EE

Internal Report
DESY M-94-09
December 1994

The Influence of Damping in Nonlinear Fields on Electron Motion in Storage Rings

by

D. P. Barber, K. Heinemann, E. Karantzoulis *, H. Mais and G. Ripken

* Sincrotrone Trieste, Padriciano 99, I-34012 Trieste

The Influence of Damping in Nonlinear Fields on Electron Motion in Storage Rings.

D.P. Barber, K. Heinemann, E. Karantzoulis *, H. Mais and G. Ripken

December, 1994

Abstract

In the following report we use a canonical dispersion formalism to derive a Fokker-Planck equation for the motion of electrons in storage rings in the presence of nonlinear damping by quadrupole - sextupole and octupole - dipole magnets. The formalism is used to show how the nonlinear damping can modify the phase space distribution.

*Sincrotrone Trieste, Padriciano 99, I-34012 Trieste

Contents

1 Introduction

In an earlier paper [1] we studied the Fokker-Planck equation for electron motion in a storage ring in the presence of damping and stochastic excitation due to synchrotron radiation using the linearized fully coupled 6-dimensional formalism.

In this paper we re-examine a topic first treated by J. Jowett [2], namely the generalization of the linearized theory in [1] to the study of nonlinear radiation damping effects caused by superpositions of nonlinear fields. As pointed out by Jowett, these nonlinear damping effects can lead to a modification of the energy distribution in the beam. As in his treatments we concentrate on the effects of quadrupole-sextupole and octupole-dipole magnets. Since in practice one would try to insert such magnets in straight sections with minimum disturbance to the optics and geometry, one would combine them into strings of magnets with alternating signs. For simplicity, we therefore refer to these systems as "nonlinear wigglers".

The present treatment contains the linear approximation of radiation damping as a special case and can handle closed orbit distortion, pointlike rf cavities and transverse coupling due to skew quadrupoles and solenoids.

We start with the fully coupled canonical 6-dimensional description of the particle motion Then the stochastic equations of motion taking into account the synchrotron radiation are derived.

After introducing the closed orbit as a new reference trajectory, new transverse coordinates are defined by a canonical transformation involving the dispersion function. This preserves the symplectic structure of the equations of motion in the absence of quantum excitation or damping effects. In this formalism the coupling between betatron and synchrotron motion appears only if there is non-vanishing dispersion in the cavities.

Using the new coordinates, action-angle variables for the synchro-betatron motion are defined and their stochastic equations of motion established. These are then used to write the Fokker-Planck equation for the phase space density in terms of action-angle variables. In Appendix D we present a convenient way of using the 6 x 6 formalism to calculate how the damping constants become modified when the frequency of the cavity fields is changed.

The stationary solution of the Fokker-Planck equation is derived and is used to construct the beam emittance matrix and energy distribution (Chapter 9). Appendix E serves to prove that the stationary solution of the Fokker-Planck equation is unique.

In Appendix F it is shown that the surfaces of constant density in the 4-dimensional transverse phase space are given by 4-dimensional ellipsoids which in turn can be described by 4 generating orbit vectors. These latter may be combined into a 4-dimensional matrix which completely defines the transverse configuration of the bunch.

The formalism developed in this report is similar to the Fokker-Planck treatment of cavity noise described in Refs. [3, 4].

The equations derived will be incorporated into the computer program "SLICK" [5].

2 The Stochastic Equations of Motion

2.1 The Hamiltonian for Coupled Synchro-Betatron Oscillations

We begin with the derivation of the equations of motion. We will use the same variables as in Refs. [1, 6]:

$$x,\ z,\ \sigma = s - c\cdot t \text{ and } \eta = \Delta E/E_0$$

where x and z are the amplitudes of transverse motion (betatron oscillations), while σ and η describe the longitudinal (synchrotron) oscillation. The quantity σ is the longitudinal separation of a particle from the centre of the bunch and η describes the energy deviation of the particle.

Starting with the Lagrangian

$$\mathcal{L} = -m_0c^2\cdot\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\cdot(\dot{\vec{r}}\cdot\vec{A}) - e\cdot\phi$$

for the motion of a relativistic charged particle with the orbit-vector $\vec{r}(x,z,s)$:

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s)\ ; \qquad (2.1a)$$

$$\begin{cases} \frac{d}{ds}\,\vec{e}_x(s) = K_x(s)\cdot\vec{e}_s(s)\ ; \\ \frac{d}{ds}\,\vec{e}_z(s) = K_z(s)\cdot\vec{e}_s(s)\ ; \\ \frac{d}{ds}\,\vec{e}_s(s) = -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s)\ ; \end{cases} \qquad (2.1b)$$

$$\vec{e}_s(s) \equiv \frac{d}{ds}\,\vec{r}_0(s)\ ;$$

$$K_x(s)\cdot K_z(s) = 0\ ; \qquad (2.1c)$$

(piecewise no torsion)

in an electromagnetic field [1] and introducing the length s along the design orbit $\vec{r}_0(s)$ as the independent variable (instead of the time t), one can construct the Hamiltonian of the orbit motion by a succession of canonical transformations. Choosing a gauge with $\phi = 0$, one then obtains in the ultrarelativistic case with $v \approx c$ [6, 7]:

$$\begin{aligned} \mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) = {} & (1+p_\sigma) - (1+p_\sigma)\cdot[1+K_x\cdot x+K_z\cdot z]\times \\ & \left[1-\frac{(p_x-\frac{e}{E_0}A_x)^2}{(1+p_\sigma)^2}-\frac{(p_z-\frac{e}{E_0}A_z)^2}{(1+p_\sigma)^2}\right]^{1/2} \\ & -[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{E_0}A_s\ . \end{aligned} \qquad (2.2)$$

The corresponding canonical equations read as:

$$\frac{d}{ds}\,x = +\frac{\partial\mathcal{H}}{\partial p_x}\ ; \quad \frac{d}{ds}\,p_x = -\frac{\partial\mathcal{H}}{\partial x}\ ; \qquad (2.3a)$$

$$\frac{d}{ds}\,z = +\frac{\partial\mathcal{H}}{\partial p_z}\ ; \quad \frac{d}{ds}\,p_z = -\frac{\partial\mathcal{H}}{\partial z}\ ; \qquad (2.3b)$$

$$\frac{d}{ds}\,\sigma = +\frac{\partial\mathcal{H}}{\partial p_\sigma}\ ; \quad \frac{d}{ds}\,p_\sigma = -\frac{\partial\mathcal{H}}{\partial \sigma} \qquad (2.3c)$$

[1] For the physical situation we examine here, torsion needed not be included, i.e. we have a piecewise flat orbit and therefore the relation (2.1c). For a more general treatment including torsion see e.g. Refs. [8, 9].

with

$$p_\sigma \equiv \eta \; . \tag{2.4}$$

Here the variables p_x, p_z and p_σ are defined [6] by:

$$p_x = \frac{c}{E_0} m_0 \gamma \, v_x + \frac{c}{E_0} A_x \; ; \tag{2.5a}$$

$$p_z = \frac{c}{E_0} m_0 \gamma \, v_z + \frac{c}{E_0} A_z \; ; \tag{2.5b}$$

$$p_\sigma \equiv \eta = \frac{\Delta E}{E_0} \; . \tag{2.5c}$$

Since $\mathcal{H}$ contains the transverse coordinates x, p_x, z, p_z as well as the longitudinal coordinates σ, p_σ we are able to handle synchrotron oscillations (longitudinal motion) and betatron oscillations (transverse motion) simultaneously.

In order to utilize this Hamiltonian, the electric field $\vec{\mathcal{E}}$ and the magnetic field $\vec{\mathcal{B}}$ or the corresponding vector potential,

$$\vec{A} = \vec{A}(x, z, s), \tag{2.6}$$

for the cavities and for commonly occurring types of accelerator magnets must be given. Once $\vec{A}$ is known the fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ can be found using the relations:

$$\vec{\mathcal{E}} = -\text{grad}\, \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \; ; \tag{2.7a}$$

$$\vec{\mathcal{B}} = \text{curl}\, \vec{A} \; . \tag{2.7b}$$

Expressed in the variables x, z, s, σ, eqns. (2.7) become (with $\phi = 0$):

$$\vec{\mathcal{E}} = \frac{\partial}{\partial \sigma} \vec{A} \tag{2.8}$$

and

$$\mathcal{B}_x = \frac{1}{[1 + K_x \cdot x + K_z \cdot z]} \cdot \left\{ \frac{\partial}{\partial z} [(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] - \frac{\partial}{\partial s} A_z \right\} \; ; \tag{2.9a}$$

$$\mathcal{B}_z = \frac{1}{[1 + K_x \cdot x + K_z \cdot z]} \cdot \left\{ \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x} [(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \right\} \; ; \tag{2.9b}$$

$$\mathcal{B}_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x \; . \tag{2.9c}$$

We assume that the ring contains bending magnets, quadrupoles, skew quadrupoles, solenoids, sextupoles, octupoles, cavities and dipoles as well as combined function magnets, quadrupole-sextupole magnets and octupole-dipole magnets [2]. Then the vector potential $\vec{A}$ can be written

[2] Skew quadrupole-sextupole magnets and skew octupole-dipole magnets are not treated.

as (see Appendix A):

$$
\begin{aligned}
\frac{e}{E_0} A_s &= -\frac{1}{2}\cdot[1 + K_x\cdot x + K_z\cdot z] + \frac{1}{2} g\cdot(z^2 - x^2) + N\cdot xz \\
&\quad -\lambda\cdot\frac{1}{6}\left(x^3 - 3xz^2\right) + \mu\cdot\frac{1}{24}\left(z^4 - 6\,x^2z^2 + x^4\right) \\
&\quad -\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right] \\
&\quad +\frac{e}{E_0}\cdot[\Delta\mathcal{B}_x\cdot z - \Delta\mathcal{B}_z\cdot x]\ ; \qquad (2.10a)
\end{aligned}
$$

$$
\frac{e}{E_0} A_x = -H\cdot z\ ; \quad \frac{e}{E_0} A_z = +H\cdot x \qquad (2.10b)
$$

(h=harmonic number) with the following abbreviations:

$$
g = \frac{e}{E_0}\cdot\left(\frac{\partial\mathcal{B}_z}{\partial x}\right)_{x=z=0}\ ; \quad \text{(strength of a quadrupole)}\ ; \qquad (2.11a)
$$

$$
N = \frac{1}{2}\cdot\frac{e}{E_0}\cdot\left(\frac{\partial\mathcal{B}_x}{\partial x} - \frac{\partial\mathcal{B}_z}{\partial z}\right)_{x=z=0}\ ; \quad \text{(strength of a skew quadrupole)}\ ; \qquad (2.11b)
$$

$$
H = \frac{1}{2}\cdot\frac{e}{E_0}\cdot\mathcal{B}_s(0,0,s)\ ; \quad \text{(strength of a solenoid)}\ ; \qquad (2.11c)
$$

$$
\lambda = \frac{e}{E_0}\cdot\left(\frac{\partial^2\mathcal{B}_z}{\partial x^2}\right)_{x=z=0}\ ; \quad \text{(strength of a sextupole)}\ ; \qquad (2.11d)
$$

$$
\mu = \frac{e}{E_0}\cdot\left(\frac{\partial^3\mathcal{B}_x}{\partial z^3}\right)_{x=z=0}\ ; \quad \text{(strength of an octupole)}\ ; \qquad (2.11e)
$$

$$
K_x = +\frac{e}{E_0}\cdot\mathcal{B}_z(0,0,s)\ ; \quad K_z = -\frac{e}{E_0}\cdot\mathcal{B}_x(0,0,s)\ ; \qquad (2.11f)
$$

(design curvature in bending magnets) .

A combined function magnet is a superposition of fields of type (2.11a) and (2.11f):

$$
\left[G_{CF}^{(x)}\right]^2 + \left[G_{CF}^{(z)}\right]^2 \neq 0
$$

with

$$
\left\{
\begin{aligned}
G_{CF}^{(x)}(s) &= g(s)\cdot K_x(s)\ ; \\
G_{CF}^{(z)}(s) &= g(s)\cdot K_z(s)\ ;
\end{aligned}
\right. \qquad (2.11g)
$$

$$
K_x(s)\cdot K_z(s) = 0\ .
$$

The quadrupole - sextupole field is a superposition of fields of type (2.11a) and (2.11d):

$$
G_{QS}(s) = g(s)\cdot\lambda(s) \neq 0\ . \qquad (2.11h)
$$

The dipole - octupole field is a superposition of a dipole field $\Delta\mathcal{B}_z$ and a field of type (2.11e):

$$
G_{DO}(s) = \frac{e}{E_0}\Delta\mathcal{B}_z(s)\cdot\mu(s) \neq 0\ . \qquad (2.11i)
$$

Then using (2.10) and (2.11a - f) the Hamiltonian (2.2) takes the form [3] :

$$
\begin{aligned}
\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) \;=\;& (1+p_\sigma) - (1+p_\sigma)\cdot[1+K_x\cdot x + K_z\cdot z]\times \\
& \left[1 - \frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{(1+p_\sigma)^2}\right]^{1/2} \\
& + \frac{1}{2}\cdot[1+K_x\cdot x + K_z\cdot z]^2 - \frac{1}{2}\cdot g\cdot(z^2 - x^2) - N\cdot xz \\
& + \lambda\cdot\frac{1}{6}\cdot\left(x^3 - 3\,xz^2\right) - \mu\cdot\frac{1}{24}\left(z^4 - 6\,x^2z^2 + x^4\right) \\
& + \frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right] \\
& - \frac{e}{E_0}\cdot[\Delta\mathcal{B}_x\cdot z - \Delta\mathcal{B}_z\cdot x]\ .
\end{aligned}
\tag{2.12}
$$

2.2 The Radiation Power

In the Hamiltonian (2.12) radiation processes have been neglected. In order to include into the equations of motion the additional forces induced by the photon emission, we adopt (as in Ref. [1]) an "ansatz" in which the radiation reaction force $\vec{R}^D$ is separated into two parts:

$$\vec{R} \;=\; \vec{R}^D + \delta\vec{R}\ , \tag{2.13}$$

a continuous part $\vec{R}^D$ describing the smoothed radiation process and a discontinuous part $\delta\vec{R}$ describing the quantum fluctuations. The explicit expression for $\vec{R}^D$ is given by [10, 11]:

$$\vec{R}^D \;=\; -\frac{2}{3}\cdot\frac{e^2}{c^5}\gamma^4\cdot\dot{\vec{r}}\cdot\left[(\ddot{\vec{r}})^2 + \frac{\gamma^2}{c^2}(\dot{\vec{r}}\cdot\ddot{\vec{r}})^2\right] \tag{2.14}$$

and we take $\delta\vec{R}$ to be a white noise process (see below, eqn. (2.22b) and the definition of $\xi(s)$ in (2.18d)).

We also introduce the radiation power:

$$P \equiv \vec{R}\cdot\dot{\vec{r}} \tag{2.15}$$

of a (ultrarelativistic) particle in a purely magnetic field.

For the case where

$$\dot{\vec{r}}\cdot\vec{\mathcal{B}} = 0 \;\Longrightarrow\; \dot{\vec{r}}\perp\vec{\mathcal{B}} \tag{2.16}$$

(a good approximation in bending magnets and the wiggler field under consideration) one then may set [2]:

---

[3] Within the quadrupole-sextupole magnet one has $g \neq 0$; $\lambda \neq 0$, within a octupole-dipole magnet $\Delta\mathcal{B}_x \neq 0$; $\mu \neq 0$ and within a combined function magnet $g \neq 0$; $K_x^2 + K_z^2 \neq 0$.

$$P(s) = E_0^2 \cdot (1+\eta)^2 \cdot \left\{ c_1 \cdot b^2 + \sqrt{c_2} \cdot |b^{3/2}| \cdot \xi(s) \right\} \tag{2.17}$$

with

$$\vec{b}(x,z,s) = \frac{e}{E_0} \cdot \vec{B}(x,z,s) \; ; \tag{2.18a}$$

$$c_1 = \frac{2r_e \cdot p_0^2}{3 \cdot (m_0 c)^3} \quad \text{with} \quad p_0 = \frac{E_0}{c} \; ; \tag{2.18b}$$

$$c_2 = \frac{55 r_e \hbar p_0^3}{24\sqrt{3} \cdot (m_0 c)^6} \; ; \tag{2.18c}$$

$$< \xi(s) \cdot \xi(s') > = \delta(s - s') \; ; \quad < \xi(s) > = 0 \, . \tag{2.18d}$$

The quantity $\xi(s)$ in (2.18d) describing a stochastic stationary process may be visualized as a random sequence of small positive and negative pulses.

