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The non-commutative geometry of stable relativistic space-time

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Abstract

Models or theories that are stable, in the sense that they do not change in a qualitative manner under a small change of parameters, are probably those with the wider range of validity. This seems to be true not only for the dissipative systems studied in non-linear dynamics but also for the fundamental theories of Nature. It is well-known that the transition from non-relativistic to relativistic and from classical to quantum mechanics may be interpreted as the stabilizing deformations of two unstable algebraic structures. When these two algebraic structures are put together one finds that the resulting relativistic quantum algebra is itself unstable and admits a two-parameter stabilizing deformation. One of the deformations is related to the fact that flat space is an isolated point in the class of constant curvature spaces, the other makes the space-time coordinates non-commuting observables and introduces a parameter with dimensions of length. In this paper some of the consequences of the deformed algebras of observables are explored, in particular the geometric aspects of non-commutative space-time.

1. Deformations and the instability of relativistic quantum mechanics

It is probable that all physical theories, developed so far, are mere approximations to Nature. Therefore the theories that have the higher probability to have a wider range of validity are those that do not change in a qualitative manner for a small change of parameters. Such theories are called ***stable*** or ***rigid***. It is unlikely that properties that are too sensitive to small changes in the theoretical model (i. e. that depend in a critical manner on particular values of the parameters) will ever be observed. Alternatively if a fine tuning is needed to reproduce some natural phenomenon, then it is most certain that the model is basically unsound and its other predictions are unreliable. It is therefore a good methodological point of view to concentrate on the robust properties of the models or, equivalently, on models which are stable.

In general, a mathematical structure is said to be ***stable*** (or ***rigid***) for a class of ***deformations*** if any deformation in this class leads to an equivalent (isomorphic) structure. The idea of stability of the structures provides a guiding principle to test either the validity or the need for generalization of a physical theory. Namely, if the mathematical structure of a given theory is not stable then, one should try to deform it until one falls into a stable one, which has a good chance of being a generalization of wider validity. The mathematical theory of deformations developed along several lines, the most developed branches being the deformations of analytic structures, the deformations of algebraic manifolds and the deformations of algebras[1-3]. In all cases the cohomology groups play a key role in characterizing the stability of the structures.

The stable-model point of view had a large impact in the field of non-linear dynamics, where it led to the notion of ***structural stability***[4]. As emphasized by Flato[5] and Faddeev[6] the same pattern seems to occur in the fundamental theories of Nature. In fact, the two most important physical revolutions of this century, namely the passage from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as the transition from two unstable theories to two stable theories. From non-relativistic to relativistic mechanics one notices that the second cohomology group of the homogeneous Galileo group does not vanish and has a deformation leading to the Lorentz algebra which, being semisimple, is stable. In turn, the transition from classical to quantum mechanics may be regarded as a deformation of the Poisson algebra of functions on phase space to the stable Moyal-Vey algebra[7]. I will refer to these two stabilizing deformations as the (1/c)-deformation and the $\hbar$-deformation. The deformed algebras are all equivalent for non-zero values of 1/c and $\hbar$. Hence relativistic mechanics and quantum mechanics might have been derived purely from considerations of stability of their algebras, but the exact values of the deformation parameters are fundamental constants to be obtained from experiment. In this sense not only deformation theory is the theory of stable theories, it also is the theory that identifies the fundamental constants.

A review of deformation theory and of the transitions from non-relativistic to relativistic and from classical to quantum mechanics as the deformation-stabilization of two unstable theories is contained in Ref.[8]. Also, it is shown there that both deformations may be studied in the context of finite-dimensional Lie algebras, which is simpler than the usual treatment of quantum mechanics as a deformation of an infinite-dimensional algebra of functions. The algebra that results from the (1/c)-deformation is the Lorentz algebra and the one coming from the $\hbar$-deformation is the Heisenberg algebra. When the two deformed algebras are put together one ends up with

$$[M_{\mu\nu},M_{\rho\sigma}] = i\left(M_{\mu\sigma}g_{\nu\rho}+M_{\nu\rho}g_{\mu\sigma}-M_{\nu\sigma}g_{\mu\rho}-M_{\mu\rho}g_{\nu\sigma}\right) \tag{1.1a}$$

$$[M_{\mu\nu},P_{\lambda}] = i\left(P_{\mu}\, g_{\nu\lambda}-P_{\nu}\, g_{\mu\lambda}\right) \tag{1.1b}$$

$$[M_{\mu\nu},x_{\lambda}] = i\left(x_{\mu}\, g_{\nu\lambda}-x_{\nu}\, g_{\mu\lambda}\right) \tag{1.1c}$$

$$[P_{\mu},P_{\nu}] = 0 \tag{1.1d}$$

$$[x_{\mu},x_{\nu}] = 0 \tag{1.1e}$$

$$[P_{\mu},x_{\nu}] = i\, g_{\mu\nu}\, \mathfrak{I} \tag{1.1f}$$

Velocities and actions are already measured in units of c and $\hbar$ (that is $c=\hbar=1$) and $\mathfrak{I}$ is the identity operator. The algebra $\mathfrak{R}_0=\left\{M_{\mu\nu},P_{\mu},x_{\mu},\mathfrak{I}\right\}$ defined by Eqs.(1.1) is the union of the Lorentz and the Heisenberg algebras together with the compatibility relations (1.1b-c), stating the four-vector nature of coordinates and momenta. $\mathfrak{R}_0$ generates the algebra of observables of relativistic quantum mechanics (as this theory is understood today). Before internal quantum numbers are introduced, all local observables are in the closure of the polynomial algebra generated by $\mathfrak{R}_0$.

