

AD

CERN LIBRARIES, GENEVA

IC/94/322



SCAN-9501064

SW 9502

**INTERNATIONAL CENTRE FOR  
THEORETICAL PHYSICS**

**EQUIVALENCE OF TWO NON-COMMUTATIVE  
GEOMETRY APPROACHES**

**Han-Ying Guo**

**Ke Wu**

**and**

**Jian-Ming Li**



**INTERNATIONAL  
ATOMIC ENERGY  
AGENCY**



**UNITED NATIONS  
EDUCATIONAL,  
SCIENTIFIC  
AND CULTURAL  
ORGANIZATION**

**MIRAMARE-TRIESTE**

International Atomic Energy Agency  
and  
United Nations Educational Scientific and Cultural Organization  
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**EQUIVALENCE OF TWO NON-COMMUTATIVE GEOMETRY  
APPROACHES**

Han-Ying Guo, Ke Wu  
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080,  
People's Republic of China

and  
Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735,  
Beijing 100080, People's Republic of China

and  
Jian-Ming Li<sup>1</sup>  
International Centre for Theoretical Physics, Trieste, Italy.

**ABSTRACT**

We show that differential calculus on discrete group  $Z_2$  is equivalent to A. Connes' approach in the case of two discrete points. They are the same theory in terms of different basis and the discrete group  $Z_2$  is the permutation group of two discrete points.

MIRAMARE - TRIESTE

October 1994

---

<sup>1</sup>Permanent Address: CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing 100080, People's Republic of China. E-mail: lijm@itp.ac.cn

## 1 Introduction

Since A. Connes introduced non-commutative geometry into particle physics[1], a lot of efforts have been made along this direction [2-6]. Soon after, Sitarz [7] proposed an attractive approach towards the construction of a pure gauge theory on arbitrary discrete groups in which Higgs appear as gauge fields on discrete groups. Though simple in its structure, it contains all the main results in [1]. But Sitarz [7] could not reach the goal towards the realistic model building in particle physics, then its results were mathematical one. Fortunately, this problem had been solved by the authors[9,10].

The main idea in these two approach is to regard Higgs field as a kind of gauge field. The first approach is to take Higgs as gauge field on discrete points and the second is on discrete group. After the physical model building, we find that the results are the same whenever either method is used. This made us think: what is the relationship between these two approaches?

In this paper, first of all, we summarize A. Connes' differential calculus on two discrete points (I) and Sitarz' differential calculus on discrete group  $Z_2$  (II). Then we show that these two approaches are equivalent to each other and the  $Z_2$  group in (II) is just the permutation group of two discrete points in (I).

## 2 Differential Calculation on two discrete points

In this section, we summarize A. Connes' non-commutative differential calculus on two discrete points. For detailed information, see [8].

Algebra  $\bar{\mathcal{A}}$  is complex number  $\mathcal{D}$  valued functions on two points space  $X = \{a, b\}$ . An element  $f \in \bar{\mathcal{A}}$  is given by two complex numbers  $f(a), f(b) \in \mathcal{D}$ . We let  $\Omega^*(\bar{\mathcal{A}})$  be the universal differential graded algebra over  $\bar{\mathcal{A}}$ . The degree

of  $\bar{\mathcal{A}}$  is 0 and elements of differential one form are generated by  $da$ ,  $a \in \bar{\mathcal{A}}$  with the following properties:

$$\begin{aligned} d(ab) &= d(a)b + adb, \quad \forall a, b \in \bar{\mathcal{A}} \\ (da)^* &= -da^* \\ d1 &= 0. \end{aligned} \quad (1)$$

The higher order differential forms are defined by

$$d(a^0 da^1 \cdots da^n) = da^0 da^1 \cdots da^n, \quad \forall a^j \in \bar{\mathcal{A}} \quad (2)$$

and satisfy

$$\begin{aligned} d^2\omega &= 0, \quad \forall \omega \in \Omega^*(\bar{\mathcal{A}}) \\ d(\omega_1\omega_2) &= (d\omega_1)\omega_2 + (-1)^{\partial\omega_1}\omega_1 d\omega_2, \quad \forall \omega_j \in \Omega^*(\bar{\mathcal{A}}). \end{aligned} \quad (3)$$

