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A Symplectic Six-Dimensional Thin-Lens Formalism for Tracking

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Abstract

In this paper we introduce a thin–lens formalism for tracking particles in accelerators and storage rings. It is shown how to solve the (six–dimensional) nonlinear canonical equations of motion for various kinds of magnets and for cavities in a straightforward manner by using symplectic kicks. In particular a thin - lens representation of the transfer matrix for solenoids is derived. The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy. This formalism has been used to extend the tracking code SIXTRACK [1] to allow the treatment of both thick and thin linear elements.

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Contents

1 Introduction

The intuitive approach to modelling an accelerator is to treat the elements like drifts, dipoles, quadrupoles and others by giving them their correct length (see [4, 5]). However, in very large machines like for instance the LHC, a hadron collider currently in its design stage, the curvature of the dipoles is very small and the length of individual elements is negligible compared to the total length of the accelerator. Moreover dynamic aperture studies are very time–consuming even on state of the art computer farms. It has therefore been desirable to approximate the long linear elements by drifts and linear point–like kicks. Of course we want to make sure that the thin linear lens approximation in the six-dimensional case uses as few operations as necessary while fulfilling the symplecticity conditions. The aim of this study is to show how this approximation can be done and to demonstrate its inherent symplecticity.

In detail, the paper is organized as follows:

In the second chapter the general canonical equations of motion are derived. Using the thin–lens approximation the equations of motion are solved for each element in chapter three. A summary of the results is presented in chapter four. Appendix A treats in detail the symplecticity condition and its relation to the canonical structure of the equations of motion. In Appendix B the tracking results are compared for a thin and thick lens lattice of the LHC. Finally Appendix C gives some useful formulae used in this paper.

2 The Canonical Equations of Motion

The aim of this chapter is to derive the canonical equations for particle motion in storage rings by a simultaneous treatment of synchrotron and betatron oscillations, taking into account all kinds of coupling induced by skew quadrupoles and solenoids (coupling of betatron motion planes) and by non-vanishing dispersion in the cavities (synchro–betatron coupling). Starting from the Hamiltonian in a fixed Cartesian coordinate system (section 2.1) and introducing the natural coordinates x, z, s, (2.2) combined with two additional variables σ and η which describe synchrotron motion, the Hamiltonian expressed in machine coordinates may be obtained by the application of suitable canonical transformations (section 2.3). The particle motion can then be conveniently calculated (2.4) to various orders of approximation by expanding this Hamiltonian into a power series. In this report we shall use an approximation in which the effect of relative energy deviation on the focusing strengths is automatically accounted for. The equations of motion for various kinds of magnets and for cavities are presented in section 2.5. The solutions of these equations in the thin–lens approximation are derived in chapter 3.

2.1 The Starting Hamiltonian

The starting point of the description of classical dynamics in storage rings will be the classical Hamiltonian[1], $\mathcal{H}$:

$$\mathcal{H}(\vec{r},\vec{P},t) \quad = \quad c \cdot \left\{ \vec{\pi}^2 + m_0^2 c^2 \right\}^{1/2} + e\phi \tag{2.1}$$

[1]In this report we use the CGS unit system.

where $\vec{r}$ and $\vec{P}$ are canonical position and momentum variables and where the kinetic momentum vector $\vec{\pi}$ is given by:

$$\vec{\pi} = \vec{P} - \frac{e}{c}\vec{A} . \tag{2.2}$$

The quantities $\vec{A}$ and ϕ appearing in eqn. (2.1) are the vector and scalar potentials from which the electric field $\vec{\mathcal{E}}$ and the magnetic field $\vec{\mathcal{B}}$ are derived as:

$$\vec{\mathcal{E}} = -\text{grad}\ \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} ; \tag{2.3a}$$

$$\vec{\mathcal{B}} = \text{curl}\ \vec{A} . \tag{2.3b}$$

In terms of the three unit cartesian coordinate vectors in the fixed laboratory frame, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ we can write $\vec{r}$ and $\vec{P}$ as:

$$\vec{r} = X_1 \cdot \vec{e}_1 + X_2 \cdot \vec{e}_2 + X_3 \cdot \vec{e}_3 ; \tag{2.4a}$$

$$\vec{P} = P_1 \cdot \vec{e}_1 + P_2 \cdot \vec{e}_2 + P_3 \cdot \vec{e}_3 . \tag{2.4b}$$

With this Hamiltonian (2.1) the orbital equations of motion are:

$$\frac{d}{dt} X_k = +\frac{\partial \mathcal{H}}{\partial P_k} ; \tag{2.5a}$$

$$\frac{d}{dt} P_k = -\frac{\partial \mathcal{H}}{\partial X_k} ; \tag{2.5b}$$

$$(k = 1, 2, 3) .$$

2.2 Reference Trajectory and Coordinate Frame

The position vector $\vec{r}$ of the particle in eqn. (2.1) refers to a fixed coordinate system with the coordinates X_1, X_2 and X_3. However, in accelerator physics, it is useful to describe the motion in terms of the natural coordinates x, z, s in a suitable curvilinear coordinate system. With this in mind we assume that an ideal closed design orbit exists describing the path of a particle of constant energy E_0 (neglecting of course energy variations due to cavities and radiation loss and assuming that there are no field errors or correction magnets). We also assume that the design orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has no torsion. The design orbit which will be used as the reference system will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}(s)$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s) . \tag{2.6}$$

The vector $\delta\vec{r}$ can as usual be described using an orthogonal coordinate system ("dreibein") accompanying the particles and comprising

$$\begin{aligned}
\text{a unit tangent vector} \quad & \vec{e}_s(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r_0}'(s) ; \\
\text{a unit normal vector} \quad & \vec{e}_N(s) ; \\
\text{and a unit binormal vector} \quad & \vec{e}_B(s) = \vec{e}_s(s) \times \vec{e}_N(s) .
\end{aligned}$$

The Serret–Frenet formulae corresponding to this dreibein read as:

$$\frac{d}{ds}\vec{e}_s = -K(s) \cdot \vec{e}_N(s) ; \tag{2.7a}$$

$$\frac{d}{ds}\vec{e}_N = +K(s) \cdot \vec{e}_s(s) ; \tag{2.7b}$$

$$\frac{d}{ds}\vec{e}_B = 0 . \tag{2.7c}$$

In this natural coordinate system we can represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r} \cdot \vec{e}_N) \cdot \vec{e}_N + (\delta\vec{r} \cdot \vec{e}_B) \cdot \vec{e}_B \tag{2.8}$$

(since the "dreibein" accompanies the design particle the $\vec{e}_s$–component of $\delta\vec{r}$ is always zero by definition).

However this representation has the disadvantage that the direction of the normal vector $\vec{e}_N$ changes discontinuously if the particle trajectory is going over from the vertical plane to the horizontal plane and vice versa. Therefore, it is advantageous to introduce new unit vectors $\vec{e}_x$, $\vec{e}_z$ and $\vec{e}_s$ which change their directions continuously. This is achieved by putting

$$\vec{e}_x(s) = \begin{cases} +\vec{e}_N(s), & \text{if the orbit lies in the horizontal plane;} \\ -\vec{e}_B(s), & \text{if the orbit lies in the vertical plane;} \end{cases}$$

$$\vec{e}_z(s) = \begin{cases} +\vec{e}_B(s), & \text{if the orbit lies in the horizontal plane;} \\ +\vec{e}_N(s), & \text{if the orbit lies in the vertical plane.} \end{cases}$$

As a result of these definitions we then obtain:

$$\begin{aligned}
\vec{e}_x(s) \times \vec{e}_z(s) &= \begin{cases} +\vec{e}_N(s) \times \vec{e}_B(s), & \text{if the orbit lies in the horizontal plane;} \\ -\vec{e}_B(s) \times \vec{e}_N(s), & \text{if the orbit lies in the vertical plane;} \end{cases} \\
&= \vec{e}_s(s) ,
\end{aligned} \tag{2.9}$$

i.e. $(\vec{e}_x(s), \vec{e}_z(s), \vec{e}_s(s))$ represents a r.h. orthonormal system, whereby $\vec{e}_x$ lies always in the horizontal plane and $\vec{e}_z$ in the vertical plane.

There is still some freedom in how to define this orthonormal system: either the tangential coordinate $\vec{e}_s(s)$ is chosen to move clockwise (in a right hand sense) around the machine, then the horizontal coordinate $\vec{e}_x(s)$ is directed outwards, i.e. away from the machine center or the tangential coordinate $\vec{e}_s(s)$ is chosen to move counter–clockwise around the machine, then the horizontal coordinate $\vec{e}_x(s)$ is directed towards the machine center. In both cases the vertical coordinate $\vec{e}_z(s)$ is pointing upwards.

The (x, z, s) coordinate system constructed above for bending magnets may also be used within a straight section where $K_x = K_z = 0$. A global and continuous coordinate system has thereby been defined under the restriction that the accelerator is torsion free.

Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x, z, s) = \vec{r}_0(s) + x(s) \cdot \vec{e}_x(s) + z(s) \cdot \vec{e}_z(s) \tag{2.10}$$

and the Serret–Frenet formulae (2.7) now read as:

$$\frac{d}{ds}\,\vec{e}_x(s) = +K_x(s) \cdot \vec{e}_s(s)\ ; \tag{2.11a}$$

$$\frac{d}{ds}\,\vec{e}_z(s) = +K_z(s) \cdot \vec{e}_s(s)\ ; \tag{2.11b}$$

$$\frac{d}{ds}\,\vec{e}_s(s) = -K_x(s) \cdot \vec{e}_x(s) - K_z(s) \cdot \vec{e}_z(s) \tag{2.11c}$$

with

$$K_x(s) \cdot K_z(s) = 0 \tag{2.12}$$

where $K_x(s)$, $K_z(s)$ denote the curvatures in the x-direction and in the z–direction respectively.

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by eqn. (2.11) and the choice of the direction of the coordinates (see above).

2.3 The Hamiltonian in Machine Coordinates

The variables x and z in eqn. (2.10) describe the amplitudes of transverse motion.

In order to provide an analytical description for longitudinal oscillations we introduce two additional small and oscillating variables σ and p_σ with

$$\sigma = s - v_0 \cdot t \tag{2.13}$$

and [2]

$$p_\sigma = \frac{1}{\beta_0^2} \cdot \eta\ , \tag{2.14}$$

where the term η is defined in (C.1).

The variable σ describes the delay in arrival time at position s of a particle and is the longitudinal separation of the particle from the center of the bunch. The quantity η is the relative energy deviation of the particle.

Starting from the orbital Hamiltonian (2.1) and introducing the length s along the design orbit as the independent variable (instead of the time t), we can construct the Hamiltonian of the orbital motion with respect to the new variables x, z, σ by a succession of canonical transformations and a scale transformation [4, 5].

Choosing a gauge with $\phi = 0$ (e.g. Coulomb gauge) we then obtain:

$$\begin{aligned}\mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s) = {} & p_\sigma - (1 + \hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \times \\ & \left\{1 - \frac{(p_x - \frac{e}{p_0 \cdot c} A_x)^2 + (p_z - \frac{e}{p_0 \cdot c} A_z)^2}{(1 + \hat{\eta})^2}\right\}^{1/2} \\ & -[1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{p_0 \cdot c} A_s\ ,\end{aligned} \tag{2.15}$$

[2] Note that in Refs. [4, 5] p_σ is defined without the scaling factor $\frac{1}{\beta_0^2}$.

where the relative momentum deviation $\hat{\eta}$ is defined in Appendix C (see eqn. (C.4)).

