CC

UNIVERSITÄT LEIPZIG

Naturwissenschaftlich-Theoretisches

Zentrum

Polynomial Lax pairs for the chiral 0(3) fields equations and the Landau-Lifshitz equation

by

L. A. Bordag[1], A. B. Yanovski[2]

[1]Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut
Augustusplatz 10
04109 Leipzig
Germany

[2]St. Klement Ohridski University
Faculty of Mathematics and Informatics
Sofia
Bulgaria

Preprint-Nr. 1/95

Abstract. A new couple of Lax pairs both for the chiral $O(3)$-fields equations and for the Landau-Lifshitz equation is found. In contrast to the already known pairs these one are polynomial in the spectral parameter λ. We found also a new 4×4-Lax pair in the case of cnoidal waves for the generalized Landau-Lifshitz equation.

1. Introduction

In the last decades considerable attention has been paid to the so called soliton equations. It is well known that their remarkable properties (see, e.g., [1]) are due to the possibility to apply to them methods of integration based on the inverse scattering problems and their modifications such as the $\overline{\partial}$ problem. The main point, however, is the possibility to cast the given evolution requation (or system of such equations) into the so called Lax form, or equivalently, into the form of the compatibility condition

$$[\partial_x - U, \partial_t - V] = \partial_t U - \partial_x V + [U, V] = 0 \tag{1}$$

of two linear systems

$$(\partial_x - U)\Psi = 0, \quad (\partial_t - V)\Psi = 0\,. \tag{2}$$

Here the matrix valued functions U and V depend on the spectral parameter λ and on a number of variables u_1, u_2, ... , usually called potential functions. The evolution equation (or system of equations) is written in terms of the potential functions. We shall deal with evolution equations with one spatial variable $x \in R$, as usual t is the time variable.

At the present time there is a number of approaches to the soliton equations based on the spectral theory of operators, on Lie group and on Lie algebra theory, on algebraic and differential geometry and others which are difficult even to list. However, we believe that there is one important problem which is open and untill now has not been paid considerable attention. The question is whether the results of the main constructions through which the soliton equations are solved such as the dressing method of Zakharov-Shabat, the Riemann-Hilbert problem or the finite gap integration method, depend or do not depend on the choice of the U-V-pair, as the constructions itself strongly depends upon this choice.

It is clear that as the compatibility condition (1) is expressed through the commutator $[U, V]$ then, if U or V belong to a certain Lie algebra g, we can write the same compatibility condition in another faithful representation of g and we shall obtain the same evolution equation. It is not evident that the constructions of exact solutions mentioned above will be compatible with such an alteration of the representation. One of the authors has paid attention to some aspects of this problem in [4] but it seems that it remains to be done for more even in the most trivial case when the pairs differ by the choice of the representation. The problem becomes more complicated if we introduce essentially different U-V pairs. It should be remarked also that finding U-V pairs is a by no meaning straightforward process and sometimes to find them some fortune is necessary. So the existence of different pairs is a quite rare phenomena and possibly this explaines that the problem we mentioned above was not given attention up to now.

2. Polynomial 6×6 pairs for the chiral O(3) fields equations and for the Landau-Lifshitz equation

The system of the chiral fields equations can be written in the form [8]

$$\left\{ \begin{array}{rcl} \vec{u}_t + \vec{u}_x - \vec{u} \times J\,\vec{v} & = & 0 \\ \vec{v}_t - \vec{v}_x - \vec{v} \times J\,\vec{u} & = & 0 \end{array} \right., \qquad \vec{u}^2 = \vec{v}^2 = 1, \qquad J = \mathrm{diag}(j_1, j_2, j_3)\,, \tag{3}$$

where $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ are the two vector functions depending on x and t, „$\times$" denotes the vector product.

The Landau-Lifshitz equation [9]

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx} + \vec{u} \times K\vec{u}, \qquad \vec{u}^2 = 1, \qquad K = \mathrm{diag}(k_1, k_2, k_3) \tag{4}$$

is written in terms of one vector function $\vec{u} = (u_1, u_2, u_3)$, where the matrix K playes the same role as J.