A 0-dimensional K cycle  $(h, D, \gamma)$  over an algebra  $\bar{\mathcal{A}}$  with involution  $*$ , is given by a representation of  $\mathcal{A}$  in the Hilbert space  $h$  corresponding to a decomposition of  $h$  as a direct sum  $h = h_a \oplus h_b$  with the action of  $\mathcal{A}$  given by  $f \in \mathcal{A} \rightarrow \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}$  and an unbounded selfadjoint operator  $D$  with compact resolvent, such that  $[D, a]$  is bounded for any  $a \in \bar{\mathcal{A}}$ . An involutive representation  $\pi$  of the universal algebra  $\forall \omega \in \Omega^*\bar{\mathcal{A}}$  in  $h$  is defined as follows:

$$\pi(a^0 da^1 \cdots da^n) = a^0 [D, a^1] \cdots [D, a^n], \quad \forall a^j \in \bar{\mathcal{A}}. \quad (4)$$

The operator  $D$  may be represented in  $h$  as a  $2 \times 2$  matrix in the following decomposition form

$$D = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix}. \quad (5)$$

We shall take  $D$  of the form

$$D = \begin{pmatrix} & D_{ab} \\ D_{ba} & \end{pmatrix} \quad (6)$$

2

because the diagonal elements commute with the action of  $\mathcal{A}$ . Since  $D$  is a selfadjoint operator, so we can introduce  $D_{ba} = D_{ab}^* = M$ , then  $D$  may be written as

$$D = \begin{pmatrix} & M^* \\ M & \end{pmatrix}. \quad (7)$$

In the following, we will discuss the simplest case  $n_a = n_b$ , where  $n_a$  and  $n_b$  are the dimension of Hilbert space  $h_a$  and  $h_b$ .

First we introduce the idempotent functional basis  $e_1, e_2 \in \mathcal{A}$ , which satisfy

$$\begin{aligned} e_1(a) &= e_2(b) = 1, \\ e_1(b) &= e_2(a) = 0, \\ e_i \cdot e_j &= \delta_{ij} e_i, \\ e_1 + e_2 &= 1 \end{aligned} \quad (8)$$

Then a function on discrete points may be written as  $f = f_1 e_1 + f_2 e_2$  and the identity  $1 = e_1 + e_2$  and we have

$$de_1 + de_2 = 0. \quad (9)$$

The space  $\Omega^1(\bar{\mathcal{A}})$  is a 2 dimensional space, which has the following basis:

$$e_1 de_1, \quad e_2 de_2. \quad (10)$$

So that every element of  $\Omega^1(\bar{\mathcal{A}})$  is of the form  $\lambda e_1 de_1 + \mu e_2 de_2$ . The differential  $d: \bar{\mathcal{A}} \rightarrow \Omega^1(\bar{\mathcal{A}})$  is the finite difference:

$$df = (\Delta f) e_1 de_1 - (\Delta f) e_2 de_2, \quad \Delta f = f(a) - f(b). \quad (11)$$

A gauge potential is given by a self adjoint element of  $\Omega^1$ ,

$$V = \Phi^\dagger e_1 de_1 + \Phi e_2 de_2, \quad (12)$$

3

then its curvature is

$$\theta = dV + V^2 = (\Phi + \Phi^\dagger)de_1de_1 - (\Phi\bar{\Phi})de_1de_1. \quad (13)$$

Under the representation  $\pi$  one has  $\pi(de_1) = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix}$  and  $\pi(de_1de_1) = \begin{pmatrix} -M^*M & 0 \\ 0 & -MM^* \end{pmatrix}$ . Therefore we can get the Yang-Mills action

$$\mathcal{L}_{YM} = \langle \theta, \theta \rangle = (|\Phi - 1|^2 - 1)^2 \langle de_1de_1, de_1de_1 \rangle \quad (14)$$

The inner product on  $\Omega^k$  is defined by

$$\langle T_1, T_2 \rangle = Tr(\pi(T_1)\pi(T_2)), \quad \forall T_i \in \Omega^k \quad (15)$$

then we have

$$\langle de_1, de_1 \rangle = Tr(\pi(de_1)\pi(de_1)) = 2Tr(MM^*), \quad \langle de_1de_1, de_1de_1 \rangle = 2Tr((MM^*)^2).$$

So we get the Lagrangian for the gauge field:

$$\mathcal{L} = 2(|\Phi - 1|^2 - 1)^2 Tr((MM^*)^2). \quad (16)$$

This is of Higgs potential type up to some coupling constants. To get the entire Lagrangian of the Higgs, we need to consider the space-time part. For detail it is discussed in [8].

### 3 Differential calculus on discrete group $Z_2$

In this section, we will outline the notion of differential calculus theory on discrete groups  $Z_2 = \{e, r\}$ . For details, it is referred to [7].