The corresponding canonical equations read as:

$$\frac{d}{ds}\,x = +\frac{\partial\mathcal{H}}{\partial p_x}\,; \quad \frac{d}{ds}\,p_x = -\frac{\partial\mathcal{H}}{\partial x}\,; \tag{2.16a}$$

$$\frac{d}{ds}\,z = +\frac{\partial\mathcal{H}}{\partial p_z}\,; \quad \frac{d}{ds}\,p_z = -\frac{\partial\mathcal{H}}{\partial z}\,; \tag{2.16b}$$

$$\frac{d}{ds}\,\sigma = +\frac{\partial\mathcal{H}}{\partial p_\sigma}\,; \quad \frac{d}{ds}\,p_\sigma = -\frac{\partial\mathcal{H}}{\partial \sigma} \tag{2.16c}$$

or, using a matrix form:

$$\frac{d}{ds}\,\vec{y} = -\underline{S}\cdot\frac{\partial\mathcal{H}}{\partial\vec{y}} \tag{2.17}$$

with

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)\ , \tag{2.18}$$

where the matrix $\underline{S}$ is given by:

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix}\ ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\ . \tag{2.19}$$

In order to utilize this Hamiltonian, the electric field $\vec{\mathcal{E}}$ and the magnetic field $\vec{\mathcal{B}}$ or the corresponding vector potential,

$$\vec{A} = \vec{A}(x, z, \sigma; s),$$

for the cavities and for commonly occurring types of accelerator magnets must be given. Once $\vec{A}$ is known the fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ may be found using the relations (2.3a, b). Expressed in the variables x, z, s, σ, eqns. (2.3a, b) become (with $\phi = 0$):

$$\vec{\mathcal{E}} = \beta_0\cdot\frac{\partial}{\partial\sigma}\,\vec{A} \tag{2.20}$$

and

$$\mathcal{B}_x = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial z}[(1+K_x\cdot x+K_z\cdot z)\cdot A_s]-\frac{\partial}{\partial s}A_z\right\}\ ; \tag{2.21a}$$

$$\mathcal{B}_z = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial s}A_x-\frac{\partial}{\partial x}[(1+K_x\cdot x+K_z\cdot z)\cdot A_s]\right\}\ ; \tag{2.21b}$$

$$\mathcal{B}_s = \frac{\partial}{\partial x}\,A_z-\frac{\partial}{\partial z}\,A_x\ . \tag{2.21c}$$

We assume that besides drift lengths the ring contains bending magnets, quadrupoles, skew quadrupoles, sextupoles, octupoles[3], solenoids and cavities. Then the vector potential $\vec{A}$ can be written as [4, 5]:

$$\begin{aligned}
\frac{e}{p_0 \cdot c} A_s &= -\frac{1}{2}[1 + K_x \cdot x + K_z \cdot z] + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz \\
&-\frac{\lambda}{6} \cdot (x^3 - 3\,xz^2) \\
&-\frac{\mu}{24} \cdot (z^4 - 6\,x^2\,z^2 + x^4) \\
&-\frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \;; \qquad (2.22a)
\end{aligned}$$

$$\frac{e}{p_0 \cdot c} A_x = -H \cdot z \;; \quad \frac{e}{p_0 \cdot c} A_z = +H \cdot x \qquad (2.22b)$$

(h = harmonic number) with the following abbreviations[4]:

$$\begin{aligned}
g &= \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0} \;; & (2.23a)\\
N &= \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial \mathcal{B}_x}{\partial x} - \frac{\partial \mathcal{B}_z}{\partial z}\right)_{x=z=0} \;; & (2.23b)\\
\lambda &= \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial^2 \mathcal{B}_z}{\partial x^2}\right)_{x=z=0} \;; & (2.23c)\\
\mu &= \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial^3 \mathcal{B}_z}{\partial x^3}\right)_{x=z=0} \;; & (2.23d)\\
H &= \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \mathcal{B}_s(0,0,s)\;. & (2.23e)
\end{aligned}$$

In detail, one has:

a)	$K_x^2 + K_z^2 \neq 0$;	$g = N = \lambda = \mu = H = V = 0:$	bending magnet;
b)	$g \neq 0$;	$K_x = K_z = N = \lambda = \mu = H = V = 0:$	quadrupole;
c)	$N \neq 0$;	$K_x = K_z = g = \lambda = \mu = H = V = 0:$	skew quadrupole;
d)	$\lambda \neq 0$;	$K_x = K_z = g = N = \mu = H = V = 0:$	sextupole;
e)	$\mu \neq 0$;	$K_x = K_z = g = N = \lambda = H = V = 0:$	octupole;
f)	$H \neq 0$;	$K_x = K_z = g = N = \lambda = \mu = V = 0:$	solenoid;
g)	$V \neq 0$;	$K_x = K_z = g = N = \lambda = \mu = H = 0:$	cavity.

[3]It has to be mentioned that the formalism can be generalized to higher order multipoles. In fact multipoles up to 10^{th} order are included in the SIXTRACK code.

[4]In the coding of SIXTRACK there is, for historical reasons, one important difference: all regular multipoles e.g. g, λ, μ and also K_x are defined opposite in sign compared to 2.23, while the skew components like N have the same sign.

Thus the Hamiltonian (2.15) takes the form:

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma - (1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\times \\ &\qquad \left\{1-\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{(1+\hat{\eta})^2}\right\}^{1/2} \\ &\quad +\frac{1}{2}\cdot[1+K_x\cdot x+K_z\cdot z]^2-\frac{1}{2}\cdot g\cdot(z^2-x^2)-N\cdot xz \\ &\quad +\frac{\lambda}{6}\cdot(x^3-3\,xz^2) \\ &\quad +\frac{\mu}{24}\cdot(z^4-6\,x^2\,z^2+x^4) \\ &\quad +\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\,. \end{aligned} \tag{2.24}$$

<u>Remarks:</u>

1) If the curvatures K_x and K_z of the design orbit appearing in (2.24) are given, the magnetic bending field

$$\vec{\mathcal{B}}^{(bend)}(s) = (\mathcal{B}_x^{(bend)}(s),\,\mathcal{B}_z^{(bend)}(s),\,0)$$

is determined by:

$$\frac{e}{p_0\cdot c}\,\mathcal{B}_x^{bend}(s) = -K_z(s)\,; \tag{2.25a}$$

$$\frac{e}{p_0\cdot c}\,\mathcal{B}_z^{bend}(s) = +K_x(s)\,. \tag{2.25b}$$

These relations may be obtained using the fact that the design orbit is a solution of the equations of motion for constant energy E_0 in the absence of cavities and correction coils [6].

2) Equation (2.24) is valid only for protons. For electrons we need the extra term in the Hamiltonian

$$\mathcal{H}_{rad} = C_1\cdot[K_x^2+K_z^2]\cdot\sigma \tag{2.26}$$

$$\left(\text{where } C_1=\frac{2}{3}e^2\frac{\gamma_0^4}{E_0}\right)$$

(for $v_0\approx c$) in order to describe the energy loss by radiation in the bending magnets [7]. In this case, the cavity phase φ in (2.22a) and (2.24) is determined by the need to replace the energy radiated in the bending magnets. Thus:

$$\underbrace{\int_{s_0}^{s_0+L} ds\cdot eV(s)\cdot\sin\varphi}_{\text{average energy uptake in the cavities}} = \underbrace{\int_{s_0}^{s_0+L} ds\cdot E_0\cdot C_1\cdot[K_x^2+K_z^2]}_{\text{average energy loss due to radiation}}\,. \tag{2.27}$$

Note that the $\mathcal{H}_{rad}$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non–symplectic terms into the equation of motion.

For those proton storage rings where radiation effects can be neglected there is no average energy gain in the cavities so that:

$$\sin\varphi = 0 \quad \Longrightarrow \quad \varphi = 0,\ \pi \tag{2.28}$$

and the choice for φ is determined by the stability condition for synchrotron motion:

$$\begin{cases} \varphi = 0 \ \text{ above "transition"} \ ; \\ \\ \varphi = \pi \ \text{ below "transition"} \ . \end{cases}$$

2.4 Series Expansion of the Hamiltonian

Since

$$\begin{aligned} |p_x + H\cdot z| &\ll 1 \ ; \\ |p_z - H\cdot x| &\ll 1 \end{aligned}$$

the square root

$$\left[1 - \frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{(1+\hat{\eta})^2}\right]^{1/2}$$

in (2.24) may be expanded in a series:

$$\left[1 - \frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{(1+\hat{\eta})^2}\right]^{1/2} = \\ 1 - \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{(1+\hat{\eta})^2} + \cdots . \tag{2.29}$$

The power at which the series is truncated defines the order of the approximation to the particle motion.

The second term on the r.h.s. of the Hamiltonian (2.24) is approximated as follows: firstly only terms of (2.29) up to quadratic in $(p_x + H\cdot z)$ and $(p_z - H\cdot x)$ will be kept, secondly of the resulting terms in the numerator only those are considered which are up to quadratic in (x, z, $\hat{\eta}$, $(p_x + H\cdot z)$ and $(p_z - H\cdot x)$) and thirdly the denominator $(1+\hat{\eta})^2$ is retained, whence:

$$\begin{aligned} \mathcal{H} \ = \ & \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1 + f(p_\sigma)]} + \\ & p_\sigma - [1 + K_x\cdot x + K_z\cdot z]\cdot f(p_\sigma) + \end{aligned}$$

$$\frac{1}{2}[K_x^2+g]\cdot x^2+\frac{1}{2}[K_z^2-g]\cdot z^2-N\cdot x\,z+$$

$$\frac{\lambda}{6}\cdot(x^3-3\,xz^2)+\frac{\mu}{24}\cdot(z^4-6\,x^2\,z^2+x^4)+$$

$$\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\ . \tag{2.30}$$

We have replaced $\hat{\eta}$ by $f(p_\sigma)$ to stress its dependence on p_σ. The power series of $f(p_\sigma)$ and its derivative $f'(p_\sigma)\equiv\frac{df(p_\sigma)}{d\sigma}$ are given in Appendix C by eqns. (C.6) and (C.9) respectively. Constant terms in the Hamiltonian with no influence on the motion have been dropped.

2.5 Equations of Motion

The Hamiltonian (2.30) now leads to the canonical equations of motion:

$$\begin{aligned}\frac{d}{ds}x &= +\frac{\partial\mathcal{H}}{\partial p_x}\\ &= \frac{p_x+H\cdot z}{[1+f(p_\sigma)]}\ ;\end{aligned} \tag{2.31a}$$

$$\begin{aligned}\frac{d}{ds}p_x &= -\frac{\partial\mathcal{H}}{\partial x}\\ &= +\frac{[p_z-H\cdot x]}{[1+f(p_\sigma)]}\cdot H-[K_x^2+g]\cdot x+N\cdot z+\ K_x\cdot f(p_\sigma)\\ &\quad -\frac{\lambda}{2}\cdot(x^2-z^2)-\frac{\mu}{6}\cdot(x^3-3\,x\,z^2)\ ;\end{aligned} \tag{2.31b}$$

$$\begin{aligned}\frac{d}{ds}z &= +\frac{\partial\mathcal{H}}{\partial p_z}\\ &= \frac{p_z-H\cdot x}{[1+f(p_\sigma)]}\ ;\end{aligned} \tag{2.31c}$$

$$\begin{aligned}\frac{d}{ds}p_z &= -\frac{\partial\mathcal{H}}{\partial z}\\ &= -\frac{[p_x+H\cdot z]}{[1+f(p_\sigma)]}\cdot H-[K_z^2-g]\cdot z+N\cdot x+K_z\cdot f(p_\sigma)\\ &\quad +\lambda\cdot xz-\frac{\mu}{6}\cdot(z^3-3\,x^2\,z)\ ;\end{aligned} \tag{2.31d}$$

$$\frac{d}{ds}\sigma = +\frac{\partial\mathcal{H}}{\partial p_\sigma}$$

$$
\begin{aligned}
&= 1-[1+K_x\cdot x+K_z\cdot z]\cdot f'(p_\sigma)\\
&\quad -\frac{1}{2}\cdot\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma)\\
&= 1-[1+K_x\cdot x+K_z\cdot z]\cdot f'(p_\sigma)\\
&\quad -\frac{1}{2}\cdot[(x')^2+(z')^2]\cdot f'(p_\sigma)\ ; \qquad (2.31e)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{ds}p_\sigma &= -\frac{\partial\mathcal{H}}{\partial\sigma}\\
&= \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\ . \qquad (2.31f)
\end{aligned}
$$

In (2.31) the first four equations describe betatron motion and the last two synchrotron oscillations. Equations (2.31f) relates to energy conservation. Note that eqns. (2.31e,f) for synchrotron motion are always nonlinear.

