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Self-dual Solitons in Dual-Transformed Abelian Gauge Theories in Curved Space*

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Abstract

A continuum version of the dual transformation is presented in Abelian gauge theories in (2+1) dimensional curved spacetime. Two representative models, one Einstein Maxwell Higgs theory and the other Einstein Chern-Simons Higgs theory, are studied within path integral formalism to understand various aspects of the dual transformation We derive the Bogomol'nyi-type bound and analyze the soliton solutions for the systems as well as the global structures of spacetime geometry of the solutions.

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For the Abelian theories whose symmetry groups are discrete, continuous or some combination of these, it has been known that the original theory formulated on a lattice or in the continuum is mapped into a dual theory [1]. When the duality transformation is successfully applied, there are a number of benefits in handling some difficulties occurring in the original theory. First, it maps a theory with a large coupling constant into a theory with a small coupling constant. So it can be useful to understand the strong coupling regime of field theories, or equivalently the high temperature limit of statistical systems and Euclidean field theories. Second, it shows well the role of topological excitations and their interaction properties for a class of theories.

In this note, we will apply the duality transformation to (2+1) dimensional scalar electrodynamics coupled to Einstein gravity in continuum and then discuss the physics related to the self-dual solitons. We select two models described by the Lagrange densities

$$\mathcal{L}_{Max} = -\frac{1}{16\pi G}R - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi - V(|\phi|) \tag{1}$$

$$\mathcal{L}_{CS} = -\frac{1}{16\pi G}R + \frac{\kappa}{2}\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}A_\mu\partial_\nu A_\rho + \frac{1}{2}g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi - V(|\phi|), \tag{2}$$

where $\phi = |\phi|e^{i\Omega}$, the gauge covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$. The form of scalar potential $V(|\phi|)$ will be fixed later so as to give suitable Bogomol'nyi equations.

From now on we reformulate the theories of our interest to dual transformed versions. Let us begin with taking into account the path integral

$$Z = \int [dg_{\mu\nu}][dA_\mu][|\phi|d|\phi|][d\Omega]e^{i\int d^3x\sqrt{g}\mathcal{L}}, \tag{3}$$

where the Lagrange density $\mathcal{L}$ is either $\mathcal{L}_{Max}$ or $\mathcal{L}_{CS}$. The procedure to achieve the dual transformation for scalar electrodynamics is divided into three Lagrangian-independent steps and one Lagrangian-dependent Gaussian integration. The first step is to linearize the interaction between scalar field and gauge field by introducing an auxiliary field C_μ

$$\begin{aligned} &\exp\left\{i\int d^3x\sqrt{g}\,\frac{1}{2}g^{\mu\nu}|\phi|^2(\partial_\mu\Omega - eA_\mu)(\partial_\nu\Omega - eA_\nu)\right\} \\ &= \int [dC_\mu]\prod_x \frac{g^{\frac{1}{4}}}{|\phi|^3}\exp\left\{i\int d^3x\,\sqrt{g}\big[-\frac{g^{\mu\nu}}{2|\phi|^2}C_\mu C_\nu + g^{\mu\nu}C_\mu(\partial_\nu\Omega - eA_\nu)\big]\right\}. \end{aligned} \tag{4}$$

The second step is to divide the path integral measure for the phase of the scalar field into two contributions

$$[d\Omega] = [d\Theta][d\eta]. \tag{5}$$

Here the first term Θ, expressed by a multi-valued function such as

$$\Theta(t,\vec{x}) = \sum_p (-1)^a \tan^{-1}\frac{x^2 - x_p^2}{x^1 - x_p^1}, \tag{6}$$

describes the configuration of vortices and antivortices, and the second term η, a single-valued function, represents the fluctuation around a given vortex sector. Then η-integration gives a restriction for C_μ

$$\int [d\eta] \exp\left\{ i \int d^3x\, \sqrt{g}\, g^{\mu\nu} C_\mu \partial_\nu \eta \right\} \approx \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu). \tag{7}$$

If we note that the auxiliary field C_μ is classically nothing but the $U(1)$ current, the above condition can be regarded as depicting a quantum conservation of current. Particularly in Chern-Simons gauge theories ($\mathcal{L} = \mathcal{L}_{CS}$), it is related to the Bianchi identity. The third step is the introduction of the dual gauge field H_μ by rewriting a part of path integral as

$$\int [dC_\mu] \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu) \cdots = \int [dH_\mu][dC_\mu] \delta(\sqrt{g}\, C^\mu - C \epsilon^{\mu\nu\rho} \partial_\nu H_\rho) \cdots, \tag{8}$$

