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The Quantum Algebraic Structure of the Twisted XXZ Chain

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ABSTRACT

We consider the Quantum Inverse Scattering Method with a new R-matrix depending on two parameters q and t . We find that the underlying algebraic structure is the two-parameter deformed algebra $SU_{q,t}(2)$ enlarged by introducing an element belonging to the centre. The corresponding Hamiltonian describes the spin-1/2 XXZ model with twisted periodic boundary conditions.

Key-words: Quantum groups; Statistical mechanics; Quantum field theory; Integrable models

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Many completely integrable one-dimensional quantum models have been treated [1] by the Quantum Inverse Scattering Method (QISM)[2, 3] which, among other achievements, led to the discovery of Quantum Groups [4] independently from Drinfeld and Jimbo[5]. In this letter we use this method to describe the relation between the spin-1/2 XXZ chain with twisted periodic boundary conditions and the two-parameter deformed algebra $SU_{q,t}(2)$ enlarged by introducing an element belonging to the centre, showing that the second parameter of the deformation, t , is linked to the twist.

The QISM introduces an auxiliary problem with the help of the so-called Lax operator. In our model this operator is

$$L_n(\lambda, t) = \begin{pmatrix} t^{Z_n - S_n^3} sh[\gamma(\lambda + iS_n^3)] & iS_n^- sin\gamma \\ iS_n^+ sin\gamma & t^{-Z_n - S_n^3} sh[\gamma(\lambda - iS_n^3)] \end{pmatrix}, \quad (1)$$

where \vec{S}_n and Z_n are operators defined on the n -th vectorial space of the periodic ($\vec{S}_{N+1} \equiv \vec{S}_1, Z_{N+1} \equiv Z_1$) chain, which in the fundamental representation are given by

$$Z_n = \frac{1}{2} \mathbb{1}_n, \vec{S}_n = \frac{1}{2} \vec{\sigma}_n, \quad (2)$$

where $\vec{\sigma}$ are the Pauli matrices and $\mathbb{1}$ is the identity operator.

The R -matrix associated to the Lax operator (1) is

$$R(\lambda, t) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & c'(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & c''(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \quad (3)$$

where

$$\begin{aligned} a(\lambda) &= sh[\gamma(\lambda + i)] \\ b(\lambda) &= i sin\gamma \\ c'(\lambda) &= tc(\lambda) \\ c''(\lambda) &= t^{-1}c(\lambda) \end{aligned} \quad (4)$$

and

$$c(\lambda) = sh \gamma \lambda ; \quad (5)$$

clearly, $R(\lambda) = R(\lambda, t = 1)$ is the appropriate matrix for the XXZ model [6]. It is easy to check that the matrix $R(\lambda, t)$ (eqs. (3-5)) satisfies the Yang-Baxter equation [7, 8]

$$R_{12}(\lambda_{12}, t)R_{13}(\lambda_{13}, t)R_{23}(\lambda_{23}, t) = R_{23}(\lambda_{23}, t)R_{13}(\lambda_{13}, t)R_{12}(\lambda_{12}, t) \quad (6)$$

and that $L_n(\lambda, t)$ (eq. (1)) obeys the Fundamental Commutation Relations (FCR)

$$R_{12}(\lambda_{12}, t)L_n^1(\lambda_1, t)L_n^2(\lambda_2, t) = L_n^2(\lambda_2, t)L_n^1(\lambda_1, t)R_{12}(\lambda_{12}, t) . \quad (7)$$

In (6) and (7), $\lambda_{ij} = \lambda_i - \lambda_j$ and

$$\begin{aligned} R_{12} &= \sum_i a_i \otimes b_i \otimes \mathbb{1} \quad , \quad R_{13} = \sum_i a_i \otimes \mathbb{1} \otimes b_i , \\ R_{23} &= \sum_i \mathbb{1} \otimes a_i \otimes b_i , \end{aligned} \quad (8)$$

with the $R(\lambda, t)$ matrix written as

$$R(\lambda, t) = \sum_i a_i \otimes b_i \quad (9)$$

and the upper indices in eq. (7) follow

$$L^1 = L \otimes \mathbb{1} \quad , \quad L^2 = \mathbb{1} \otimes L. \quad (10)$$

We notice that the R -matrix is defined on the tensor product of two auxiliary spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the L -matrix is defined on the tensor product of the auxiliary space \mathbb{C}^2 and the internal space \mathbb{C}^d , with d the dimension of the representation of the associated algebra satisfied by the operators in the elements of the matrix L .