Remark:

If the variables $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$ at position s are known, one obtains the terms $x'(s)$, $z'(s)$, and $\eta(s)$ by the relations:

$$x'(s) = \frac{p_x+H\cdot z}{[1+f(p_\sigma)]}\ ; \qquad (2.32a)$$

$$z'(s) = \frac{p_z-H\cdot x}{[1+f(p_\sigma)]} \qquad (2.32b)$$

and

$$\eta(s) = \beta_0^2\cdot p_\sigma(s) \qquad (2.32c)$$

(see eqns. (2.31a), (2.31c) and (2.14)).

3 Thin–Lens Approximation

The canonical equations of motion (2.31) shall now be solved for various kinds of magnets and for cavities using the thin–lens approximation. The symplecticity condition is checked in all cases using the Jacobian matrix.

3.1 Bending Magnet

3.1.1 Canonical Equations of Motion

For a bending magnet we have:

$$K_x^2 + K_z^2 \neq 0; \quad K_x \cdot K_z = 0$$

and

$$g = N = \lambda = \mu = H = V = 0 .$$

Writing for a bending magnet at position s_0:

$$K_{x,z}^2(s) = K_{x,z}(s_0) \cdot K_{x,z}(s)$$

and assuming $K_{x,z}(s)$ to be taken in the form (thin–lens approximation):

$$K_{x,z}(s) = K_{x,z}(s_0) \cdot \Delta s \cdot \delta(s - s_0) ,$$

whereby Δs denotes the length of the bending magnet we obtain from (2.31):

$$\frac{d}{ds}\, x = \frac{p_x}{[1 + f(p_\sigma)]} ; \tag{3.1a}$$

$$\frac{d}{ds}\, p_x = -[K_x(s_0)]^2 \cdot \Delta s \cdot \delta(s - s_0) \cdot x + K_x(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot f(p_\sigma) ; \tag{3.1b}$$

$$\frac{d}{ds}\, z = \frac{p_z}{[1 + f(p_\sigma)]} ; \tag{3.1c}$$

$$\frac{d}{ds}\, p_z = -[K_z(s_0)]^2 \cdot \Delta s \cdot \delta(s - s_0) \cdot z + K_z(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot f(p_\sigma) ; \tag{3.1d}$$

$$\begin{aligned} \frac{d}{ds}\, \sigma &= 1 - f'(p_\sigma) - [K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot \delta(s - s_0) \cdot f'(p_\sigma) \\ &\quad - \frac{1}{2} \cdot [(x')^2 + (z')^2] \cdot f'(p_\sigma) ; \end{aligned} \tag{3.1e}$$

$$\frac{d}{ds}\, p_\sigma = 0 . \tag{3.1f}$$

3.1.2 Solution of the Equations of Motion

Equations (3.1) can be solved by integrating both sides from

$$s_0 - \epsilon \ \text{to} \ s_0 + \epsilon$$

with

$$0 < \epsilon \longrightarrow 0$$

leading to [5] :

$$x^f = x^i \ ; \tag{3.2a}$$

$$p_x^f = p_x^i - [K_x(s_0)]^2 \cdot \Delta s \cdot x^i + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \ ; \tag{3.2b}$$

$$z^f = z^i \ ; \tag{3.2c}$$

$$p_z^f = p_z^i - [K_z(s_0)]^2 \cdot \Delta s \cdot z^i + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \ ; \tag{3.2d}$$

$$\sigma^f = \sigma^i - [K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot f'(p_\sigma^i) \ ; \tag{3.2e}$$

$$p_\sigma^f = p_\sigma^i \tag{3.2f}$$

with

$$\begin{aligned} y^i &\equiv y(s_0 - 0) \ ; \\ y^f &\equiv y(s_0 + 0) \ ; \end{aligned}$$

$$(y = x, \, p_x, \, z, \, p_z, \, \sigma, \, p_\sigma) \ .$$

3.1.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.2) reads as:

$$\underline{\mathcal{J}}_{bend} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)} \equiv \begin{pmatrix} \frac{\partial x^f}{\partial x^i} & \frac{\partial x^f}{\partial p_x^i} & \frac{\partial x^f}{\partial z^i} & \frac{\partial x^f}{\partial p_z^i} & \frac{\partial x^f}{\partial \sigma^i} & \frac{\partial x^f}{\partial p_\sigma^i} \\ \frac{\partial p_x^f}{\partial x^i} & \frac{\partial p_x^f}{\partial p_x^i} & \frac{\partial p_x^f}{\partial z^i} & \frac{\partial p_x^f}{\partial p_z^i} & \frac{\partial p_x^f}{\partial \sigma^i} & \frac{\partial p_x^f}{\partial p_\sigma^i} \\ \frac{\partial z^f}{\partial x^i} & \frac{\partial z^f}{\partial p_x^i} & \frac{\partial z^f}{\partial z^i} & \frac{\partial z^f}{\partial p_z^i} & \frac{\partial z^f}{\partial \sigma^i} & \frac{\partial z^f}{\partial p_\sigma^i} \\ \frac{\partial p_z^f}{\partial x^i} & \frac{\partial p_z^f}{\partial p_x^i} & \frac{\partial p_z^f}{\partial z^i} & \frac{\partial p_z^f}{\partial p_z^i} & \frac{\partial p_z^f}{\partial \sigma^i} & \frac{\partial p_z^f}{\partial p_\sigma^i} \\ \frac{\partial \sigma^f}{\partial x^i} & \frac{\partial \sigma^f}{\partial p_x^i} & \frac{\partial \sigma^f}{\partial z^i} & \frac{\partial \sigma^f}{\partial p_z^i} & \frac{\partial \sigma^f}{\partial \sigma^i} & \frac{\partial x^f}{\partial p_\sigma^i} \\ \frac{\partial p_\sigma^f}{\partial x^i} & \frac{\partial p_\sigma^f}{\partial p_x^i} & \frac{\partial p_\sigma^f}{\partial z^i} & \frac{\partial p_\sigma^f}{\partial p_z^i} & \frac{\partial p_\sigma^f}{\partial \sigma^i} & \frac{\partial p_\sigma^f}{\partial p_\sigma^i} \end{pmatrix}$$

[5]Note that the factors in (3.1b, d, e) which multiply the δ–function are continuous functions of s at s_0.

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -[K_x(s_0)]^2 \cdot \Delta s & 1 & 0 & 0 & 0 & +Q_x \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -[K_z(s_0)]^2 \cdot \Delta s & 1 & 0 & +Q_z \\ -Q_x & 0 & -Q_z & 0 & 1 & Q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3a}
$$

with

$$
\begin{aligned} Q &= -[K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot f''(p_\sigma^i) ; \\ Q_x &= +K_x(s_0) \cdot \Delta s \cdot f'(p_\sigma^i) ; \\ Q_z &= +K_z(s_0) \cdot \Delta s \cdot f'(p_\sigma^i) . \end{aligned} \tag{3.3b}
$$

Using eqn. (2.12) it can be verified that $\underline{\mathcal{J}}_{bend}$ obeys the symplecticity condition

$$
\underline{\mathcal{J}}_{bend}^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_{bend} = \underline{S} . \tag{3.4}
$$

Equation (3.4) proves that the transformation

$$
\vec{y}^i \longrightarrow \vec{y}^f
$$

described by (3.2a–f) is indeed symplectic (see Appendix A).

3.2 Quadrupole

3.2.1 Canonical Equations of Motion

For a quadrupole we have:

$$
g \neq 0
$$

and

$$
K_x = K_z = N = \lambda = \mu = H = V = 0 .
$$

Using thin–lens approximation we write for a quadrupole of length Δs at position s_0:

$$
g(s) = g(s_0) \cdot \Delta s \cdot \delta(s - s_0) .
$$

Then we obtain from (2.31):

$$
\frac{d}{ds} x = \frac{p_x}{[1 + f(p_\sigma)]} ; \tag{3.5a}
$$

$$
\frac{d}{ds} p_x = -g(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot x ; \tag{3.5b}
$$

$$\frac{d}{ds}\, z \;=\; \frac{p_z}{[1+f(p_\sigma)]}\;; \tag{3.5c}$$

$$\frac{d}{ds}\, p_z \;=\; +g(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot z\;; \tag{3.5d}$$

$$\frac{d}{ds}\, \sigma \;=\; 1-f'(p_\sigma)-\frac{1}{2}\cdot[(x')^2+(z')^2]\cdot f'(p_\sigma)\;; \tag{3.5e}$$

$$\frac{d}{ds}\, p_\sigma \;=\; 0\,. \tag{3.5f}$$

3.2.2 Solution of the Equations of Motion

The solution of eqn. (3.5) reads as:

$$x^f \;=\; x^i\;; \tag{3.6a}$$

$$p_x^f \;=\; p_x^i - g(s_0)\cdot\Delta s\cdot x^i\;; \tag{3.6b}$$

$$z^f \;=\; z^i\;; \tag{3.6c}$$

$$p_z^f \;=\; p_z^i + g(s_0)\cdot\Delta s\cdot z^i\;; \tag{3.6d}$$

$$\sigma^f \;=\; \sigma^i\;; \tag{3.6e}$$

$$p_\sigma^f \;=\; p_\sigma^i\,. \tag{3.6f}$$

3.2.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.6) takes the form:

$$\underline{\mathcal{J}}_{qua} \;=\; \frac{\partial(x^f,\, p_x^f,\, z^f,\, p_z^f,\, \sigma^f,\, p_\sigma^f)}{\partial(x^i,\, p_x^i,\, z^i,\, p_z^i,\, \sigma^i,\, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -g(s_0)\cdot\Delta s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & +g(s_0)\cdot\Delta s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

From eqn. (3.7) it can be verified that $\underline{\mathcal{J}}_{qua}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}^T_{qua}\cdot\underline{S}\cdot\underline{\mathcal{J}}_{qua} \;=\; \underline{S}\,. \tag{3.8}$$

3.3 Synchrotron–Magnet

3.3.1 Canonical Equations of Motion

For a synchrotron magnet [6] we have:

$$g \neq 0; \quad K_x^2 + K_z^2 \neq 0 \ \text{with} \ K_x \cdot K_z = 0 \tag{3.9}$$

and

$$N = \lambda = \mu = H = V = 0 \, .$$

Writing:

$$K_{x,z}^2(s) = K_{x,z}(s_0) \cdot K_{x,z}(s)$$

and assuming $K_{x,z}(s)$ and $g(s)$ to be taken in the form (thin–lens approximation):

$$K_{x,z}(s) = K_{x,z}(s_0) \cdot \Delta s \cdot \delta(s - s_0) \, ;$$

$$g(s) = g(s_0) \cdot \Delta s \cdot \delta(s - s_0)$$

we obtain from (2.31):

$$\frac{d}{ds}\, x = \frac{p_x}{[1 + f(p_\sigma)]} \, ; \tag{3.10a}$$

$$\frac{d}{ds}\, p_x = -G_1(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot x + K_x(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot f(p_\sigma) \, ; \tag{3.10b}$$

$$\frac{d}{ds}\, z = \frac{p_z}{[1 + f(p_\sigma)]} \, ; \tag{3.10c}$$

$$\frac{d}{ds}\, p_z = -G_2(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot z + K_z(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot f(p_\sigma) \, ; \tag{3.10d}$$

$$\frac{d}{ds}\, \sigma = 1 - f'(p_\sigma) - [K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot \delta(s - s_0) \cdot f'(p_\sigma) - \frac{1}{2} \cdot [(x')^2 + (z')^2] \cdot f'(p_\sigma) \, ; \tag{3.10e}$$

$$\frac{d}{ds}\, p_\sigma = 0 \tag{3.10f}$$

with

$$G_1 = K_x^2 + g; \quad G_2 = K_z^2 - g \, . \tag{3.11}$$

[6]Note that due to the condition (3.9) cross–terms of g and $K_{x,z}$ exist that lead to sextupole and higher order terms in the Hamiltonian (2.24). This terms are considered small and are omitted in the treatment of synchrotron magnets.