here the constant C is chosen to be 1 for the Maxwell Higgs theory, Eq.(1), and to be $\frac{\kappa}{e}$ for the Chern-Simons Higgs theory, Eq.(2), for later convenience. Now that the action at this stage is quadratic both in the auxiliary field C_μ and in the gauge field A_μ, the path integrals for those fields are Gaussians and can be carried out in closed form. We now obtain the dual-transformed version of Einstein Maxwell Higgs theory

$$\begin{aligned} Z \;=\; & \int [g dg_{\mu\nu}][dH_\mu][|\phi|^{-2} d|\phi|][d\Theta][d\chi] \\ & \exp\, i \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) \right. \\ & \left. - \frac{1}{4|\phi|^2} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma} + \frac{1}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_{\mu\nu} \partial_\rho \Theta + \frac{e^2}{2} g^{\mu\nu} (H_\mu - \frac{1}{e} \partial_\mu \chi)(H_\nu - \frac{1}{e} \partial_\nu \chi) \right\}, \end{aligned} \tag{9}$$

and that of Einstein Chern-Simons Higgs theory

$$\begin{aligned} Z &= \int [g^{\frac{3}{4}} dg_{\mu\nu}][dH_\mu][|\phi|^{-2} d|\phi|][d\Theta] \\ & \exp i \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) \right. \\ & \left. -\frac{\kappa^2}{4e^2 |\phi|^2} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma} - \frac{\kappa}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_\mu \partial_\nu H_\rho + \frac{\kappa}{2e} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_{\mu\nu} \partial_\rho \Theta \right\}, \end{aligned} \quad (10)$$

where a single-valued field χ is introduced to guarantee the gauge invariance of Maxwell Higgs theory, and $H_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu$. Note that this transformation can be generalized to almost any Abelian theory in any number of dimensions. In $(D+1)$ dimensions we have only to replace dual vector field to an antisymmetric tensor field of rank $(D-1)$ $H_{\mu_1\mu_2\cdots\mu_{D-1}}$ defined by

$$C^\mu = \frac{\epsilon^{\mu\nu_1\cdots\nu_D}}{\sqrt{g}} \partial_{\nu_1} H_{\nu_2\nu_3\cdots\nu_D}, \quad (11)$$

and, for the theory with couplings to an arbitrary number of Abelian gauge fields, we may appropriately introduce several dual fields of which the number is the same as that of conserved currents.

From the dual-transformed actions in Eq.(9) and Eq.(10), we easily read the contents of gauge bosons in Higgs phase; an even-parity photon of mass ev in massive vector theory (Eq.(9)) and an odd-parity helicity-one photon of mass $\frac{e^2v^2}{\kappa}$ in topologically massive gauge theory from (Eq.(10)) [2]. Gauge coupling e is inversely multiplied to the interaction term between the gauge field and the Higgs field, which looks like the strong coupling expansion being done. However, when Higgs effects are important, one must take into account the nonpolynomial interaction in the Maxwell-like term and the Jacobian in Higgs measure. Note that the path integral measure of gravitational field has nontrivial Jacobian compared with that of original theory, and it depends on the form of gauge interaction.

Abelian gauge theories coupled to a complex scalar field in (2+1) spacetime dimensions produce solitonic excitations and, for the specific form of scalar potential, they saturate so-called Bogomol'nyi limit which shows interesting aspects in both theory and application. Examples are Nielsen-Olesen vortices in Abelian Higgs model [3, 4], and topological and nontopological Chern-Simons solitons in Chern-Simons Higgs model [5, 6]. They have been examined also with the inclusion of gravity [7, 8]. The role of such excitations in Abelian Higgs model have been studied in dual-transformed lattice model with the Higgs degrees frozen out [9]. The dual transformation of both Abelian Higgs model and Chern-Simons Higgs model was derived in continuum [10].

From now on we explore the classical self-dual soliton solutions in continuum version of dual-transformed Einstein Maxwell Higgs theory in Eq.(9) and Einstein Chern-Simons Higgs theory in Eq.(10). We take the Euler invariant defined by $E = \frac{1}{16\pi G}\int d^2x\sqrt{\gamma}\,{}^2R$ as the total energy of the system constituted by static matter fields and the stationary metric

$$ds^2 = N^2(dt + K_i dx^i)^2 - \gamma_{ij}dx^i dx^j, \tag{12}$$

where the components of the metric, $N, K_i,$ and γ_{ij}, $(i,j = 1,2)$, are functions of spatial coordinates only.