Writing

$$\begin{aligned} C_1 \equiv \frac{E_0}{c} \cdot c_1 &= \frac{2r_e \cdot E_0^3}{3c^3 \cdot (m_0 c)^3} \quad \text{with} \quad r_e = \frac{e^2}{m_0 c^2} \\ &= \frac{2}{3} e^2 \cdot \frac{E_0^3}{(m_0 c)^4} \\ &= \frac{2}{3} e^2 \cdot \frac{\gamma_0^4}{E_0} \; ; \end{aligned} \tag{2.19}$$

$$\begin{aligned} C_2 \equiv \frac{E_0^2}{c^2} \cdot c_2 &= \frac{55 r_e \hbar \cdot c^6}{24\sqrt{3} \cdot (m_0 c)^6} \cdot \frac{E_0^5}{c^5} \\ &= \frac{55 r_e \hbar \cdot \gamma_0^5}{24\sqrt{3} \cdot (m_0 c)} \end{aligned} \tag{2.20}$$

we obtain:

$$P(s) = P^D(s) + \delta P(s) \tag{2.21}$$

with

$$\frac{P^D(s)}{E_0 \cdot c} = (1+\eta)^2 \cdot C_1 \cdot b^2 \; ; \tag{2.22a}$$

$$\frac{\delta P(s)}{E_0 \cdot c} = (1+\eta)^2 \cdot \sqrt{C_2} \cdot |b^{3/2}| \cdot \xi(s) \, . \tag{2.22b}$$

Using the equations for the magnetic fields of the different kinds of lenses (which may be calculated by eqns. (2.9) and (2.10) [1, 6]; see also Appendix A) the term b^2 appearing in (2.22a, b) can be written:

a) Dipole:

$$b^2 = \left(\frac{e}{E_0}\right)^2 \cdot \left[(\Delta B_x)^2 + (\Delta B_z)^2 \right] \; ; \tag{2.23a}$$

b) Bending magnet:

$$b^2 = b_x^2 + b_z^2 = K_x^2 + K_z^2 \; ; \qquad (2.23b)$$

c) Quadrupole:

$$b^2 = b_x^2 + b_z^2 = g^2 \cdot (x^2 + z^2) \; ; \qquad (2.23c)$$

d) Skew quadrupole:

$$b^2 = b_x^2 + b_z^2 = N^2 \cdot (x^2 + z^2) \; ; \qquad (2.23d)$$

e) Sextupole:

$$b^2 = b_x^2 + b_z^2 = \lambda^2 \cdot \left[x^2 z^2 + \frac{1}{4} (x^2 - z^2)^2 \right] \; ; \qquad (2.23e)$$

f) Octupole:

$$b^2 = b_x^2 + b_z^2 = \mu^2 \cdot \frac{1}{36} \left[(z^3 - 3\,x^2 z)^2 + (3\,x z^2 - x^3)^2 \right] \; ; \qquad (2.23e)$$

g) Combined function magnet (horizontal bend):

$$\begin{aligned} b^2 = b_x^2 + b_z^2 &= [g \cdot z - K_z]^2 + [g \cdot x + K_x]^2 \\ &= g^2 \cdot (x^2 + z^2) + [K_x^2 + K_z^2] + 2\,g \cdot [K_x \cdot x - K_z \cdot z] \\ &= g^2 \cdot (x^2 + z^2) + [K_x^2 + K_z^2] + 2\,G_{CF}^{(x)} \cdot x - 2\,G_{CF}^{(z)} \cdot z \,. \end{aligned} \qquad (2.23f)$$

h) Quadrupole - sextupole magnet:

$$\begin{aligned} b^2 = b_x^2 + b_z^2 &= [g \cdot z + \lambda \cdot xz]^2 + \left[g \cdot x + \frac{1}{2} \lambda \cdot (x^2 - z^2) \right]^2 \\ &= g^2 \cdot (x^2 + z^2) + \lambda^2 \cdot \left[x^2 z^2 + \frac{1}{4} (x^2 - z^2)^2 \right] \\ &\quad + G_{QS} \cdot \left[x^3 + x z^2 \right] \,. \end{aligned} \qquad (2.23f)$$

i) Dipole - octupole magnet (horizontal bend):

$$\begin{aligned} b^2 = b_x^2 + b_z^2 &= \left[\mu \cdot \frac{1}{6} (z^3 - 3\,x^2 z) \right]^2 + \left[\frac{e}{E_0} \cdot \Delta B_z(s) + \mu \cdot \frac{1}{6} (3\,x z^2 - x^3) \right]^2 \\ &= \left(\frac{e}{E_0} \right)^2 \cdot (\Delta B_z)^2 + \mu^2 \cdot \frac{1}{36} \left[(z^3 - 3\,x^2 z)^2 + (3\,x z^2 - x^3)^2 \right] \\ &\quad + G_{DO} \cdot \frac{1}{3} \left[3\,x z^2 - x^3 \right] \,. \end{aligned} \qquad (2.23f)$$

Note that if there are no electric fields and (2.16) is valid, (2.22a) can be derived directly from (2.14).

The photons are emitted in the direction of the momentum of the particle with an opening angle of order

$$\Theta \approx \frac{m_0 c^2}{E} \equiv \frac{1}{\gamma} \qquad (2.24)$$

so that at high energy we are justified in taking the radiation reaction force to be collinear with $\dot{\vec{r}}$ [12].

For a longitudinal field:

$$\dot{\vec{r}} \parallel \vec{B} \tag{2.25a}$$

one has

$$P(s) = 0 \tag{2.25b}$$

(see eqn. (2.14)).

2.3 Motion under the Influence of Radiation Forces

The direction of the photon emission is given approximately by (see eqn. (2.24)):

$$\vec{\tau} \equiv \frac{d\vec{r}}{dl} = \frac{dx}{dl} \cdot \vec{e}_x + \frac{dz}{dl} \cdot \vec{e}_z + [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{ds}{dl} \cdot \vec{e}_s \tag{2.26}$$

with (see eqn. (2.1))

$$dl = | d\vec{r} | \ . \tag{2.27}$$

Thus, taking into account conservation of energy and momentum during to the radiation process, the quantities

$$p_x' \equiv \frac{d}{ds} p_x \ ; \ p_z' \equiv \frac{d}{ds} p_z \ ; \ p_\sigma' \equiv \frac{d}{ds} p_\sigma = \eta'$$

in eqn. (2.3) must be replaced using the relationships [4] :

$$\Delta\eta = \eta'(s) \cdot \Delta s \longrightarrow \Delta\eta - \frac{1}{E_0} \cdot \Delta t \cdot P(s) = \Delta s \cdot \left[\eta'(s) - \frac{t'(s)}{E_0} \cdot P(s)\right] ; \tag{2.28a}$$

$$\begin{aligned} \Delta p_x = p_x'(s) \cdot \Delta s \longrightarrow \Delta p_x - \frac{c}{E_0} \cdot \frac{P(s) \cdot \Delta t}{c} \cdot \tau_x & \\ = \Delta p_x - \frac{P(s) \cdot \Delta t}{E_0} \cdot \frac{dx}{dl} & \\ = \Delta p_x - \frac{P(s) \cdot \Delta t}{E_0} \cdot \frac{x'(s) \cdot \Delta s}{c \cdot \Delta t} & \quad \text{with} \quad dl = c \cdot dt \\ = \Delta s \cdot \left[p_x'(s) - \frac{P(s)}{E_0 \cdot c} \cdot x'(s)\right] ; & \end{aligned} \tag{2.28b}$$

$$\begin{aligned} \Delta p_z = p_z'(s) \cdot \Delta s \longrightarrow \Delta p_z - \frac{c}{E_0} \cdot \frac{P(s) \cdot \Delta t}{c} \cdot \tau_z & \\ = \Delta s \cdot \left[p_z'(s) - \frac{P(s)}{E_0 \cdot c} \cdot z'(s)\right] & \end{aligned} \tag{2.28c}$$

[4] In Ref. [1] we used other transverse variables at this point.

(see eqn. (2.5a, b)).

Using the relation:

$$\sigma(s) \;=\; s - c\cdot t(s) \;\Longrightarrow\; \sigma'(s) \;=\; 1 - c\cdot t'(s) \tag{2.29}$$

the term $t'(s)$ appearing in (2.28a) can be written as:

$$c\cdot t'(s) \;=\; 1-\sigma'(s)\,. \tag{2.30}$$

The stochastic equations of motion then read as:

$$\frac{d}{ds}\,x \;=\; +\frac{\partial\mathcal{H}}{\partial p_x}\,; \quad \frac{d}{ds}\,p_x = -\frac{\partial\mathcal{H}}{\partial x} - \frac{P(s)}{E_0\cdot c}\cdot\frac{\partial\mathcal{H}}{\partial p_x}\,; \tag{2.31a}$$

$$\frac{d}{ds}\,z \;=\; +\frac{\partial\mathcal{H}}{\partial p_z}\,; \quad \frac{d}{ds}\,p_z = -\frac{\partial\mathcal{H}}{\partial z} - \frac{P(s)}{E_0\cdot c}\cdot\frac{\partial\mathcal{H}}{\partial p_z}\,; \tag{2.31b}$$

$$\frac{d}{ds}\,\sigma \;=\; +\frac{\partial\mathcal{H}}{\partial p_\sigma}\,; \quad \frac{d}{ds}\,p_\sigma = -\frac{\partial\mathcal{H}}{\partial \sigma} - \frac{P(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right]\,. \tag{2.31c}$$

Instead of p_x and p_z we now introduce the variables [5] :

$$\hat{p}_x \;=\; x' - H\cdot z\,; \tag{2.32a}$$

$$\hat{p}_z \;=\; z' + H\cdot x \tag{2.32b}$$

and from the Hamiltonian in (2.12) using (2.31) we obtain:

$$\begin{aligned}
\hat{p}_x + H\cdot z \;\equiv\; x' \;&=\; \frac{\partial\mathcal{H}}{\partial p_x}\\
&=\; [1+K_x\cdot x+K_z\cdot z]\\
&\quad\times\left[(1+p_\sigma)^2-[p_x+H\cdot z]^2-[p_z-H\cdot x]^2\right]^{-1/2}\cdot[p_x+H\cdot z]\,;
\end{aligned} \tag{2.33a}$$

$$\begin{aligned}
\hat{p}_z - H\cdot x \;\equiv\; z' \;&=\; \frac{\partial\mathcal{H}}{\partial p_z}\\
&=\; [1+K_x\cdot x+K_z\cdot z]\\
&\quad\times\left[(1+p_\sigma)^2-[p_x-H\cdot z]^2-[p_z+H\cdot x]^2\right]^{-1/2}\cdot[p_z-H\cdot x]
\end{aligned} \tag{2.33b}$$

($\hat{p}_x$ and $\hat{p}_z$ are the canonical momenta of the linear theory obtained from the Hamiltonian $\mathcal{H}$ in eqn. (2.12) when terms of third and higher order are neglected; see below, eqn. (2.43a)).

Then using (2.31):

$$\begin{aligned}
\hat{p}_x' \;&=\; \frac{d}{ds}\frac{\partial\mathcal{H}}{\partial p_x} - H'\cdot z - H\cdot z'\\
&=\; \frac{\partial^2\mathcal{H}}{\partial x\partial p_x}\cdot x' + \frac{\partial^2\mathcal{H}}{\partial p_x^2}\cdot p_x' + \frac{\partial^2\mathcal{H}}{\partial z\partial p_x}\cdot z' + \frac{\partial^2\mathcal{H}}{\partial p_z\partial p_x}\cdot p_z'
\end{aligned}$$

[5] Using these new variables we get the linear theory of damping constants as a special case by neglecting nonlinear damping terms (see chapter 8).

$$+\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_x}\cdot\sigma'+\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_x}\cdot p_\sigma'+\frac{\partial^2\mathcal{H}}{\partial s\partial p_x}-H'\cdot z-H\cdot z'$$

$$\begin{aligned}
&= \frac{\partial^2\mathcal{H}}{\partial x\partial p_x}\cdot[\hat{p}_x+H\cdot z]-\frac{\partial^2\mathcal{H}}{\partial p_x^2}\cdot\frac{\partial\mathcal{H}}{\partial x}+\left[\frac{\partial^2\mathcal{H}}{\partial z\partial p_x}-H\right]\cdot[\hat{p}_z-H\cdot x]\\
&\quad -\frac{\partial^2\mathcal{H}}{\partial p_z\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial z}+\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial p_\sigma}-\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial\sigma}\\
&\quad +\frac{\partial^2\mathcal{H}}{\partial s\partial p_x}-H'\cdot z+r_x\ ; \qquad (2.34a)
\end{aligned}$$

$$\begin{aligned}
\hat{p}_z' &= \frac{d}{ds}\frac{\partial\mathcal{H}}{\partial p_z}+H'\cdot x+H\cdot x'\\
&= \frac{\partial^2\mathcal{H}}{\partial x\partial p_z}\cdot x'+\frac{\partial^2\mathcal{H}}{\partial p_x\partial p_z}\cdot p_x'+\frac{\partial^2\mathcal{H}}{\partial z\partial p_z}\cdot z'+\frac{\partial^2\mathcal{H}}{\partial p_z^2}\cdot p_z'\\
&\quad +\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_z}\cdot\sigma'+\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_z}\cdot p_\sigma'+\frac{\partial^2\mathcal{H}}{\partial s\partial p_z}+H'\cdot x+H\cdot x'\\
&= \left[\frac{\partial^2\mathcal{H}}{\partial x\partial p_z}+H\right]\cdot[\hat{p}_x+H\cdot z]-\frac{\partial^2\mathcal{H}}{\partial p_x\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial x}+\frac{\partial^2\mathcal{H}}{\partial z\partial p_x}\cdot[\hat{p}_z-H\cdot x]\\
&\quad -\frac{\partial^2\mathcal{H}}{\partial p_z^2}\cdot\frac{\partial\mathcal{H}}{\partial z}+\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial p_\sigma}-\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial\sigma}\\
&\quad +\frac{\partial^2\mathcal{H}}{\partial s\partial p_z}+H'\cdot x+r_z \qquad (2.34b)
\end{aligned}$$

where we have gathered the radiation terms (appearing in p_x', and p_z') in r_x and r_z:

$$r_x = -\frac{P(s)}{E_0\cdot c}\left\{\frac{\partial^2\mathcal{H}}{\partial p_x^2}\cdot\frac{\partial\mathcal{H}}{\partial p_x}+\frac{\partial^2\mathcal{H}}{\partial p_z\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial p_z}+\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_x}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right]\right\}\ ; \qquad (2.35a)$$

$$r_z = -\frac{P(s)}{E_0\cdot c}\left\{\frac{\partial^2\mathcal{H}}{\partial p_z^2}\cdot\frac{\partial\mathcal{H}}{\partial p_z}+\frac{\partial^2\mathcal{H}}{\partial p_x\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial p_x}+\frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_z}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right]\right\}\ . \qquad (2.35b)$$

Using eqns. (2.33a,b) and the relations:

$$\begin{aligned}
\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] &= [1+K_x\cdot x+K_z\cdot z]\cdot(1+p_\sigma)\\
&\quad\times\left[(1+p_\sigma)^2-[p_x+H\cdot z]^2-[p_z-H\cdot x]^2\right]^{-\frac{1}{2}}\ ; \qquad (2.36a)\\
\frac{\partial^2\mathcal{H}}{\partial p_x^2} &= [1+K_x\cdot x+K_z\cdot z]\cdot\left[(1+p_\sigma)^2-p_z^2\right]\\
&\quad\times\left[(1+p_\sigma)^2-[p_x+H\cdot z]^2-[p_z-H\cdot x]^2\right]^{-\frac{3}{2}}\ ; \qquad (2.36b)\\
\frac{\partial^2\mathcal{H}}{\partial p_z\partial p_x} &= [1+K_x\cdot x+K_z\cdot z]\cdot p_xp_z\\
&\quad\times\left[(1+p_\sigma)^2-[p_x+H\cdot z]^2-[p_z-H\cdot x]^2\right]^{-\frac{3}{2}}\ ; \qquad (2.36c)\\
\frac{\partial^2\mathcal{H}}{\partial p_x\partial p_\sigma} &= -[1+K_x\cdot x+K_z\cdot z]\cdot(1+p_\sigma)\cdot p_x\\
&\quad\times\left[(1+p_\sigma)^2-[p_x+H\cdot z]^2-[p_z-H\cdot x]^2\right]^{-\frac{3}{2}} \qquad (2.36d)
\end{aligned}$$

resulting from (2.2) we have [6] :

$$r_x = r_z = 0 \; . \tag{2.37}$$

Thus eqns. (2.31) can be replaced by:

$$\begin{aligned}
\frac{d}{ds}\, x &= \hat{p}_x + H\cdot z \; ; \\
\frac{d}{ds}\, \hat{p}_x &= \frac{\partial^2\mathcal{H}}{\partial x\partial p_x}\cdot[\hat{p}_x + H\cdot z] - \frac{\partial^2\mathcal{H}}{\partial p_x^2}\cdot\frac{\partial\mathcal{H}}{\partial x} + \left[\frac{\partial^2\mathcal{H}}{\partial z\partial p_x} - H\right]\cdot[\hat{p}_z - H\cdot x] \\
&\quad - \frac{\partial^2\mathcal{H}}{\partial p_z\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial z} + \frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial p_\sigma} - \frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial\sigma} + \frac{\partial^2\mathcal{H}}{\partial s\partial p_x} - H'\cdot z \; ; \qquad (2.38a) \\
\frac{d}{ds}\, z &= \hat{p}_z - H\cdot x \; ; \\
\frac{d}{ds}\, \hat{p}_z &= \left[\frac{\partial^2\mathcal{H}}{\partial x\partial p_z} + H\right]\cdot[\hat{p}_x + H\cdot z] - \frac{\partial^2\mathcal{H}}{\partial p_x\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial x} + \frac{\partial^2\mathcal{H}}{\partial z\partial p_z}\cdot[\hat{p}_z - H\cdot x] \\
&\quad - \frac{\partial^2\mathcal{H}}{\partial p_z^2}\cdot\frac{\partial\mathcal{H}}{\partial z} + \frac{\partial^2\mathcal{H}}{\partial\sigma\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial p_\sigma} - \frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial\sigma} + \frac{\partial^2\mathcal{H}}{\partial s\partial p_z} + H'\cdot x \; ; \qquad (2.38b) \\
\frac{d}{ds}\, \sigma &= +\frac{\partial\mathcal{H}}{\partial p_\sigma} \; ; \\
\frac{d}{ds}\, p_\sigma &= -\frac{\partial\mathcal{H}}{\partial\sigma} - \frac{P(s)}{E_0\cdot c}\cdot\left[1 - \frac{\partial\mathcal{H}}{\partial p_\sigma}\right] \; . \qquad (2.38c)
\end{aligned}$$

Note that eqns. (2.38) are exact corresponding to the exact Hamiltonian (2.2) or (2.12).

The next step would be to express the r.h.s. of (2.38a, b, c) in terms of $\hat{p}_x$, $\hat{p}_z$ instead of p_x, p_z. The variables p_x, p_z can in principle be eliminated exactly using the equations:

$$\begin{aligned}
p_x + H\cdot z &= (1+p_\sigma)\cdot[\hat{p}_x + H\cdot z] \\
&\quad \times \left[(1+K_x\cdot x + K_z\cdot z)^2 + [\hat{p}_x + H\cdot z]^2 + [\hat{p}_z - H\cdot x]^2\right]^{-\frac{1}{2}} \; ; \\
&\qquad\qquad (2.38d) \\
p_z - H\cdot x &= (1+p_\sigma)\cdot[\hat{p}_z - H\cdot x] \\
&\quad \times \left[(1+K_x\cdot x + K_z\cdot z)^2 + [\hat{p}_x + H\cdot z]^2 + [\hat{p}_z - H\cdot x]^2\right]^{-\frac{1}{2}}
\end{aligned}$$

which may be obtained from (2.33a,b) using the relation:

$$\begin{aligned}
& [1+K_x\cdot x + K_z\cdot z]^2 + [\hat{p}_x + H\cdot z]^2 + [\hat{p}_z - H\cdot x]^2 \\
= \; & [1+K_x\cdot x + K_z\cdot z]^2\cdot\frac{(1+p_\sigma)^2}{(1+p_\sigma)^2 - [p_x + H\cdot z]^2 - [p_z - H\cdot x]^2} \; ; \\
\Longrightarrow \quad & [1+K_x\cdot x + K_z\cdot z]\cdot\left\{(1+p_\sigma)^2 - [p_x + H\cdot z]^2 - [p_z - H\cdot x]^2\right\}^{-1/2} \\
= \; & \frac{\{(1+K_x\cdot x + K_z\cdot z)^2 + [\hat{p}_x + H\cdot z]^2 + [\hat{p}_z - H\cdot x]^2\}^{1/2}}{(1+p_\sigma)} \; .
\end{aligned}$$

[6] This result is due to the fact that the photons are emitted in the direction of the tangent to the trajectory so that x' and z' remain unchanged during a radiation process.

which also comes from (2.33a, b).