For convenience we shall write down the equations (3) and (4) in a different form. Let us introduce the linear mapping $\mathbf{M} : \mathbb{R}^3 \to \mathbf{so}(3)$ where $\mathbf{so}(3)$ is the Lie algebra of 3×3 schief-symmetric matrices

$$\mathcal{M}(\vec{u}) = \mathcal{M}(u_1, u_2, u_3) = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}. \tag{5}$$

Clearly,

$$[\mathcal{M}(\vec{u}), \mathcal{M}(\vec{v})] = -\mathcal{M}(\vec{u} \times \vec{v}) \tag{6}$$

and therefore we cast (3) and (4) into the equivalent form:

A) Chiral fields equations (CFE)

$$\left\{ \begin{array}{rcl} \mathcal{M}(\vec{u})_t + \mathcal{M}(\vec{u})_x + [\mathcal{M}(\vec{u}), \mathcal{M}(J\vec{v})] & = & 0 \\ \mathcal{M}(\vec{v})_t - \mathcal{M}(\vec{v})_x + [\mathcal{M}(\vec{v}), \mathcal{M}(J\vec{u})] & = & 0 \end{array} \right., \qquad \vec{u}^2 = \vec{v}^2 = 1\,. \tag{7}$$

B) Landau-Lifshitz equation (LLE)

$$\mathcal{M}(\vec{u})_t + [\mathcal{M}(\vec{u}), \mathcal{M}(\vec{u})_{xx}] + [\mathcal{M}(\vec{u}), \mathcal{M}(K\vec{u})] = 0, \qquad \vec{u}^2 = 1\,. \tag{8}$$

An inspiration to construct 6×6 pairs we got from reading the last lines in ref. [5], where the authors claimed that there exist U-V pairs which are linear in λ for CFE and LLE and gave formulas for the U matrices in both cases. However, as fas as we know, the final answer has not been obtained there or elsewhere. Besides we failed to obtain a pair linear in λ for LLE. The pair we obtained depends quadratically on λ. So, it is completely new.

Below we shall write the 6×6 matrices in the block form $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a, b, c, d are 3×3 matrices. Using these notations we present the following U-V pairs:

A. The pair for the chiral O(3) fields equations

$$U = \frac{\lambda}{2}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & -\mathcal{M}(\vec{v}) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathcal{M}(J\vec{v}) & 0 \\ 0 & -\mathcal{M}(J\vec{u}) \end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & \mathcal{M}(\vec{v}) \end{pmatrix}$$

$$V = -\frac{\lambda}{2}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & \mathcal{M}(\vec{v}) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathcal{M}(J\vec{v}) & 0 \\ 0 & \mathcal{M}(J\vec{u}) \end{pmatrix} - \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & \mathcal{M}(\vec{v}) \end{pmatrix} \quad (9)$$

where $\hat{J} = \left(\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right)$ and $\mathrm{ad}_{\hat{J}}A$ means $[\hat{J}, A]$.

B. The pair for the Landau-Lifshitz equation

$$U = \frac{\lambda}{2}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\mathcal{M}(J\vec{u}) \end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \frac{\lambda^2}{4}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & 0 \end{pmatrix} + \frac{\lambda}{2}\left\{\begin{pmatrix} \mathcal{M}(\vec{u}\times\vec{u}_x) & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & 0 \end{pmatrix}\right\}$$
$$-\frac{1}{4}\begin{pmatrix} \mathcal{M}(J^2\vec{u}) & 0 \\ 0 & \mathcal{M}(J(\vec{u}\times\vec{u}_x)) \end{pmatrix} + \frac{1}{4}\mathrm{ad}_{\hat{J}}\begin{pmatrix} \mathcal{M}(\vec{u}\times\vec{u}_x) & 0 \\ 0 & 2\mathcal{M}(J\vec{u}) \end{pmatrix}. \quad (10)$$

It is clear that these pairs are polynomial in contrast to the already known pair of Sklianin-Borovik [2, 3] and Cherednik [6] which are elliptic in λ.

In order to obtain from this pair the LLE one has to put in addition $k^s = \frac{1}{2}\sqrt{-j_s^2}$, $s = 1,2,3$. It should be mentioned that for $j_s = 0$, $s = 1,2,3$ the pair (10) becomes equivalent to the pair

$$U = \frac{\lambda}{2}\mathcal{M}(\vec{u})$$
$$V = \frac{\lambda^2}{4}\mathcal{M}(\vec{u}) + \frac{\lambda}{2}\mathcal{M}(\vec{u}\times\vec{u}_x). \quad (11)$$