Two algebras, obtained through deformations that stabilize previously unstable theories, are put together in Eqs.(1.1). A natural question is whether the whole algebra is now stable or whether there are still non-trivial deformations leading to qualitatively different theories. The answer is that the algebra in Eqs.(1.1) is not stable because there is a 2-parameter deformation leading to

$$[M_{\mu\nu},M_{\rho\sigma}] = i\left(M_{\mu\sigma}g_{\nu\rho}+M_{\nu\rho}g_{\mu\sigma}-M_{\nu\sigma}g_{\mu\rho}-M_{\mu\rho}g_{\nu\sigma}\right) \tag{1.2a}$$

$$[M_{\mu\nu},P_\lambda] = i\left(P_\mu\, g_{\nu\lambda}-P_\nu\, g_{\mu\lambda}\right) \tag{1.2b}$$

$$[M_{\mu\nu},x_\lambda] = i\left(x_\mu\, g_{\nu\lambda}-x_\nu\, g_{\mu\lambda}\right) \tag{1.2c}$$

$$[P_\mu,P_\nu] = -i\,\frac{\epsilon'}{R^2}\,M_{\mu\nu} \tag{1.2d}$$

$$[x_\mu,x_\nu] = -i\,\epsilon\,\ell^2\,M_{\mu\nu} \tag{1.2e}$$

$$[P_\mu,x_\nu] = i\,g_{\mu\nu}\,\mathfrak{I} \tag{1.2f}$$

$$[P_\mu,\mathfrak{I}] = -i\,\frac{\epsilon'}{R^2}\,x_\mu \tag{1.2g}$$

$$[x_\mu,\mathfrak{I}] = i\,\epsilon\,\ell^2\,P_\mu \tag{1.2h}$$

$$[M_{\mu\nu},\mathfrak{I}] = 0 \tag{1.2i}$$

The algebra $\mathfrak{R}_{\ell,R}$, defined by Eqs.(1.2), is not equivalent to the one in Eqs.(1.1) and, being isomorphic to the conformal algebra, it is stable. The algebra $\mathfrak{R}_{\ell,R}$ is obtained from $\mathfrak{R}_0$ by a (ϵ'/R^2)-deformation and a $(\epsilon\ell^2)$-deformation, R and ℓ being length parameters and ϵ,ϵ' plus or minus signs. Notice also that the operator $\mathfrak{I}$ which previously was a trivial center of the Heisenberg algebra becomes now a non-trivial operator. Therefore if, once more, the stable-theory point of view is adopted we are led to postulate that $\mathfrak{R}_{\ell,R}$ has a wider range of validity than $\mathfrak{R}_0$. Hence $\mathfrak{R}_{\ell,R}$, or some approximation thereof, should be adopted as the "true" algebra of relativistic quantum mechanics.

To understand the role of the two deformation parameters consider first the Poincaré subalgebra $\mathfrak{P}=\left\{M_{\mu\nu},P_\mu\right\}$ of $\mathfrak{R}_0$. It is well known that already this subalgebra is not stable and may be deformed[5,9] to the stable simple algebras of the De Sitter groups O(4,1) or O(3,2). This is the deformation that leads to the commutation relation (1.2d). This instability of the Poincaré algebra is however physically harmless, at least before general relativity effects are taken into account. It simply means that flat space is an isolated point in the set of constant curvature spaces. As long as the Poincaré group is used as the kinematical group of the tangent space to the space-time manifold, and not as a group of motions in the manifold itself, it is perfectly consistent to take $R\to\infty$ and this deformation goes away. In contrast, the $(\epsilon\ell^2)$-deformation is not removable by any such considerations. In conclusion: the algebra $\mathfrak{R}_{\ell,\infty}$ defined by

$$[M_{\mu\nu},M_{\rho\sigma}] = i\left(M_{\mu\sigma}g_{\nu\rho}+M_{\nu\rho}g_{\mu\sigma}-M_{\nu\sigma}g_{\mu\rho}-M_{\mu\rho}g_{\nu\sigma}\right) \tag{1.3a}$$

$$[M_{\mu\nu},P_\lambda] = i\left(P_\mu\, g_{\nu\lambda}-P_\nu\, g_{\mu\lambda}\right) \tag{1.3b}$$

$$[M_{\mu\nu},x_\lambda] = i\left(x_\mu\, g_{\nu\lambda}-x_\nu\, g_{\mu\lambda}\right) \tag{1.3c}$$

