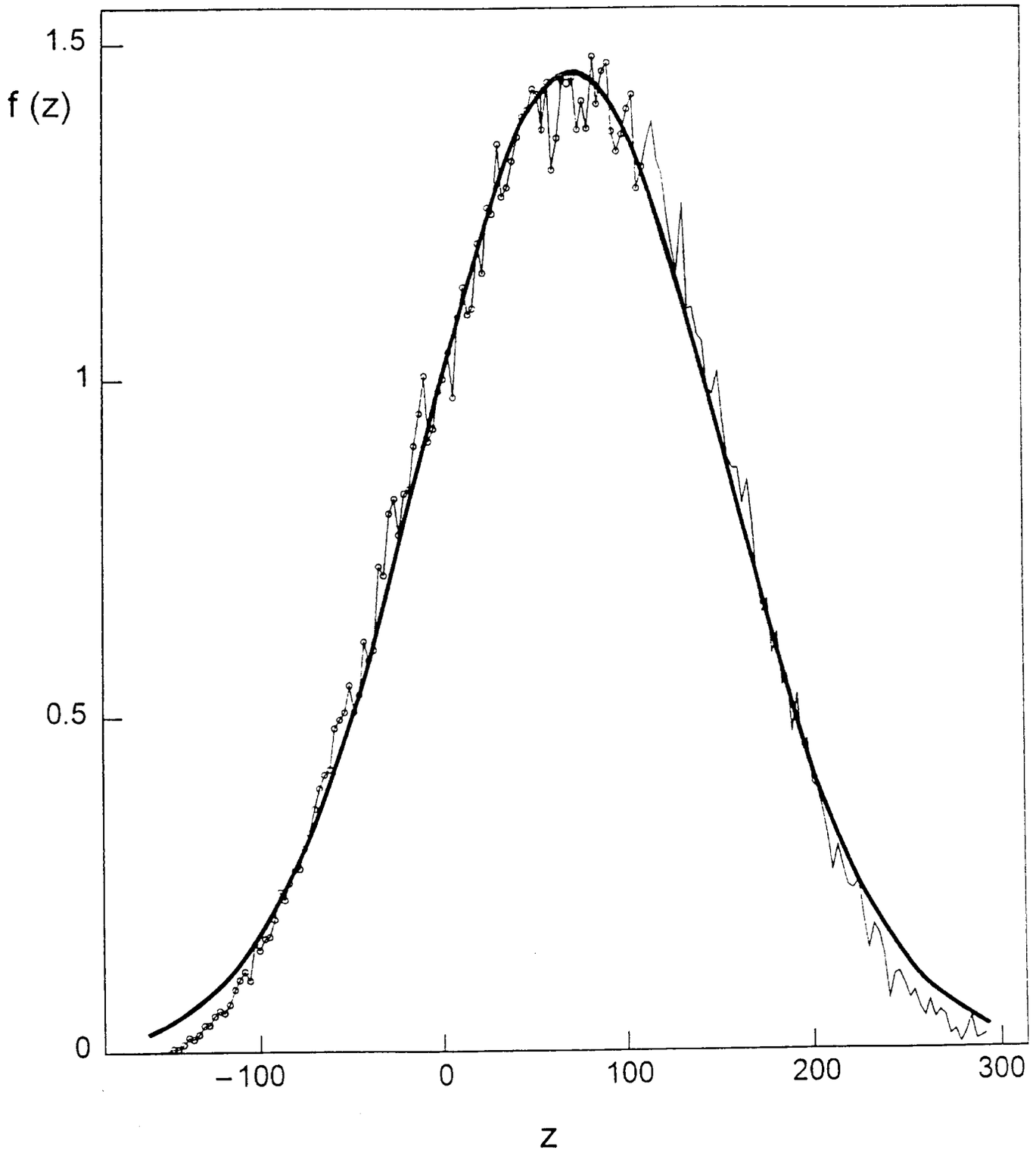

Fig. 5