Let  $\mathcal{A}$  be the algebra of the all complex valued functions on  $Z_2$ . The right action of  $Z_2$  group on  $\mathcal{A}$  read as

$$(R_e f)(g) = f(g), \quad (R_r f)(g) = f(g \odot r), \quad g \in Z_2, \quad \forall f \in \mathcal{A}, \quad (17)$$

where  $\odot$  denotes the group multiplication. The derivative is defined as

$$\partial_g f = f - R_g f, \quad g \in Z_2. \quad (18)$$

It is easy to see  $\partial_e f = 0$  and the only nontrivial derivative is  $\partial_r$ , which satisfies

$$\partial_r \cdot \partial_r = 2\partial_r. \quad (19)$$

The first order differential calculus  $(\Omega^1, d)$  may be given by the definition of its dual space, one dimensional vector space on  $\mathcal{A}$  with basis  $\partial_r$  as follows:

$$\chi(\partial_r) = 1, \quad (20)$$

where  $\chi$  is the basis of differential one form.

The definition for higher order forms is natural and we take  $\Omega^n$  to be the tensor product of  $n$  copies of  $\Omega^1$ ,  $\Omega^n = (\Omega^1)^{\otimes n}$  and  $\Omega^0 = \mathcal{A}$ . To complete the construction of the differential algebra  $\Omega^* = \bigoplus_n \Omega^n$ , we need to define the exterior derivative  $d$ ,  $d: \Omega^n \rightarrow \Omega^{n+1}$  whose action on  $\mathcal{A}$  is defined by

$$df = \partial_r f \chi. \quad (21)$$

It is easy to prove the following lemma [7]:

*There exists exactly one linear exterior derivative operator  $d$  such that it satisfies*

$$(i) \quad d^2 = 0,$$

$$(ii) \quad d(fg) = df \cdot g + (-1)^{\deg f} f \cdot dg, \quad \forall f, g \in \Omega^*,$$

*provided that  $\chi$  satisfy the following two conditions*

$$\begin{aligned} \chi f &= (R_r f)\chi, \quad f \in \mathcal{A}, \\ d\chi &= -2\chi\chi. \end{aligned} \quad (22)$$

The involution on the differential algebra agrees with the complex conjugation on  $\mathcal{A}$  and (graded) commutes with  $d$ , i.e.  $d(\omega^*) = (-1)^{\deg \omega} (d\omega)^*$ . Again, it is sufficient to calculate it if we set the involution to  $\chi$ , the basis of one-forms,  $(\chi)^* = -\chi$ .

The Haar integral on discrete group  $Z_2$  is introduced as a complex valued linear functional on  $\mathcal{A}$  that remains invariant under the action of  $R_g$ ,

$$\int_{Z_2} f = \frac{1}{2}(f(e) + f(\tau)), \quad (23)$$

which is normalized such that  $\int_{Z_2} 1 = 1$ .

Let us consider the case that there are Lie group transformations among the elements of the function space and those transformations also depend on the elements of the discrete group. Then the derivatives introduced above are no longer covariant. In order to get meaningful differential calculus in this case, the connection one form is needed to define the covariant exterior differential:

$$D = d_G + A, \quad (24)$$

where the connection one form  $A$  is a self adjoint element in  $\Omega^1$  and may be written as

$$A = \phi\chi. \quad (25)$$

Then the generalized curvature two form is

$$F = dA + A^2 = (-\phi - R_r\phi + \phi R_r\phi)\chi\chi. \quad (26)$$

The self adjoint character of  $A$  requires that  $\phi^\dagger = R_r\phi$ , then we have

$$F = (-\phi - \phi^\dagger + \phi\phi^\dagger)\chi\chi. \quad (27)$$

After introducing the metric, we can get the Lagrangian for the theory.

In the  $Z_2$  case, we can define the metric as

$$\langle \chi, \chi \rangle = \eta, \quad \langle \chi \otimes \chi, \chi \otimes \chi \rangle = \eta^2, \quad (28)$$

where  $\eta$  is a positive number. Therefore

$$\mathcal{L} = \langle \bar{F}, F \rangle = 2\eta^2(|\Phi - 1|^2 - 1)^2. \quad (29)$$

This is the same type of Higgs potential as we get in A. Connes' approach. The entire Lagrangian of the Higgs may be introduced also if we consider the space-time part. For detail it is referred to [9, 10].