$$[P_\mu,P_\nu] = 0 \tag{1.3d}$$

$$[x_\mu,x_\nu] = -i\,\epsilon\,\ell^2\,M_{\mu\nu} \tag{1.3e}$$

$$[P_\mu,x_\nu] = i\,g_{\mu\nu}\,\mathfrak{I} \tag{1.3f}$$

$$[P_\mu,\mathfrak{I}] = 0 \tag{1.3g}$$

$$[x_\mu,\mathfrak{I}] = i\,\epsilon\,\ell^2\,P_\mu \tag{1.3h}$$

$$[M_{\mu\nu},\mathfrak{I}] = 0 \tag{1.3i}$$

seems to be a minimal candidate for a *stable algebra of relativistic quantum mechanics*, because it contains the smallest changes that are required if stability of the algebra of observables (in the tangent plane) is a good guiding principle. The main features are the non-commutativity of the x_μ coordinates and the fact that $\mathfrak{I}$, previously a trivial center of the Heisenberg algebra, becomes now a non-trivial operator. The algebra $\mathfrak{R}_{\ell,\infty}$ is isomorphic to the algebra of the pseudo-Euclidian groups E(1,4) or E(2,3), depending on whether ϵ is -1 or $+1$. In the following Sections one explores some of the consequences of using $\mathfrak{R}_{\ell,\infty}$ to generate the algebra of observables in relativistic quantum mechanics and the non-commutative geometry aspects of deformed space-time.

2. Representations of the deformed canonical commutation relations

The first step, to characterize the properties implied by $\mathfrak{R}_{\ell,\infty}$, is to construct the Hilbert space representations of a set of bounded operators formed from the unbounded generators. This is the same procedure as when, in ordinary quantum mechanics, one constructs the (unique) unitary irreducible representations of the canonical commutation relations. Let us consider only one of the space dimensions and the modified canonical commutation relations following from (1.3), namely

$$[p,x] = -i\, \mathfrak{I} \tag{2.1a}$$

$$[p,\mathfrak{I}] = 0 \tag{2.1b}$$

$$[x,\mathfrak{I}] = i\, \epsilon\, \ell^2\, p \tag{2.1c}$$

which imply the modified Weyl relations

$$e^{itp}\, e^{isx} = e^{is(x+\mathfrak{I}t)}\, e^{itp} \tag{2.2a}$$

$$e^{i\alpha\mathfrak{I}}\, e^{isx} = e^{is(x+\epsilon\ell^2\alpha p)}\, e^{i\alpha\mathfrak{I}} \tag{2.2b}$$

The unitary representations of (2.2) are obtained by noticing that (2.1) is isomorphic to the Lie algebra of the Euclidian (or pseudo-Euclidian) group in two dimensions and the representations of this group are easily obtained by the induced representation method. p and $\mathfrak{I}$ are generators of translations in a 2-dimensional space (ξ,η) and x is the generator of rotations $(\epsilon=-1)$ or hyperbolic rotations $(\epsilon=+1)$ in the (ξ,η) space.

To obtain the representations of (2.2) consider a basis of simultaneous generalized eigenstates of p and $\mathfrak{I}$, which may be realized as exponentials of (ξ,η)

$$|\pi\nu\rangle = e^{i(\pi\xi+\nu\eta)}$$

The generalized eigenstates $|\pi\nu\rangle$ are dense on the Hilbert space of square-integrable $f(\xi,\eta)$ functions. One has two classes of inequivalent representations, those associated to $\pi=\nu=0$ (class A) and those for which π or $\nu\neq 0$ (class B). In the first case the isotropy group of $|00\rangle$ is exp(isx) which acts as

$$e^{isx}\, |0\,0\,\mu\rangle = e^{i\mu x}\, |0\,0\,\mu\rangle \tag{2.3}$$

and in the second case the isotropy group of $|\pi\nu\rangle$ is the identity, exp(isx) acting on $|\pi\nu\rangle$ by

$$|\pi\nu\rangle \rightarrow |\pi'\nu'\rangle$$

with

$$\pi' = \pi\cos(\sqrt{-\epsilon}\,\ell s) - \frac{\nu}{\ell\sqrt{-\epsilon}}\sin(\sqrt{-\epsilon}\,\ell s)$$

$$\nu' = \pi\sqrt{-\epsilon}\,\ell\sin(\sqrt{-\epsilon}\,\ell s) + \nu\cos(\sqrt{-\epsilon}\,\ell s)$$

One concludes that the most general representation of the commutation relations (2.1) is

$$p \sim -i\frac{\partial}{\partial\xi} \;;\quad x \sim -i\,\ell\left(\xi\frac{\partial}{\partial\eta} + \epsilon\,\eta\frac{\partial}{\partial\xi}\right) + \ell\Sigma \;;\quad \mathfrak{I} \sim -i\,\ell\,\frac{\partial}{\partial\eta}$$

Σ being the phase operator associated to the representations of class A.