The reason for $R(\lambda, t)$ to satisfy eq. (6) is that it can be written in terms of $R(\lambda)$ as¹

$$R_{12}(\lambda, t) = g^1(g^2)^{-1}R_{12}(\lambda)g^1(g^2)^{-1} , \quad (11)$$

¹Equivalently one could write

$$g^1(g^2)^{-1} = t^{S^3 \otimes Z - Z \otimes S^3} ,$$

with S^3 and Z given by eq. (2). This form is more appropriate if one wishes to compare the algebraic structure here presented with ref. [9]. In a forthcoming paper we shall discuss this subject, as well as the relationship of our approach with the one in ref. [10].

where

$$g^1 = g \otimes \mathbb{1} \quad , \quad g^2 = \mathbb{1} \otimes g \quad , \quad g = t^{\frac{1}{2}} S^3 \quad (12)$$

and

$$[g^1 g^2, R_{12}(\lambda)] = 0. \quad (13)$$

Moreover, eq. (7) follows from eq. (6), because (for $d=2$)

$$L_n(\lambda, t) = R_{o,n}(\lambda - \frac{i}{2}, t), \quad (14)$$

where "o" labels the auxiliary space. We also observe that

$$R(0, t) = P, \quad (15)$$

where P is the permutation operator on the tensor product of the two spaces where the R -matrix is defined.

According to the standard procedure of the QISM, eq. (7) allows one to build an infinite set of commuting operators

$$F(\lambda, t) = Tr[L_N(\lambda, t) \cdots L_2(\lambda, t) L_1(\lambda, t)], \quad (16)$$

where both the matrix product and the trace are performed in the auxiliary space.

In our case the Bethe Ansatz equations for the fundamental representation are given by

$$\left(\frac{\alpha(\lambda_\beta)}{\delta(\lambda_\beta)} \right)^N = t^{-N} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^M \left\{ \frac{a(\lambda_\beta - \lambda_\alpha) c(\lambda_\alpha - \lambda_\beta)}{a(\lambda_\alpha - \lambda_\beta) c(\lambda_\beta - \lambda_\alpha)} \right\}; \quad \beta = 1, \dots, M \leq N, \quad (17)$$

with

$$\begin{aligned} \alpha(\lambda) &= sh \left[\gamma \left(\lambda + \frac{i}{2} \right) \right] \\ \delta(\lambda) &= sh \left[\gamma \left(\lambda - \frac{i}{2} \right) \right], \end{aligned} \quad (18)$$

explicitly showing the contribution due to the parameter t , as α , δ , a and c are the same functions appearing in the XXZ model.

In order to show the algebraic structure underlying the R and L matrices defined in eqs. (1) and (3-5) we perform a suitable similarity transformation [3] on (6) and (7) which permits us to have the following decomposition:

$$\begin{aligned}\tilde{L}_n(\lambda, t) &= \frac{1}{2}(e^{\lambda\gamma} L_+ - e^{-\lambda\gamma} L_-) \\ \tilde{R}(\lambda_{ij}, t) &= e^{\gamma\lambda_{ij}} R_+ - e^{-\gamma\lambda_{ij}} R_- ;\end{aligned}\tag{19}$$

where

$$\begin{aligned}L_+ &= \begin{pmatrix} q^{S^3} t^{Z-S^3} & \Omega S^- \\ 0 & q^{-S^3} t^{-Z-S^3} \end{pmatrix} \\ L_- &= \begin{pmatrix} q^{-S^3} t^{Z-S^3} & 0 \\ -\Omega S^+ & q^{S^3} t^{-Z-S^3} \end{pmatrix}\end{aligned}\tag{20}$$

and

$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & t & \Omega & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix},\tag{21}$$

with $\Omega = q - q^{-1}$ and $R_- = P R_+^{-1} P$.