The nonlinear evolution equation which corresponds to this pair is the Heisenberg ferrromagnet equation (HFE):

$$\vec{u}_t = \vec{u}_x\vec{u}_{xx}.$$

The pair (11), taking into account the well known isomorphism between the algebras **so**$(3, R)$ and **su**(2) could be written in terms of 2×2 matrices. In that way we obtain the well known pair for HFE [6]. There are other questions arising from the pair (10). As we already mentioned the pair which was used up to now for LLE is the Sklianin-Borovik pair [2, 3] containing elliptic functions in λ. It is known, however, that the Heisenberg ferromagnet equation is gauge equivalent to the nonlinear Schrödinger equation (see [6]) and the first operator in the pair for the Heisenberg ferromagnet equation is gauge equivalent to the Zakharov-Shabat linear problem. It is natural to ask whether the Landau-Lifshitz equation is gauge equivalent to some Schrödinger like equation. It is evident that the operator $\partial_x - U$ in the Cherednik pair cannot be gauge equivalent to the Zakharov-Shabat type linear problem. As this difficulty and as the eigenvalues of $\begin{pmatrix} \mathcal{M}(\vec{u}) & 0 \\ 0 & 0 \end{pmatrix}$ do not depend on $\vec{u}$ it is possible that through (10) one can establish the mentioned gauge equivalence.

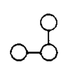
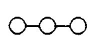
3. Polynomial 4×4 pairs for CFE and LLE

The pairs (9) and (10) have an important property which allows us to write them in tems of 4×4 matrices. Both they belong to the Lie algebra $\mathbf{so}_{\mathbb{C}}(3,3)$ – the complexification of $\mathbf{so}(3,3)$:

$$\mathbf{so}_{\mathbb{C}}(3,3) = \left\{\mathcal{A} : \mathcal{A} \in \mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^6);\ \mathcal{A}^{\top}\varphi + \varphi\mathcal{A} = 0\right\},\ \varphi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}.$$

If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belongs to $\mathbf{so}_{\mathbb{C}}(3,3)$ then $\alpha^{\top} = -\alpha$, $\delta^{\top} = -\delta$, $\beta^{\top} = \gamma$ and vice versa. A simple similarity transformation establishes the isomorphism between $\mathbf{so}_{\mathbb{C}}(3,3)$ and $\mathbf{so}(6,\mathbb{C})$. Indeed, if we introduce the matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ -i\mathbb{I} & -i\mathbb{I} \end{pmatrix}$$

then the similarity transformation $A \to T^{-1}AT$ converts $\mathbf{so}_{\mathbb{C}}(3,3)$ into $\mathbf{so}(6,\mathbb{C})$. It is well known that $\mathbf{so}(6,\mathbb{C})$ is isomorphic to $\mathbf{sl}(4,\mathbb{C})$ – the algebra of 4×4 traceless matrices. Thus it is possible to write the pairs (9), (10) in 4×4 form. However, one needs to construct explicitely the isomorphism between $\mathbf{so}_{\mathbb{C}}(3,3)$ and $\mathbf{sl}(4,\mathbb{C})$. We could not find the explicite form of this isomorphism which is of course trivial in terms of Dynkin diagrams:  $\sim$ . This is why we shall briefly sketch how one can obtain isomorphic Cartan-Weyl bases for these algebras. In order to present the Cartan subalgebra with diagonal matrices we shall use another representation of $\mathbf{so}_{\mathbb{C}}(3,3) \sim \mathbf{so}(6,\mathbb{C})$. Let us introduce

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix} = R^{-1}.$$

Then it is easy to see that if $RBR \in \mathbf{so}_{\mathbb{C}}(3,3)$ then $B \in \widetilde{\mathbf{so}(6)}$ and vice versa, where

$$\widetilde{\mathbf{so}(6)} = \left\{ A\ :\ A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^{\top} \end{pmatrix},\ \beta^{\top} = -\beta,\ \gamma^{\top} - -\gamma \right\}.$$

Now the Cartan subalgebra h can be defined as

$$h = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix},\ \xi = \mathrm{diag}(\xi_1, \xi_2, \xi_3) \right\} \subset \widetilde{\mathbf{so}(6)}\,.$$

We shall represent every element $\begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}$ from the Cartan subalgebra by the vector $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$. The Killing form of $\widetilde{\mathbf{so}(6)}$ is well known:

$$B(x,y) \equiv tr(\mathrm{ad}_X\ \mathrm{ad}_Y) = 4\mathrm{tr}XY; \quad X, Y \in \mathbf{so(6)}\,.$$

Let the elements ϵ_i from the dual space h^* been given by

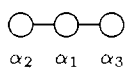
$$\epsilon_i(\xi) = \xi_i\,,\ i = 1,2,3\,.$$