2.4 Radiation Power and Series Expansion of the Hamiltonian

Now, since

$$\begin{aligned} |p_x + H \cdot z| &\ll 1 \, ; \\ |p_z - H \cdot x| &\ll 1 \end{aligned}$$

the square root

$$\left[1 - \frac{[p_x + H \cdot z]^2 + [p_z - 2H \cdot x]^2}{(1+p_\sigma)^2}\right]^{1/2}$$

in the Hamiltonian (2.12) may be expanded in a series:

$$\left[1 - \frac{[p_x + H \cdot z]^2 + [p_z - 2H \cdot x]^2}{(1+p_\sigma)^2}\right]^{1/2} = \\ 1 - \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+p_\sigma)^2} + \cdots$$

so that in practice the particle motion can be conveniently calculated to various orders of approximation.

As we will demonstrate below the special dissipative effects described in this paper originate in third order terms appearing in the equations of motion in the wigglers. Thus it is only necessary to expand the Hamiltonian up to fourth order. Furthermore, within this framework and in order to simplify the presentation, we only expand the Hamiltonian up to quadratic terms in p_x and p_z [7]. Then we obtain from eqn. (2.12):

$$\begin{aligned} \mathcal{H} &= -\sigma \cdot \frac{eV}{E_0} \sin\varphi - \frac{e}{E_0} \cdot [\Delta\mathcal{B}_x \cdot z - \Delta\mathcal{B}_z \cdot x] \\ &\quad + \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+p_\sigma)} \\ &\quad + \mathcal{H}_0 + \mathcal{H}_1 \end{aligned} \tag{2.39}$$

[7] The additional third and fourth order terms

$$+\frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+p_\sigma)} \cdot [K_x \cdot x + K_z \cdot z]$$

and

$$-\frac{1}{8} \cdot (1+p_\sigma) \cdot \left\{\frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+p_\sigma)^2}\right\}^2$$

which would appear in the Hamiltonian (2.39) after expanding (2.12) and the relations for $[p_x + H \cdot z]$ and $[p_z - H \cdot x]$ in eqn. (2.38d) as far as needed to get all third order terms in x, z, $\hat{p}_x$, $\hat{p}_z$ in the equations of motion, contribute second and third order terms on the r.h.s of of eqns. (2.44a-e). Of these, the second order terms will vanish by phase averaging (in Chapter 7). The third order terms are oscillatory and can be neglected. Note, that eqn. (2.44f) which includes the radiation damping term is unaffected by these higher order terms of the Hamiltonian. See also Footnote # 8.

with

$$\begin{aligned}
\mathcal{H}_0(x,p_x,z,p_z,\sigma,p_\sigma;s) &= \frac{1}{2}\cdot\left\{[K_x^2+g]\cdot x^2+[K_z^2-g]\cdot z^2\right\}-N\cdot xz \\
&\quad -\frac{1}{2}\sigma^2\cdot\frac{eV}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi-[K_x\cdot x+K_z\cdot z]\cdot p_\sigma\ ; \qquad (2.39a)
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_1(x,p_x,z,p_z;\sigma,p_\sigma;s) &= \mathcal{H}_{11}+\mathcal{H}_{12}\ ; \qquad (2.39b) \\
&\quad \mathcal{H}_{11} = +\lambda\cdot\frac{1}{6}(x^3-3xz^2)\ ; \\
&\quad \mathcal{H}_{12} = -\mu\cdot\frac{1}{24}\left(z^4-6\,x^2z^2+x^4\right)\ .
\end{aligned}$$

(constant terms in the Hamiltonian, which have no influence on the motion have been dropped).

By applying the relations:

$$\hat{p}_x+H\cdot z \equiv x' = \frac{\partial\mathcal{H}}{\partial p_x}\ ; \qquad (2.40a)$$

$$\hat{p}_z-H\cdot x \equiv z' = \frac{\partial\mathcal{H}}{\partial p_z} \qquad (2.40b)$$

(see eqns.(2.31) and (2.32)) to the Hamiltonian in the approximate [8] form (2.39) we may write:

$$[\hat{p}_x+H\cdot z] = \frac{p_x+H\cdot z}{(1+p_\sigma)}\ ; \qquad (2.41a)$$

$$[\hat{p}_z-H\cdot x] = \frac{p_z-H\cdot x}{(1+p_\sigma)}\ . \qquad (2.41b)$$

As a device to separate the r.h.s. of eqns. (2.38a, b, c) into symplectic and nonsymplectic components, we now define an artificial Hamiltonian $\hat{\mathcal{H}}$ in the variables x, $\hat{p}_x$, z, $\hat{p}_z$, σ, p_σ by

$$\hat{\mathcal{H}} = \frac{1}{2}\cdot\left\{[\hat{p}_x+H\cdot z]^2+[\hat{p}_z-H\cdot x]^2\right\}+\hat{\mathcal{H}}_0+\hat{\mathcal{H}}_1 \qquad (2.42)$$

with

$$\hat{\mathcal{H}}_0(x,\hat{p}_x,z,\hat{p}_z,\sigma,p_\sigma;s) \equiv \mathcal{H}_0\ ; \qquad (2.43a)$$

$$\begin{aligned}
\hat{\mathcal{H}}_1(x,\hat{p}_x,z,\hat{p}_z;\sigma,p_\sigma;s) &= \hat{\mathcal{H}}_{11}+\hat{\mathcal{H}}_{12}\ ; \qquad (2.43b) \\
&\quad \hat{\mathcal{H}}_{11} \equiv \mathcal{H}_{11}\ ; \\
&\quad \hat{\mathcal{H}}_{12} \equiv \mathcal{H}_{12}\ .
\end{aligned}$$

Then, using eqns. (2.40a, b) together with the relations

$$\frac{\partial\mathcal{H}}{\partial p_x} = \frac{\partial\hat{\mathcal{H}}}{\partial\hat{p}_x}\ ;$$

[8] This approximation is fully consistent with that already discussed in Footnote # 7.

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial p_z} &= \frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_z} \; ; \\
\frac{\partial \mathcal{H}}{\partial x} &= [\hat{p}_z - H \cdot x] \cdot (-H) \\
&\quad + \frac{e}{E_0} \cdot \Delta \mathcal{B}_z + [K_x^2 + g] \cdot x - N \cdot z - K_x \cdot p_\sigma \\
&\quad + \lambda \cdot \frac{1}{2} (x^2 - z^2) - \mu \cdot \frac{1}{6} (x^3 - 3xz^2) \\
&= \frac{\partial \hat{\mathcal{H}}}{\partial x} + \frac{e}{E_0} \cdot \Delta \mathcal{B}_z \; ; \\
\frac{\partial \mathcal{H}}{\partial z} &= \frac{\partial \hat{\mathcal{H}}}{\partial z} - \frac{e}{E_0} \cdot \Delta \mathcal{B}_z \; ; \\
\frac{\partial \mathcal{H}}{\partial p_\sigma} &= -[K_x \cdot x + K_z \cdot z] - \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1 + p_\sigma)^2} \\
&= \frac{\partial \hat{\mathcal{H}}}{\partial p_\sigma} - \frac{1}{2} \cdot \left\{ [\hat{p}_x + H \cdot z]^2 + [\hat{p}_z - H \cdot x]^2 \right\} \; ; \\
\frac{\partial \mathcal{H}}{\partial \sigma} &= \frac{\partial \hat{\mathcal{H}}}{\partial \sigma} - \frac{eV}{E_0} \sin \varphi \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial p_x^2} &= \frac{\partial^2 \mathcal{H}}{\partial p_z^2} = \frac{1}{1 + p_\sigma} \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial x \partial p_z} &= -\frac{\partial^2 \mathcal{H}}{\partial z \partial p_x} = \frac{H}{1 + p_\sigma} \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial s \partial p_x} &= + \frac{H' \cdot z}{(1 + p_\sigma)} \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial s \partial p_z} &= - \frac{H' \cdot x}{(1 + p_\sigma)} \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial p_\sigma \partial p_x} \cdot \frac{\partial \mathcal{H}}{\partial \sigma} &= \frac{p_x + H \cdot z}{(1 + p_\sigma)^2} \cdot \frac{eV}{E_0} \sin \varphi \; ; \\
\frac{\partial^2 \mathcal{H}}{\partial p_\sigma \partial p_z} \cdot \frac{\partial \mathcal{H}}{\partial \sigma} &= \frac{p_z - H \cdot x}{(1 + p_\sigma)^2} \cdot \frac{eV}{E_0} \sin \varphi
\end{aligned}
$$

(no solenoid fields in the cavities $\Longrightarrow V(s) \cdot H(s) = 0$) which result from the Hamiltonian (2.39), eqn. (2.38) leads to:

$$
\frac{d}{ds} x = + \frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_x} \; ; \tag{2.44a}
$$

$$
\frac{d}{ds} \hat{p}_x = - \frac{\partial \hat{\mathcal{H}}}{\partial x} - \frac{eV(s)}{E_0} \sin \varphi \cdot \hat{p}_x - \frac{e}{E_0} \cdot \Delta \mathcal{B}_z \; ; \tag{2.44b}
$$

$$
\frac{d}{ds} z = + \frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_z} \; ; \tag{2.44c}
$$

$$
\frac{d}{ds} \hat{p}_z = - \frac{\partial \hat{\mathcal{H}}}{\partial z} - \frac{eV(s)}{E_0} \sin \varphi \cdot \hat{p}_z + \frac{e}{E_0} \cdot \Delta \mathcal{B}_x \; ; \tag{2.44d}
$$

$$
\frac{d}{ds} \sigma = + \frac{\partial \hat{\mathcal{H}}}{\partial p_\sigma} \; ; \tag{2.44e}
$$

$$
\begin{aligned}
\frac{d}{ds}\, p_\sigma &= -\frac{\partial\hat{\mathcal{H}}}{\partial\sigma} + \frac{eV(s)}{E_0}\sin\varphi \\
&\quad -\frac{P^D(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] - \frac{\delta P(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] \qquad (2.44f)
\end{aligned}
$$

where for simplicity we have neglected several small nonsymplectic terms in eqn. (2.38) in order to make the theory more transparent, i.e. we approximate:

$$
\begin{aligned}
\frac{\partial^2\mathcal{H}}{\partial p_x^2} &= \frac{\partial^2\mathcal{H}}{\partial p_z^2} = \frac{1}{1+p_\sigma} \approx 1\;; \\
\frac{\partial^2\mathcal{H}}{\partial x\partial p_z} &= -\frac{\partial^2\mathcal{H}}{\partial z\partial p_x} = \frac{H}{1+p_\sigma} \approx H\;; \\
\frac{\partial\mathcal{H}}{\partial p_\sigma} &= -[K_x\cdot x + K_z\cdot z] - \frac{1}{2}\cdot\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{(1+p_\sigma)^2} \\
&= \frac{\partial\hat{\mathcal{H}}}{\partial p_\sigma} - \frac{1}{2}\cdot\left\{[\hat{p}_x+H\cdot z]^2+[\hat{p}_z-H\cdot x]^2\right\} \\
&\approx \frac{\partial\hat{\mathcal{H}}}{\partial p_\sigma}\;; \\
\frac{\partial^2\mathcal{H}}{\partial p_x^2} &= \frac{\partial^2\mathcal{H}}{\partial p_z^2} = \frac{1}{1+p_\sigma} \approx 1\;; \\
\frac{\partial^2\mathcal{H}}{\partial x\partial p_z} &= -\frac{\partial^2\mathcal{H}}{\partial z\partial p_x} = \frac{H}{1+p_\sigma} \approx H\;; \\
\frac{\partial^2\mathcal{H}}{\partial s\partial p_x} &= +\frac{H'\cdot z}{(1+p_\sigma)} \approx +H'\cdot z\;; \\
\frac{\partial^2\mathcal{H}}{\partial s\partial p_z} &= -\frac{H'\cdot x}{(1+p_\sigma)} \approx -H'\cdot x\;; \\
\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_x}\cdot\frac{\partial\mathcal{H}}{\partial\sigma} &= \frac{p_x+H\cdot z}{(1+p_\sigma)^2}\cdot\frac{eV}{E_0}\sin\varphi \approx p_x\cdot\frac{eV}{E_0}\sin\varphi\;; \\
\frac{\partial^2\mathcal{H}}{\partial p_\sigma\partial p_z}\cdot\frac{\partial\mathcal{H}}{\partial\sigma} &= \frac{p_z-H\cdot x}{(1+p_\sigma)^2}\cdot\frac{eV}{E_0}\sin\varphi \approx p_z\cdot\frac{eV}{E_0}\sin\varphi\,.
\end{aligned}
$$

These terms could in principle be taken into account in a straightforward manner but only produce a small shift of the nonlinear closed orbit and make a small correction to the linear damping behaviour (chapter 3). Note that the radiation terms which appeared for p'_x, p'_z in (2.31) no longer appear for $\hat{p}'_x$, $\hat{p}'_x$.

For the radiation term

$$
\begin{aligned}
-\frac{P^D(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] &= -\frac{P^D(s)}{E_0\cdot c}\cdot[1+K_x\cdot x+K_z\cdot z] \\
&\quad\times\left[1-\frac{[p_x+H\cdot z]^2}{(1+p_\sigma)^2}-\frac{[p_z-H\cdot x]^2}{(1+p_\sigma)^2}\right]^{-\frac{1}{2}}
\end{aligned}
$$

(see eqn. (2.36a)) appearing in (2.44f) we obtain from (2.23a) and (2.24a-f):

$$
-\frac{P^D(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] = -C_1\cdot[1+K_x\cdot x+K_z\cdot z]\cdot(1+2p_\sigma+p_\sigma^2)
$$

$$
\begin{aligned}
& \times \Big\{ (g^2 + N^2) \cdot (x^2 + z^2) \\
& \quad + [K_x^2 + K_z^2] + \left(\frac{e}{E_0}\right)^2 \cdot \left[\Delta \mathcal{B}_x^2 + \Delta \mathcal{B}_z^2\right] \\
& \quad + 2\, G_{CF}^{(x)} \cdot x - 2\, G_{CF}^{(z)} \cdot z \\
& \quad + G_{QS} \cdot [x^3 + x z^2] + \cdots \Big\} \\
& \quad + G_{DO} \cdot \frac{1}{3} [3\, x z^2 - x^3] + \cdots \Big\} \\
\approx \; & -C_1 \cdot \left\{ [K_x^2 + K_z^2] + \left(\frac{e}{E_0}\right)^2 \left[\Delta \mathcal{B}_x^2 + \Delta \mathcal{B}_z^2\right] \right\} \\
& -C_1 \cdot [K_x^2 + K_z^2] \cdot [K_x \cdot x + K_z \cdot z + 2\, p_\sigma] \\
& -C_1 \cdot (g^2 + N^2) \cdot (x^2 + z^2) \cdot (1 + 2 p_\sigma) \\
& -C_1 \cdot 2\, G_{CF}^{(x)} \cdot x + C_1 \cdot 2\, G_{CF}^{(z)} \cdot z \\
& -C_1 \cdot G_{QS} \cdot [x^3 + x z^2] \\
& -C_1 \cdot G_{DO} \cdot \frac{1}{3} [3\, x z^2 - x^3] \qquad (2.45)
\end{aligned}
$$

where we again only retain terms up to third order.

The quantity

$$
-\frac{\delta P(s)}{E_0 \cdot c} \cdot \left[1 - \frac{\partial \mathcal{H}}{\partial p_\sigma}\right]
$$

in (2.44f) should be evaluated on the actual orbit $\vec{y}(s)$ leading to a multiplicative stochastic process, but for simplicity we approximate it by (see eqns. (2.22b) and (2.23b)):

$$
-\frac{\delta P(s)}{E_0 \cdot c} \cdot \left[1 - \frac{\partial \mathcal{H}}{\partial p_\sigma}\right] = \sqrt{\omega} \cdot \xi(s) \qquad (2.46)
$$

with ω given by:

$$
\omega \;=\; C_2 \cdot (|K_x|^3 + |K_z|^3) + C_2 \cdot \left(\left|\frac{e}{E_0}\, \Delta \mathcal{B}_x\right|^{3/2} + \left|\frac{e}{E_0}\, \Delta \mathcal{B}_z\right|^{3/2} \right), \qquad (2.47)
$$

replacing $\vec{y}(s)$ by the design orbit [9] .

In this way we avoid the difficulties due to calculating multiplicative stochastic processes and we only have to treat stochastic differential equations with additive noise. These are much easier to handle.

By taking into account the relations (2.45) and (2.46) the final equations of motion up to third order can be written in matrix form as [10] :

$$
\frac{d}{ds}\, \vec{y} \;=\; \vec{F}(\vec{y}) + \delta \vec{c} \qquad (2.48a)
$$

[9] A more precise calculation would take the field on the closed orbit intoduced in chapter 3.

[10] We neglect the contribution of the octupole component $\tilde{\mathcal{H}}_{12}$ introduced in eqn. (2.43) assuming that the octupole - dipole field has alternating signs thereby minimizing the optical distortion. The same assumption is made for the quadrupole - sextupole wiggler.

with

$$\vec{F}(\vec{y}) \;=\; \underline{A}\cdot\vec{y} + \delta\underline{A}\cdot\vec{y} + \vec{c}_0 + \vec{c}_1 + \vec{c}_{qua} + \vec{c}_{sqd} + \vec{c}_{sex} + \vec{c}_w \qquad (2.48b)$$

and with

$$\vec{y} = \begin{pmatrix} x \\ \hat{p}_x \\ z \\ \hat{p}_z \\ \sigma \\ p_\sigma \end{pmatrix} ; \quad \delta\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \delta c \end{pmatrix} ; \qquad (2.49a)$$

$$\delta c = \sqrt{\omega}\cdot\xi(s) ; \qquad (2.49b)$$

$$\vec{c}_0^{\,T} = \left(0,\,0,\,0,\,0,\,0,\,\frac{eV}{E_0}\sin\varphi - C_1\cdot\left\{[K_x^2+K_z^2] + \left(\frac{e}{E_0}\right)^2\left[(\Delta\mathcal{B}_x)^2+(\Delta\mathcal{B}_z)^2\right]\right\}\right) ; \qquad (2.49c)$$

$$\vec{c}_1^{\,T} = \frac{e}{E_0}\cdot(0,\,-\Delta\mathcal{B}_z,\,0,\,\Delta\mathcal{B}_x,\,0,\,0) ; \qquad (2.49d)$$

$$\vec{c}_{qua}^{\,T} = -C_1\cdot g^2\cdot\left(x^2+z^2\right)\cdot(0,\,0,\,0,\,0,\,0,\,1)\cdot(1+2p_\sigma) ; \qquad (2.49e)$$

$$\vec{c}_{sqd}^{\,T} = -C_1\cdot N^2\cdot\left(x^2+z^2\right)\cdot(0,\,0,\,0,\,0,\,0,\,1)\cdot(1+2p_\sigma) ; \qquad (2.49f)$$

$$\vec{c}_{sex}^{\,T} = \frac{1}{2}\lambda(s)\cdot(0,\,z^2-x^2,\,0,\,2xz,\,0,\,1) ; \qquad (2.49g)$$

$$\vec{c}_w^{\,T} = -C_1\cdot\left\{G_{QS}\cdot[x^3+xz^2] + G_{DO}\cdot\frac{1}{3}[3\,xz^2-x^3]\right\}\cdot(0,\,0,\,0,\,0,\,0,\,1) ; \qquad (2.49h)$$

$$\underline{A}(s) = \begin{pmatrix} & \underline{G} & & & \vec{O}_4 & \vec{K} \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi & 0 \end{pmatrix} \qquad (2.50)$$

and

$$\begin{aligned}
\delta\underline{A} &= ((\delta A_{ik})) ; \\
\delta A_{22} &= -\frac{eV(s)}{E_0}\cdot\sin\varphi ; \\
\delta A_{44} &= \delta A_{22} ; \\
\delta A_{61} &= -C_1\cdot(K_x^2+K_z^2)\cdot K_x - C_1\cdot 2\,G_{CF}^{(x)} ; \\
\delta A_{63} &= -C_1\cdot(K_x^2+K_z^2)\cdot K_z + C_1\cdot 2\,G_{CF}^{(z)} ; \\
\delta A_{66} &= -2\,C_1\cdot[K_x^2+K_z^2] - 2\,C_1\cdot\left(\frac{e}{E_0}\right)^2\left[(\Delta\mathcal{B}_x)^2+(\Delta\mathcal{B}_z)^2\right] ; \\
\delta A_{ik} &= 0 \quad \text{otherwise}
\end{aligned} \qquad (2.51)$$

and with

$$\underline{G}(s) = \begin{pmatrix} 0 & 1 & H & 0 \\ -(K_x^2 + g + H^2) & 0 & N & H \\ -H & 0 & 0 & 1 \\ N & -H & -(K_z^2 - g + H^2) & 0 \end{pmatrix} ; \tag{2.52a}$$

$$\vec{K}^T = (0,\, K_x,\, 0,\, K_z) \ . \tag{2.52b}$$

Here the matrix $\underline{A}(s)$ represents the symplectic part of the linear motion resulting from the Hamiltonian $\hat{\mathcal{H}}_0$ (see eqn. (2,43a)):

$$\underline{A} \cdot \vec{y} = -\underline{S} \cdot \frac{\partial \hat{\mathcal{H}}_{(0)}}{\partial \vec{y}} \tag{2.53}$$

and the vector $\vec{c}_{sex}$ is the symplectic part of the nonlinear motion resulting from the sextupole component $\hat{\mathcal{H}}_{11}$ of the Hamiltonian in eqn. (2.43). The matrix $\underline{S}$ is given by:

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \tag{2.54}$$

and in $\delta\underline{A}(s)$ we have gathered the nonsymplectic linear terms.