3.3.2 Solution of the Equations of Motion

Equations (3.10) can be solved by integrating both sides from

$$s_0 - \epsilon \text{ to } s_0 + \epsilon$$

with

$$0 < \epsilon \longrightarrow 0$$

leading to:

$$x^f = x^i ; \tag{3.12a}$$

$$p_x^f = p_x^i - G_1(s_0) \cdot \Delta s \cdot x^i + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma^i) ; \tag{3.12b}$$

$$z^f = z^i ; \tag{3.12c}$$

$$p_z^f = p_z^i - G_2(s_0) \cdot \Delta s \cdot z^i + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma^i) ; \tag{3.12d}$$

$$\sigma^f = \sigma^i - [K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot f'(p_\sigma^i) ; \tag{3.12e}$$

$$p_\sigma^f = p_\sigma^i . \tag{3.12f}$$

3.3.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.12) reads as:

$$\underline{\mathcal{J}}_{syn} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -G_1(s_0) \cdot \Delta s & 1 & 0 & 0 & 0 & +Q_x \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -G_2(s_0) \cdot \Delta s & 1 & 0 & +Q_z \\ -Q_x & 0 & -Q_z & 0 & 1 & Q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.13a}$$

with

$$\begin{aligned} Q &= -[K_x \cdot x + K_z \cdot z] \cdot \Delta s \cdot f''(p_\sigma^i) ; \\ Q_x &= +K_x(s_0) \cdot \Delta s \cdot f'(p_\sigma^i) ; \\ Q_z &= +K_z(s_0) \cdot \Delta s \cdot f'(p_\sigma^i) . \end{aligned} \tag{3.13b}$$

Using eqn. (3.13) it can be verified that $\underline{\mathcal{J}}_{syn}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}_{syn}^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_{syn} = \underline{S} . \tag{3.14}$$

3.4 Skew Quadrupole

3.4.1 Canonical Equations of Motion

For a skew quadrupole we have:

$$N \neq 0$$

and

$$K_x = K_z = g = \lambda = \mu = H = V = 0 .$$

Using thin-lens approximation we write:

$$N(s) = N(s_0) \cdot \Delta s \cdot \delta(s - s_0) .$$

Then we obtain from (2.31):

$$\frac{d}{ds}\, x = \frac{p_x}{[1 + f(p_\sigma)]} ; \tag{3.15a}$$

$$\frac{d}{ds}\, p_x = N(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot z ; \tag{3.15b}$$

$$\frac{d}{ds}\, z = \frac{p_z}{[1 + f(p_\sigma)]} ; \tag{3.15c}$$

$$\frac{d}{ds}\, p_z = N(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot x ; \tag{3.15d}$$

$$\frac{d}{ds}\, \sigma = 1 - f'(p_\sigma) - \frac{1}{2} \cdot [(x')^2 + (z')^2] \cdot f'(p_\sigma) ; \tag{3.15e}$$

$$\frac{d}{ds}\, p_\sigma = 0 . \tag{3.15f}$$

3.4.2 Solution of the Equations of Motion

The solution of eqn. (3.15) reads as:

$$x^f = x^i ; \tag{3.16a}$$

$$p_x^f = p_x^i + N(s_0) \cdot \Delta s \cdot z^i ; \tag{3.16b}$$

$$z^f = z^i ; \tag{3.16c}$$

$$p_z^f = p_z^i + N(s_0) \cdot \Delta s \cdot x^i ; \tag{3.16d}$$

$$\sigma^f = \sigma^i ; \tag{3.16e}$$

$$p_\sigma^f = p_\sigma^i . \tag{3.16f}$$

3.4.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.16) takes the form:

$$\underline{\mathcal{J}}_{sqd} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & N(s_0)\cdot\Delta s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ N(s_0)\cdot\Delta s & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.17}$$

From eqn. (3.17) it can be verified that $\underline{\mathcal{J}}_{sqd}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}^T_{sqd} \cdot \underline{S} \cdot \underline{\mathcal{J}}_{sqd} = \underline{S} . \tag{3.18}$$

3.5 Sextupole

3.5.1 Canonical Equations of Motion

For a sextupole we have:

$$\lambda \neq 0$$

and

$$K_x = K_z = g = N = \mu = H = V = 0 .$$

Using thin-lens approximation we write for a sextupole of length Δs at position s_0:

$$\lambda(s) = \lambda(s_0)\cdot\Delta s\cdot\delta(s-s_0) .$$

Then we obtain from (2.31):

$$\frac{d}{ds}\, x = \frac{p_x}{[1+f(p_\sigma)]} ; \tag{3.19a}$$

$$\frac{d}{ds}\, p_x = -\frac{1}{2}\,\lambda(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot[x^2-z^2] ; \tag{3.19b}$$

$$\frac{d}{ds}\, z = \frac{p_z}{[1+f(p_\sigma)]} ; \tag{3.19c}$$

$$\frac{d}{ds}\, p_z = +\lambda(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot x\, z ; \tag{3.19d}$$

$$\frac{d}{ds}\, \sigma = 1 - f'(p_\sigma) - \frac{1}{2}\cdot[(x')^2+(z')^2]\cdot f'(p_\sigma) ; \tag{3.19e}$$

$$\frac{d}{ds}\, p_\sigma = 0 . \tag{3.19f}$$

3.5.2 Solution of the Equations of Motion

The solution of eqn. (3.19) reads as:

$$x^f = x^i ; \tag{3.20a}$$

$$p_x^f = p_x^i - \frac{1}{2}\lambda(s_0)\cdot\Delta s\cdot[(x^i)^2 - (z^i)^2] ; \tag{3.20b}$$

$$z^f = z^i ; \tag{3.20c}$$

$$p_z^f = p_z^i + \lambda(s_0)\cdot\Delta s\cdot x^i\, z^i ; \tag{3.20d}$$

$$\sigma^f = \sigma^i ; \tag{3.20e}$$

$$p_\sigma^f = p_\sigma^i \tag{3.20f}$$

(see also Refs. [4, 5]).

3.5.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.20) takes the form:

$$\underline{\mathcal{J}}_{sext} = \frac{\partial(x^f,\, p_x^f,\, z^f,\, p_z^f,\, \sigma^f,\, p_\sigma^f)}{\partial(x^i,\, p_x^i,\, z^i,\, p_z^i,\, \sigma^i,\, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda(s_0)\cdot\Delta s\cdot x^i & 1 & +\lambda(s_0)\cdot\Delta s\cdot z^i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ +\lambda(s_0)\cdot\Delta s\cdot z^i & 0 & +\lambda(s_0)\cdot\Delta s\cdot x^i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.21}$$

From eqn. (3.21) it can be verified that $\underline{\mathcal{J}}_{sext}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}_{sext}^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_{sext} = \underline{S} . \tag{3.22}$$

3.6 Octupole

3.6.1 Canonical Equations of Motion

For an octupole we have:

$$\mu \neq 0$$

and

$$K_x = K_z = g = N = \lambda = H = V = 0\,.$$

Using thin–lens approximation we write for a sextupole of length Δs at position s_0:

$$\mu(s) = \mu(s_0)\cdot\Delta s\cdot\delta(s-s_0)\,.$$

Then we obtain from (2.31):

$$\frac{d}{ds}\,x = \frac{p_x}{[1+f(p_\sigma)]}\,; \tag{3.23a}$$

$$\frac{d}{ds}\,p_x = -\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot[x^3-3\,x\,z^2]\,; \tag{3.23b}$$

$$\frac{d}{ds}\,z = \frac{p_z}{[1+f(p_\sigma)]}\,; \tag{3.23c}$$

$$\frac{d}{ds}\,p_z = -\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot[z^3-3\,x^2\,z]\,; \tag{3.23d}$$

$$\frac{d}{ds}\,\sigma = 1-f'(p_\sigma)-\frac{1}{2}\cdot[(x')^2+(z')^2]\cdot f'(p_\sigma)\,; \tag{3.23e}$$

$$\frac{d}{ds}\,p_\sigma = 0\,. \tag{3.23f}$$

3.6.2 Solution of the Equations of Motion

The solution of eqn. (3.23) reads as:

$$x^f = x^i\,; \tag{3.24a}$$

$$p_x^f = p_x^i-\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\left[(x^i)^3-3\,(x^i)\,(z^i)^2\right]\,; \tag{3.24b}$$

$$z^f = z^i\,; \tag{3.24c}$$

$$p_z^f = p_z^i-\frac{1}{6}\,\mu(s_0)\cdot\Delta s\cdot\left[(z^i)^3-3\,(x^i)^2\,(z^i)\right]\,; \tag{3.24d}$$

$$\sigma^f = \sigma^i\,; \tag{3.24e}$$

$$p_\sigma^f = p_\sigma^i \tag{3.24f}$$

(see also Refs. [4, 5]).

3.6.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.24) takes the form:

$$\underline{\mathcal{J}}_{oct} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\mu(s_0)}{2}\cdot\Delta s\cdot[(x^i)^2-(z^i)^2] & 1 & +\mu(s_0)\cdot\Delta s\cdot x^i\,z^i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ +\mu(s_0)\cdot\Delta s\cdot x^i\,z^i & 0 & +\frac{\mu(s_0)}{2}\cdot\Delta s\cdot[(x^i)^2-(z^i)^2] & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \quad (3.25)$$

From eqn. (3.25) it can be verified that $\underline{\mathcal{J}}_{oct}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}^T_{oct}\cdot\underline{S}\cdot\underline{\mathcal{J}}_{oct} = \underline{S} . \quad (3.26)$$

3.7 Solenoid

3.7.1 Canonical Equations of Motion

For a solenoid we have:

$$H \neq 0$$

and

$$K_x = K_z = g = N = \lambda = \mu = V = 0 .$$

Writing:

$$[H(s)]^2 = H(s_0)\cdot H(s) \quad (3.27a)$$

and assuming $H(s)$ to be taken in the form (thin-lens approximation):

$$H(s) = H(s_0)\cdot\Delta s\cdot\delta(s-s_0) \quad (3.27b)$$

we obtain from (2.31) the equations of motion for a solenoid in the form:

$$\frac{d}{ds}\,x = \frac{p_x + H(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot z}{[1+f(p_\sigma)]} ; \quad (3.28a)$$

$$\frac{d}{ds}\,p_x = +\frac{[p_z - H(s_0)\cdot x]}{[1+f(p_\sigma)]}\cdot H(s_0)\cdot\Delta s\cdot\delta(s-s_0) ; \quad (3.28b)$$

$$\frac{d}{ds}\,z = \frac{p_z - H(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot x}{[1+f(p_\sigma)]} ; \quad (3.28c)$$

$$\frac{d}{ds}\, p_z \;=\; -\frac{[p_x + H(s_0)\cdot z]}{[1+f(p_\sigma)]}\cdot H(s_0)\cdot \Delta s\cdot \delta(s-s_0)\;; \tag{3.28d}$$

$$\frac{d}{ds}\,\sigma \;=\; 1 - f'(p_\sigma) - \frac{1}{2}\cdot [(x')^2 + (z')^2]\cdot f'(p_\sigma)\;; \tag{3.28e}$$

$$\frac{d}{ds}\, p_\sigma \;=\; 0 \tag{3.28f}$$

resulting from the Hamiltonian

$$\mathcal{H}_{sol} \;=\; \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]} \tag{3.29}$$

(see eqn. (2.31)).