By the unitary transformation

$$\underset{\sim}{Q} \rightarrow e^{-i\eta/\ell}\, \underset{\sim}{Q}\, e^{i\eta/\ell}$$

one obtains

$$p = -i \frac{\partial}{\partial \xi} \tag{2.4a}$$

$$x = \xi + i \ell \left(\xi \frac{\partial}{\partial \eta} + \epsilon\, \eta \frac{\partial}{\partial \xi} \right) + \ell\, \Sigma \tag{2.4b}$$

$$\mathfrak{I} = 1 + i \ell \frac{\partial}{\partial \eta} \tag{2.4c}$$

which reduces to the usual representation of the Heisenberg algebra in the $\ell \to 0$ limit. In this notation, representations of class A act on vectors of the type $\exp(-i\eta/\ell)|\mu\rangle$ and those of class B on functions $\exp(-i\eta/\ell)\ f(\xi,\eta)$. When $\ell = 0$, $\mathfrak{I}$ cannot have a zero eigenvalue and the representations of class A cannot exist. Therefore for ℓ=0 there is (up to unitary equivalence) only one type of representation, consistent with the von Neumann uniqueness theorem. For $\ell \neq 0$ Eqs(2.4) contain the most general representation of the commutation relations. In this case there are several non-equivalent unitary irreducible representations. The non-uniqueness is associated to the phase operator Σ. A second source of freedom, in the implementation of the modified canonical commutation relations, is the topology of the (ξ,η) space. ξ is associated by (2.4) and the $\ell = 0$ limit to the "coarse grained" topology of the physical configuration space, but the η coordinate is not constrained. In particular we may distinguish the compact and the non-compact cases for the η-coordinate.

The representation (2.4) is easily generalized to the full algebra $\mathfrak{R}_{\ell,\infty}$ as follows

$$P_\mu = i \frac{\partial}{\partial \xi^\mu} + i\, D_{P_\mu} \tag{2.5a}$$

$$M_{\mu\nu} = i \left(\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu} \right) + \Sigma_{\mu\nu} \tag{2.5b}$$

$$x_\mu = \xi_\mu + i \ell \left(\xi_\mu \frac{\partial}{\partial \xi^4} - \epsilon\, \xi_4 \frac{\partial}{\partial \xi^\mu} \right) + \ell\, \Sigma_{\mu 4} \tag{2.5c}$$

$$\mathfrak{I} = 1 + i \ell \frac{\partial}{\partial \xi^4} + i \ell\, D_{\xi^4} \tag{2.5d}$$

where, for later convenience, I have replaced $\eta = \xi^4$ and $\epsilon\eta = \xi_4$. The set $(\Sigma_{\mu\nu},\Sigma_{\mu 4})$ is an "internal spin" operator for the groups O(4,1) (if $\epsilon = -1$) or O(3,2) (if $\epsilon = +1$) and D_{P_μ} and D_{ξ^4} are derivations operating in the space where $(\Sigma_{\mu\nu},\Sigma_{\mu 4})$ acts.

3. Some simple consequences of the commutation relations

The non-commuting operator $\mathfrak{I}$, that in the algebra $\mathfrak{R}_{\ell,\infty}$ replaces the center of the Heisenberg algebra, has no familiar physical interpretation. However the following commutators and double commutators involve only familiar measurable observables

$$[x_\mu, x_\nu] = -i\,\epsilon\,\ell^2\,M_{\mu\nu} \tag{1.3e}$$

$$[[P_\mu, x_\nu], x_\alpha] = \epsilon\,\ell^2\,g_{\mu\nu}\,P_\alpha \tag{3.1}$$

$$[[x_\mu, x_\nu], x_\alpha] = \epsilon\,\ell^2\,(g_{\nu\alpha}\,x_\mu - g_{\mu\alpha}\,x_\nu) \tag{3.2}$$

I have already mentioned in Ref.[8] that, by taking the expectation value of Eq.(3.1) in a normalized state ψ for $\mu = \nu = \alpha = 1,2$ or 3 , one obtains a dipole momentum-type sum rule

$$\int dk\, k \left\{ |\langle\psi|\, x\, |k\rangle|^2 - \mathrm{Re}\,(\langle\psi|\, x^2\, |k\rangle\langle k|\psi\rangle) \right\} = \frac{\epsilon}{2}\,\ell^2 \langle\psi|p|\psi\rangle \tag{3.3}$$

where generalized momentum eigenvectors were used for the decompositions of the unit $\int dk\ |k\rangle\langle k|$. If the state ψ has a large momentum component, the right hand side becomes large and this sum rule may lead to observable effects.

Similarly, on a basis of angular-momentum eigenstates, one obtains from Eq.(1.3e)

$$2\,\mathrm{Im} \sum_{m'} \langle m|x^1|m'\rangle\langle m'|x^2|m\rangle = -\epsilon\,\ell^2\,m \tag{3.4}$$

m being the quantum number of angular momentum along the $3-$axis.

Both in (3.3) and (3.4) the deviations from the usual quantum mechanical results depend on the magnitude of the length parameter ℓ and are expected to be very small. Therefore it would be interesting to look for effects which are not explicitly dependent on the size of ℓ but only on the fact that $\ell \neq 0$. For this purpose one starts by discussing the eigenstates of the coordinate operators x^μ.