After arranging the terms of energy, we obtain the Bogomol'nyi-type bound of Einstein Maxwell Higgs theory under $A_0 = 0$ gauge

$$\begin{aligned} E_{Max} &= \int d^2x\sqrt{\gamma}\left\{\frac{1}{4}\gamma^{ik}\gamma^{jl}(F_{ij} \mp \sqrt{\gamma}\epsilon_{ij}\sqrt{2V_{Max}})(F_{kl} \mp \sqrt{\gamma}\epsilon_{kl}\sqrt{2V_{Max}})\right. \\ &\qquad + \frac{1}{4}(\overline{D_i\phi \mp i\sqrt{\gamma}\epsilon_{ik}\gamma^{kl}D_l\phi})(D_j\phi \mp i\sqrt{\gamma}\epsilon_{jm}\gamma^{mn}D_\nu\phi) \\ &\qquad \left.\pm \frac{\epsilon^{ij}}{2\sqrt{\gamma}}F_{ij}\Big[\sqrt{2V_{Max}} - \frac{e}{2}(|\phi|^2 - v^2)\Big]\right\} \\ &\quad \pm \frac{ev^2}{2}\Phi \\ &\quad \pm \frac{i}{4}\int d^2x\ \epsilon^{ij}\partial_i(\bar{\phi}D_j\phi - \overline{D_j\phi}\phi), \end{aligned} \tag{13}$$

and that for Einstein Chern-Simons Higgs theory

$$\begin{aligned} E_{CS} &= \int d^2x\sqrt{\gamma}\left\{\frac{e^2}{2}|\phi|^2\Big(A_0 \pm \frac{U}{e|\phi|}\Big)^2\right. \\ &\qquad + \frac{1}{4}\gamma^{ij}(\overline{\tilde{D}_i\phi \mp i\sqrt{\gamma}\epsilon_{ik}\gamma^{kl}\tilde{D}_l\phi})(\tilde{D}_j\phi \mp i\sqrt{\gamma}\epsilon_{jm}\gamma^{mn}\tilde{D}_n\phi) \\ &\qquad \mp \frac{e}{2}(|\phi|^2 - v^2)\Big[\frac{\epsilon^{ij}}{\sqrt{\gamma}}\partial_i(A^j - K_jA_0) + KA_0 + \frac{e^2}{\kappa}A_0|\phi|^2\Big] \\ &\qquad \mp eA_0|\phi|\Big[U - \frac{e^2}{2\kappa}|\phi|(|\phi|^2 - v^2)\Big] \end{aligned} \tag{14}$$

$$+\Big[W+\frac{ev^2}{2}K\big(\pm\frac{A_0}{v^2}(|\phi|^2-v^2)+\frac{ev^2}{2\kappa}(1-\phi_\infty^2)\big)-\frac{3}{32\pi G}K^2\Big]\Big\}$$

$$\pm\frac{ev^2}{2}\Phi$$

$$+\int d^2x\ \partial_i\Big[\epsilon^{ij}\big(\pm\frac{ev^2}{2}K_j(A_0\mp\frac{ev^2}{2\kappa}(1-\phi_\infty^2))\pm\frac{i}{4}(\bar\phi\tilde D_j\phi-\phi\overline{\tilde D_j\phi})\big)\Big],$$

where $\tilde D_i\phi=(\partial_i-ieA_i+ieK_iA_0)\phi$, $K=\frac{\epsilon^{ij}}{\sqrt\gamma}\partial_iK_j$, $V_{CS}=\frac{1}{2}U^2+W$, Φ is the magnetic flux defined by

$$\Phi=-\frac{1}{2}\int d^2x\ \epsilon^{ij}F_{ij}, \tag{15}$$

and the value of scalar amplitude at spatial infinity divided by the vacuum expectation value v, ϕ_∞, is introduced to let the integral in the last line finite. Hence, for the specific scalar potentials

$$V_{Max}=\frac{e^2}{8}(|\phi|^2-v^2)^2, \tag{16}$$

$$V_{CS}=\frac{e^4}{8\kappa^2}\left\{|\phi|^2(|\phi|^2-v^2)^2-\pi G(|\phi|^4-2v^2|\phi|^2+v^4\phi_\infty^2)^2\right\}, \tag{17}$$

the energy is proportional to the magnetic flux Φ and then multi-vortex solutions are supported in these systems. In case of $U(1)$ Chern-Simons gauge theory, the flux carrying objects are charged because of the Gauss' law ($U(1)$ charge) $=-\frac{\kappa}{e}\Phi$. When the energy is not larger than meson mass times charge, this Chern-Simons Higgs model can support nontopological solitons. Another characteristic to distinguish Chern-Simons solitons from Nielsen-Olesen vortices is that they carry non-zero spin; the former carry the spin $J=\frac{1}{8\pi G}\oint_{|\vec x|\to\infty}dx^iK_i$ and the latter are spinless.