In this form eqns. (3.28) cannot be solved by integrating both sides from

$$s_0 - \epsilon \ \text{ to } \ s_0 + \epsilon$$

with

$$0 < \epsilon \ \longrightarrow \ 0$$

since the factors $x(s)$ and $z(s)$ of the δ–function in (3.28b) and (3.28d) are not continuous, as can be seen from (3.28a, c).

In order to simplify eqn. (3.28) we introduce a new set of canonical variables

$$(\hat{x},\, \hat{p}_x,\, \hat{z},\, \hat{p}_z,\, \hat{\sigma},\, \hat{p}_\sigma)$$

using the generating function:

$$F_3 \;=\; -[\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\cdot p_x - [-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta]\cdot p_z - \hat{\sigma}\cdot p_\sigma \tag{3.30}$$

with

$$\Theta(s) \;=\; \frac{1}{[1+f(p_\sigma)]}\cdot\int_{s_1}^{s} d\tilde{s}\cdot H(\tilde{s}) \tag{3.31}$$

which leads to:

$$\begin{aligned} x &= -\frac{\partial F_3}{\partial p_x} \\ &= +[\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\;; \end{aligned}$$

$$\begin{aligned} \hat{p}_x &= -\frac{\partial F_3}{\partial \hat{x}} \\ &= +[p_x\cdot\cos\Theta - p_z\cdot\sin\Theta]\;; \end{aligned}$$

$$z \;=\; -\frac{\partial F_3}{\partial p_z}$$

$$= +[-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta] \ ;$$

$$\begin{aligned}
\hat{p}_z &= -\frac{\partial F_3}{\partial \hat{z}} \\
&= +[p_x\cdot\sin\Theta + p_z\cdot\cos\Theta] \ ;
\end{aligned}$$

$$\begin{aligned}
\sigma &= -\frac{\partial F_3}{\partial p_\sigma} \\
&= \hat{\sigma} - \{-[-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta]\cdot p_x + [\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\cdot p_z\}\cdot\frac{\partial\Theta}{\partial p_\sigma} \\
&= \hat{\sigma} + \{-[-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta]\cdot p_x + [\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\cdot p_z\}\cdot\frac{\int_{s_1}^{s} d\tilde{s}\cdot H(\tilde{s})}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) \ ;
\end{aligned}$$

$$\hat{p}_\sigma = -\frac{\partial F_3}{\partial \hat{\sigma}} = p_\sigma$$

or

$$x = \hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta \ ; \tag{3.32a}$$

$$p_x = \hat{p}_x\cdot\cos\Theta + \hat{p}_z\cdot\sin\Theta \ ; \tag{3.32b}$$

$$z = -\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta \ ; \tag{3.32c}$$

$$p_z = -\hat{p}_x\cdot\sin\Theta + \hat{p}_z\cdot\cos\Theta \ ; \tag{3.32d}$$

$$\begin{aligned}
\sigma &= \hat{\sigma} + \{-[-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta]\cdot p_x \\
&\qquad + [\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\cdot p_z\}\cdot\frac{\int_{s_1}^{s} d\tilde{s}\cdot H(\tilde{s})}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) \\
&= \hat{\sigma} + \{-[-\hat{x}\cdot\sin\Theta + \hat{z}\cdot\cos\Theta]\cdot[\hat{p}_x\cdot\cos\Theta + \hat{p}_z\cdot\sin\Theta] \\
&\qquad + [\hat{x}\cdot\cos\Theta + \hat{z}\cdot\sin\Theta]\cdot[-\hat{p}_x\cdot\sin\Theta + \hat{p}_z\cdot\cos\Theta]\} \\
&\qquad \times\frac{\int_{s_1}^{s} d\tilde{s}\cdot H(\tilde{s})}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) \\
&= \hat{\sigma} + \{\hat{x}\cdot\hat{p}_z - \hat{z}\cdot\hat{p}_x\}\cdot\frac{\int_{s_1}^{s} d\tilde{s}\cdot H(\tilde{s})}{[1+f(\hat{p}_\sigma)]^2}\cdot f'(\hat{p}_\sigma) \ ;
\end{aligned} \tag{3.32e}$$

$$p_\sigma = \hat{p}_\sigma \ . \tag{3.32f}$$

The new Hamiltonian reads as:

$$\hat{\mathcal{H}}_{sol} = \mathcal{H}_{sol} + \frac{\partial F_3}{\partial s}$$

$$= \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]} + \frac{\partial F_3}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial s}$$

$$= \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]}$$

$$+ \{[\hat{x} \cdot \sin\Theta - \hat{z} \cdot \cos\Theta] \cdot p_x + [\hat{x} \cdot \cos\Theta + \hat{z} \cdot \sin\Theta] \cdot p_z\} \cdot \frac{H(s)}{[1 + f(p_\sigma)]}$$

$$= \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]} + \{-z \cdot p_x + x \cdot p_z\} \cdot \frac{H(s)}{[1 + f(p_\sigma)]}$$

$$= \frac{1}{[1 + f(p_\sigma)]} \cdot \frac{1}{2} \left\{[p_x^2 + p_z^2] + H^2 \cdot [x^2 + z^2]\right\}$$

$$= \frac{1}{[1 + f(\hat{p}_\sigma)]} \cdot \frac{1}{2} \left\{[\hat{p}_x^2 + \hat{p}_z^2] + H^2 \cdot [\hat{x}^2 + \hat{x}^2]\right\} \tag{3.33}$$

and the corresponding canonical equations take the form:

$$\frac{d}{ds}\hat{x} = +\frac{\partial \hat{\mathcal{H}}_{Sol}}{\partial \hat{p}_x}$$

$$= \frac{\hat{p}_x}{[1 + f(\hat{p}_\sigma)]} ; \tag{3.34a}$$

$$\frac{d}{ds}\hat{p}_x = -\frac{\partial \hat{\mathcal{H}}_{Sol}}{\partial \hat{x}}$$

$$= -\frac{\hat{x}}{[1 + f(\hat{p}_\sigma)]} \cdot [H(s_0)]^2 \cdot \Delta s \cdot \delta(s - s_0) ; \tag{3.34b}$$

$$\frac{d}{ds}\hat{z} = +\frac{\partial \hat{\mathcal{H}}_{Sol}}{\partial \hat{p}_z}$$

$$= \frac{\hat{p}_z}{[1 + f(\hat{p}_\sigma)]} ; \tag{3.34c}$$

$$\frac{d}{ds}\hat{p}_z = -\frac{\partial \hat{\mathcal{H}}_{Sol}}{\partial \hat{z}}$$

$$= -\frac{\hat{z}}{[1 + f(\hat{p}_\sigma)]} \cdot [H(s_0)]^2 \cdot \Delta s \cdot \delta(s - s_0) ; \tag{3.34d}$$

$$\frac{d}{ds}\hat{\sigma} = +\frac{\partial \hat{\mathcal{H}}_{Sol}}{\partial \hat{p}_\sigma}$$

$$= -\frac{f'(\hat{p}_\sigma)}{[1 + f(\hat{p}_\sigma)]^2}$$

$$\times\frac{1}{2}\left\{\left[\hat{p}_x^2+\hat{p}_z^2\right]+\left[H(s_0)\right]^2\cdot\Delta s\cdot\delta(s-s_0)\cdot\left[\hat{x}^2+\hat{z}^2\right]\right\} \; ; \qquad (3.34e)$$

$$\frac{d}{ds}\hat{p}_\sigma = -\frac{\partial\hat{\mathcal{H}}_{Sol}}{\partial\hat{\sigma}}$$

$$= 0 \; . \qquad (3.34f)$$

3.7.2 Solution of the Equations of Motion

Equations (3.34) can now be solved by integrating both sides from

$$s_0-\epsilon \text{ to } s_0+\epsilon$$

with

$$0<\epsilon \longrightarrow 0$$

leading to:

$$\hat{x}^f = \hat{x}^i \; ; \qquad (3.35a)$$

$$\hat{p}_x^f = \hat{p}_x^i - \frac{\hat{x}^i}{[1+f(\hat{p}_\sigma^i)]}\cdot[H(s_0)]^2\cdot\Delta s \; ; \qquad (3.35b)$$

$$\hat{z}^f = \hat{z}^i \; ; \qquad (3.35c)$$

$$\hat{p}_z^f = \hat{p}_z^i - \frac{\hat{z}^i}{[1+f(\hat{p}_\sigma^i)]}\cdot[H(s_0)]^2\cdot\Delta s \; ; \qquad (3.35d)$$

$$\hat{\sigma}^f = \hat{\sigma}^i - \frac{f'(\hat{p}_\sigma^i)}{[1+f(\hat{p}_\sigma^i)]^2}\cdot[H(s_0)]^2\cdot\Delta s\cdot\frac{1}{2}\left[(\hat{x}^i)^2+(\hat{z}^i)^2\right] \; ; \qquad (3.35e)$$

$$\hat{p}_\sigma^f = \hat{p}_\sigma^i \; . \qquad (3.35f)$$

Choosing in eqn. (3.31) the lower limit of integration s_1 as

$$s_1 = s_0 - 0$$

we furthermore obtain from (3.32):

$$x^i = \hat{x}^i \; ; \qquad (3.36a)$$

$$p_x^i = \hat{p}_x^i \; ; \qquad (3.36b)$$

$$z^i = \hat{z}^i \; ; \tag{3.36c}$$

$$p_z^i = \hat{p}_z^i \; ; \tag{3.36d}$$

$$\sigma^i = \hat{\sigma}^i \; ; \tag{3.36e}$$

$$p_\sigma^i = \hat{p}_\sigma^i \tag{3.36f}$$

and

$$x^f = \hat{x}^f \cdot \cos\Delta\Theta + \hat{z}^f \cdot \sin\Delta\Theta \; ; \tag{3.37a}$$

$$p_x^f = \hat{p}_x^f \cdot \cos\Delta\Theta + \hat{p}_z^f \cdot \sin\Delta\Theta \; ; \tag{3.37b}$$

$$z^f = -\hat{x}^f \cdot \sin\Delta\Theta + \hat{z}^f \cdot \cos\Delta\Theta \; ; \tag{3.37c}$$

$$p_z^f = -\hat{p}_x^f \cdot \sin\Delta\Theta + \hat{p}_z^f \cdot \cos\Delta\Theta \; ; \tag{3.37d}$$

$$\begin{aligned} \sigma^f &= \hat{\sigma}^f + \left\{\hat{x}^f \cdot \hat{p}_z^f - \hat{z}^f \cdot \hat{p}_x^f\right\} \cdot \frac{\int_{s_0-0}^{s_0+0} d\tilde{s} \cdot H(\tilde{s})}{\left[1 + f(\hat{p}_\sigma^f)\right]^2} \cdot f'(\hat{p}_\sigma^f) \\ &= \hat{\sigma}^f + \left\{\hat{x}^f \cdot \hat{p}_z^f - \hat{z}^f \cdot \hat{p}_x^f\right\} \cdot \frac{H(s_0) \cdot \Delta s}{\left[1 + f(\hat{p}_\sigma^f)\right]^2} \cdot f'(\hat{p}_\sigma^f) \; ; \end{aligned} \tag{3.37e}$$

$$p_\sigma^f = \hat{p}_\sigma^f \tag{3.37f}$$

with

$$\Delta\Theta = \frac{H(s_0) \cdot \Delta s}{[1 + f(\hat{p}_\sigma^i)]} \; , \tag{3.38}$$

whereby we have used eqns. (3.27b) and (3.31).