Finally we remark that the cavity phase φ in (2.49) and (2.50) is to be determined by the condition that the averaged energy radiated away in the bending magnets:

$$U_{Bend} = E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \left[K_x^2 + K_z^2 \right] \ , \tag{2.55}$$

in the dipole correction magnets:

$$U_{Dip} = E_0 \cdot \sum_{dipole\ correction\ magnets} C_1 \cdot \left(\frac{e}{E_0} \right)^2 \left[(\Delta \mathcal{B}_x)^2 + (\Delta \mathcal{B}_z)^2 \right] \cdot \Delta s \tag{2.56}$$

and in the wigglers ($= U_{Wiggler}$) [11] must be compensated by the averaged energy gain in the cavities:

$$E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{eV(s)}{E_0} \sin\varphi = U_{Cav} \ , \tag{2.57}$$

i.e. we have to require that

$$U_0 + U_{Wiggler} = U_{Cav} \tag{2.58}$$

with

$$U_0 = U_{Bend} + U_{Dip} \ . \tag{2.59}$$

Remark:

In eqn. (2.48b) we have neglected nonsymplectic third order radiation terms resulting from bending magnets, quadrupoles, skew quadrupoles and sextupoles (eqn. (2.45)). These terms can in principle be treated in the same way as the wiggler term $\vec{c}_w$. See also Footnote # 13.

[11] The energy radiated in the wiggler will increase with the transverse beam size.

3 Introduction of the Closed Orbit

Under the influence of radiation and nonlinear damping by wigglers the particle motion in electron storage rings can be completely described by eqn. (2.48) together with eqns. (2.49 - 52) but the equations must be solved in several steps. First of all it is necessary to eliminate the inhomogeneous terms $\vec{c}_0$ and $\vec{c}_1$. This is achieved in the usual way by writing the general solution as

$$\vec{y} = \vec{y}_0 + \vec{\bar{y}} \tag{3.1}$$

where $\vec{y}_0$ is the (unique) periodic solution of eqn. (2.48) (without $\delta\vec{c}$) i.e.

$$\frac{d}{ds}\,\vec{y}_0 = \vec{F}(\vec{y}_0)\ ; \tag{3.2a}$$

$$\vec{y}_0\,(s_0 + L) = \vec{y}_0\,(s_0)\ . \tag{3.2b}$$

with $\vec{F}$ given by (2.48b). (For a solution of eqn. (3.2a) see Appendix B.)

By substituting (3.1) into (2.48) we then obtain the equation for the orbit vector $\vec{\bar{y}}$ valid up to third order and describing the oscillations around the closed orbit $\vec{y}_0$:

$$\begin{aligned} \frac{d}{ds}\,\vec{\bar{y}} &= \vec{F}(\vec{y}) - \vec{F}(\vec{y}_0) + \delta\vec{c} \\ &= \underline{\bar{A}}\cdot\vec{\bar{y}} + \delta\underline{\bar{A}}\cdot\vec{\bar{y}} + \vec{c}_2 + \vec{\bar{c}}_w + \delta\vec{c}\,(\vec{y}_0) \end{aligned} \tag{3.3}$$

with

$$\vec{\bar{c}}_w^{\,T} = -C_1\cdot\left\{G_{QS}\cdot[\bar{x}^3 + \bar{x}\bar{z}^2] + G_{DO}\cdot\frac{1}{3}\,[3\,\bar{x}\bar{z}^2 - \bar{x}^3]\ .\right\}\cdot(0,\,0,\,0,\,0,\,0,\,1)\ ; \tag{3.4a}$$

$$\underline{\bar{A}}(s) = \begin{pmatrix} & \underline{\bar{G}} & & & \vec{O}_4 & \vec{K} \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi & 0 \end{pmatrix} \tag{3.4b}$$

and

$$\begin{aligned} \delta\underline{\bar{A}} &= ((\delta\bar{A}_{ik}))\ ; \\ \delta\bar{A}_{22} &= -\frac{eV(s)}{E_0}\cdot\sin\varphi\ ; \\ \delta\bar{A}_{44} &= -\frac{eV(s)}{E_0}\cdot\sin\varphi\ ; \\ \delta\bar{A}_{61} &= \delta A_{61} - 2\,C_1\cdot(g^2+N^2)\cdot x_0\cdot(1+2\eta_0) - C_1\cdot\left\{G_{QS}\cdot[3x_0^2+z_0^2] + G_{DO}\cdot[z_0^2-x_0^2]\right\}\ ; \\ \delta\bar{A}_{63} &= \delta A_{63} - 2\,C_1\cdot(g^2+N^2)\cdot z_0\cdot(1+2\eta_0) - C_1\cdot\{G_{QS}\cdot 2x_0\,z_0 + G_{DO}\cdot 2x_0\,z_0\}\ ; \\ \delta\bar{A}_{66} &= \delta A_{66} - 2\,C_1\cdot(g^2+N^2)\cdot(x_0^2+z_0^2) - 2\,C_1\cdot\left(\frac{e}{E_0}\right)^2\cdot\left[(\Delta\mathcal{B}_x)^2 + (\Delta\mathcal{B}_z)^2\right]\ ; \\ \delta\bar{A}_{ik} &= 0 \quad \text{otherwise} \end{aligned} \tag{3.4c}$$

and with

$$\bar{\underline{G}} = \underline{G} + \underline{G}_{sex} ; \tag{3.5a}$$

$$\underline{G}_{sex} = \lambda(s) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_0 & 0 & z_0 & 0 \\ 0 & 0 & 0 & 0 \\ z_0 & 0 & x_0 & 0 \end{pmatrix} \tag{3.5b}$$

whereby in $\vec{c}_2$ we have gathered second order terms in $\bar{x}$, $\bar{p}_x$, $\bar{z}$, $\bar{p}_z$, $\bar{\sigma}$, $\bar{p}_\sigma$. However, as we shall see in chapter 7, it is not necessary to know the special form of $\vec{c}_2$.

Equation (3.4c) demonstrates that closed orbit distortions can change the damping behaviour. See Appendix D also [12] .

4 Introduction of the Dispersion via a Canonical Transformation

Equation (3.3) describes nonlinear coupled synchro-betatron motion in the presence of synchrotron radiation. The linear symplectic part of this equation reads as:

$$\frac{d}{ds}\,\vec{\bar{y}} = \bar{\underline{A}} \cdot \vec{\bar{y}} \equiv -\underline{S} \cdot \frac{\partial \mathcal{K}}{\partial \vec{\bar{y}}} \tag{4.1}$$

with

$$\begin{aligned} \mathcal{K}(\bar{x}, \bar{p}_x, \bar{z}, \bar{p}_z, \bar{\sigma}, \bar{p}_\sigma; s) = &\ \frac{1}{2} \cdot \left\{ [\bar{p}_x + H \cdot \bar{z}]^2 + [\bar{p}_z - H \cdot \bar{x}]^2 \right\} \\ &+ \frac{1}{2} \cdot \left\{ [K_x^2 + \hat{g}] \cdot \bar{x}^2 + [K_z^2 - \hat{g}] \cdot \bar{z}^2 \right\} - \hat{N} \cdot \bar{x}\bar{z} \\ &- \frac{1}{2}\bar{\sigma}^2 \cdot \frac{eV}{E_0} \cdot h \cdot \frac{2\pi}{L} \cos\varphi - [K_x \cdot \bar{x} + K_z \cdot \bar{z}] \cdot \bar{p}_\sigma \end{aligned} \tag{4.2a}$$

and

$$\hat{g} = g + \lambda \cdot x_0 ; \quad \hat{N} = N + \lambda \cdot z_0 \tag{4.2b}$$

or, in component form:

$$\frac{d}{ds}\,\bar{x} = \bar{p}_x + H \cdot \bar{z} ; \tag{4.3a}$$

$$\frac{d}{ds}\,\bar{p}_x = +[\bar{p}_z - H \cdot \bar{x}] \cdot H - [K_x^2 + \hat{g}] \cdot \bar{x} + \hat{N} \cdot \bar{z} + K_x \cdot \bar{\eta} ; \tag{4.3b}$$

$$\frac{d}{ds}\,\bar{z} = \bar{p}_z - H \cdot \bar{x} ; \tag{4.3c}$$

$$\frac{d}{ds}\,\bar{p}_z = -[\bar{p}_x + H \cdot \bar{z}] \cdot H + \hat{N} \cdot \bar{x} - [K_z^2 - \hat{g}] \cdot \bar{z} + K_z \cdot \bar{\eta} ; \tag{4.3d}$$

[12] We have neglected curvature terms resulting from dipole correction coils, quadrupoles, skew quadrupoles and sextupoles. But these additional terms generating spurious dispersion can be taken into account in a straightforward manner. For more details see Ref. [9].

$$\frac{d}{ds}\,\bar{\sigma} \;=\; -[K_x \cdot \bar{x} + K_z \cdot \bar{z}]\,; \tag{4.3e}$$

$$\frac{d}{ds}\,\bar{\eta} \;=\; \bar{\sigma} \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi\,. \tag{4.3f}$$

Note that the linear betatron oscillations (eqns. (4.3a - d)) and the longitudinal motion (eqns. (4.3e, f)) are coupled by the term

$$-[K_x \cdot \bar{x} + K_z \cdot \bar{z}] \tag{4.4}$$

appearing in (4.3e) which depends on the curvature of the orbit in the bending magnets.

We now introduce dispersion:

$$\vec{D} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} \tag{4.5}$$

and replace the quantities $\bar{x}$, $\bar{p}_x$, $\bar{z}$, $\bar{p}_z$, $\bar{\sigma}$, $\bar{p}_\sigma$ by the new variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ defined by:

$$\tilde{x} = \bar{x} - \bar{p}_\sigma \cdot D_1\,; \tag{4.6a}$$

$$\tilde{p}_x = \bar{p}_x - \bar{p}_\sigma \cdot D_2\,; \tag{4.6b}$$

$$\tilde{z} = \bar{z} - \bar{p}_\sigma \cdot D_3\,; \tag{4.6c}$$

$$\tilde{p}_z = \bar{p}_z - \bar{p}_\sigma \cdot D_4\,, \tag{4.6d}$$

where the components D_k, $(k = 1, 2, 3, 4)$ of the dispersion vector $\vec{D}$ are (in the ultrarelativistic case) given by the periodic solutions of the equations (resulting from (4.3a - d) for $\eta = 1$):

$$\frac{d}{ds}\,D_1 = D_2 + H \cdot D_3\,; \tag{4.7a}$$

$$\frac{d}{ds}\,D_2 = +[D_4 - H \cdot D_1] \cdot H - [K_x^2 + \hat{g}] \cdot D_1 + \hat{N} \cdot D_3 + K_x\,; \tag{4.7b}$$

$$\frac{d}{ds}\,D_3 = D_4 - H \cdot D_1\,; \tag{4.7c}$$

$$\frac{d}{ds}\,D_4 = -[D_2 + H \cdot D_3] \cdot H + \hat{N} \cdot D_1 - [K_z^2 - \hat{g}] \cdot D_3 + K_z\,. \tag{4.7d}$$

(For a calculation of dispersion see chapter B.2 in Appendix B.)

This replacement:

$$(\bar{x},\, \bar{p}_x,\, \bar{z},\, \bar{p}_z,\, \bar{\sigma},\, \bar{p}_\sigma) \longrightarrow (\tilde{x},\, \tilde{p}_x,\, \tilde{z},\, \tilde{p}_z,\, \tilde{\sigma},\, \tilde{p}_\sigma) \tag{4.8}$$

can be achieved using the generating function [13, 14]:

$$\begin{aligned} F_2(\bar{x},\, \bar{z},\, \bar{\sigma},\, \tilde{p}_x,\, \tilde{p}_z,\, \tilde{p}_\sigma) &= \tilde{p}_x \cdot [\bar{x} - \tilde{p}_\sigma \cdot D_1] + \tilde{p}_\sigma \cdot D_2 \cdot \bar{x} \\ &+ \tilde{p}_z \cdot [\bar{z} - \tilde{p}_\sigma \cdot D_3] + \tilde{p}_\sigma \cdot D_4 \cdot \bar{z} \\ &- \frac{1}{2} \cdot [D_1 \cdot D_2 + D_3 \cdot D_4] \cdot \tilde{p}_\sigma^2 + \tilde{p}_\sigma \cdot \bar{\sigma} \end{aligned} \tag{4.9}$$

with the result that :

$$\begin{aligned}
\tilde{x} &= \frac{\partial F_2}{\partial \tilde{p}_x} = \bar{x} - \tilde{p}_\sigma \cdot D_1 \; ; && (4.10a)\\
\bar{p}_x &= \frac{\partial F_2}{\partial \bar{x}} = \tilde{p}_x + \tilde{p}_\sigma \cdot D_2 \; ; && (4.10b)\\
\tilde{z} &= \frac{\partial F_2}{\partial \tilde{p}_z} = \bar{z} - \tilde{p}_\sigma \cdot D_3 \; ; && (4.10c)\\
\bar{p}_z &= \frac{\partial F_2}{\partial \bar{z}} = \tilde{p}_z + \tilde{p}_\sigma \cdot D_4 \; . && (4.10d)\\
\tilde{\sigma} &= \frac{\partial F_2}{\partial \tilde{p}_\sigma} = \bar{\sigma} - \tilde{p}_x \cdot D_1 + \bar{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \bar{z} \cdot D_4 \\
&\qquad -[D_1 \cdot D_2 + D_3 \cdot D_4] \cdot \tilde{p}_\sigma \\
&= \bar{\sigma} - \tilde{p}_x \cdot D_1 + [\bar{x} - D_1 \cdot \tilde{p}_\sigma] \cdot D_2 \\
&\qquad -\tilde{p}_z \cdot D_3 + [\bar{z} - D_3 \cdot \tilde{p}_\sigma] \cdot D_4 \\
&= \bar{\sigma} - \tilde{p}_x \cdot D_1 + \tilde{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \tilde{z} \cdot D_4 \\
&= \bar{\sigma} - \bar{p}_x \cdot D_1 + \bar{x} \cdot D_2 - \bar{p}_z \cdot D_3 + \bar{z} \cdot D_4 \; ; && (4.10e)\\
\bar{p}_\sigma &= \frac{\partial F_2}{\partial \bar{\sigma}} = \tilde{p}_\sigma && (4.10f)
\end{aligned}$$

and

$$\bar{\mathcal{K}} = \mathcal{K} + \frac{\partial F_2}{\partial s} \qquad (4.11)$$

in agreement with eqn. (4.6).

In matrix form eqns. (4.10a - f) read as :

$$\vec{\bar{y}} = \underline{F} \cdot \vec{\tilde{y}} \; ; \quad \vec{\tilde{y}} = \underline{F}^{-1} \cdot \vec{\bar{y}} \qquad (4.12)$$

with

$$\vec{\tilde{y}} = \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \\ \tilde{\sigma} \\ \tilde{p}_\sigma \end{pmatrix} \qquad (4.13)$$

and

$$\underline{F}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -D_1 \\ 0 & 1 & 0 & 0 & 0 & -D2 \\ 0 & 0 & 1 & 0 & 0 & -D3 \\ 0 & 0 & 0 & 1 & 0 & -D4 \\ +D_2 & -D_1 & +D_4 & -D_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \qquad (4.14a)$$

$$\underline{F}^{-1}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & +D_1 \\ 0 & 1 & 0 & 0 & 0 & +D2 \\ 0 & 0 & 1 & 0 & 0 & +D3 \\ 0 & 0 & 0 & 1 & 0 & +D4 \\ -D_2 & +D_1 & -D_4 & +D_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.14b}$$

The new Hamiltonian (4.11) has the form:

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 + \tilde{\mathcal{K}}_1 \tag{4.15}$$

with

$$\begin{aligned} \tilde{\mathcal{K}}_0 &= \frac{1}{2} \cdot \left\{ [\tilde{p}_x + H \cdot \tilde{z}]^2 + [\tilde{p}_z - H \cdot \tilde{x}]^2 \right\} \\ &+ \frac{1}{2} \cdot \left\{ [K_x^2 + \tilde{g}] \cdot \tilde{x}^2 + [K_z^2 - \tilde{g}] \cdot \tilde{z}^2 \right\} - \hat{N} \cdot \tilde{x}\tilde{z} \\ &- \frac{1}{2} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV}{E_0} \cos\varphi \cdot \tilde{\sigma}^2 \\ &- \frac{1}{2} \cdot \tilde{p}_\sigma^2 \cdot [K_x \cdot D_1 + K_z \cdot D_3] ; \end{aligned} \tag{4.15a}$$

$$\begin{aligned} \tilde{\mathcal{K}}_1 &= -\frac{1}{2} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV}{E_0} \cos\varphi \cdot [\tilde{\sigma} + \tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4]^2 \\ &+ \frac{1}{2} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV}{E_0} \cos\varphi \cdot \tilde{\sigma}^2 . \end{aligned} \tag{4.15b}$$

In this Hamiltonian the coupling term (4.4) which arose from the orbit curvature no longer appears. Instead, there appears a new term $\tilde{\mathcal{K}}_1$ for the cavities representing a coupling between the longitudinal and transverse motion which disappears if

$$V(s) \cdot D_1 = V(s) \cdot D_3 = 0 ; \tag{4.16a}$$
$$V(s) \cdot D_2 = V(s) \cdot D_4 = 0 \tag{4.16b}$$

(i.e. no dispersion in the cavities).