It is easy to see that these two approaches are similar whatever the structures or results. Then a stimulating question may be raised as to whether we can build a bridge between these two approaches? In next section, we will find that the answer is yes.

## 4 Equivalence of Two Approaches

To illustrate the equivalence between these two approaches, we start from A. Connes' derivative on two discrete points. Let  $f = f_1e_1 + f_2e_2$  be a function on two discrete point space  $X$ , then

$$df = (f_1 - f_2)(e_1de_1 - e_2de_2). \quad (30)$$

We know that the permutation group of two points is a  $Z_2$  group. Then we can define the  $Z_2$  action on the two points according to permutation as follows:

$$R_e e_i = e_i, \quad R_r e_1 = e_2, \quad R_r e_2 = e_1. \quad (31)$$

We find that the derivative (30) may be written as

$$df = (f - R_r f)(e_1de_1 + e_2de_2). \quad (32)$$

If we introduce

$$\bar{\partial}_r f = f - R_r f, \quad \bar{\chi} = (e_1 de_1 + e_2 de_2), \quad (33)$$

we have

$$df = \bar{\partial}_r f \cdot \bar{\chi} \quad (34)$$

and every element of  $\Omega^1(\bar{\mathcal{A}})$  may be in terms of  $\bar{\chi}$  as,

$$V = \lambda e_1 de_1 + \mu e_2 de_2 = f \cdot \bar{\chi}$$

where  $f = \lambda e_1 + \mu e_2$ .

To verify (34) is just the differential calculus in the case of  $Z_2$  in Sitarz' approach, we should check the relations (19) and (22). In doing so, it is useful to write out some relations, which result from the equations (8),

$$\begin{aligned} de_1 e_2 &= -e_1 de_2, & de_2 e_1 &= -e_2 de_1 \\ de_1 e_1 &= e_2 de_1, & e_1 de_1 &= de_1 e_2 \\ de_2 e_2 &= e_1 de_2, & e_2 de_2 &= de_2 e_1 \end{aligned} \quad (35)$$

From direct calculation, it is easy to show that:

$$\partial_r \cdot \bar{\partial}_r = 2\partial_r, \quad \bar{\chi} f = R_r f \bar{\chi}, \quad d\bar{\chi} = -2\bar{\chi} \bar{\chi},$$

The involution on the algebra  $\bar{\mathcal{A}}$  is just the complex conjugation when it valued on discrete points,

$$f^*(X) = (f(X))^*, \quad (36)$$

then, we have  $e_1^* = e_1$ ,  $e_2^* = e_2$ , from which we also have  $(de_1)^* = de_1$ ,  $(de_2)^* = de_2$ . Using these relations, we can get the equation

$$(\bar{\chi})^* = -\bar{\chi}. \quad (37)$$

So far, it is shown that  $\bar{\partial}_r$  and  $\bar{\chi}$  satisfy all the relations in (19) and (22).

Hence, we conclude that A. Connes' differential calculus on two discrete points is equivalent to Sitarz' approach in the case of  $Z_2$ .

To discuss the Yang-Mills action, first we should study the metric. From the definition of inner product on  $\Omega^k$ , we have

$$\langle \bar{\chi}, \bar{\chi} \rangle = Tr\pi(\bar{\chi})^2 = 2Tr(MM^*), \quad \langle \bar{\chi}\bar{\chi}, \bar{\chi}\bar{\chi} \rangle = Tr\pi(\bar{\chi})^4 = 2Tr(MM^*)^2. \quad (38)$$

This metric appears a little different from (28) in Sitarz' approach. However they may be consistent with each other if we redefine the definition of inner product on  $\Omega^k$  as

$$\langle T_1, T_2 \rangle = \alpha Tr(\pi(T_1)\pi(T_2)), \quad \forall T_i \in \Omega^k, \quad (39)$$

where  $\alpha = \frac{Tr(MM^*)^2}{(TrMM^*)^2}$ . Then we get

$$\langle \bar{\chi}, \bar{\chi} \rangle = 2\bar{\eta}, \quad \langle \bar{\chi}\bar{\chi}, \bar{\chi}\bar{\chi} \rangle = 2\bar{\eta}^2, \quad (40)$$

where  $\bar{\eta} = \frac{Tr(MM^*)^2}{(TrMM^*)^2}$  is a positive number. These results correspondence to the formulas in (28) after integrating over discrete group  $Z_2$ .

$$\int_{Z_2} \langle \chi, \chi \rangle = 2\eta, \quad \int_{Z_2} \langle \chi\chi, \chi\chi \rangle = 2\eta^2. \quad (41)$$