Notice that, because of the non-commutativity in Eq.(1.3e), only one of the components of x^μ may be sharply defined each time. Consider a space component x^1 and take, for definiteness, $\epsilon = -1$. In the representation (2.4b), acting on functions $\exp(-i\eta/\ell)\,f(\xi,\eta)$, the eigenstates of x^1 are

$$|x^i> = |n\ell> \; = \; e^{i\left(n \tan^{-1}\left(\frac{\xi^i}{\eta}\right) - \frac{\eta}{\ell}\right)} \tag{3.5}$$

the eigenvalue being $n\ell$ ($x^i|n\ell> = n\ell|n\ell>$). By unicity of the angle in the (ξ^i,η) plane, n can only take integer values and the spectrum of the position coordinate is discrete. This happens because, for $\epsilon = -1$, x^i contains a rotation in the (ξ^i,η)-plane. For the time coordinate x^0, one has an hyperbolic rotation and there is no discreteness constraint in the spectrum. Conversely, for $\epsilon = +1$, it is the x^0 coordinate that will have a discrete spectrum and the position coordinates a continuous spectrum. Later, in the fully relativistic treatment of Sect.4, we will see that the discrete eigenvalues of the coordinate components may actually be either an integer or an half-integer multiple of ℓ.

Because of the discreteness of the spectrum the localized states $|x^i>$ (localized in one coordinate only) may be normalized. Taking the expectation value of the commutators (1.3c), (3.1) and (3.2) in such a state one obtains

$$0 \; = \; \ell^2 <x^i| \, M_{ij} \, |x^i> \tag{3.6a}$$

$$0 \; = \; \ell^2 <x^i| \, P^i \, | \, x^i> \tag{3.6b}$$

$$0 \; = \; \ell^2 <x^i| \, x^k \, |x^i> \qquad k \neq i \tag{3.6c}$$

That is, when a particular space coordinate i is sharply localized, the resulting state has vanishing expectation values for the angular momenta along the directions orthogonal to i, for the momentum along i and for the other space components. These conclusions do not depend on the magnitude of ℓ, depend only on ℓ being different from zero. In the case $\ell = 0$ there are no similar restrictions on the corresponding expectation values.

If ℓ is very small one cannot, in practice, localize a state at a precise x^i value. Therefore, we should concern ourselves with effects for states localized in an interval Δx. Consider a plane wave (momentum eigenstate) $|k> = e^{ik\xi}$ that is going to be localized in the interval $(-\Delta x/2, \Delta x/2)$, as in a slit diffraction experiment. For simplicity use the representation $x = -\, i\,\ell\left(\xi\,\frac{\partial}{\partial\eta} - \eta\,\frac{\partial}{\partial\xi}\right)$, which is unitarily equivalent to (2.4b). Denote $\phi = \tan^{-1}(\xi/\eta)$, $\rho = \sqrt{\xi^2+\eta^2}$ and by $P_{\Delta x}$ the projector that filters the states to the interval $(-\Delta x/2, \Delta x/2)$. Then

$$P_{\Delta x}\,|k> = J_0(k\rho) + 2\sum_{n=1}^{[\frac{\Delta x}{4\ell}]} J_{2n}(k\rho)\,\cos(2n\phi) + i2\sum_{n=0}^{[\frac{\Delta x}{4\ell}]} J_{2n+1}(k\rho)\,\sin((2n+1)\phi) \tag{3.7}$$

$[\frac{\Delta x}{4\ell}]$ denoting the integer part of $\frac{\Delta x}{4\ell}$. The momentum content of the filtered state is obtained by projecting on a generalized momentum eigenstate, $<k'|P_{\Delta x}|k> = \int d\xi \exp(-ik'\xi)\, P_{\Delta x}|k>$. Using the addition theorem for Bessel functions and performing the ξ – integration

$$<k'|P_{\Delta x}|k> = \frac{2}{\sqrt{k^2-k'^2}}\Theta(|k|-|k'|)\left\{J_0(k\eta)+2\sum_{j=1}^{\infty}J_{2j}(k\eta)T_{2j}\left(\frac{k'}{k}\right)+2\sum_{n=1}^{[\frac{\Delta x}{4\ell}]}\sum_{j=-\infty}^{j=\infty}J_{2j+1}(k\eta)T_j\left(\frac{k'}{k}\right)\right\} \tag{3.8}$$

where $T_n(z) = \cos(n \arccos z)$.