To proceed we will treat $\tilde{\mathcal{K}}_0$ as the unperturbed part of the Hamiltonian (4.15) and $\tilde{\mathcal{K}}_1$ (representing the synchro-betatron coupling) as a perturbation.

In terms of the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ eqn. (4.1) now takes the form:

$$\frac{d}{ds} \vec{\tilde{y}} = \underline{\tilde{A}} \cdot \vec{\tilde{y}} + \Delta\underline{\tilde{A}} \cdot \vec{\tilde{y}} \tag{4.17}$$

with

$$\underline{\tilde{A}} \cdot \vec{\tilde{y}} \equiv -\underline{S} \cdot \frac{\partial \tilde{\mathcal{K}}_0}{\partial \vec{\tilde{y}}} ; \tag{4.17a}$$

$$\Delta\underline{\tilde{A}} \cdot \vec{\tilde{y}} \equiv -\underline{S} \cdot \frac{\partial \tilde{\mathcal{K}}_1}{\partial \vec{\tilde{y}}} . \tag{4.17b}$$

In particular one obtains from (4.15a) and (4.17a):

$$\underline{\tilde{A}} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & \hat{N} & H & 0 & 0 \\ -H & 0 & 0 & 1 & 0 & 0 \\ \hat{N} & -H & -(G_2+H^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -[K_x \cdot D_1 + K_z \cdot D_3] \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot h \cdot \frac{2\pi}{L}\cos\varphi & 0 \end{pmatrix} ; \tag{4.18a}$$

$$(G_1 = K_x^2 + \hat{g} \; ; \;\; G_2 = K_z^2 - \hat{g})$$

and from (4.15b) and (4.17b):

$$\Delta\underline{\tilde{A}}(s) = \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cos\varphi \times \begin{pmatrix} D_2 \cdot \vec{D} & -D_1 \cdot \vec{D} & D_4 \cdot \vec{D} & -D_3 \cdot \vec{D} & -\vec{D} & \vec{0}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -D_2 & D_1 & -D_4 & D_3 & 0 & 0 \end{pmatrix} . \tag{4.18b}$$

The r.h.s. of eqn. (4.17) includes only the (linear) symplectic part of the motion. The whole equation (3.3) including also the nonsymplectic part of the motion now reads in terms of the new variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ as:

$$\begin{aligned} \frac{d}{ds}\vec{\tilde{y}} &= \frac{d}{ds}\left[\underline{F}\cdot\vec{y}\right] \\ &= \underline{F}'\cdot\vec{y} + \underline{F}\cdot\vec{y}\,' \\ &= \underline{F}'\cdot\vec{y} + \underline{F}\cdot\left[\underline{\bar{A}}\cdot\vec{y} + \delta\underline{\bar{A}}\cdot\vec{y} + \vec{c}_2 + \vec{\bar{c}}_w + \delta\vec{c}\right] \\ &= \left[\underline{F}' + \underline{F}\cdot\underline{\bar{A}}\right]\cdot\underline{F}^{-1}\vec{\tilde{y}} + \underline{F}\cdot\delta\underline{\bar{A}}\cdot\underline{F}^{-1}\cdot\vec{\tilde{y}} + \underline{F}\cdot\left[\vec{c}_2 + \vec{\bar{c}}_w + \delta\vec{c}\right] \\ &= \left[\underline{\tilde{A}} + \Delta\underline{\tilde{A}}\right]\cdot\vec{\tilde{y}} + \delta\underline{\tilde{A}}\cdot\vec{\tilde{y}} + \vec{\tilde{c}}_2 + \vec{\tilde{c}}_w + \delta\vec{\tilde{c}} \end{aligned} \tag{4.19}$$

with

$$\left[\underline{\tilde{A}} + \Delta\underline{\tilde{A}}\right] = \left[\underline{F}' + \underline{F}\cdot\underline{\bar{A}}\right]\cdot\underline{F}^{-1}$$

given by eqn. (4.17) and

$$\delta\underline{\tilde{A}} = \underline{F}\cdot\delta\underline{\bar{A}}\cdot\underline{F}^{-1} \; ; \tag{4.20a}$$

$$\vec{\tilde{c}}_2 = \underline{F}\cdot\vec{c}_2 \; ; \tag{4.20b}$$

$$\vec{\tilde{c}}_w = \underline{F}\cdot\vec{\bar{c}}_w \; ; \tag{4.20c}$$

$$\delta\vec{\tilde{c}} = \underline{F}\cdot\delta\vec{c} \, . \tag{4.20d}$$

Writing $\delta\underline{\bar{A}}$ and $\underline{F}$ in the form:

$$\delta\underline{\bar{A}} = \begin{pmatrix} \delta\underline{\bar{A}}_{(4\times4)} & \vec{0}_4 & \vec{0}_4 \\ \vec{0}_4^{\,T} & 0 & 0 \\ \vec{c}_4^{\,T} & 0 & \delta\bar{A}_{66} \end{pmatrix} ; \tag{4.21a}$$

$$\underline{F} = \begin{pmatrix} \underline{1} & \vec{0}_4 & -\vec{D} \\ +\vec{D}^T\cdot\underline{S}_4 & 1 & 0 \\ \vec{0}_4^{\,T} & 0 & 1 \end{pmatrix} ; \quad \underline{F}^{-1} = \begin{pmatrix} \underline{1} & \vec{0}_4 & +\vec{D} \\ -\vec{D}^T\cdot\underline{S}_4 & 1 & 0 \\ \vec{0}_4^{\,T} & 0 & 1 \end{pmatrix} \tag{4.21b}$$

with

$$\vec{c}_4^{\,T} = (\delta\bar{A}_{61},\, \delta\bar{A}_{62},\, \delta\bar{A}_{63},\, \delta\bar{A}_{64}) \; ; \tag{4.22a}$$

$$\underline{S}_4 = \begin{pmatrix} \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{S}_2 \end{pmatrix} \tag{4.22b}$$

we obtain for $\delta\underline{\tilde{A}}$:

$$\delta\underline{\tilde{A}} = \begin{pmatrix} \delta\underline{\bar{A}}_{(4\times4)} - \vec{D}\cdot\vec{c}_4^{\,T} & \vec{0}_4 & \delta\underline{\bar{A}}_{(4\times4)}\cdot\vec{D} - \vec{D}\cdot\vec{c}_4^{\,T}\cdot\vec{D} - \vec{D}\cdot\delta\bar{A}_{66} \\ -\vec{D}^T\cdot\underline{S}_4\cdot\underline{\bar{A}}_{(4\times4)} & 0 & \vec{D}^T\cdot\underline{S}_4\cdot\delta\underline{\bar{A}}_{(4\times4)}\cdot\vec{D} \\ \vec{c}_4^{\,T} & 0 & \vec{c}_4^{\,T}\cdot\vec{D} + \delta\bar{A}_{66} \end{pmatrix} \tag{4.23}$$

and for $\vec{\tilde{c}}_w$ and $\delta\vec{\tilde{c}}$ we get:

$$\vec{\tilde{c}}_w = \vec{\hat{D}}\cdot C_1\cdot\left\{G_{QS}\cdot[\bar{x}^3 + \bar{x}\bar{z}^2] + G_{DO}\cdot\frac{1}{3}[3\,\bar{x}\bar{z}^2 - \bar{x}^3]\right\} \; ; \tag{4.24a}$$

$$\delta\vec{\tilde{c}} = -\delta c\cdot\vec{\hat{D}} = -\sqrt{\omega(s)}\cdot\xi(s)\cdot\vec{\hat{D}} \tag{4.24b}$$

with

$$\begin{aligned} \bar{x} &= \tilde{x} + \tilde{p}_\sigma\cdot D_1 \; ; \\ \bar{p}_x &= \tilde{p}_x + \tilde{p}_\sigma\cdot D_2 \; ; \\ \bar{z} &= \tilde{z} + \tilde{p}_\sigma\cdot D_3 \; ; \\ \bar{p}_z &= \tilde{p}_z + \tilde{p}_\sigma\cdot D_4 \end{aligned}$$

where we have introduced in (4.24a, b) the vector

$$\vec{\hat{D}} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ 0 \\ -1 \end{pmatrix} . \tag{4.25}$$

In the absence of the terms $\vec{\tilde{c}}_2$ and $\vec{\tilde{c}}_w$ the stochastic differential equation (4.19) would be a Langevin equation [1, 15] with periodic s dependent coefficients and linear drift terms and the moments of the statistical distribution of the orbit variables could be found by standard techniques [1, 15, 16, 17, 18, 19, 20]. However in the presence of nonlinear terms $\vec{\tilde{c}}_2$, $\vec{\tilde{c}}_w$ an alternative approach is needed in which the canonical variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ are reexpressed in terms of action - angle variables as described in the next chapter.

Later on we will need the relation:

$$[\Delta\underline{\tilde{A}}(s)]^T\cdot\underline{S} + \underline{S}\cdot\Delta\underline{\tilde{A}}(s) = 0 \tag{4.26}$$

resulting from (4.17b) [21].

Remark:

The Matrix $\Delta\underline{\tilde{A}}$ in (4.18b) results from the synchro-betatron coupling induced by non-vanishing dispersion in the cavities.

The relations (4.18) can be used to calculate the 6×6 transfer matrix of a cavity in the version of dispersion formalism in which all synchro-betatron coupling terms are retained [13].

For this purpose one has to investigate the solution of the equation of motion (4.17) for a cavity. For a pointlike cavity at position $s = s_0$:

$$V(s) = \hat{V} \cdot \delta(s - s_0) \tag{4.27}$$

one then obtains:

$$\frac{d}{ds}\vec{\tilde{y}} = \frac{e\hat{V}}{E_0} \cdot h \cdot \frac{2\pi}{L} \cos\varphi \cdot \delta(s - s_0) \cdot \underline{K}(s)\,\vec{\tilde{y}} \tag{4.28}$$

with

$$\underline{K} = \begin{pmatrix} D_2 \cdot \vec{D} & -D_1 \cdot \vec{D} & D_4 \cdot \vec{D} & -D_3 \cdot \vec{D} & -\vec{D} & \vec{0}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -D_2 & D_1 & -D_4 & D_3 & 1 & 0 \end{pmatrix} . \tag{4.29}$$

In solving eqn. (4.28) it is important that the term $\underline{K}(s)\,\vec{\tilde{y}}(s)$ which multiplies the δ-function is a continous function of s at s_0 although $\vec{\tilde{y}}(s)$ changes discontinously at s_0 as may be seen from (4.28).

The continuity can be proven easily if one takes into account the fact that $\tilde{\sigma}(s)$ and

$$\vec{\tilde{y}}_\perp = \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \end{pmatrix}$$

are continous at $s = s_0$ (see eqn. (3.4b)) and that

$$\vec{D}^T \cdot \underline{S}_4 \cdot \vec{D} = 0 \implies \vec{D}^T \cdot \underline{S}_4 \cdot \vec{\tilde{y}}_\perp = \vec{D}^T \cdot \underline{S}_4 \cdot \vec{y}_\perp$$

with

$$\vec{y}_\perp = \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \end{pmatrix}$$

(see eqn. (4.10)) and if one rewrites

$$\begin{aligned} \underline{K}\,\vec{\tilde{y}} &= \begin{pmatrix} D_2 \cdot \vec{D} & -D_1 \cdot \vec{D} & D_4 \cdot \vec{D} & -D_3 \cdot \vec{D} & -\vec{D} & \vec{0}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -D_2 & D_1 & -D_4 & D_3 & 1 & 0 \end{pmatrix} \vec{\tilde{y}} \\ &= \begin{pmatrix} \vec{D} \cdot \left[\vec{D}^T \cdot \underline{S}_4 \cdot \vec{\tilde{y}}_\perp\right] - \vec{D} \cdot \tilde{\sigma} \\ 0 \\ -\left[\vec{D}^T \cdot \underline{S}_4 \cdot \vec{\tilde{y}}_\perp\right] + \vec{D} \cdot \tilde{\sigma} \end{pmatrix} \end{aligned}$$

in the form:

$$\underline{K}\,\vec{y} = \begin{pmatrix} \vec{D}\cdot\left[\vec{D}^T\cdot\underline{S}_4\cdot\vec{y}_\perp\right] - \vec{D}\cdot\vec{\sigma} \\ 0 \\ -\left[\vec{D}^T\cdot\underline{S}_4\cdot\vec{y}_\perp\right] + \vec{\sigma} \end{pmatrix} .$$

Now, integrating both sides of eqn. (4.28) from $s_0 - \epsilon$ to $s_0 + \epsilon$ one immediately obtains ($\epsilon \to 0$):

$$\vec{y}(s_0+0) = \left[\underline{1} + \frac{e\hat{V}}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi\cdot\underline{K}(s_0)\right]\vec{y}(s_0-0) . \tag{4.30}$$

From eqn. (4.30) one can extract the six-dimensional transfer matrix of a cavity which reads as [9, 13]:

$$\underline{M}(s_0+0, s_0-0) =$$

$$\underline{1} + \frac{e\hat{V}}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\begin{pmatrix} D_2\cdot\vec{D} & -D_1\cdot\vec{D} & D_4\cdot\vec{D} & -D_3\cdot\vec{D} & -\vec{D} & \vec{0}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -D_2 & D_1 & -D_4 & D_3 & 1 & 0 \end{pmatrix} . \tag{4.31}$$

Note that $\underline{M}(s_0+0, s_0-0)$ in (4.31) resulting from the Hamiltonian (4.15) is a symplectic matrix.

5 The Unperturbed Problem

In order to investigate the particle motion under the influence of synchrotron radiation we begin by neglecting in a first approximation the small terms $\Delta\underline{\tilde{A}}$, $\delta\underline{\tilde{A}}$, $\vec{\tilde{c}}_2$, $\vec{\tilde{c}}_w$ and $\delta\vec{\tilde{c}}$ in eqn. (4.19) and consider only the "unperturbed problem":

$$\frac{d}{ds}\vec{y} = \underline{\tilde{A}}\cdot\vec{y} . \tag{5.1}$$

The synchro-betatron coupling described by $\Delta\underline{\tilde{A}}$ and the radiative perturbations described by $\delta\underline{\tilde{A}}$, $\vec{\tilde{c}}_2$, $\vec{\tilde{c}}_w$ and $\delta\vec{\tilde{c}}$ will then be treated in a second step with perturbation theory.

Since eqn. (5.1) is linear and homogeneous, the solution can be written in the form:

$$\vec{y}(s) = \underline{\tilde{M}}(s, s_0)\cdot\vec{y}(s_0) , \tag{5.2}$$

which defines the transfer matrix $\underline{\tilde{M}}(s, s_0)$ of the motion.

To come further we need the eigenvalues and the eigenvectors of the matrix $\underline{\tilde{M}}(s+L, s)$:

$$\underline{\tilde{M}}(s+L, s)\,\vec{v}_\mu(s) = \lambda_\mu\cdot\vec{v}_\mu(s) \tag{5.3}$$

in order to study the normal modes. We proceed in the usual way [1]:

The vector $\vec{v}_\mu(s)$ in eqn. (5.3) is an eigenvector of the matrix $\underline{\tilde{M}}(s + L, s)$ at point s with the eigenvalue λ_μ. The eigenvalues are independent of s.

If the eigenvector $\vec{v}_\mu(s_0)$ at a fixed point s_0 is known, the eigenvector at an arbitrary point s may be obtained by:

$$\vec{v}_\mu(s) \quad = \quad \underline{\tilde{M}}(s,\, s_0)\, \vec{v}_\mu(s_0)\ . \tag{5.4}$$

Since $\underline{M}(s + L, s)$ is symplectic and we assume stability, the eigenvectors $\vec{v}_\mu(s)$ come in complex conjugate pairs

$$(\vec{v}_k,\, \vec{v}_{-k} = \vec{v}_k^{\,*})\,; \quad (k = I,\ II,\ III)$$

with complex conjugate eigenvalues.

In the following we put:

$$\begin{cases} \lambda_k = e^{-i \cdot 2\pi Q_k}\ ; \\ \lambda_{-k} = e^{-i \cdot 2\pi Q_{-k}}\ ; \end{cases} \tag{5.5}$$

$$(k = I,\ II,\ III)$$

with

$$Q_{-k} \quad = \quad -Q_k \tag{5.6}$$

where Q_k is a real number.

Defining $\vec{\tilde{v}}_\mu(s)$ by

$$\vec{v}_\mu(s) = \vec{\tilde{v}}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \tag{5.7a}$$

we find:

$$\vec{\tilde{v}}_\mu(s + L) = \vec{\tilde{v}}_\mu(s)\ . \tag{5.7b}$$

Equation (5.7a, b) is a statement of the Floquet theorem: vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (5.1) which can be expressed as the product of a periodic function $\vec{\tilde{v}}_\mu(s)$ and a harmonic function

$$e^{-i \cdot 2\pi Q_\mu \cdot (s/L)}\ .$$

The general solution of the equation of motion (5.1) is a linear combination of the special solutions (5.7a) and can therefore be written as:

$$\vec{y}(s) = \sum_{k=I,II,III} \left\{ A_k \cdot \vec{\tilde{v}}_k(s) \cdot e^{-i \cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \vec{\tilde{v}}_{-k}(s) \cdot e^{+i \cdot 2\pi Q_{-k} \cdot (s/L)} \right\}. \tag{5.8}$$

We have the orthogonality relations:

$$\begin{cases} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) \neq 0 \; ; \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{for } \mu \neq \nu \; ; \end{cases}$$

$$(k = I,\ II,\ III) \; .$$

Furthermore the terms $\vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\mu(s)$ in the last equation are pure imaginary:

$$\begin{aligned} \left[\vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\mu(s)\right]^+ &= \vec{v}_\mu^{\,+}(s) \cdot \underline{S}^+ \cdot \vec{v}_\mu(s) \\ &= -\left[\vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\mu(s)\right] \end{aligned}$$

(since $\underline{S}^+ = -\underline{S}$). We choose to normalise the vectors $\vec{v}_k(s)$ and $\vec{v}_{-k}(s)$ at a fixed point s_0 as:

$$\vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i \; ;$$

$$(k = I,\ II,\ III) \; .$$

This normalisation is valid for all s if we use the definition in eqn. (5.4) for $\vec{v}_\mu(s)$. Thus we obtain:

$$\begin{cases} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i \; ; \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{for } \mu \neq \nu \; . \end{cases} \tag{5.9}$$

Note that the Floquet-vectors

$$\vec{\tilde{v}}_\mu(s) = \vec{v}_\mu(s) \cdot e^{+i \cdot 2\pi Q_\mu \cdot (s/L)}$$

then fulfill the same relationships:

$$\begin{cases} \vec{\tilde{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_k(s) = -\vec{\tilde{v}}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_{-k}(s) = i \; ; \\ \vec{\tilde{v}}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_\nu(s) = 0 \quad \text{for } \mu \neq \nu \; . \end{cases} \tag{5.10}$$

Using these results we are now able to introduce a new set of canonical variables J_k, Φ_k which will be important for further investigations.