Then it is easy to see that the set of simple roots $\alpha_1, \alpha_3, \alpha_3$ is given by

$$\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3,\ \alpha_3 = \epsilon_2 + \epsilon_3$$

and the set of all roots is then

$$\Delta = \{\pm\alpha_1,\ \pm\alpha_2,\ \pm(\alpha_1 + \alpha_2),\ \pm(\alpha_1 + \alpha_3),\ \pm(\alpha_1 + \alpha_2 + \alpha_3)\}\ .$$

The Dynkin diagram is clearly 

as the elements corresponding to α_i through the isomorphism established by R are

$$H_{\alpha_1} = \frac{1}{8}(1,-1,0),\ H_{\alpha_2} = \frac{1}{8}(0,1,-1),\ H_{\alpha_3} = \frac{1}{8}(1,1,0)$$

and it is easy to calculate that

$$\langle\alpha_3, \alpha_3\rangle = \frac{1}{4},\ \langle\alpha_1, \alpha_2\rangle = -\frac{1}{8},\ \langle\alpha_1, \alpha_3\rangle = -\frac{1}{8},\ \langle\alpha_2, \alpha_3\rangle = 0$$

(Here and below we use the notations and normalizations of [7] which are universally accepted.)

Of course, the Dynkin diagram is isomorphic to the diagram of the algebra $\mathbf{sl}(4,\mathbb{C}) \sim \mathrm{A}_3$. As it is well known, for this algebra it holds

$$B(x,y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y) = 8\ \mathrm{tr}XY; \quad X, Y \in \mathbf{sl}(4,\mathbb{C})\,.$$

The Cartan subalgebra is

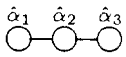
$$\hat{h} = \left\{ \mathrm{diag}(h_1, h_2, h_3, h_4),\ \sum_{i=1}^{4} h_i = 0 \right\}\ .$$

If we introduce $\hat{\epsilon}_i \in \hat{h}^*$ by

$$\hat{\epsilon}_i\ (\mathrm{diag}(h_1, h_2, h_3, h_4)) = h_i\,.$$

then the set of roots $\hat{\Delta}$ is $\hat{\Delta} = \{\hat{\epsilon}_i - \hat{\epsilon}_j,\ i \neq j\},\ i,j = 1,2,3,4$ and the set of simple roots is given by

$$\hat{\alpha_1} = \hat{\epsilon_1} - \hat{\epsilon_2},\ \hat{\alpha_2} = \hat{\epsilon_2} - \hat{\epsilon_3},\ \hat{\alpha_3} = \hat{\epsilon_3} - \hat{\epsilon_4}\,.$$

The Dynkin diagram is 

. Then we can define an isomorphism ψ of the root systems supporting that for the simple roots we have

$$\psi(\alpha_1) = \hat{\alpha}_2,\ \psi(\alpha_2) = \hat{\alpha}_1,\ \psi(\alpha_3) = \hat{\alpha}_3$$

extending ψ by linearity. It is well known that the mapping ψ then generates the isomorphism of the algebras. We shall denote it by Ψ. Finally we have

$$\Psi(E_\alpha) = \hat{E}_{\psi(\alpha)},\quad \alpha \in \Delta,$$

$$\Psi(H_{\alpha_i}) = H_{\psi(\alpha_i)},\quad i = 1,2,3,$$

where $\{E_\alpha, H_{\alpha_i}, \alpha \in \Delta,\ i = 1,2,3\}$ and $\left\{\hat{E}_{\hat{\alpha}}, H_{\hat{\alpha}_i}, \hat{\alpha} \in \hat{\Delta},\ i = 1,2,3\right\}$ are the Cartan-Weyl basis for $\mathbf{so}_{\mathbb{C}}(3,3) \sim \mathbf{so}(6,\mathbb{C})$ and $\mathbf{sl}(4,\mathbb{C})$, respectively. Using the explicit expressions for this basises in [7] we can construct the needed isomorphism. We put aside the cumbersome but straightforward calculations and present only the final results:

1. U-V pair for the chiral O(3) fields equations

$$U = \frac{-1}{2}\,(A_1 - A_2)\,(\lambda + \widetilde{J})$$
$$V = \frac{1}{2}\,(A_1 + A_2)\,(\lambda + \widetilde{J}) \tag{12}$$

where

$$A_1 = \frac{1}{2}\begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tag{13}$$

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} \tag{14}$$

$$\widetilde{J} = \begin{pmatrix} -j_1 - j_2 + j_3 & 0 & 0 & 0 \\ 0 & -j_1 + j_2 - j_3 & 0 & 0 \\ 0 & 0 & j_1 - j_2 - j_3 & 0 \\ 0 & 0 & 0 & j_1 + j_2 + j_3 \end{pmatrix} \tag{15}$$