For a strictly localized state, the result that the expectation value of the momentum vanishes should be recovered from (3.8). Consider, for example, that the interval Δx contains only the state localized at x=0

$$<k'|P_0|k> = \frac{2}{\sqrt{k^2-k'^2}}\,\Theta(|k|-|k'|)\left\{J_0(k\eta)+2\sum_{j=1}^{\infty}J_{2j}(k\eta)\,T_{2j}\left(\frac{k'}{k}\right)\right\} \tag{3.9}$$

One sees that, after the localization at $x=0$, the intensity of the diffracted wave is the same for k' and $-k'$, hence the expectation value of the momentum is zero. The intensity is peaked both at $k=k'$ and $k=-k'$. A backward peak in the momentum distribution is only expected for localization in small intervals Δx. For large Δx, many terms contribute to the sums in (3.8) and their interference destroys the backward peak. When $\Delta x \to \infty$

$$\lim_{\Delta x \to \infty} <k'|P_{\Delta x}|k> = \delta(k'-k)$$

Similarly, for the $\ell=0$ limit (with fixed Δx), one has the usual quantum mechanical result $\sin((k-k')\frac{\Delta x}{2})/(k-k')$ and no backward peak. Hence, the occurrence of peaks both at $k=k'$ and $k=-k'$ for small Δx would be a signature of the non-vanishing of the length parameter ℓ.

4. The geometry of stable relativistic space-time

In the $\mathcal{R}_{\ell,\infty}$ algebra the space-time coordinates x_μ are no longer a set of commuting variables, therefore the geometry of the space-time manifold is a non-commutative geometry. In quantum theory the appearance of non-commutative geometry is not new because already the phase-space of conventional quantum mechanics is a non-commutative space. Non-commutative geometry formulations have recently provide new insights in solid-state physics[10] and possibly in particle physics as well[11]. However, the main reason why the non-commutative nature of quantum geometry is usually not emphasized is because the explicit realization of the momentum operators as differential operators puts the emphasis on the Hilbert space of functions on a (commutative) configuration space. When the coordinates in configuration space are themselves non-commuting entities, the non-commutative geometry aspect has to come to the forefront.

Every geometrical property of an ordinary (commutative) manifold M may be expressed as a property of the commutative C^*-algebra $C_0(M)$ of continuous functions on M vanishing at infinity. For example, there is a one-to-one correspondence between the characters of $C_0(M)$ and the points of the manifold M, regular Borel measures on M correspond to positive linear functionals on $C_0(M)$, complex vector bundles over M are given by the finite projective modules over $C_0(M)$, etc. Similarly in non-commutative geometry one starts from a non-commutative C^*-algebra and uses the same correspondence as in the commutative case to characterize the geometric properties of the non-commutative space[12,13].

In the commutative case the points of the manifold M play the double role of support of the pure states and of being in a one-to-one correspondence with the character representations of the algebra. In non-commutative geometry the pure states are associated to the rays in Hilbert space representations, the rays of unitary representations playing the role of points in the non-commutative space.

Snyder[14] was probably the first to propose a definite non-commuting algebra for the space-time coordinates[15]. The structure proposed by Snyder for the momentum operators and the Heisenberg algebra is different from the $\mathfrak{R}_{\ell,\infty}$ – algebra defined in Eqs.(1.3). However, when restricted to set $\mathcal{A}_S = \{x_\mu, M_{\alpha\beta}\}$, the commutation relations coincide. Therefore, in the present context, I will refer to $\mathcal{A}_S$ as the Snyder algebra. Actually, for $\epsilon = +1$ and $\epsilon = -1$ respectively, the Snyder algebra is isomorphic to the algebra of the DeSitter groups O(3,2) and O(4,1). In $\mathfrak{R}_{\ell,\infty}$, the set $\mathcal{A}_S$ is the minimal algebraically closed set containing the space-time coordinate operators. It is therefore their representations that define the basic structure of non-commutative space-time. To have a manifold structure we need however a family of derivations to play the role of vector fields. Therefore we should also concern ourselves with the representations of the

P_μ and $\mathfrak{I}$ operators.

The operators in $\mathfrak{R}_{\ell,\infty}$ are not bounded operators in a general representation. However, once a representation of $\mathfrak{R}_{\ell,\infty}$ is obtained, there are standard ways to construct bounded operators from unbounded ones Γ in the universal enveloping algebra of $\mathfrak{R}_{\ell,\infty}$. For example,

$$\Gamma \to e^{i\alpha\Gamma} \tag{4.1a}$$

or

$$\Gamma \to \Gamma\,(\,1 + \Gamma^*\Gamma)^{-\frac{1}{2}} \tag{4.1b}$$

and to construct from the latter, by norm-completion, the associated C^*-algebra. Therefore, for simplicity, the discussion of the representations will be carried out at the $\mathfrak{R}_{\ell,\infty}$ – algebra level.

Consider the general representation of the operators given in Eqs.(2.5). They are considered as operating in a tensor product Hilbert space

$$\mathcal{H} = \left\{ f(\xi) \otimes |\underline{m}> : f(\xi) \in \mathcal{L}^2(M_5) \ , |\underline{m}> \in V_{SO(4,1)} \right\}$$

where $|\underline{m}> = |m_{51} z_{52} m_{41} m_{42} m_{31} m_{21}>$ is a vector in the representation space $V_{SO(4,1)}$ of SO(4,1) where $(\Sigma_{\mu\nu}, \Sigma_{\mu 4})$ acts. For definiteness I am considering $\epsilon = -1$ and for the representations of SO(4,1) the notation of Ref.[16] is used.