Substituting eq. (19) in the Y-B equation (6) and in the FCR, eq. (7), one gets the following independent equations:

$$\begin{aligned}R_+ L_\epsilon^1 L_\epsilon^2 &= L_\epsilon^2 L_\epsilon^1 R_+ \quad (\epsilon = \pm 1) \\ R_+ L_+^1 L_-^2 &= L_-^2 L_+^1 R_+ ,\end{aligned}\tag{22}$$

which imply that the operators in the entries of L must satisfy

$$[S^3, Z] = [S^\pm, Z] = 0\tag{23.a}$$

$$[S^3, S^\pm] = \pm S_\pm\tag{23.b}$$

$$t^{-1} S^+ S^- - t S^- S^+ = t^{-2S^3} [2S^3]_q ,\tag{23.c}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ with $q = \exp(i\gamma)$. Eqs. (23) are the commutation relations of the two-parametric deformed $SU(2)$ [9, 11, 12] with Z , an element of the center of the resulting algebra. The coproduct is obtained by considering the product of two L_z acting on two internal spaces and we find:

$$\Delta S^3 = S^3 \otimes \mathbb{1} + \mathbb{1} \otimes S^3 \quad (24.a)$$

$$\Delta Z = Z \otimes \mathbb{1} + \mathbb{1} \otimes Z \quad (24.b)$$

$$\Delta S^\pm = q^{S^3} t^{\mp Z - S^3} \otimes S^\pm + S^\pm \otimes q^{-S^3} t^{\pm Z - S^3}. \quad (24.c)$$

The coproduct (24c) is related to the one in ref. [12] by a similarity transformation generated by the operator $t^{S^3 \otimes Z - Z \otimes S^3}$.

Following the QISM, a local Hamiltonian can be written as

$$H \propto \frac{\partial}{\partial \lambda} \ell g F(\lambda, t) |_{\lambda = \frac{i}{2}} \quad (25)$$

and thanks to eqs. (14-16), it becomes for the fundamental representation of the algebra

$$H = \sum_{i=1}^N H_{i,i+1} \quad (N+1 \equiv 1) \quad (26)$$

$$H_{i,i+1} = \frac{J \sin \gamma}{i \gamma} \frac{\partial}{\partial \lambda} \hat{R}_{i,i+1}(\lambda, t) |_{\lambda=0},$$

where $\hat{R} = PR$ and $R(\lambda, t)$ is given by eqs. (3-5). In the above equation $R_{i,i+1}(\lambda, t)$ acts on the two internal spaces $(i, i+1)$ instead of acting on two auxiliary spaces.

Substituting $R(\lambda, t)$ given by eqs. (3-5) in eq. (26), apart from an additive constant, we get

$$H = \frac{J}{2} \sum_{i=1}^N \left[2t^{-1} \sigma_i^+ \sigma_{i+1}^- + 2t \sigma_i^- \sigma_{i+1}^+ + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right], \quad (27)$$

where $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$.

Such a Hamiltonian is very similar to the XXZ model with periodic boundary conditions but for each pair of sites $(i, i+1)$, the site $(i+1)$ is rotated of an angle α ($t = e^{i\alpha}$) in the $x - y$ plane with respect to the site i .

The similarity transformation generated by $\exp\{-i\frac{\alpha}{2}\sum_{\ell=1}^N(\ell-1)\sigma_{\ell}^z\}$ takes the Hamiltonian (eq. (27)) to

$$H = \frac{J}{2} \left[\sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos\gamma \sigma_n^z \sigma_{n+1}^z) + \cos\gamma \sigma_N^z \sigma_1^z + 2t^{-N} \sigma_N^+ \sigma_1^- + 2t^N \sigma_N^- \sigma_1^+ \right] , \quad (28)$$

which is the well-known [13] Hamiltonian for the XXZ chain with twisted periodic boundary conditions.

It is amusing to observe that, thanks to eq. (13) and following the procedure of ref. [10], the Hamiltonian (eq. (27)) could also be obtained from the R -matrix of the XXZ model $R(\lambda) = R(\lambda, t = 1)$, using $L'_n(\lambda, t) = t^{S^3} L_n(\lambda, t = 1) \neq L_n(\lambda, t)$. Conversely, the untwisted XXZ model can be built from $R(\lambda, t)$ (eqs. (3-5)) and $L''_n(\lambda, t) = t^{-S^3} L_n(\lambda, t)$. All these topics will be discussed in detail in a forthcoming paper.

Finally, we would like to point out that by introducing a central element Z which enlarges the $SU_{q,t}(2)$ algebra, we make appear the underlying algebraic structure of the so-called twisted XXZ model.

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