For this we write for the coefficients $A_k,\ A_{-k}\ \ (k = I, II, III)$ in eqn. (5.8):

$$A_k = \sqrt{J_k} \cdot e^{-i[\Phi_k - 2\pi Q_k \cdot s/L]} \; ; \tag{5.11a}$$

$$A_{-k} = \sqrt{J_k} \cdot e^{+i[\Phi_k - 2\pi Q_k \cdot s/L]} \; . \tag{5.11b}$$

Then eqn. (5.8) takes the form:

$$\vec{y}(s) = \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \vec{\tilde{v}}_k(s) \cdot e^{-i\Phi_k} + \vec{\tilde{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} \; . \tag{5.12}$$

From (5.12) we now have:

$$\frac{\partial\vec{\tilde{y}}}{\partial\Phi_k} = -i\cdot\sqrt{J_k}\cdot\left\{\vec{\tilde{v}}_k(s)\cdot e^{-i\Phi_k} - \vec{\tilde{v}}_{-k}(s)\cdot e^{+i\Phi_k}\right\}; \tag{5.13a}$$

$$\frac{\partial\vec{\tilde{y}}}{\partial J_k} = +\frac{1}{2\sqrt{J_k}}\cdot\left\{\vec{\tilde{v}}_k(s)\cdot e^{-i\Phi_k} + \vec{\tilde{v}}_{-k}(s)\cdot e^{+i\Phi_k}\right\}. \tag{5.13b}$$

Taking into account the relations (5.10) one obtains the equations:

$$\frac{\partial\vec{\tilde{y}}}{\partial J_k}^T\cdot\underline{S}\cdot\frac{\partial\vec{\tilde{y}}}{\partial\Phi_l} = -\frac{\partial\vec{\tilde{y}}}{\partial\Phi_l}^T\cdot\underline{S}\cdot\frac{\partial\vec{\tilde{y}}}{\partial J_k} = \delta_{kl}; \tag{5.14a}$$

$$\frac{\partial\vec{\tilde{y}}}{\partial J_k}^T\cdot\underline{S}\cdot\frac{\partial\vec{\tilde{y}}}{\partial I_l} = \frac{\partial\vec{\tilde{y}}}{\partial\Phi_k}^T\cdot\underline{S}\cdot\frac{\partial\vec{\tilde{y}}}{\partial\Phi_l} = 0 \tag{5.14b}$$

which can be combined into the matrix form [22]:

$$\underline{\mathcal{J}}^T\cdot\underline{S}\cdot\underline{\mathcal{J}} = \underline{S} \tag{5.15}$$

where $\underline{\mathcal{J}}$ signifies the Jacobian matrix:

$$\underline{\mathcal{J}} = \left(\frac{\partial\vec{\tilde{y}}}{\partial\Phi_I}, \frac{\partial\vec{\tilde{y}}}{\partial J_I}, \frac{\partial\vec{\tilde{y}}}{\partial\Phi_{II}}, \frac{\partial\vec{\tilde{y}}}{\partial J_{II}}, \frac{\partial\vec{\tilde{y}}}{\partial\Phi_{III}}, \frac{\partial\vec{\tilde{y}}}{\partial J_{III}}\right) \tag{5.16}$$

being a 6×6-matrix just written as a row of column vectors $(\partial\vec{\tilde{y}}/\partial\Phi_I)$ etc.

Equation (5.15) proves that eqn. (5.12) represents a canonical transformation [22]

$$\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma \longrightarrow \Phi_I,\ J_I,\ \Phi_{II},\ J_{II},\ \Phi_{III},\ J_{III} \tag{5.17}$$

and that Φ_k, J_k $(k = I, II, III)$ are indeed canonical variables which can now be interpreted as action-angle variables since

$$\frac{dJ_k}{ds} = 0 \Longrightarrow J_k = const; \tag{5.18a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L}Q_k \Longrightarrow \Phi_k = \frac{2\pi}{L}Q_k\cdot s + const\,. \tag{5.18b}$$

These variables may also be used to describe the orbital motion.

Later we will need the fact that the revolution matrix $\underline{\tilde{M}}(s+L,s)$ has (eqns. (4.18) and (5.1)) the simple block diagonal form:

$$\underline{\tilde{M}}(s+L,s) = \begin{pmatrix} \underline{M}^{(\beta)}_{(4\times 4)}(s+L,s) & \underline{0}_{(4\times 2)} \\ \underline{0}_{(2\times 4)} & \underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s) \end{pmatrix} \tag{5.19}$$

where $\underline{M}^{(\beta)}_{(4\times 4)}(s+L,s)$ corresponds to the (transverse) betatron motion and $\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s)$ to the (longitudinal) synchrotron oscillations.

Furthermore, the 2-dimensional revolution matrix $\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s)$ which is defined by the equations of synchrotron motion:

$$\frac{d}{ds}\,\tilde{\sigma} \;=\; -[K_x \cdot D_x + K_z \cdot D_z] \cdot \tilde{p}_\sigma \;; \tag{5.20a}$$

$$\frac{d}{ds}\,\tilde{p}_\sigma \;=\; h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi \cdot \tilde{\sigma} \tag{5.20b}$$

(see eqns. (4.12), (4.18) and (5.1)) can be represented in the form:

$$\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_\sigma + \alpha_\sigma(s) \cdot \sin 2\pi Q_\sigma & \beta_\sigma(s) \cdot \sin 2\pi Q_\sigma \\ -\gamma_\sigma(s) \cdot \sin 2\pi Q_\sigma & \cos 2\pi Q_\sigma + \alpha_\sigma(s) \cdot \sin 2\pi Q_\sigma \end{pmatrix} \tag{5.21}$$

with

$$\beta_\sigma \cdot \gamma_\sigma = \alpha_\sigma^2 + 1 \,. \tag{5.22}$$

From these equations one sees that for the eigenvectors $\vec{v}_k(s)$ one can write:

$$\vec{v}_k \;=\; \begin{pmatrix} \vec{v}_k^{\,(\beta)} \\ \vec{0}_2 \end{pmatrix} \;;\quad (k = I,\ II)\;; \tag{5.23a}$$

$$\vec{v}_{III} \;=\; \begin{pmatrix} \vec{0}_4 \\ \vec{w}_\sigma \end{pmatrix} \;;\quad \vec{w}_\sigma = \frac{1}{\sqrt{2\beta_\sigma(s)}} \cdot \begin{pmatrix} \beta_\sigma(s) \\ -[\alpha_\sigma(s) + i] \end{pmatrix} \cdot e^{-i \cdot \varphi_\sigma(s)} \tag{5.23b}$$

where, in the case that the betatron oscillations are decoupled:

$$\underline{M}^{(\beta)}_{(4\times 4)}(s+L,s) = \begin{pmatrix} \underline{M}^{(x)}_{(2\times 2)}(s+L,s) & \underline{0}_{(2\times 2)} \\ \underline{0}_{(2\times 2)} & \underline{M}^{(z)}_{(2\times 2)}(s+L,s) \end{pmatrix}; \tag{5.24a}$$

$$\underline{M}^{(y)}_{(2\times 2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_y + \alpha_y \sin 2\pi Q_y & \beta_y \sin 2\pi Q_y \\ -\gamma_y \sin 2\pi Q_y & \cos 2\pi Q_y + \alpha_y \sin 2\pi Q_y \end{pmatrix}; \tag{5.24b}$$

$$\beta_y \cdot \gamma_y = \alpha_y^2 + 1 \;;\quad (y \equiv x,\ z) \tag{5.24c}$$

the vectors $\vec{v}_I$ and $\vec{v}_{II}$ take a form similar to $\vec{v}_{III}$:

$$\vec{v}_I^{\,(\beta)} \;=\; \begin{pmatrix} \vec{w}_x \\ \vec{0}_2 \end{pmatrix} \;;\quad \vec{v}_{II}^{\,(\beta)} \;=\; \begin{pmatrix} \vec{0}_2 \\ \vec{w}_z \end{pmatrix} \;; \tag{5.25a}$$

$$\vec{w}_y \;=\; \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} \cdot e^{-i \cdot \varphi_y(s)} \;; \tag{5.25b}$$

$$(y \equiv x,\ z)\,.$$

<u>Remark:</u>

An approximate form for the matrix $\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s)$ can be established if, in the equation of motion (5.20a, b), the coefficients of $\tilde{\sigma}$ and $\tilde{p}_\sigma$ are averaged over one turn (oscillator-model):

$$[K_x \cdot D_x + K_z \cdot D_z] \;\longrightarrow\; \kappa = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [K_x \cdot D_x + K_z \cdot D_z] \tag{5.26a}$$

(momentum compaction factor);

$$h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi \longrightarrow \kappa = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi$$
$$= \frac{1}{L} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{\cos\varphi}{\sin\varphi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{eV(\tilde{s})}{E_0} \cdot \sin\varphi$$
$$= \frac{\Omega^2}{\kappa} \tag{5.26b}$$

with

$$\Omega^2 = \frac{\kappa}{L} \cdot h \cdot \frac{2\pi}{L} \cdot \text{ctg}\,\varphi \cdot \frac{U_{Cav}}{E_0} \tag{5.27}$$

where U_{Cav} is given by eqn. (2.57). Thus, eqn. (5.20) transforms to

$$\frac{d}{ds}\tilde{\sigma} = -\kappa \cdot \tilde{p}_\sigma ; \tag{5.28a}$$

$$\frac{d}{ds}\tilde{p}_\sigma = \frac{\Omega^2}{\kappa} \cdot \tilde{\sigma} \tag{5.28b}$$

with the solution:

$$\begin{pmatrix} \tilde{\sigma}(s) \\ \tilde{p}_\sigma(s) \end{pmatrix} = \begin{pmatrix} \cos\Omega(s-s_0) & -(\kappa/\Omega)\cdot\sin\Omega(s-s_0) \\ (\kappa/\Omega)\cdot\sin\Omega(s-s_0) & \cos\Omega(s-s_0) \end{pmatrix} \begin{pmatrix} \tilde{\sigma}(s_0) \\ \tilde{p}_\sigma(s_0) \end{pmatrix} . \tag{5.29}$$

Using this "oscillator-model", the one turn matrix is given by:

$$\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s) = \begin{pmatrix} \cos\Omega L & -(\kappa/\Omega)\cdot\sin\Omega L \\ (\kappa/\Omega)\cdot\sin\Omega L & \cos\Omega L \end{pmatrix} \begin{pmatrix} \tilde{\sigma}(s_0) \\ \tilde{p}_\sigma(s_0) \end{pmatrix} \tag{5.30}$$

and by the comparison of (5.30) with (5.21) we find ($\beta_\sigma > 0$):

$$2\pi Q_\sigma = -\Omega \cdot L ; \tag{5.31a}$$

$$\beta_\sigma = \frac{\kappa}{\Omega} ; \tag{5.31b}$$

$$\alpha_\sigma = 0 ; \tag{5.31c}$$

$$\gamma_\sigma = \frac{\Omega}{\kappa} = \frac{1}{\beta_\sigma} \tag{5.31d}$$

where the quantities Ω and κ are taken from (5.26a) and (5.27). In particular by substituting (5.30a) and (5.31) in (5.35b) one obtains:

$$\beta_\sigma^2 = \frac{L}{2\pi} \cdot \frac{1}{h} \cdot \frac{\kappa \cdot \text{tg}\varphi}{\int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \cdot [K_x^2(\tilde{s}) + K_z^2(\tilde{s})]} . \tag{5.32}$$

6 The Perturbed Problem

The general solution of the unperturbed equation of motion (5.1) can be written in the form

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k \cdot \vec{v}_k(s) + A_{-k} \cdot \vec{v}_{-k}(s)\}$$

with A_k, A_{-k} being constants of integration ($k = I,\ II,\ III$).

In order to solve the perturbed problem (4.19) we now make the following "ansatz" (variation of constants):

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k(s) + A_{-k}(s) \cdot \vec{v}_{-k}(s)\} \ . \tag{6.1}$$

Inserting (6.1) into (4.19) one obtains:

$$\begin{aligned}\sum_{k=I,II,III} \{A_k'(s) \cdot \vec{v}_k + A_{-k}'(s) \cdot \vec{v}_{-k}\} &= [\Delta \underline{\tilde{A}} + \delta \underline{\tilde{A}}] \cdot \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k + A_{-k}(s) \cdot \vec{v}_{-k}\} \\ &\quad + \vec{\tilde{c}}_2 + \vec{\tilde{c}}_w + \delta \vec{\tilde{c}} \ .\end{aligned} \tag{6.2}$$

Using the orthogonality conditions (5.10) and the relations (4.24a, b) one gets from (6.2) for $k = I,\ II,\ III$:

$$\begin{aligned} A_k'(s) &= X_k(s) - i \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \vec{\tilde{c}}_2 \\ &\quad - i \cdot C_1 \cdot \left\{ G_{QS} \cdot [\tilde{x}^3 + \tilde{x}\tilde{z}^2] + G_{DO} \cdot \frac{1}{3} [3\, \tilde{x}\tilde{z}^2 - \tilde{x}^3]\ . \right\} \cdot \vec{v}_k^{\,+}(s)\, \underline{S}\, \vec{\tilde{D}}(s) \\ &\quad + i \cdot \sqrt{\omega(s)} \cdot \xi(s) \cdot \vec{v}_k^{\,+}(s)\, \underline{S}\, \vec{\tilde{D}}(s)\ ; \end{aligned} \tag{6.3a}$$

$$A_{-k}'(s) = [A_k'(s)]^* \tag{6.3b}$$

with

$$\begin{aligned} X_k(s) &= \sum_{l=I,II,III} A_l(s) \cdot (-i) \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \left[\Delta \underline{\tilde{A}} + \delta \underline{\tilde{A}}\right] \cdot \vec{v}_l \\ &+ \sum_{l=I,II,III} A_{-l}(s) \cdot (-i) \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \left[\Delta \underline{\tilde{A}} + \delta \underline{\tilde{A}}\right] \cdot \vec{v}_{-l}\ . \end{aligned} \tag{6.4}$$

Taking into account eqn. (2.54) and the defining equation (4.25) for $\vec{\tilde{D}}$ one can write for the term $\vec{v}_k^{\,+}\, \underline{S}\, \vec{\tilde{D}}$ appearing on the r.h.s. of (6.3a):

$$\vec{v}_k^{\,+}\, \underline{S}\, \vec{\tilde{D}} = v_{k2}^* \cdot D_1 - v_{k1}^* \cdot D_2 + v_{k4}^* \cdot D_3 - v_{k3}^* \cdot D_4 + v_{k5}^*$$

or, using (5.23):

$$\vec{v}_k^{\,+}\underline{S}\vec{\tilde{D}} = \begin{cases} v_{k2}^* \cdot D_1 - v_{k1}^* \cdot D_2 + v_{k4}^* \cdot D_3 - v_{k3}^* \cdot D_4 & \text{for } k = I,\ II\ ; \\ +v_{k5}^* & \text{for } k = III\ . \end{cases} \tag{6.5}$$

7 Stochastic Equations for the Variables $J_k(s)$ and $\Phi_k(s)$

Writing $A_k(s)$ in the form (5.11), we obtain for the derivative $A'_k(s)$:

$$
\begin{aligned}
A'_k(s) &= \frac{1}{2}\cdot\frac{J'_k}{\sqrt{J_k}}\cdot e^{-i\cdot[\Phi_k - 2\pi Q_k\cdot s/L]} - i\cdot\left[\Phi'_k - \frac{2\pi}{L}Q_k\right]\cdot A_k \\
&= A_k\cdot\left\{\frac{1}{2}\cdot\frac{J'_k}{J_k} - i\cdot\left[\Phi'_k - \frac{2\pi}{L}Q_k\right]\right\}
\end{aligned}
$$

and for the derivatives $J'_k(s)$ and $\Phi'_k(s)$ of the action - angle variables $J_k(s)$ and $\Phi_k(s)$:

$$
\begin{aligned}
J'_k(s) &= \frac{d}{ds}\left[A_k(s)\cdot A_{-k}(s)\right] \\
&= A'_k(s)\cdot A_{-k}(s) + A_k(s)\cdot A'_{-k}(s) \\
&= 2\cdot\Re e\left\{A'_k(s)\cdot A_{-k}(s)\right\}\ ; \qquad (7.1a)
\end{aligned}
$$

$$
\begin{aligned}
\Phi'_k(s) - \frac{2\pi}{L}\cdot Q_k &= -\frac{A'_k(s)\cdot A_{-k}(s)}{i\cdot J_k(s)} + \frac{1}{2}\cdot\frac{A'_k(s)\cdot A_{-k}(s) + A_k(s)\cdot A'_{-k}(s)}{i\cdot J_k(s)} \\
&= -\frac{1}{2}\cdot\frac{A'_k(s)\cdot A_{-k}(s) - A_k(s)\cdot A'_{-k}(s)}{i\cdot J_k(s)} \\
&= -\frac{1}{J_k(s)}\cdot\Im m\left\{A'_k(s)\cdot A_{-k}(s)\right\}\ . \qquad (7.1b)
\end{aligned}
$$