3.7.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqns. (3.35), (3.36), and (3.37) read as:

$$\begin{aligned} \underline{\mathcal{J}}_{sol} &= \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)} \\ &= \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(\hat{x}^f, \hat{p}_x^f, \hat{z}^f, \hat{p}_z^f, \hat{\sigma}^f, \hat{p}_\sigma^f)} \\ &\times \frac{\partial(\hat{x}^f, \hat{p}_x^f, \hat{z}^f, \hat{p}_z^f, \hat{\sigma}^f, \hat{p}_\sigma^f)}{\partial(\hat{x}^i, \hat{p}_x^i, \hat{z}^i, \hat{p}_z^i, \hat{\sigma}^i, \hat{p}_\sigma^i)} \end{aligned}$$

$$\times \quad \frac{\partial(\hat{x}^i,\, \hat{p}_x^i,\, \hat{z}^i,\, \hat{p}_z^i,\, \hat{\sigma}^i,\, \hat{p}_\sigma^i)}{\partial(x^i,\, p_x^i,\, z^i,\, p_z^i,\, \sigma^i,\, p_\sigma^i)}$$

$$= \underline{\mathcal{J}}_1 \cdot \underline{\mathcal{J}}_2 \cdot \underline{\mathcal{J}}_3 \tag{3.39}$$

with [7]

$$\underline{\mathcal{J}}_1 \equiv \frac{\partial(x^f,\, p_x^f,\, z^f,\, p_z^f,\, \sigma^f,\, p_\sigma^f)}{\partial(\hat{x}^f,\, \hat{p}_x^f,\, \hat{z}^f,\, \hat{p}_z^f,\, \hat{\sigma}^f,\, \hat{p}_\sigma^f)}$$

$$= \begin{pmatrix} \cos\Delta\Theta & 0 & \sin\Delta\Theta & 0 & 0 & -z^f \cdot Z \\ 0 & \cos\Delta\Theta & 0 & \sin\Delta\Theta & 0 & -p_z^f \cdot Z \\ -\sin\Delta\Theta & 0 & \cos\Delta\Theta & 0 & 0 & +x^f \cdot Z \\ 0 & -\sin\Delta\Theta & 0 & \cos\Delta\Theta & 0 & +p_x^f \cdot Z \\ \hat{p}_z^f \cdot Z & -\hat{z}^f \cdot Z & -\hat{p}_x^f \cdot Z & \hat{x}^f \cdot Z & 1 & Z_0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \tag{3.40a}$$

$$\underline{\mathcal{J}}_2 \equiv \frac{\partial(\hat{x}^f,\, \hat{p}_x^f,\, \hat{z}^f,\, \hat{p}_z^f,\, \hat{\sigma}^f,\, \hat{p}_\sigma^f)}{\partial(\hat{x}^i,\, \hat{p}_x^i,\, \hat{z}^i,\, \hat{p}_z^i,\, \hat{\sigma}^i,\, \hat{p}_\sigma^i)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ Q & 1 & 0 & 0 & 0 & +R_x \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & Q & 1 & 0 & +R_z \\ -R_x & 0 & -R_z & 0 & 1 & W \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \tag{3.40b}$$

$$\underline{\mathcal{J}}_3 \equiv \frac{\partial(\hat{x}^i,\, \hat{p}_x^i,\, \hat{z}^i,\, \hat{p}_z^i,\, \hat{\sigma}^i,\, \hat{p}_\sigma^i)}{\partial(x^i,\, p_x^i,\, z^i,\, p_z^i,\, \sigma^i,\, p_\sigma^i)} = \underline{1} \tag{3.40c}$$

and with

$$Z = +\frac{f'(\hat{p}_\sigma^f)}{\left[1+f(\hat{p}_\sigma^f)\right]^2} \cdot H(s_0) \cdot \Delta s\,; \tag{3.41a}$$

$$Z_0 = -\frac{2 \cdot \left[f'(\hat{p}_\sigma^f)\right]^2 - f''(\hat{p}_\sigma^f) \cdot \left[1+f(\hat{p}_\sigma^f)\right]}{\left[1+f(\hat{p}_\sigma^f)\right]^3} \cdot H(s_0) \cdot \Delta s \cdot \left[\hat{x}^f \cdot \hat{p}_z^f - \hat{z}^f \cdot \hat{p}_x^f\right]; \tag{3.41b}$$

$$Q = -\frac{1}{[1+f(\hat{p}_\sigma^i)]} \cdot [H(s_0)]^2 \cdot \Delta s\,; \tag{3.41c}$$

$$R_x = +\frac{f'(\hat{p}_\sigma^i)}{[1+f(\hat{p}_\sigma^i)]^2} \cdot [H(s_0)]^2 \cdot \Delta s \cdot \hat{x}^i\,; \tag{3.41d}$$

[7] Equations (3.39) and (3.40a, b, c) correspond to the usual factorization of the transfer matrix for solenoids into a rotation and focussing part.

$$R_z = + \frac{f'(\hat{p}^i_\sigma)}{[1+f(\hat{p}^i_\sigma)]^2} \cdot [H(s_0)]^2 \cdot \Delta s \cdot \hat{z}^i \, ; \tag{3.41e}$$

$$W = + \frac{[f'(\hat{p}^i_\sigma)]^2 - \frac{1}{2} \cdot f''(\hat{p}^i_\sigma) \cdot [1+f(\hat{p}^i_\sigma)]}{[1+f(\hat{p}^i_\sigma)]^3} \cdot [H(s_0)]^2 \cdot \Delta s \cdot \left[(\hat{x}^i)^2 + (\hat{z}^i)^2\right] . \tag{3.41f}$$

Using eqn. (3.40) it can be shown that:

$$\underline{\mathcal{J}}_1^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_1 = \underline{\mathcal{J}}_2^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_2 = \underline{S} \, ;$$

$$\Longrightarrow \quad \underline{\mathcal{J}}_{sol}^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_{sol} = \underline{S} \, , \tag{3.42}$$

i.e. $\underline{\mathcal{J}}_{sol} = \underline{\mathcal{J}}_1 \cdot \underline{\mathcal{J}}_2$ is symplectic.

3.8 Cavity

3.8.1 Canonical Equations of Motion

For a cavity we have:

$$V \neq 0$$

and

$$K_x = K_z = g = N = \lambda = \mu = H = 0 \, .$$

Using thin–lens approximation we write for a cavity of length Δs at position s_0:

$$V(s) = V(s_0) \cdot \Delta s \cdot \delta(s - s_0) \, .$$

Then we obtain from (2.31):

$$\frac{d}{ds} x = \frac{p_x}{[1+f(p_\sigma)]} \, ; \tag{3.43a}$$

$$\frac{d}{ds} p_x = 0 \, ; \tag{3.43b}$$

$$\frac{d}{ds} z = \frac{p_z}{[1+f(p_\sigma)]} \, ; \tag{3.43c}$$

$$\frac{d}{ds} p_z = 0 \, ; \tag{3.43d}$$

$$\frac{d}{ds} \sigma = 1 - f'(p_\sigma) - \frac{1}{2} \cdot [(x')^2 + (z')^2] \cdot f'(p_\sigma) \, ; \tag{3.43e}$$

$$\frac{d}{ds} p_\sigma = \frac{1}{\beta_0^2} \cdot \frac{eV(s_0)}{E_0} \cdot \Delta s \cdot \delta(s - s_0) \cdot \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \, . \tag{3.43f}$$

3.8.2 Solution of the Equations of Motion

The solution of eqn. (3.43) reads as:

$$x^f = x^i ; \tag{3.44a}$$

$$p_x^f = p_x^i ; \tag{3.44b}$$

$$z^f = z^i ; \tag{3.44c}$$

$$p_z^f = p_z^i ; \tag{3.44d}$$

$$\sigma^f = \sigma^i ; \tag{3.44e}$$

$$p_\sigma^f = p_\sigma^i + \frac{1}{\beta_0^2} \cdot \frac{eV(s_0)}{E_0} \cdot \Delta s \cdot \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma^i + \varphi\right] \tag{3.44f}$$

(see also Refs. [4, 5]).

3.8.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.44) takes the form:

$$\underline{\mathcal{J}}_{cav} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & Q & 1 \end{pmatrix} . \tag{3.45}$$

with

$$Q = h \cdot \frac{2\pi}{L} \cdot \frac{1}{\beta_0^2} \cdot \frac{eV(s_0)}{E_0} \cdot \Delta s \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma^i + \varphi\right] . \tag{3.46}$$

From eqn. (3.45) it can be verified that $\underline{\mathcal{J}}_{cav}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}_{cav}^T \cdot \underline{S} \cdot \underline{\mathcal{J}}_{cav} = \underline{S} . \tag{3.47}$$

3.9 Drift Space

Up to now all elements have been kicks of zero length. The actual length of the machine is equal to the sum of the drift spaces which are in between the various kicks. Of course we cannot ignore the length in this case. The treatment of the long drift element concludes our chapter on magnet elements in the thin–lens approximation.

3.9.1 Canonical Equations of Motion

For a drift space we have:

$$K_x = K_z = g = N = \lambda = \mu = H = V = 0 .$$

Then we obtain from (2.31):

$$\frac{d}{ds}\, x = \frac{p_x}{[1+f(p_\sigma)]} ; \tag{3.48a}$$

$$\frac{d}{ds}\, p_x = 0 ; \tag{3.48b}$$

$$\frac{d}{ds}\, z = \frac{p_z}{[1+f(p_\sigma)]} ; \tag{3.48c}$$

$$\frac{d}{ds}\, p_z = 0 ; \tag{3.48d}$$

$$\frac{d}{ds}\, \sigma = 1 - f'(p_\sigma) - \frac{1}{2} \cdot \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \cdot f'(p_\sigma) ; \tag{3.48e}$$

$$\frac{d}{ds}\, p_\sigma = 0 . \tag{3.48f}$$

These (nonlinear) differential equations describe the motion of the particles in the space between the point-like lenses.

3.9.2 Solution of the Equations of Motion

The solution of eqn. (3.48) reads as:

$$x^f = x^i + \frac{p_x^i}{[1+f(p_\sigma^i)]} \cdot l ; \tag{3.49a}$$

$$p_x^f = p_x^i ; \tag{3.49b}$$

$$z^f = z^i + \frac{p_z^i}{[1+f(p_\sigma^i)]} \cdot l ; \tag{3.49c}$$

$$p_z^f = p_z^i ; \tag{3.49d}$$

$$\sigma^f = \sigma^i + \left\{ 1 - f'(p_\sigma^i) - \frac{1}{2} \cdot \frac{(p_x^i)^2 + (p_z^i)^2}{[1+f(p_\sigma^i)]^2} \cdot f'(p_\sigma^i) \right\} \cdot l ; \tag{3.49e}$$

$$p_\sigma^f = p_\sigma^i . \tag{3.49f}$$

3.9.3 Jacobian Matrix and Symplecticity Condition

The Jacobian matrix resulting from eqn. (3.49) takes the form:

$$\underline{\mathcal{J}}_{drift} = \frac{\partial(x^f, p_x^f, z^f, p_z^f, \sigma^f, p_\sigma^f)}{\partial(x^i, p_x^i, z^i, p_z^i, \sigma^i, p_\sigma^i)}$$

$$= \begin{pmatrix} 1 & \frac{l}{[1+f(p_\sigma^i)]} & 0 & 0 & 0 & -\frac{p_x^i}{[1+f(p_\sigma^i)]^2}\cdot f'(p_\sigma^i)\cdot l \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{l}{[1+f(p_\sigma^i)]} & 0 & -\frac{p_z^i}{[1+f(p_\sigma^i)]^2}\cdot f'(p_\sigma^i)\cdot l \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{p_x^i}{[1+f(p_\sigma^i)]^2}\cdot f'(p_\sigma^i)\cdot l & 0 & -\frac{p_z^i}{[1+f(p_\sigma^i)]^2}\cdot f'(p_\sigma^i)\cdot l & 1 & Q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.50}$$

with

$$Q = -l\cdot\left\{f''(p_\sigma^i) - \frac{(p_x^i)^2+(p_z^i)^2}{[1+f(p_\sigma^i)]^3}\cdot\left[f'(p_\sigma^i)\right]^2 + \frac{1}{2}\,\frac{(p_x^i)^2+(p_z^i)^2}{[1+f(p_\sigma^i)]^2}\cdot f''(p_\sigma^i)\right\} \tag{3.51}$$

and

$$y^i \equiv y(s_0)\;;$$
$$y^f \equiv y(s_0+l)\;;$$

$$(y = x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma)\,.$$

From eqn. (3.50) it can be verified that $\underline{\mathcal{J}}_{drift}$ obeys the symplecticity condition

$$\underline{\mathcal{J}}_{drift}^T\cdot\underline{S}\cdot\underline{\mathcal{J}}_{drift} = \underline{S}\,. \tag{3.52}$$

4 Summary

We have shown how to solve the nonlinear canonical equations of motion in the framework of the fully the six–dimensional formalism for various kinds of magnets (bending magnets, quadrupoles, synchrotron magnets, skew quadrupoles, sextupoles, octupoles, solenoids) and for cavities by using symplectic kicks, taking into account the energy dependence of the focusing strength.

