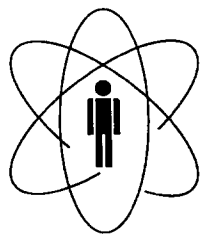


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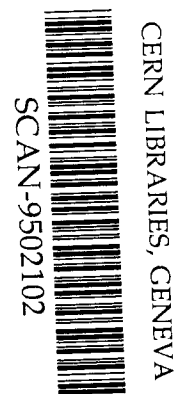
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# Virial Expansion for an $\varepsilon$ -Deformed System

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## ABSTRACT

We compute the virial expansion of an ideal quantum  $q$ -gas equation of state for small values of the deformation parameter.

**Key-words:** Statistical mechanics; Virial expansion; Quantum groups;  $q$ -oscillators.

The interest in  $q$ -oscillators comes specially from their connection with quantum algebras [1] and superalgebras [2]. Quantum groups [3-5] are non-trivial generalizations of Lie algebras and groups and have left their trace in several areas of physics [6]. Also, simple physical systems present quantum group symmetry [7, 8].

It has recently been shown [9] that ideal quantum  $q$ -gases [10, 11] present interesting features: Bose-Einstein condensation,  $\lambda$ -point discontinuity for the specific heat and a general trend to favour criticality. These results were found for real  $|q| > 1$  deformed systems and due to the method employed they are not extensible to  $0 \leq |q| < 1$ .

In this short note, we explore an ideal quantum  $q$ -gas for values of  $q$  smaller than one. We find the virial expansion for its equation of state in the high-temperature (or low density) regime.

A bosonic  $q$ -oscillator is the associative algebra generated by the elements  $A, A^+$  and  $N$  satisfying the relations [1, 2, 12]:

$$\begin{aligned} [N, A^+] &= A^+ & [N, A] &= -A \\ AA^+ - q^2 A^+ A &= 1 \end{aligned} \quad (1)$$

where we are taking  $q$  a real parameter.

It is possible to construct representations of the relations (2.2) in the Fock space  $\mathcal{F}$  spanned by the normalized eigenstates  $|n\rangle$  of the number operator  $N$  as

$$\begin{aligned} A|0\rangle &= 0, & N|n\rangle &= n|n\rangle \quad n = 0, 1, 2, \dots \\ |n\rangle &= \frac{1}{\sqrt{[n]!}} (A^+)^n |0\rangle \end{aligned} \quad (2)$$

where  $[n] = (q^{2n} - 1)/(q^2 - 1)$  and  $[n]! = [n] \cdots [1]$ .

In the Fock space  $\mathcal{F}$  it is possible to express the deformed oscillators in terms of the standard bosonic ones  $b, b^+$  as [12]

$$A = \left( \frac{[N+1]}{N+1} \right)^{1/2} b, \quad A^+ = b^+ \left( \frac{[N+1]}{N+1} \right)^{1/2}; \quad (3)$$

it can easily be shown in  $\mathcal{F}$  that

$$AA^+ = [N+1], \quad A^+A = [N], \quad (4)$$

and as expected the standard bosonic algebra is obtained in the  $q \rightarrow 1$  limit.

We have previously investigated highly deformed  $q$ -bosons [9] and we observed that in the limit  $q = \infty$  the statistical properties are those of fermions while for  $q = 1$  the usual bosonic behaviour is recovered [13].

Our ideal deformed system is described by the Hamiltonian

$$H = \sum_i \omega_i A_i^+ A_i = \sum_i \omega_i [N_i], \quad (5)$$

where  $A_i, A_i^\dagger$  and  $N_i$  obey algebra (1) and are the annihilation, creation and occupation number operators of particles in level  $i$  with energy  $\omega_i$ . The total number operator is then  $N = \sum_i N_i$ . We note that different modes of bosonic  $q$ -oscillators commute among themselves being thus different from the alternative deformed commutation relations called quon algebra [14, 15]. With  $\mu$  the chemical potential and  $\Omega$  the grand canonical potential, our grand canonical partition function is given by

$$Z = Tr \exp[-\beta(H - \mu N)] = \exp(-\beta\Omega), \quad (6)$$

where  $\beta = 1/kT$ , with  $k$  the Boltzman constant.