Here the terms $(A'_k\cdot A_{-k})$ appearing in (7.1a,b) are given by:

$$
\begin{aligned}
A'_k(s)\cdot A_{-k}(s) &= Y_k(s) + Z_k(s) \qquad (7.2a) \\
Y_k(s) &= Y_k^{(1)}(s) + Y_k^{(2)}(s) + Y_k^{(3)}(s) \qquad (7.2b)
\end{aligned}
$$

where we have separated Y_k into first, second and third order terms with

$$
\begin{aligned}
Y_k^{(1)}(s) &= X_k(s)\cdot A_{-k}(s) \\
&= \sum_{l=I,II,III}\sqrt{J_l}\cdot\sqrt{J_k}\cdot(-i)\cdot\vec{\hat{v}}_k^{\,+}\cdot\underline{S}\cdot\left[\Delta\underline{\tilde{A}} + \delta\underline{\tilde{A}}\right]\cdot\vec{\hat{v}}_l\cdot e^{i\cdot[\Phi_k - \Phi_l]} \\
&+ \sum_{l=I,II,III}\sqrt{J_l}\cdot\sqrt{J_k}\cdot(-i)\cdot\vec{\hat{v}}_k^{\,+}\cdot\underline{S}\cdot\left[\Delta\underline{\tilde{A}} + \delta\underline{\tilde{A}}\right]\cdot\vec{\hat{v}}_{-l}\cdot e^{i\cdot[\Phi_k + \Phi_l]}\ ; \qquad (7.3a)
\end{aligned}
$$

$$
\begin{aligned}
Y_k^{(2)}(s) &= -i\cdot\left[\vec{v}_k^{\,+}(s)\cdot A_{-k}(s)\right]\cdot\underline{S}\cdot\vec{\tilde{c}}_2 \\
&= -i\cdot\sqrt{J_k}\cdot e^{i\Phi_k(s)}\cdot\vec{\hat{v}}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{\tilde{c}}_2\ ; \qquad (7.3b)
\end{aligned}
$$

$$
\begin{aligned}
Y_k^{(3)}(s) &= -i\cdot C_1\cdot\left\{G_{QS}\cdot[\bar{x}^3 + \bar{x}\bar{z}^2] + G_{DO}\cdot\frac{1}{3}[3\,\bar{x}\bar{z}^2 - \bar{x}^3]\,.\right\} \\
&\quad\times\left[\vec{v}_k^{\,+}(s)\cdot A_{-k}(s)\right]\cdot\underline{S}\cdot\vec{\tilde{D}} \\
&= -i\cdot C_1\cdot\left\{G_{QS}\cdot[\bar{x}^3 + \bar{x}\bar{z}^2] + G_{DO}\cdot\frac{1}{3}[3\,\bar{x}\bar{z}^2 - \bar{x}^3]\,.\right\} \\
&\quad\times\sqrt{J_k}\cdot e^{i\Phi_k(s)}\cdot\vec{\hat{v}}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{\tilde{D}} \qquad (7.3c)
\end{aligned}
$$

and

$$\begin{aligned} Z_k(s) &= i \cdot \sqrt{\omega(s)} \cdot \xi(s) \cdot \left[\vec{v}_k^{+}(s) \cdot A_{-k}(s)\right] \cdot \underline{S} \cdot \hat{\vec{D}} \\ &= i \cdot \sqrt{\omega(s)} \cdot \xi(s) \cdot \sqrt{J_k} \cdot e^{i \cdot \Phi_k(s)} \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \hat{\vec{D}} \, . \end{aligned} \tag{7.4}$$

If we write then $J_k'(s)$ and $\Phi_k'(s)$ in the form:

$$J_k'(s) = K_J^{(k)}(\Phi_k, J_k) + Q_J^{(k)}(\Phi_k, J_k) \cdot \xi(s) \; ; \tag{7.5a}$$

$$\Phi_k'(s) = K_\Phi^{(k)}(\Phi_k, J_k) + Q_\Phi^{(k)}(\Phi_k, J_k) \cdot \xi(s) \tag{7.5b}$$

we obtain:

$$\begin{aligned} K_J^{(k)} &= 2 \cdot \Re e\left\{Y_k(s)\right\} \; ; \\ Q_J^{(k)} &= -i \cdot \sqrt{\omega} \cdot \sqrt{J_k} \cdot \left\{\vec{v}_k^{+} \underline{S} \hat{\vec{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{v}_k^{+} \underline{S} \hat{\vec{D}}\right]^* \cdot e^{-i \cdot \Phi_k}\right\} \; ; \end{aligned} \tag{7.6a}$$

$$\begin{aligned} K_\Phi^{(k)} &= +\frac{2\pi}{L} \cdot Q_k - \frac{1}{J_k(s)} \cdot \Im m\left\{Y_k(s)\right\} \; ; \\ Q_\Phi^{(k)} &= +\sqrt{\omega} \cdot \frac{1}{2\sqrt{J_k}} \cdot \left\{\vec{v}_k^{+} \underline{S} \hat{\vec{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{v}_k^{+} \underline{S} \hat{\vec{D}}\right]^* \cdot e^{-i \cdot \Phi_k}\right\} \, . \end{aligned} \tag{7.6b}$$

In making the transformation from the variables $(\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma)$ to (J_k, Φ_k) we have used the usual rules of algebra i.e. in our classical model of photon emission we are interpreting the Langevin equations according to the Stratanovich convention (see for example [15, 23]).

A comparison of eqns. (7.5) and (7.6) with eqn. (5.18) shows explicitely how the J_k' and Φ_k' are modified by radiation effects. Note also from (7.5) that the stochastic motions of J_k and Φ_k are driven from a common noise source. Furthermore, it is clear that the variables (J_k, Φ_k) which by construction originally described uncoupled normal modes, have become coupled via the radiation emission. Also, in contrast to the use of $(\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma)$ variables in eqn. (5.16), the Langevin equations for the (J_k, Φ_k) are nonlinear.

The relations (7.5) and (7.6) now provide the basis for a Fokker-Planck [15] treatment of orbit motion.

8 The Fokker-Planck Equation of Stochastic Orbital Motion

Now that the stochastic differential equations (7.5a, b) of the orbital motion have been established the Fokker-Planck (F-P) equation for the orbit phase space density function $W(J, \Phi; s)$ can be written without further ado as [2]:

$$\begin{aligned} \frac{\partial W}{\partial s} &= \sum_{k=I,II,III} \left\{-\frac{\partial}{\partial J_k}[D_I^{(k)} \cdot W] - \frac{\partial}{\partial \Phi_k}[D_\Phi^{(k)} \cdot W]\right\} \\ &+ \sum_{k,l=I,II,III} \left\{\frac{1}{2}\frac{\partial^2}{\partial J_k \partial J_l}[Q_J^{(k)} \cdot Q_J^{(l)} \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[Q_J^{(k)} \cdot Q_\Phi^{(l)} \cdot W] \right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \cdot W]\right\} \end{aligned} \tag{8.1}$$

with the drift coefficients given by

$$D_J^{(k)} = K_J^{(k)} + \tilde{K}_J^{(k)} ; \tag{8.2a}$$
$$D_\Phi^{(k)} = K_\Phi^{(k)} + \tilde{K}_\Phi^{(k)} \tag{8.2b}$$

and where the quantities $\tilde{K}_J^{(k)}$ and $\tilde{K}_\Phi^{(k)}$ are the artificial drift terms which arise when using the Stratanovich interpretation of eqn. (4.19):

$$\tilde{K}_J^{(k)} = \frac{1}{2}\frac{\partial Q_J^{(k)}}{\partial J_k} \cdot Q_J^{(k)} + \frac{1}{2}\frac{\partial Q_J^{(k)}}{\partial \Phi_k} \cdot Q_\Phi^{(k)} ; \tag{8.3a}$$
$$\tilde{K}_\Phi^{(k)} = \frac{1}{2}\frac{\partial Q_\Phi^{(k)}}{\partial J_k} \cdot Q_J^{(k)} + \frac{1}{2}\frac{\partial Q_\Phi^{(k)}}{\partial \Phi_k} \cdot Q_\Phi^{(k)} . \tag{8.3b}$$

(Note that $Q_J^{(k)}$ and $Q_\Phi^{(k)}$ only contain the two variables J_k and Φ_k; see eqn. (7.6).)

From (8.3a,b) one has, taking into account (7.6):

$$\begin{aligned} \tilde{K}_J^{(k)} &= \omega(s) \cdot |\vec{\bar{v}}_k^{\,+} \underline{S} \vec{\tilde{D}}|^2 \\ &= \omega(s) \cdot |\vec{v}_k^{\,+} \underline{S} \vec{\tilde{D}}|^2 ; \end{aligned} \tag{8.4a}$$

$$\tilde{K}_\Phi^{(k)} = i \cdot \frac{\omega(s)}{4J_k} \cdot \left\{ \left[\vec{\bar{v}}_k^{\,+} \underline{S} \vec{\tilde{D}}\right]^2 e^{i \cdot 2\Phi_k} - \left[\left(\vec{\bar{v}}_k^{\,+} \underline{S} \vec{\tilde{D}}\right)^*\right]^2 e^{-i \cdot 2\Phi_k} \right\} . \tag{8.4b}$$

It is clear that the F-P equation (8.1) is very complicated and that the drift and diffusion coefficients are oscillating functions in s. But as in the previous paper [1] we will be interested in the long time (asymptotic) equilibrium behaviour and therefore it will be sufficient to deal with the distribution of quantities averaged over times on the scale of damping times [2]. Denoting the one-turn averages by the bracket $\langle\quad\rangle$, we therefore write the F-P equation in the form:

$$\begin{aligned} \frac{\partial W}{\partial s} = & \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k}[\langle D_J^{(k)}\rangle \cdot W] - \frac{\partial}{\partial \Phi_k}[\langle D_\Phi^{(k)}\rangle \cdot W] \right\} \\ + & \sum_{k,l=I,II,III} \left\{ \frac{1}{2}\frac{\partial^2}{\partial J_k \partial J_l}[\langle Q_J^{(k)} \cdot Q_J^{(l)}\rangle \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[\langle Q_J^{(k)} \cdot Q_\Phi^{(l)}\rangle \cdot W] \right. \\ & \left. + \frac{1}{2}\frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)}\rangle \cdot W] \right\} \end{aligned} \tag{8.5}$$

whereby oscillating terms of the integrand due to the (linear) s-dependence of the angle variables Φ_k:

$$\Phi_k(s) \approx \Phi_k(0) + (s - s_0) \cdot \frac{2\pi}{L} Q_k$$

(see eqn. (5.18b)) may be neglected since they are approximately averaged away by integration.

To do that, we first introduce the following abbreviations:

$$\begin{aligned} \Delta Q_k &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{\bar{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \Delta\underline{\tilde{A}}(s) \cdot \vec{v}_{\bar{k}}(s) \\ &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \Delta\underline{\tilde{A}}(s) \cdot \vec{v}_k(s) ; \end{aligned} \tag{8.6a}$$

and

$$\begin{aligned}\delta Q_k &= \frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} ds\cdot \vec{\tilde{v}}_k^{\,+}(s)\cdot \underline{S}\cdot \delta\underline{\tilde{A}}(s)\cdot \vec{v}_k(s)\\ &= \frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} ds\cdot \vec{v}_k^{\,+}(s)\cdot \underline{S}\cdot \delta\underline{\tilde{A}}(s)\cdot \vec{v}_k(s)\ ; \qquad (8.6b)\\ &(k = I,\ II,\ III)\end{aligned}$$

and the quantities (the "damping constants"; see forward to eqn. (8.19)):

$$\alpha_k = -2\pi\cdot \Im m\{\delta Q_k\}\ . \qquad (8.7)$$

Note that ΔQ_k is real (eqn. (4.26)).

Thus by using (7.3a, b, c):

$$\begin{aligned}\langle Y_k^{(1)}\rangle &= J_k\cdot(-i)\cdot\frac{2\pi}{L}[\Delta Q_k+\delta Q_k]\\ &= J_k\cdot(-i)\cdot\frac{2\pi}{L}\cdot\left[\Re e\{\Delta Q_k+\delta Q_k\}-i\cdot\frac{1}{2\pi}\alpha_k\right]\ ; \qquad (8.8a)\end{aligned}$$

$$\langle Y_k^{(2)}\rangle = -i\cdot\sqrt{J_k}\cdot\langle e^{i\Phi_k(s)}\cdot \vec{\tilde{v}}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{\tilde{c}}_2\rangle\ ; \qquad (8.8b)$$

$$\begin{aligned}\langle Y_k^{(3)}\rangle &= -i\cdot C_1\cdot\sqrt{J_k}\ \times\\ &\left\langle\left\{G_{QS}\cdot[\tilde{x}^3+\tilde{x}\tilde{z}^2]+G_{DO}\cdot\frac{1}{3}[3\,\tilde{x}\tilde{z}^2-\tilde{x}^3]\right\}\cdot e^{i\Phi_k(s)}\cdot\vec{\tilde{v}}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{\tilde{D}}\right\rangle\ . \qquad (8.8c)\end{aligned}$$

In Appendix C it is shown that the quantities ΔQ_k and δQ_k appearing in (8.8a) are just the (complex) Q-shifts of the k-th oscillation mode ($k = I, II, III$) caused by the (linear) perturbations $\Delta\underline{A}$ and $\delta\underline{A}$ (see also Ref. [1]).

Away from resonances only the non - oscillating terms inside the $\langle\ \rangle$ survive. In particular, since $\vec{\tilde{c}}_2$ only contains quadratic terms, $\langle Y_k^{(2)}\rangle$ vanishes:

$$\langle Y_k^{(2)}\rangle = 0\ , \qquad (8.9)$$

leaving contributions from just the first and third orders. The quantity $\langle Y_k^{(1)}\rangle$ is the usual linear damping term and contains contributions due to the closed orbit distortions. The next non - vanishing order is $\langle Y_k^{(3)}\rangle$.

In order to exhibit the essentials of the damping effects due to the nonlinear wigglers we now adopt the smooth approximation:

$$\tilde{x} \approx D_1\cdot\tilde{p}_\sigma\ ; \qquad (8.10a)$$

$$\tilde{z} \approx D_3\cdot\tilde{p}_\sigma \qquad (8.10b)$$

$$\begin{aligned}\Longrightarrow\quad [\tilde{x}^3+\tilde{x}\tilde{z}^2] &\approx [D_1^3+D_1D_3^2]\cdot\tilde{p}_\sigma^3\\ &= [D_1^3+D_1D_3^2]\cdot J_{III}^{3/2}\cdot\left\{\tilde{v}_{III6}(s)\cdot e^{-i\Phi_{III}}+\tilde{v}_{-III6}(s)\cdot e^{+i\Phi_{III}}\right\}^3\ ;\end{aligned}$$

$$\begin{aligned}[3\,\tilde{x}\tilde{z}^2-\tilde{x}^3] &\approx [3\,D_1D_3^2-D_1^3]\cdot\tilde{p}_\sigma^3\\ &= [3\,D_1D_3^2-D_1^3]\cdot J_{III}^{3/2}\cdot\left\{\tilde{v}_{III6}(s)\cdot e^{-i\Phi_{III}}+\tilde{v}_{-III6}(s)\cdot e^{+i\Phi_{III}}\right\}^3\end{aligned}$$

and from (8.8c) we get :

$$\langle Y_k^{(3)} \rangle = 0 \qquad \text{for } k = I, II \; ; \tag{8.11a}$$

$$\begin{aligned}
\langle Y_k^{(3)} \rangle &= -i \cdot C_1 \cdot J_k^2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot 3\hat{v}_{k5}^*(\tilde{s}) \cdot \hat{v}_{k6}(\tilde{s}) \cdot |\hat{v}_{k6}(\tilde{s})|^2 \\
&\qquad \times \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [3\, D_1 D_3^2 - D_1^3] \right\} \\
&= -i \cdot C_1 \cdot J_k^2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot 3v_{k5}^*(\tilde{s}) \cdot v_{k6}(\tilde{s}) \cdot |v_{k6}(\tilde{s})|^2 \\
&\qquad \times \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [3\, D_1 D_3^2 - D_1^3] \right\} \\
&= +\frac{3}{4} \cdot i \cdot C_1 \cdot J_k^2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \gamma_\sigma(\tilde{s}) \cdot [\alpha_\sigma(\tilde{s}) + i] \\
&\qquad \times \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [3\, D_1 D_3^2 - D_1^3] \right\} \\
&\qquad \text{for } k = III \; .
\end{aligned} \tag{8.11b}$$

Then we obtain from (8.2), (7.6) and (8.4):

$$\begin{aligned}
\langle D_J^{(k)} \rangle &= -J_k \cdot \frac{2}{L} \alpha_k + 2 \cdot \Re e\{Y_k^{(3)}(s)\} \\
&\quad + \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \omega(\tilde{s}) \; ; \\
&= -J_k \cdot \frac{2}{L} \alpha_k \\
&\quad -\frac{3}{2} \cdot C_1 \cdot J_k^2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \gamma_\sigma \\
&\qquad \times \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [D_1^3 - D_1 D_3^2] \right\} \\
&\quad + \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \omega(\tilde{s}) \; ;
\end{aligned} \tag{8.12a}$$

$$\langle D_\Phi^{(k)} \rangle = \frac{2\pi}{L} \cdot [Q_k + \Re e\{\Delta Q_k + \delta Q_k\}] \; ; \tag{8.12b}$$

$$\langle (Q_J^{(k)})^2 \rangle = \frac{2J_k}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \omega(\tilde{s}) \; ; \tag{8.12c}$$

$$\langle (Q_\Phi^{(k)})^2 \rangle = \frac{1}{2J_k \cdot L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \omega(\tilde{s}) \tag{8.12d}$$

and

$$\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle = 0 \text{ for } k \neq l \; ; \tag{8.13a}$$

$$\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \text{ for } k \neq l \; ; \tag{8.13b}$$

$$\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \; . \tag{8.13c}$$

Introducing the constants :

$$a_k = \alpha_k \cdot \frac{1}{L} \; ; \tag{8.14a}$$

$$M_k = \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot|\vec{v}_k^{\,+}\underline{S}\hat{\vec{D}}|^2\cdot\omega(\tilde{s}) \tag{8.14b}$$

$$b_k = 2\pi\cdot\frac{1}{L}\cdot\hat{Q}_k\ ; \tag{8.14c}$$

$$d_k = \begin{cases} 0 \quad \text{for } k = I,\ II\ ; \\ 3\cdot C_1\cdot\frac{1}{L}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\gamma_\sigma(\tilde{s}) \\ \quad\times\left\{G_{QS}\cdot[D_1^3 + D_1 D_3^2] + G_{DO}\cdot\frac{1}{3}[D_1^3 - D_1 D_3^2]\right\} \quad \text{for } k = III \end{cases} \tag{8.14d}$$

with

$$\hat{Q}_k = Q_k + \Re e\{\Delta Q_k + \delta Q_k\} \tag{8.15}$$

we can finally write:

$$\begin{aligned}\frac{\partial W}{\partial s} &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}\left[(-2a_k\cdot J_k + M_k - \frac{d_k}{2}\cdot J_{III}^2)\cdot W\right] - \frac{\partial}{\partial\Phi_k}[b_k\cdot W]\right.\\ &\qquad\left. +\frac{1}{2}\frac{\partial^2}{\partial J_k^2}[2J_k\cdot M_k\cdot W] + \frac{1}{2}\frac{\partial^2}{\partial\Phi_k^2}\left[\frac{1}{2J_k}\cdot M_k\cdot W\right]\right\}\\ &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}\left[-2a_k\cdot J_k\cdot W - \frac{d_k}{2}\cdot J_{III}^2\cdot W - M_k\cdot J_k\cdot\frac{\partial W}{\partial J_k}\right]\right.\\ &\qquad\left. -\frac{\partial}{\partial\Phi_k}\left[b_k\cdot W - \frac{M_k}{4J_k}\cdot\frac{\partial W}{\partial\Phi_k}\right]\right\}\ . \end{aligned} \tag{8.16}$$

This equation determines the averaged charge distribution of the particles in a bunch. On comparison with eqn. (8.1) we see that the s-dependent coefficients have been replaced by s-independent constants given by the one turn averages and that the r.h.s. has separated into a sum of three terms, one for each pair of action-angle variables.