From now on we fix the gauge for γ_{ij} as the conformal gauge

$$\gamma_{ij}=\delta_{ij}b(x^i) \tag{18}$$

and that for K_i as the Coulomb gauge $\nabla_iK^i=0$ which enables K_i to be expressed by a function ψ

$$K^i=\frac{\kappa}{e^2v^2}\frac{\epsilon^{ij}}{\sqrt\gamma}\partial_j\ln\psi. \tag{19}$$

Solving the equations of motion, we obtain the Bogomol'nyi equation for the Maxwell system

$$\partial^2 \ln \frac{|\phi|^2}{v^2} = e^2 v^4 e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}}}{\prod_{p=1}^{n} |z - z_p|^2} \right)^{4\pi G v^2} \left(\frac{|\phi|^2}{v^2} - 1 \right) \mp 2\epsilon^{ij} \partial_i \partial_j \Omega, \tag{20}$$

and two remaining equations for Chern-Simons system

$$\begin{aligned} \partial^2 \ln \frac{|\phi|^2}{v^2} &= \frac{e^4 v^4}{\kappa^2} e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}} \psi^{\phi_\infty^2 - 1}}{\prod_{p=1}^{n} |z - z_p|^2} \right)^{4\pi G v^2} \left(\frac{|\phi|^2}{v^2} - 1 \right) \\ &\times \left(\frac{|\phi|^2}{v^2} - 2\pi G v^2 \Big(\frac{|\phi|^4}{v^4} - 2\frac{|\phi|^2}{v^2} + \phi_\infty^2 \Big) \right) \mp 2\epsilon^{ij} \partial_i \partial_j \Omega, \end{aligned} \tag{21}$$

$$\partial^2 \ln \psi = -2\pi G \frac{e^4 v^6}{\kappa^2} e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}} \psi^{\phi_\infty^2 - 1}}{\prod_{p=1}^{n} |z - z_p|^2} \right)^{4\pi G v^2} \Big(\frac{|\phi|^4}{v^4} - 2\frac{|\phi|^2}{v^2} + \phi_\infty^2 \Big) \tag{22}$$

where ∂^2 is the flat-space Laplacian.

Suppose that the following conditions hold: 1. There exist nonsingular and finite-energy solutions 2. the base manifold defined by the solution is smooth. We look for the rotationally symmetric solutions. The results are given as follows*. Though the former system described by Eq.(20) allows the boundary condition $|\phi| \neq v$ at $|x^i| = \infty$, the only solution is that which makes the space either a cone asymptotically (when $4\pi n G v^2 < 2$) or a cylinder asymptotically (when $4\pi n G v^2 = 2$). For Chern-Simons case there is also no need to fix the boundary value of scalar field, ϕ_∞ in deriving the Bogomol'nyi limit, but an appropriate consideration on the shape of the spatial manifold restricts it to 0 or 1. In addition to topological vortices ($\phi_\infty = 1$) and nontopological solitons ($\phi_\infty = 0$) in open spaces, the system contains the solution with $\phi_\infty = 0$ which corresponds two-sphere space where the vortices are placed both at north and south poles.

A final comment is in order. Our formalism shows clearly that the interaction term between the multi-valued scalar phase Θ and the gauge field A_μ is Chern-Simons-like term irrespective of the dynamics of gauge field. However, the physics of nonzero topological part, for example the phase transition of the above theories induced by topological excitations, needs further study.

*For the detailed analysis see our papers in Refs[7, 8]

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References

[1] For a review, see R. Savit, Rev. Mod. Phys. **52** (1980) 453, and the references therein.

[2] S. Deser, and R. Jackiw, Phys. Lett. **B139** (1984) 371; S. Deser, and Z. Yang, Mod. Phys. Lett. **A4** (1989) 2123.

[3] A. A. Abrikosov, Sov. Phys. JETP **5** (1957) 1174; H. B. Nielsen, and P. Olesen, Nucl. Phys. **B61** (1973) 45.

[4] E. Bogomol'nyi, Sov. J. Nucl. Phys. **24** (1976) 449.

[5] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64** (1990) 2230; R. Jackiw, and E. J. Weinberg, *ibid* **64** (1990) 2234.

[6] R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. **D42** (1990) 3488.

[7] B. Linet, Gen. Rel. Grav. **20** (1988) 451; A. Comtet, and G. W. Gibbons, Nucl. Phys. **B299** (1988) 719; C. Kim, and Y. Kim, preprint KHTP-93-05.

[8] P. Valtancoli, Int. J. Mod. Phys. **A18** (1992) 4335; Cangemi, and C. Lee, Phys. Rev. **D46** (1992) 4768; B. Chung, J.-M. Chung, and Y. Kim, preprint KHTP-93-06.

[9] M. B. Einhorn, and R. Savit, Phys. Rev. **D17** (1978) 2583; M. E. Peskin, Ann. Phys. **113** (1978) 122.

[10] Y. Kim, and K. Lee, preprint CU-TP-574.