2. U-V pair for the Landau-Lifshitz equations

$$
\begin{aligned}
U &= \frac{1}{2}A_1(\lambda + \widetilde{J}) \\
V &= \frac{1}{2}\left(2\lambda A_1 - [A_1, A_{1x}] + A_{2J}\right)(\lambda + \widetilde{J})
\end{aligned}
\tag{16}
$$

where

$$
A_{2J} = \frac{1}{2}\begin{pmatrix} 0 & j_1u_1 & j_2u_2 & j_3u_3 \\ -j_1u_1 & 0 & j_3u_3 & -j_2u_2 \\ -j_2u_2 & -j_3u_3 & 0 & j_1u_1 \\ -j_3u_3 & j_2u_2 & -j_1u_1 & 0 \end{pmatrix}, \tag{17}
$$

$\widetilde{J}$ is given by (15) and A_1 by (13).

We shall make only one comment about the pair for LLE. When $j_s = 0$, $s = 1,2,3$ the matrixes in this pair become elements of the subalgebra $\mathbf{so}(4) \subset \mathbf{sl}(4)$. As it is well known, $\mathbf{so}(4)$ is isomorphic to $\mathbf{so}(3) \times \mathbf{so}(3)$. Then for $j_s = 0$ the pair is equivalent to the pair (11) or to the well known 2×2 pair for HFE.

4. A new 4×4 pair for the generalized LLE in the case of cnoidal waves

The next in turn of the simplest generalizations of LLE has the form

$$
\begin{cases} \vec{u}_t = \vec{u} \times \vec{u}_{xx} + \vec{u} \times J\vec{u}, \quad J = \mathrm{diag}(j_1, j_2, j_3) \\ \vec{v}_t = \vec{v} \times \vec{v}_{xx} + \vec{v} \times J\vec{v}, \, . \end{cases} \tag{18}
$$

It describes the anisotropic interaction of two isotropic ferromagnetic lattices. The Lax pair for this system was not known up to now, except for the case $u(x,t) = u(x - at)$, $v(x,t) = v(x - at)$ known as the case of cnoidal waves. For this case the Lax pair of the classic kind

$$
\dot{L} = [L, A] \tag{19}
$$

was found in [10] with matrices 6×6 belonging to $\mathbf{so}(3,3)$. Using the same isomorphism as in the forgoing sections we rewrite this Lax pair in a new form in terms of 4×4 matrices belonging to $\mathbf{sl}(4, \mathbb{C})$.

The result of this straightforward calculations can be represented in its final form. The Lax pair for the generalized LLE is

$$
\begin{aligned}
L &= -2\lambda^2 A_1 A_2 + \lambda(a(A_1 + A_2) + [(A_1 + A_2), (A_1 + A_2)_\xi]) - \widetilde{J} \\
A &= -2\lambda A_1 A_2
\end{aligned}
$$

with $\xi = x - at$ and the operators A_1, A_2, $\widetilde{J}$ have the same form as in (13)-(15).

We hope that this pair, which looks much more simpler, will allow for finding of new classes of solutions and stimulate the search for new Lax pairs in the general case.

Aknowledgements

One of the authors, A.B.Y., has the pleasure to thank the Institut for Mathematics of Leipzig University for the kind hospitality during his stay at it in 1994 when this paper had been finished.

This work is partially supported by the contract MM 428/94 with the Ministry of Science and Education of Bulgaria, by the DAAD (Germany) and by the NTZ of Leipzig University.

References

[1] Faddeev L D , Takhtadjan 1987 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer Verlag).

[2] Sklianin A E 1979 LOMI Preprint E-3-79.

[3] Borovik A E 1978 *Lett JETP* **28**,10 629-632

[4] Gerdjikov V S and Yanovski A B 1994 *J. Math. Phys.* **35**,7 3687-3723

[5] Sidorenko U N 19.. *Zap. LOMI* **161** 76-87

[6] Zakharov V E and Takhtadjan 1979 *Teor. i Mat. Fis.* **38** 26-37

[7] Goto M and Grosshans F 1978 *Semisimple Lie algebras* Lecture Notes in Pure and Applied Mathematics 38, (New York and Basel: M. Dekker Inc.)

[8] Cherednik I V 1981 *Journal of Nucl. Phys.* **33** 278-282

[9] Landau L D and Lifshitz E M 1935 *Phys. Z. Sow.* **8** 153

[10] Veselov A P 1984 *DAN SSSR* **276** 590-593