For simplicity take $D_{P_\mu} = D_{\xi^4} = 0$ in $V_{SO(4,1)}$. $\mathcal{L}^2(M_5)$ is a space of square-integrable functions in a 5-dimensional pseudo-Riemanian manifold of local metric $(1,-1,-1,-1,-1)$. Notice however that the geometry of the non-commutative space-time manifold is <u>not</u> obtained from the geometry of M_5 (by dimensional reduction, for example) but from the action on $\mathcal{H}$ of the operators (2.5). $\mathcal{L}^2(M_5)$ is not irreducible for $\mathfrak{R}_{\ell,\infty}$ but this is as in the commutative case where points are associated to irreducible representations but the whole manifold is not irreducible for the algebra.

The elementary geometric entities in the non-commutative case are the rays in $\mathcal{H}$. From the representation space point of view the local points in the commutative case are generalized states $\delta^n(x-a) \in \mathcal{S}'$, in a Gelfand triplet formulation $\mathcal{S}' \supset \mathcal{H} \supset \mathcal{S}$. In our non-commutative case the entity closest to a "local point" is also a generalized ray, as in Eq.(3.5), which however is only localized in one of the coordinates.

Given the representation space $\mathcal{H}$ and the explicit representation (2.5) for the basic algebra elements, one has the necessary elements to construct all mathematical structures in non-commutative space-time:

1. Let us denote by $\mathcal{A}$ the universal enveloping algebra of $\mathfrak{R}_{\ell,\infty}$ and by $\mathcal{C}_{\mathcal{A}}$ the associative normed algebra of bounded operators formed from the elements of $\mathcal{A}$ by the exponentiation (4.1a). The norm is obtained from the sup norm of the elements of $\mathcal{C}_{\mathcal{A}}$ as operators in $\mathcal{H}$, induced by the Hermitean representation (2.5).

$$||T|| = \mathrm{Sup}\left\{||T\eta|| : \eta \in \mathcal{H} \;\; ||\eta|| \leq 1\right\} \tag{4.3}$$

Non-commutative space-time is the pair $X = \{\mathcal{C}_{\mathcal{A}}, \mathcal{H}\}$. A *probability measure on X* is a *state*, that is a linear form $\chi : \mathcal{C}_{\mathcal{A}} \to C$ such that $\chi(T^*T) \geq 0 \;\; \forall T \in \mathcal{C}_{\mathcal{A}}$ and $\chi(1) = 1$. The simplest probability measures are the *pure states*

$$\chi_\eta = (\eta, T\eta) \qquad T \in \mathcal{C}_{\mathcal{A}} \qquad \eta \in \mathcal{H} \qquad ||\eta|| = 1 \tag{4.4}$$

2. *Vector bundle* is an important mathematical structure that corresponds to the physical notion of quantum fields. In the commutative case a quantum field is a section of a vector bundle and the space of sections is a representation space for the algebra of functions on the base manifold (more precisely a projective module). The notion is therefore carried over to the non-commutative case as follows. Let

$$\mathcal{E}_\Pi = \left\{\psi \in \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A} : \Pi\psi = \psi\right\} \tag{4.5}$$

The *non-commutative version of a section* (n-component quantum field) is an element of the n-fold tensor product of the enveloping algebra restricted by the projector relation $\Pi\psi = \psi$. $(\Pi - 1)\psi = 0$ is the equivalent to a field equation.

Example: *Scalar field*

$$\phi \in \mathcal{A} \;\; : \qquad [P_\mu,[P^\mu,\phi]] - m^2\, S(\mathfrak{I}^2\, \phi) = 0 \tag{4.6}$$

S is a symmetrization operation on the operators of $\mathfrak{R}_{\ell,\infty}$ defined for arbitrary elements of $\mathcal{A}$ through term by term symmetrization of a series expansion. For example

$$S(\mathfrak{I}^2\, x^\sigma x_\sigma) = \tfrac{1}{6}\,(\mathfrak{I}^2\, x^\sigma x_\sigma + \mathfrak{I}\, x^\sigma \mathfrak{I}\, x_\sigma + \mathfrak{I}\, x^\sigma x_\sigma \mathfrak{I} + x^\sigma\, \mathfrak{I}^2\, x_\sigma + x^\sigma \mathfrak{I}\, x_\sigma \mathfrak{I} + x^\sigma x_\sigma \mathfrak{I}^2) \tag{4.7a}$$

$$S(x^\mu x_\mu x^\sigma x_\sigma) = \tfrac{1}{3}\,(x^\mu x_\mu x^\sigma x_\sigma + x^\mu x^\sigma x_\sigma x_\mu + x^\mu x^\sigma x_\mu x_\sigma) \tag{4.7b}$$

The field equation (4.6) has the operator solution $\exp(i\; k^\mu x_\mu)$ with $k^\mu \in C$ and $k^\mu k_\mu = m^2$. The solution that corresponds to the invariant function solution of the commutative case is

$$\phi(x^\mu x_\mu) = \sum_{n=0} \frac{(-1)^n}{n+1} \left(\frac{m^2}{4}\right)^n \left(\frac{1}{n!}\right)^2 S(x^\mu x_\mu)^n \tag{4.8}$$

which may formally be written as

$$\phi(x^\mu x_\mu) = 2\, S\left(\frac{J_1(m\sqrt{x^\mu x_\mu})}{m\sqrt{x^\mu x_\mu}}\right) \tag{4.9}$$

Similarly *free spinor fields* are to be defined by

$$\psi \in \mathcal{A}^{\otimes 4} : \qquad \gamma^\mu[P_\mu,\psi] - m\, S(\mathfrak{I}\psi) = 0 \tag{4.10}$$

the *Dirac operator*

$$D = \gamma^\mu[P_\mu, \cdot\,] \tag{4.11}$$

being the usual one because $\mathfrak{R}_{\ell,\infty}$ preserves the Poincaré group structure.