As  $Z$  factorizes for the above system, the grand canonical potential is given by a sum over single level partition functions [11]

$$\Omega = -\frac{1}{\beta} \sum_i \ln Z_i^0(\omega_i, \beta, \mu), \quad (7)$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i[n] - \mu n)}. \quad (8)$$

For a non-relativistic  $q$ -boson, the energy is  $\omega_i = \vec{p}^2/2m$  and following the usual procedure we enclose our system in a large volume  $V$ , allowing for the sum over levels to be replaced by an integral over the  $p$ -space:

$$\sum_i \rightarrow \frac{V}{(2\pi\hbar)^3} \int d^3p. \quad (9)$$

We shall now consider a small  $q$  ( $q = \varepsilon$ ) for which the partition function (8) is

$$Z_i^0 = 1 + ze^{-\beta\omega} + \frac{z^2}{1-z} e^{-\beta\omega} (1 - \beta\omega\varepsilon^2) + O(\varepsilon^4) \quad (10)$$

where  $z$  is the fugacity,  $z = e^{\beta\mu}$  and we have kept terms up to  $\varepsilon^2$  order.

Assuming the fugacity  $z$  small compared to one and integrating by parts, we obtain the pressure  $P = -\Omega/V$ :

$$P = \beta^{-1} \Lambda^{-3} z \left[ 1 + z(-2^{-5/2} + 1 - \frac{3}{2}\varepsilon^2) + z^2 \left( 3^{-5/2} + 1 - 2^{-3/2} + \varepsilon^2 \left( -\frac{3}{2} + 3 \times 2^{-7/2} \right) \right) + O(z^3) \right], \quad (11)$$

where  $\Lambda = (\hbar^2\beta/2\pi m)^{1/2}$  is the thermal wavelength.

The  $q$ -boson density  $n = \frac{\partial P}{\partial \mu}|_{T,V}$  is then

$$n = \Lambda^{-3} z \left\{ 1 + 2z(-2^{-5/2} + 1 - \frac{3}{2}\varepsilon^2) + 3z^2 \left[ 3^{-5/2} + 1 - 2^{-3/2} + \varepsilon^2 \left( -\frac{3}{2} + 3 \times 2^{-7/2} \right) \right] + \dots \right\}. \quad (12)$$

Inverting the power series above and expanding  $z$  in powers of  $n\Lambda^3$  in (11), we obtain the virial expansion of the equation of state

$$P = \frac{n}{\beta} \left\{ 1 + (2^{-5/2} - 1 + \frac{3}{2}\epsilon^2)n\Lambda^3 + 2 \left[ 2^{-4} - 3^{-5/2} - 2^{-1/2} + 1 + \frac{3}{2}\epsilon^2 (3 + 2^{-5/2}) \right] n^2\Lambda^6 + \dots \right\}. \quad (13)$$

We note that the approximations made here are valid for large  $V$ ,  $z \ll 1$  and  $n\Lambda^3 \ll 1$ , implying large-temperature (or low density) regime.

From expression (13) we see immediately that a small deformation increases the pressure relative to  $q = 0$ . Besides, a comparison of (13) with the virial expansion for a highly deformed system, namely [9]

$$P = \frac{n}{(2\pi)^3\beta} \left[ 1 + \left( \frac{1}{2^{5/2}} - \frac{1}{(q^2 + 1)^{3/2}} \right) n\Lambda^3 + \left( \frac{1}{2^3} - \frac{2}{3^{5/2}} - \frac{4}{(q^2 + 1)^3} - \frac{4}{(2q^2 + 2)^{3/2}} + \frac{2}{(q^2 + 2)^{3/2}} \right) n^2\Lambda^6 + \dots \right], \quad (14)$$

shows that: 1)  $P$  attains its minimum for  $q = 0$ ; 2)  $P$  attains its maximum for  $q = \infty$ , which is the fermionic gas; 3)  $P$  has an intermediate value for  $q = 1$ . As a matter of fact, the pressure grows with  $q$  for the whole range  $0 \leq |q| < \infty$ .

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