We have checked in each case the symplecticity condition with the help of the Jacobian matrix.

The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy.

Almost all these elements including higher order kicks up to $10th$ order are available in SIXTRACK using this formalism. One exception is the solenoid element which can, however, easily be added.

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Appendix A: The Symplecticity Condition

The canonical equations of motion can be written as

$$\frac{d}{ds}\vec{y} \;=\; \underline{S}\cdot\frac{\partial}{\partial\vec{y}}\mathcal{H}(\vec{y};\,s) \tag{A.1}$$

or in component form as:

$$\frac{d}{ds}y_i \;=\; \sum_k S_{ik}\cdot\frac{\partial}{\partial y_k}\mathcal{H}(\vec{y};\,s) \tag{A.2}$$

with the notation

$$\begin{aligned}\vec{y}^{\,T} &= (y_1, y_2, y_3, y_4, y_5, y_6)\\ &\equiv (x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma)\;.\end{aligned}$$

We now introduce the Jacobian matrix:

$$\underline{\mathcal{J}} \;=\; ((\mathcal{J}_{ik}))\;;\quad \mathcal{J}_{ik}(s,s_0) \;=\; \frac{\partial y_i(s)}{\partial y_k(s_0)}\;. \tag{A.3}$$

Then it follows that:

$$\begin{aligned}\frac{d}{ds}\mathcal{J}_{ik}(s,s_0) &= \frac{\partial}{\partial y_k(s_0)}\,\frac{d}{ds}\,y_i(s)\\ &= \sum_n \frac{\partial}{\partial y_k(s_0)}\left[S_{in}\cdot\frac{\partial}{\partial y_n(s)}\mathcal{H}(\vec{y};\,s)\right]\\ &= \sum_{n,l}\frac{\partial y_l(s)}{\partial y_k(s_0)}\cdot\frac{\partial}{\partial y_l(s)}\left[S_{in}\cdot\frac{\partial}{\partial y_n(s)}\mathcal{H}(\vec{y};\,s)\right]\\ &= \sum_{n,l}\mathcal{J}_{lk}(s,s_0)\cdot S_{in}\cdot\frac{\partial^2}{\partial y_l(s)\partial y_n(s)}\mathcal{H}(\vec{y};\,s)\\ &= \sum_{n,l} S_{in}\cdot\mathcal{H}_{nl}\cdot\mathcal{J}_{lk}\end{aligned} \tag{A.4}$$

with

$$\mathcal{H}_{nl} \;=\; \frac{\partial^2}{\partial y_l(s)\partial y_n(s)}\mathcal{H}(\vec{y};\,s) \tag{A.5}$$

or that

$$\underline{\mathcal{J}}'(s,s_0) \;=\; \underline{S}\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}}(s,s_0) \tag{A.6}$$

with

$$\underline{\mathcal{H}} \;=\; ((\mathcal{H}_{ik}))\;.$$

Thus we have:

$$\begin{aligned}
\frac{d}{ds}\left\{\underline{\mathcal{J}}^T(s,s_0)\cdot\underline{S}\cdot\underline{\mathcal{J}}(s,s_0)\right\} &= \{\underline{S}\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}}(s,s_0)\}^T\cdot\underline{S}\cdot\underline{\mathcal{J}}+\underline{\mathcal{J}}^T(s,s_0)\cdot\underline{S}\cdot\{\underline{S}\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}}(s,s_0)\} \\
&= \underline{\mathcal{J}}^T(s,s_0)\cdot\underline{\mathcal{H}}^T\cdot\underline{S}^T\cdot\underline{S}\cdot\underline{\mathcal{J}}+\underline{\mathcal{J}}^T(s,s_0)\cdot\underline{S}^2\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}} \\
&= \underline{\mathcal{J}}^T(s,s_0)\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}}(s,s_0)-\underline{\mathcal{J}}^T(s,s_0)\cdot\underline{\mathcal{H}}\cdot\underline{\mathcal{J}}(s,s_0) \\
&= 0\,, \qquad \text{(A.7)}
\end{aligned}$$

where we have used the relations

$$\begin{aligned}
\underline{S}^T &= -\underline{S}\,; \\
\underline{S}^2 &= -\underline{1}\,; \\
\underline{\mathcal{H}}^T &= \underline{\mathcal{H}}\,.
\end{aligned}$$

From (A.7) we obtain:

$$\begin{aligned}
\underline{\mathcal{J}}^T(s,s_0)\cdot\underline{S}\cdot\underline{\mathcal{J}}(s,s_0) &= \text{const.} \\
&= \underline{\mathcal{J}}^T(s_0,s_0)\cdot\underline{S}\cdot\underline{\mathcal{J}}(s_0,s_0) \\
&= \underline{S} \qquad \text{(A.8)}
\end{aligned}$$

(see also Ref. [6]).

If the Hamiltonian is quadratic in y_i, $(i = 1,\cdots 6)$, one has according to (A.3):

$$\underline{\mathcal{J}}(s,s_0) = \underline{M}(s,s_0)\,. \qquad \text{(A.9)}$$

In this case eqn. (A.8) reads as

$$\underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0) = \underline{S} \qquad \text{(A.10)}$$

representing the "symplecticity–condition" for the (linear) transfer matrix $\underline{M}(s,s_0)$.

We thus have proved:

Theorem I: The canonical structure of the equations of motion implies the symplecticity of the Jacobian matrices.

We now show that the converse of theorem I is also true.

Theorem II: The symplecticity of the Jacobian matrix implies that the equations of motion can be written in canonical form.

Supposition: The Jacobian matrix $\mathcal{J}(s, s_0)$ with

$$\underline{\mathcal{J}} = ((\mathcal{J}_{ik})) ;$$
$$\mathcal{J}_{ik}(s, s_0) = \frac{\partial y_i(s)}{\partial y_k(s_0)}$$

satisfies the symplecticity condition

$$\underline{\mathcal{J}}^T(s, s_0) \cdot \underline{S} \cdot \underline{\mathcal{J}}(s, s_0) = \underline{S} . \tag{A.11}$$

Proposition: There is a function $\mathcal{H}(q_i, p_i; s)$ so that the equations of motion can be written in the canonical form:

$$\frac{d}{ds} q_k = + \frac{\partial}{\partial p_k} \mathcal{H} ; \tag{A.12a}$$
$$\frac{d}{ds} p_k = - \frac{\partial}{\partial q_k} \mathcal{H} \tag{A.12b}$$

with the notation

$$\vec{y}^{\,T} \equiv (x, p_x, z, p_z, \sigma, p_\sigma)$$
$$= (q_1, p_1, q_2, p_2, q_3, p_3) .$$

Proof:

From eqn. (A.11) we get:

$$\underline{\mathcal{J}}\,\underline{S} \cdot \underline{\mathcal{J}}^T\,\underline{S}\,\underline{\mathcal{J}} \cdot \underline{\mathcal{J}}^{-1}\,\underline{S}^T = \underline{\mathcal{J}}\,\underline{S} \cdot \underline{S} \cdot \underline{\mathcal{J}}^{-1}\,\underline{S}^T$$

or

$$\underline{\mathcal{J}}\,\underline{S}\,\underline{\mathcal{J}}^T = \underline{S} . \tag{A.13}$$

Taking into account this relationship we obtain for the Poisson–brackets [8]:

$$[y_m(s), y_n(s)]_{\vec{y}(s_0)} = \sum_{i,k=1}^{3} S_{ik} \cdot \frac{\partial y_m(s)}{\partial y_i(s_0)} \cdot \frac{\partial y_n(s)}{\partial y_k(s_0)}$$

[8] The Poisson–brackets for two arbitrary functions $f[\vec{y}(s)]$, $g[\vec{y}(s)]$ are defined by:

$$\begin{aligned}
[f[\vec{y}(s)], g[\vec{y}(s)]]_{\vec{y}(s_0)} &= \left[\frac{\partial f[\vec{y}(s)]}{\partial p_x(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial x(s_0)} - \frac{\partial f[\vec{y}(s)]}{\partial x(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial p_x(s_0)}\right] \\
&+ \left[\frac{\partial f[\vec{y}(s)]}{\partial p_z(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial z(s_0)} - \frac{\partial f[\vec{y}(s)]}{\partial z(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial p_z(s_0)}\right] \\
&+ \left[\frac{\partial f[\vec{y}(s)]}{\partial p_\sigma(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial \sigma(s_0)} - \frac{\partial f[\vec{y}(s)]}{\partial \sigma(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial p_\sigma(s_0)}\right] \\
&\equiv \sum_{i,k=1}^{3} S_{ik} \cdot \frac{\partial f[\vec{y}(s)]}{\partial y_i(s_0)} \cdot \frac{\partial g[\vec{y}(s)]}{\partial y_k(s_0)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,k=1}^{3} S_{ik} \cdot \mathcal{J}_{mi}(s, s_0) \cdot \mathcal{J}_{nk}(s, s_0) \\
&= \sum_{i,k=1}^{3} \mathcal{J}_{mi}(s, s_0) \cdot S_{ik} \cdot \mathcal{J}^T_{kn}(s, s_0) \\
&= S_{mn} \qquad \text{(A.14)}
\end{aligned}
$$

or

$$
\begin{aligned}
[p_i(s),\, p_k(s)]_{\vec{y}(s_0)} &= 0\,; && \text{(A.15a)} \\
[q_i(s),\, q_k(s)]_{\vec{y}(s_0)} &= 0\,; && \text{(A.15b)} \\
[p_i(s),\, q_k(s)]_{\vec{y}(s_0)} &= \delta_{ik}\,. && \text{(A.15c)}
\end{aligned}
$$

From (A.15) it follows by differentiation that:

$$
\begin{aligned}
[p'_i(s),\, p_k(s)]_{\vec{y}(s_0)} + [p_i(s),\, p'_k(s)]_{\vec{y}(s_0)} &= 0\,; && \text{(A.16a)} \\
[q'_i(s),\, q_k(s)]_{\vec{y}(s_0)} + [q_i(s),\, q'_k(s)]_{\vec{y}(s_0)} &= 0\,; && \text{(A.16b)} \\
[p'_i(s),\, q_k(s)]_{\vec{y}(s_0)} + [p_i(s),\, q'_k(s)]_{\vec{y}(s_0)} &= 0\,. && \text{(A.16c)}
\end{aligned}
$$