In A. Connes' approach, a gauge potential is given by a self adjoint element of  $\Omega^1$  as

$$V = \Phi^\dagger e_1 de_1 + \Phi e_2 de_2. \quad (42)$$

In terms of  $\bar{\chi}$  the gauge potential may be written as  $V = f \cdot \bar{\chi}$ , where  $f = \Phi^\dagger e_1 + \Phi e_2$ . Using the formulas in Sitarz' approach, we obtain the curvature

$$\theta = dV + V^2 = (-\Phi - \Phi^* + \Phi\Phi^*)\bar{\chi}\bar{\chi}. \quad (43)$$

Therefore, the Yang-Mills action is

$$\mathcal{L} = \langle \theta, \theta \rangle = 2\bar{\eta}^2(|\Phi - 1|^2 - 1). \quad (44)$$

It is easy to see that whenever either way is used, we get the same results.

If starting from differential calculus on discrete group  $Z_2$ , we can get A. Connes's approach also. For this purpose, we should introduce the idempotent basis  $e_e, e_r \in \mathcal{A}$ , which satisfy

$$\begin{aligned} e_e(e) &= e_r(r) = 1 \\ e_e(r) &= e_r(e) = 0 \\ e_g \cdot e_h &= \delta_g^h e_h, \quad g, h \in Z_2 \\ e_e + e_r &= 1 \end{aligned}$$

Then a function on discrete group  $Z_2$  may be written as  $f = f_e e_e + f_r e_r$  and the right action of the group may be introduced as  $R_r \cdot e_e = e_r, R_r \cdot e_r = e_e$ . It is easy to show that  $(R_r \cdot f)(g) = f(g \cdot r)$ . The more detailed calculation is similar as the previous part of this section.

From a mathematical point of view, We have shown that A. Connes' differential calculus on two discrete points is equivalent to Sitarz' approach in the case of  $Z_2$ . The basis of differential one form in the second approach is a recombination of the first one, and the discrete group is nothing but the permutation group of the discrete points. In [11], we have shown that another known non-commutative geometry approach proposed by R. Coquereaux et.al.[3] is the matrix representation of the gauge theory on discrete group  $Z_2$ . Therefore, these three approaches of non-commutative geometry are consistent with each other.

## Acknowledgements

One of the authors (J.M.L.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He would also like to thank Dr. Y.K. Lau for his helpful discussion.

## References

1. A. Connes, in: *The Interface of Mathematics and Particle Physics*, eds. D. Quillen, G. Segal and S. Tsou (Oxford U. P, Oxford 1990); A. Connes and J. Lott, *Nucl. Phys. (Proc. Suppl.)* **B18**, 44 (1990); A. Connes and Lott, *Proceedings of 1991 Cargese Summer Conference*;
2. D. Kastler, Marseille, CPT preprint **CPT-91/P.2610**, **CPT-91/P.2611**.
3. R. Coquereaux, G. Esposito-Farèse and G Vaillant, *Nucl Phys* **B353** 689 (1991).
4. A. H. Chamseddine, G Felder and J. Fröhlich, *Phys. Lett.* **296B** (1993) 109.
5. B. S. Balakrishna, F Gürsey and K. C. Wali, *Phys Lett* **B254**, 430 (1991).
6. N. A. Papadopoulos, J. Plass and F. Scheck, *Phys Lett* **B324**, 380 (1994).
7. A. Sitarz, *Non-commutative Geometry and Gauge Theory on Discrete Groups*, preprint **TPJU-7/1992**; A.Sitarz, *Phys. Lett* **308B**(1993) 311.
8. A. Connes, *Non-Commutative Geometry*. **IHES/M/93/12**.
9. Haogang Ding, Hanying Guo, Jianming Li and Ke Wu, *Commun. Theor. Phys.* **21** (1994) 85-94; *J. Phys. A:Math.Gen.* **27**(1994) L75-L79; *J. Phys. A:Math.Gen.* **27** (1994)L231-L236.
10. Hao-Gang Ding, Han-Ying Guo, Jian-Ming Li and Ke Wu, *Higgs as Gauge Fields On Discrete Groups and Standard Model For Electroweak and Electroweak-Strong Interactions*, preprint **ASITP-93-23, CCAST-93-5**, To appear in *Z. Physik C*.
11. Jianming Li, *Matrix Realization of Gauge Theory on Discrete Group  $Z_2$* , *ASITP-94-9*, hep-th/9408157.