3. To a derivation ∂ we can associate a connection in $\mathcal{E}_{II}$ by a linear map $\nabla:\mathcal{E}_{II}\to\mathcal{E}_{II}$ such that

$$\nabla(av) = \partial(a)v + a\nabla(v) \qquad a \in \mathcal{A} \qquad v \in \mathcal{E}_{II}$$

Of particular interest are the connections associated to the P_μ-derivations. I will discuss this in more detail in the context of the definition of ***gauge fields in the non-commutative space-time.*** To define parallel transport and local gauge transformations we need a notion of ***displacement*** in the non-commutative manifold X. The group generated by the P_μ operators induces an automorphism in $\mathcal{A}$ that generalizes the (commutative) notion of displacement. It acts on the x_μ operators as follows.

$$e^{i\alpha^\mu P_\mu} x_\nu\, e^{-i\alpha^\mu P_\mu} = x_\nu + \alpha_\nu \mathfrak{I} \tag{4.12}$$

Let now $\psi \in \mathcal{A}^{\otimes n}$ be a n-component field. By definition an ***infinitesimal displacement in X*** is

$$\psi \to \psi_\alpha = \psi + i\,\alpha^\mu P_\mu \psi \tag{4.13a}$$

with P_μ acting componentwise

$$P_\mu\ a_1 \otimes \cdots \otimes a_n = [P_\mu, a_1] \otimes \cdots \otimes [P_\mu, a_n] \tag{4.13b}$$

$a_i \in \mathcal{A}$.

P_μ being also an element of $\mathcal{A}$ it acts on a_i by commutation. Let now A_μ be an element of another algebra $\mathcal{G}$, also acting on $\mathcal{A}^{\otimes n}$, which defines ***parallel transport in X***.

$$\psi \to \psi + i\, \alpha^\mu A_\mu \psi \tag{4.14}$$

Similarly one defines ***gauge transformations*** by

$$\psi \to \psi' = \psi + i\, \Lambda\, \psi \tag{4.15}$$

with $\Lambda \in \mathcal{G}$.

in the non-commutative context a gauge transformation is said to be ***local*** if it transforms non-trivially under displacements, that is

$$(1 + i\, \Lambda_\alpha) = (1 + i\, \alpha^\mu P_\mu)\,(1 + i\Lambda)\,(1 - i\, \alpha^\mu P_\mu) \neq (1 + i\Lambda) \tag{4.16}$$

$(1 + i\, \Lambda_\alpha)$ acts on the displaced field ψ_α. Comparing

$$(1 + i\, \Lambda_\alpha)\,(1 + i\, \alpha^\mu A_\mu)\,\psi = (1 + i\, \alpha^\mu A'_\mu)\,(1 + i\Lambda)\,\psi$$

one obtains the transformation of the gauge connection A_μ under gauge transformations

$$A'_\mu = A_\mu + i[P_\mu , \Lambda] - i\,[A_\mu , \Lambda] \tag{4.17}$$

Although the space-time manifold X is non-commutative, the displacement group (4.12) is commutative. Therefore the gauge curvature may be obtained simply by comparing the parallel transport along direction μ followed by ν with ν followed by μ, without having to subtract torsion terms. The result is the ***gauge field***

$$F_{\mu\nu} = i\,[P_\mu , A_\nu] - i[P_\nu , A_\mu] + i[A_\mu , A_\nu] \tag{4.18}$$

as an operator acting on $\mathcal{A}^{\otimes n}$. This construction of gauge fields in non-commutative space-time is actually a particular instance of the pointless generalization of Yang-Mills theory discussed by Hong-Mo and Tsun[17].

4. With the above constructions and the definition of the Dirac operator (4.11) other mathematical structures may be defined on $X = (\mathcal{C}_{\mathcal{A}}, \mathcal{H})$ by the standard techniques of non-commutative geometry[12]. For example the ***distance*** between two states on $\mathcal{C}_{\mathcal{A}}$

$$d(\chi,\zeta) = \mathrm{Sup}\left\{ |\chi(a) - \zeta(a)| : a \in \mathcal{C}_{\mathcal{A}} ,\ \| [D,a] \| \le 1 \right\} \tag{4.19}$$

Concerning integration in X, for trace-class operators it may be defined as a trace. For operators that are not trace-class one may use the Dixmier trace.

References and footnotes

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