Putting $s = s_0$, the relations (A.16) lead to:

$$
\begin{aligned}
-\frac{\partial p'_i(s)}{\partial q_k(s)} + \frac{\partial p'_k(s)}{\partial q_i(s)} &= 0\,; && \text{(A.17a)} \\
\frac{\partial q'_i(s)}{\partial p_k(s)} - \frac{\partial q'_k(s)}{\partial p_i(s)} &= 0\,; && \text{(A.17b)} \\
\frac{\partial p'_i(s)}{\partial p_k(s)} + \frac{\partial q'_k(s)}{\partial q_i(s)} &= 0\,. && \text{(A.17c)}
\end{aligned}
$$

Equation (A.17a) implies that the 3 functions $p'_i(s)$ $(i = 1, 2, 3)$ form an irrotational vector field in the space of the q_k so that they can be expressed in this space as a gradient of a function $F(q, p)$ [8]:

$$
p'_i(s) = \frac{\partial}{\partial q_i} F(q, p)\,. \qquad \text{(A.18a)}
$$

Because of eqn. (A.17b) a similar expression holds for the 3 functions $q'_i(s)$ in the space of the p_k:

$$
q'_i(s) = \frac{\partial}{\partial p_i} G(q, p)\,. \qquad \text{(A.18b)}
$$

Substituting (A.18a, b) into the remaining expression (A.17c) we get:

$$
\frac{\partial^2}{\partial p_k\, \partial q_i}(F + G) = 0 \qquad \text{(A.18c)}
$$

which means that $(F + G)$ can be written in the form:

$$
(F + G) = f(q) + g(p)\,. \qquad \text{(A.19)}
$$

Thus in eqn. (A.18a) we can express F in terms of G:

$$\begin{aligned} p_i'(s) &= \frac{\partial}{\partial q_i} \left[f(q) + g(p) - G(q, p)\right] \\ &= \frac{\partial}{\partial q_i} \left[f(q) - G(q, p)\right] \ . \end{aligned}$$

Since eqn. (A.18b) can be replaced by

$$q_i'(s) = -\frac{\partial}{\partial p_i} \left[f(q) - G(q, p)\right]$$

we may finally write:

$$p_i{}'(s) = +\frac{\partial}{\partial q_i} \mathcal{H} \ ; \qquad \text{(A.20a)}$$

$$q_i{}'(s) = -\frac{\partial}{\partial p_i} \mathcal{H} \qquad \text{(A.20b)}$$

with a single function

$$\mathcal{H} = f(q) - G(q, p) \qquad \text{(A.21)}$$

which proves the canonical structure of the equations of motion (see also Ref. [9]).

Appendix B: Thin–Lens Formalism in SIXTRACK

The single particle code SIXTRACK is based on A. Wrulich's RACETRACK code [2] which has been extended to six dimensions in a symplectic manner in 1987. Many new features and extensions have been added since then, such as for instance the production of differential algebra maps 'a la BERZ [3]. Due to its simplicity, user–friendliness and a considerable post–processing package SIXTRACK now has many users around the world.

In most cases the accelerator structures to be studied are given as a sequence of thick lens elements (drifts, dipoles and quadrupoles) interleaved by thin non–linear (or linear) kicks. In fact SIXTRACK has been mostly used in this mode and runs at very high speed, in particular when using the vectorized version (at present no faster code is known to the authors). For the planned new accelerators like the LHC, however, this performance may still not be good enough. For these machines tracking runs are necessary which take single particles over millions of turns which may take months of CPU–time even on the most advanced computers available today. This thin–lens formalism has therefore been welcome for speeding up tracking runs without losing symplecticity.

Of course it is also mandatory that the thin lens version allows to predict the same long-term behaviour as the thick lens version and that both versions give about the same dynamic aperture. The symplecticity has been proven in this report, however, the question of the quality of the thin–lens approximation is more difficult to answer. As there is no direct proof or easy approximation we restrict ourselves to one relevant and complex example to obtain at least a rough estimate of the loss in precision and the gain in tracking speed.

As an example we have chosen a model of the LHC lattice (version 2) which has 1280 sets of dipole multipolar errors (with standard values [10] of up to order 9) and 384 sextupoles for the chromaticity correction. The lattices include the interaction zones properly, closed orbit resulting from dipole kicks have been ignored, for simplicity the correction schemes to correct the detuning due to the sextupole and decapole components of the dipole field have not been applied, only one random seed has been tested, special care has been taken to ensure that both models have exactly the same sequence of errors and a momentum deviation of about two thirds of the bucket half–height has been considered. Note that the full nonlinear equation (3.44f) is used in SIXTRACK.

In Fig. 1 the survival turn numbers (before reaching 10^5 turns) of the thick and thin lens lattice (40 twin initial conditions each) are shown together with the border of the onset of chaotic motion in the full six-dimensional phase space. The agreement of the survival turn numbers is very satisfactory and the difference of the borders of the onset of chaos for the two cases will probably become smaller for a finer stepsize of initial conditions. The gain in speed is 33% which is the maximum that can expected from the ratio of calculation time between the 6d linear and non-linear part respectively.

We conclude that the thin lens lattice can reproduce with good precision the long-term behavior of a large and complex machine. We have tested only one case, but as the linear part is almost identical and the machine is very non-linear (e.g. chromaticity is dominated by the b_3 components of the dipole magnets) an accidental agreement seems unlikely. The gain of a third in tracking speed is relevant when CPU times of weeks or months are considered. Moreover there seems to be no penalty to be payed, neither in terms of symplecticity nor in terms of precision.

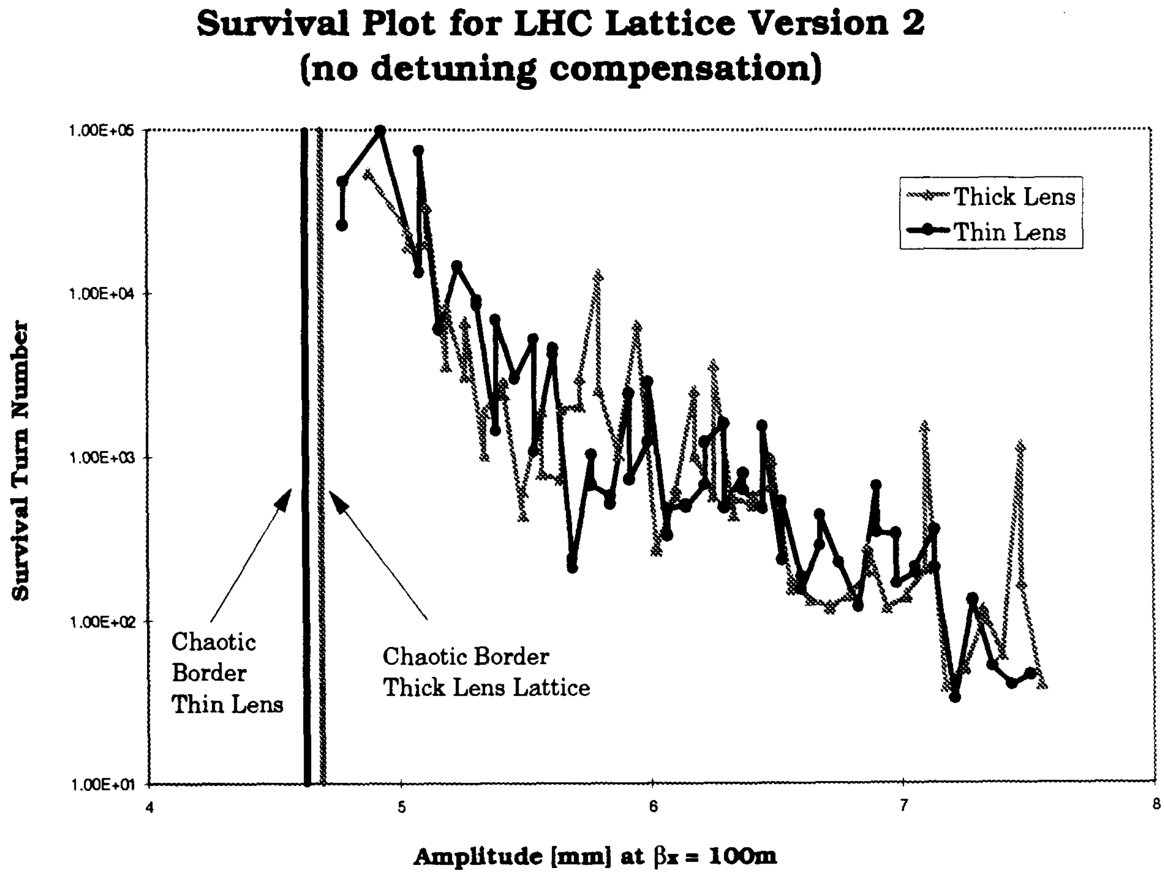

Figure 1: Comparison of survial times and borders of the onset of chaotic motion for thick and thin lens LHC lattices (version 2).

Appendix C: A Collection of Useful Formulae

The following abbreviations have been used:

t = time
e = charge of the particle
c = velocity of light
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
$E = m_0\gamma\, c^2$ = energy
s = longitudinal position
m_0 = rest mass of the particle
$\beta = \sqrt{1-\left(\frac{m_0c^2}{E}\right)^2}$
$v_0 = c\beta_0$ = design velocity
$p = m_0\gamma\, v$ = momentum

The relative energy deviation is defined as:

$$\eta \;=\; \frac{\Delta E}{E_0}\,. \tag{C.1}$$

The canonical coordinates of the longitudinal oscillations are:

$$\sigma \;=\; s - v_0\cdot t \tag{C.2}$$

and

$$p_\sigma \;=\; \frac{1}{\beta_0^2}\cdot\eta\,. \tag{C.3}$$

The relative momentum deviation is:

$$\hat{\eta} \;=\; \frac{p}{p_0} - 1 = \frac{p-p_0}{p_0} = \frac{\Delta p}{p_0}\,; \tag{C.4}$$

$$(1+\hat{\eta}) \;=\; \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - (\frac{m_0c^2}{E_0})^2} = \frac{1}{\beta_0}\cdot\frac{p\cdot c}{E_0} = \frac{p}{p_0}\,. \tag{C.5}$$

To stress that $\hat{\eta}$ depends on the longitudinal canonical variable p_σ (see C.3) we define $f(p_\sigma)$ as follows:

$$\begin{aligned} f(p_\sigma) \;\equiv\; \hat{\eta} \;&=\; \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - \left(\frac{m_0c^2}{E_0}\right)^2} - 1 \\ &=\; \frac{1}{\beta_0}\sqrt{\left(1+\beta_0^2\cdot p_\sigma\right)^2 - \left(\frac{m_0c^2}{E_0}\right)^2} - 1\,. \end{aligned} \tag{C.6}$$

A series expansion of $f(p_\sigma)$

$$f(p_\sigma) \;=\; f(0) + f'(0)\cdot p_\sigma + f''(0)\cdot\frac{1}{2}p_\sigma^2 + \cdots \tag{C.7}$$

leads to:

$$f(p_\sigma) \;=\; p_\sigma - \frac{1}{\gamma_0^2}\cdot\frac{1}{2}p_\sigma^2 \pm \cdots\,; \tag{C.8}$$

whereby we have used:

$$f'(p_\sigma) = \frac{\beta_0 \cdot (1 + \beta_0^2 \cdot p_\sigma)}{\sqrt{(1 + \beta_0^2 \cdot p_\sigma)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2}} \equiv \frac{v_0}{v} ; \tag{C.9}$$

$$\Longrightarrow \quad f'(0) = 1$$

and

$$f''(p_\sigma) = -\frac{\beta_0^3}{\gamma_0^2 \cdot \left(\sqrt{(1 + \beta_0^2 \cdot p_\sigma)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2}\right)^3} \equiv -\frac{1}{\gamma_0^2} \cdot \left(\frac{p_0}{p}\right)^3 ; \tag{C.10}$$

$$\Longrightarrow \quad f''(0) = -\frac{1}{\gamma_0^2} .$$

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and literature cited therein.
The updated manual can be retrieved from the WWW at the location: *http://hpariel.cern.ch/frs/Documentation/six.ps.*

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