<u>Remarks:</u>

1) The averaging procedure indicated by the bracket $\langle\ \ \rangle$ only results in the forms (8.12) and (8.13) away from the linear and second order resonances

$$\begin{aligned} n_1\cdot Q_I + n_2\cdot Q_{II} + n_3\cdot Q_{III} &\approx \textit{integer}\ ;\\ |n_1| + |n_2| + |n_3| &\leq 4\ . \end{aligned}$$

On resonance the common noise source would cause the modes to be correlated and also extra non-oscillating terms would appear [19]. But on resonance the particle motion can be unstable so that this case is of no interest here.

2) In eqn. (7.5) the first terms on the r.h.s. describe the influence of the continuous emission of synchrotron radiation on the synchro-betatron oscillations and the second terms the influence of quantum fluctuations of the radiation field (function $\xi(s)$ in (7.5)). If the quantum fluctuation term is neglected and if one takes into account eqns. (7.6a) and (8.8) by

neglecting the influence of wigglers (i.e. the term $Y_k^{(3)}$), eqn. (7.5a) may (approximately) be written in the form:

$$J'_k(s) \;=\; -\frac{2}{L}\alpha_k \cdot J_k(s) \; . \tag{8.17}$$

Equation (8.17) can be integrated with the result:

$$\begin{aligned} J_k(s) &= J_k(s_0) \cdot e^{-2\alpha_k \cdot (s - s_0)/L} \\ \Longrightarrow \quad \sqrt{J_k(s)} &= \sqrt{K_k(s_0)} \cdot e^{-\alpha_k \cdot (s - s_0)/L} . \end{aligned} \tag{8.18}$$

Since $\sqrt{J_k}$ represents the amplitude for the k-th mode of the synchro-betatron oscillations, the quantity α_k may be interpreted as the damping constant of the k-th mode [20, 24]. From (8.6b) and (8.7) we have:

$$\begin{aligned} \alpha_k &= -\Im m \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \delta\underline{\bar{A}}(s) \cdot \vec{v}_k(s) \\ &= \frac{i}{2} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{+}(s) \cdot \left[\underline{S} \cdot \delta\underline{A}(s) + \delta\underline{A}^T(s) \cdot \underline{S}\right] \cdot \vec{v}_k(s) \; . \end{aligned} \tag{8.19}$$

This formula may be used for a calculation of the damping constants.

By using (4.23) and (5.23) we get:

$$\alpha_k = -\Im m \int_{s_0}^{s_0+L} ds \cdot [\vec{v}_k^{\,(\beta)}(s)]^+ \cdot \underline{S}_4 \cdot \left[\delta\underline{\bar{A}}_{(4\times4)}(s) - \vec{D}(s) \cdot \vec{c}_4^{\,T}(s)\right] \cdot \vec{v}_k^{\,(\beta)}(s) \quad \text{for} \quad k = I,\ II \; ;$$

$$\alpha_{III} = -\Im m \int_{s_0}^{s_0+L} ds \cdot [\vec{w}_\sigma(s)]^+ \cdot \underline{S}_2 \cdot \begin{pmatrix} 0 & \vec{D}^T \cdot \underline{S}_4 \cdot \delta\underline{\bar{A}}_{(4\times4)} \cdot \vec{D} \\ 0 & \vec{c}_4^{\,T} \cdot \vec{D} + \delta\bar{A}_{66} \end{pmatrix} \cdot \vec{w}_\sigma(s)$$

or using (4.22a, b), (3.4c), (2.55) and (2.56):

$$\begin{aligned} \alpha_k &= \frac{1}{2} \cdot \frac{U_{Cav}}{E_0} - \Im m \int_{s_0}^{s_0+L} ds \cdot [v_{k1}^* \cdot D_2 - v_{k2}^* \cdot D_1 + v_{k3}^* \cdot D_4 - v_{k4}^* \cdot D_3] \\ &\qquad \times \sum_{n=1}^{4} \delta\bar{A}_{6n} \cdot v_{kn} \quad \text{for} \quad k = I,\ II \; ; \end{aligned} \tag{8.20a}$$

$$\begin{aligned} \alpha_{III} &= \frac{U_{Bend}}{E_0} + C_1 \cdot \int_{s_0}^{s_0+L} ds \left\{ \left[g^2 + N^2\right] \cdot (x_0^2 + z_0^2) + \left(\frac{e}{E_0}\right)^2 \left[(\Delta\mathcal{B}_x)^2 + \Delta\mathcal{B}_z)^2\right] \right\} \\ &\quad -\frac{1}{2} \cdot \int_{s_0}^{s_0+L} ds \cdot \sum_{n=1}^{4} \delta\bar{A}_{6n}(s) \cdot D_n(s) \; . \end{aligned} \tag{8.20b}$$

These results are already derived in Ref. [13].

For the sum

$$\alpha_I + \alpha_{II} + \alpha_{III}$$

one obtains from eqn. (8.20):

$$\alpha_I + \alpha_{II} + \alpha_{III}$$

$$= C_1 \cdot \int_{s_0}^{s_0+L} ds \left\{ \left[g^2 + N^2\right] \cdot (x_0^2 + z_0^2) + \left(\frac{e}{E_0}\right)^2 \left[(\Delta\mathcal{B}_x)^2 + \Delta\mathcal{B}_z)^2\right] \right\}$$

$$+ \frac{U_{Cav}}{E_0} + \frac{U_{Bend}}{E_0} + \frac{1}{2} \cdot \sum_{n=1}^{4} \int_{s_0}^{s_0+L} ds \cdot \delta\bar{A}_{6n}(s)$$

$$\times \left\{ -D_n + \sum_{k=I,II} \left[-i \cdot \left([\vec{v}_k^{\,(\beta)}]^+ \cdot \underline{S}_4 \cdot \vec{D} \right) \cdot v_{kn}^{(\beta)} + i \cdot \left([\vec{v}_k^{\,(\beta)}]^+ \cdot \underline{S}_4 \cdot \vec{D} \right)^* \cdot [v_{kn}^{(\beta)}]^* \right] \right\} . \quad (8.21)$$

The dispersion vector $\vec{D}$ may now be expanded in terms of the eigenvectors

$$\vec{v}_k^{\,(\beta)} \quad \text{and} \quad \vec{v}_{-k}^{\,(\beta)} = [\vec{v}_k^{\,(\beta)}]^*$$

$(k = I,\ II)$:

$$\vec{D} = \sum_{k=I,II} \left[c_k \cdot \vec{v}_k^{\,(\beta)} + c_{-k} \cdot \vec{v}_{-k}^{\,(\beta)} \right] .$$

Since the coefficients c_k and c_{-k} according to (5.10) and (5.23a) are given by:

$$c_k = -i \cdot \left([\vec{v}_k^{\,(\beta)}]^+ \cdot \underline{S}_4 \cdot \vec{D} \right) ;$$
$$c_{-k} = c_k^*$$

one has:

$$\vec{D} = \sum_{k=I,II} \left[-i \cdot \left([\vec{v}_k^{\,(\beta)}]^+ \cdot \underline{S}_4 \cdot \vec{D} \right) \vec{v}_k^{\,(\beta)} + i \cdot \left([\vec{v}_k^{\,(\beta)}]^+ \cdot \underline{S}_4 \cdot \vec{D} \right)^* \vec{v}_{-k}^{\,(\beta)} \right] . \quad (8.22)$$

It is then clear from (8.22) that the second summand on the right sight of (8.21) vanishes so that finally

$$\alpha_I + \alpha_{II} + \alpha_{III} = \frac{U_{Cav}}{E_0} + \frac{U_{Bend}}{E_0}$$
$$+ C_1 \cdot \int_{s_0}^{s_0+L} ds \left\{ \left[g^2 + N^2\right] \cdot (x_0^2 + z_0^2) + \left(\frac{e}{E_0}\right)^2 \left[(\Delta\mathcal{B}_x)^2 + \Delta\mathcal{B}_z)^2\right] \right\} \quad (8.23)$$

results.

In the absence of wigglers one has

$$U_{Cav} = U_0$$

(see eqn. (2.58) with $U_{Wiggler} = 0$) with U_0 given by (2.56), (2.57) and (2.59) and thus:

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2\,\frac{U_0}{E_0} + C_1 \cdot \int_{s_0}^{s_0+L} ds \cdot \left[g^2 + N^2\right] \cdot (x_0^2 + z_0^2) . \quad (8.24)$$

Since, in general

$$C_1 \cdot \int_{s_0}^{s_0+L} ds \cdot \left[g^2 + N^2\right] \cdot (x_0^2 + z_0^2) \ll \frac{U_0}{E_0}$$

(due to the contribution of the quadrupoles to the radiation damping) we find:

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2\frac{U_0}{E_0} . \tag{8.25}$$

This relation is known as the Robinson Theorem and in the absence of wigglers it allows one of the damping constants to be defined in terms of the other two.

3) Taking into account also the influence of wigglers (i.e. the term $Y_k^{(3)}$), an additional term appears on the r.h.s. of eqn. (8.17):

$$J_k'(s) = -\frac{2}{L}\alpha_k \cdot J_k(s) - \frac{d_k}{2} \cdot J_k^2(s) \tag{8.26}$$

describing a nonlinear damping proportional J_k^2. Thus the quantities d_k can be interpreted as nonlinear damping constants.

In our approximation (8.10) only d_k for $k = III$ (i.e. for the synchrotron oscillation) is different from zero.

4) In order to optimize the design of the wiggler at a prescribed value of

$$d_{III} > 0$$

($d_{III} < 0$ leads to antidamping) it is convenient to have a positive integrand in (8.14d), i.e. one has to choose the sign of G_{QS} and G_{DO} and the local dispersion in such a way that the terms $G_{QS} \cdot [D_1^3 + D_1 D_3^2]$ and $G_{DO} \cdot [D_1^3 - D_1 D_3^2]$ in (8.14d) become positive at every position within the wiggler [13] .

5) Equation (8.26) can be generalized if we calculate the derivative of the average $< J_k >$ of the action variable J_k, using the Fokker-Planck equation (8.16).

Then we obtain:

$$\begin{aligned}
\frac{d}{ds} < J_k(s) > &= \frac{d}{ds} \int \cdots \int dJ_I\, dJ_{II}\, dJ_{III}\, d\Phi_I\, d\Phi_{II}\, d\Phi_{III} \cdot J_k \cdot W \\
&= \int \cdots \int dJ_I\, dJ_{II}\, dJ_{III}\, d\Phi_I\, d\Phi_{II}\, d\Phi_{III} \cdot J_k \cdot \frac{\partial}{\partial s} W \\
&= \int \cdots \int dJ_I\, dJ_{II}\, dJ_{III}\, d\Phi_I\, d\Phi_{II}\, d\Phi_{III} \cdot J_k
\end{aligned}$$

[13] Taking into account nonsymplectic third order radiation terms in the equation of motion (4.19) resulting from bending magnets, quadrupoles, skew quadrupoles and sextupoles, we would obtain an additional integrand in (8.14d) which takes positive and negative values around the ring. The contribution of these oscillating quantities to nonlinear damping has been neglected since they have in practice the tendency to average away by integration around the circumference of the ring. In principle the omitted terms could be included in a straightforward manner.

$$\times \sum_{l=I,II,III} \left\{ -\frac{\partial}{\partial J_l} \left[\left(-2a_l \cdot J_l + M_l - \frac{d_l}{2} \cdot J_l^2 \right) \cdot W - \frac{\partial}{\partial J_l} (J_l \cdot M_l \cdot W) \right] - \frac{\partial}{\partial \Phi_l} \left[b_l \cdot W - \frac{1}{4J_l} \cdot M_l \cdot \frac{\partial}{\partial \Phi_l} W \right] \right\}$$

$$= \int \cdots \int dJ_I \, dJ_{II} \, dJ_{III} \, d\Phi_I \, d\Phi_{II} \, d\Phi_{III} \cdot J_k \times \left\{ -\frac{\partial}{\partial J_k} \left[-2a_k \cdot J_k \cdot W - \frac{d_k}{2} \cdot J_k^2 \cdot W - M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right] \right\}$$

$$= \int \cdots \int dJ_I dJ_{II} dJ_{III} d\Phi_I d\Phi_{II} d\Phi_{III} \times \left[-2a_k \cdot J_k \cdot W - \frac{d_k}{2} \cdot J_k^2 \cdot W - M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right]$$

$$= \int \cdots \int dJ_I dJ_{II} dJ_{III} d\Phi_I d\Phi_{II} d\Phi_{III} \times \left[-2a_k \cdot J_k \cdot W - \frac{d_k}{2} \cdot J_k^2 \cdot W + M_k \cdot W \right]$$

$$= -2a_k \cdot < J_k > - \frac{d_k}{2} \cdot < J_k^2 > + M_k \tag{8.27}$$

(see also Ref. [2]).

In this equation there appear similar damping terms

$$-2a_k \cdot < J_k > = -\frac{2\,\alpha_k}{L} \cdot < J_k >$$

and

$$-\frac{d_k}{2} \cdot < J_k^2 >$$

as on the r.h.s. of (8.26) and an additional term M_k due to the influence of quantum fluctuation on the orbit motion (characterized by the function $\omega(s)$ in eqns. (8.14b)). Thus the constant M_k which is proportional to $\hbar$ is a measure of the stochastic excitation rate of orbit motion.

6) Inspection of eqn. (8.16) shows that for the angle variables there is no analogue of the coefficients α_k which lead to damping of the action variables. Thus the angle variables are only subject to diffusion and we can thus assume that the angles Φ_k are uniformly distributed in $[0, 2\pi]$ [2].

7) In Appendix D it is shown that the damping behaviour of the beam characterised by the linear and nonlinear damping constants can be modified by making slight changes in the frequency of the accelerating fields and how the dependence of damping constants and bunch lengths on frequency deviation may be computed.

8) Assuming that the phases Φ_k are uniformly distributed (see Appendix E), the phase space density function W becomes independent of the Φ_k and we may write:

$$W(J_k, \Phi_k) = \left(\frac{1}{2\pi}\right)^3 \cdot \hat{W}(J_I, J_{II}, J_{III}) \, ; \tag{8.28a}$$

$$\frac{\partial}{\partial s}\hat{W} = \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k} \left[-2a_k \cdot J_k \cdot \hat{W} - \frac{d_k}{2} \cdot J_{III}^2 \cdot \hat{W} - M_k \cdot J_k \cdot \frac{\partial}{\partial J_k} \hat{W} \right] \right\}. \quad (8.28b)$$

The relation (8.27b) has the form of a continuity equation:

$$\frac{\partial}{\partial s}\hat{W} + \sum_{k=I,II,III} \frac{\partial}{\partial J_k} \Im_k = 0 \quad (8.29a)$$

with

$$\Im_k = -2a_k \cdot J_k \cdot \hat{W} - \frac{d_k}{2} \cdot J_{III}^2 \cdot \hat{W} - M_k \cdot J_k \cdot \frac{\partial}{\partial J_k} \hat{W} \; . \quad (8.29b)$$

Thus $\Im_k$ may be interpreted as a current density for the probability $\hat{W}$.

9 Solution of the Fokker-Planck Equation

In order to solve the Fokker-Planck equation (8.16) we make the ansatz:

$$W = w_I(J_I, \Phi_I) \cdot w_{II}(J_{II}, \Phi_{II}) \cdot w_{III}(J_{III}, \Phi_{III}) \quad (9.1)$$

and obtain

for $k = I, \; II$:

$$\begin{aligned} \frac{\partial}{\partial s} w_k &= \frac{\partial}{\partial J_k} \left[2a_k \cdot J_k \cdot w_k + M_k \cdot J_k \cdot \frac{\partial}{\partial J_k} w_k \right] \\ &\quad - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot w_k - \frac{M_k}{4J_k} \cdot \frac{\partial}{\partial \Phi_k} w_k \right] \end{aligned} \quad (9.2a)$$

and for $k = III$:

$$\begin{aligned} \frac{\partial}{\partial s} w &= +\frac{\partial}{\partial J} \left[2a \cdot J \cdot w + \frac{d}{2} \cdot J^2 \cdot w + M \cdot J \cdot \frac{\partial}{\partial J} w \right] \\ &\quad - \frac{\partial}{\partial \Phi} \left[b \cdot w - \frac{M}{4J} \cdot \frac{\partial}{\partial \Phi} w \right] \end{aligned} \quad (9.2b)$$

with

$$\begin{aligned} w &\equiv w_{III} \; ; \\ J &\equiv J_{III} \; ; \\ \Phi &\equiv \Phi_{III} \; ; \\ a &\equiv a_{III} \; ; \\ b &\equiv b_{III} \; ; \\ M &\equiv M_{III} \end{aligned}$$

and

$$d \equiv d_{III} = 3 \cdot C_1 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \gamma_\sigma(\tilde{s}) \times \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [3\, D_1 D_3^2 - D_1^3] \right\} \tag{9.3a}$$

(see eqn. (8.14d)) or, using the oscillator model (see eqn. (5.31)):

$$d = 3 \cdot C_1 \cdot \frac{\Omega}{\kappa} \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \left\{ G_{QS} \cdot [D_1^3 + D_1 D_3^2] + G_{DO} \cdot \frac{1}{3} [3\, D_1 D_3^2 - D_1^3] \right\} . \tag{9.3b}$$

We are only interested in the stationary distribution:

$$\frac{\partial}{\partial s} w_k = 0 . \tag{9.4}$$

In this case eqn. (9.2) can easily be integrated. One gets

for $k = I,\ II$:

$$w_k = C_k \cdot \exp\left[-J_k \cdot \frac{2a_k}{M_k} \right] . \tag{9.5a}$$

and for $k = III$:

$$w = C \cdot \exp\left[-J \cdot \frac{2a}{M} - J^2 \cdot \frac{d}{4M} \right] . \tag{9.5b}$$

Thus in these approximations the nonlinear wigglers modify the longitudinal phase space distribution so that it is no longer Gaussian. The deviation from a Gaussian form depends on the size of the integral in eqn. (9.3a) over the wiggler strengths and the dispersions. As has been shown in Ref. [2], if d is large enough and positive the tails of the energy distribution can be reduced.

Here the factor C_k is fixed by the normalization condition:

$$\int_0^{2\pi} d\Phi_k \int_0^\infty dJ_k \cdot w_k(J_k, \Phi_k) = 1 \tag{9.6}$$

which leads to

$$C_k = \frac{1}{2\pi} \cdot \frac{1}{\hat{J}_k} \quad \text{for} \quad k = I,\ II \tag{9.7a}$$

and

$$C \equiv C_{III} = \frac{1}{2\pi} \cdot \frac{1}{\hat{J}} \tag{9.7b}$$

with

$$\hat{J}_k = \frac{M_k}{2a_k} = \frac{1}{2\alpha_k} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{\,+} \underline{S} \vec{\hat{D}}|^2 \cdot \omega(\tilde{s}) \quad \text{for} \ k = I,\ II \tag{9.8a}$$

and

$$\hat{J} = \int_0^\infty dJ \cdot \exp\left[-J \cdot \frac{2a}{M} - J^2 \cdot \frac{d}{4M}\right] \quad \text{for } k = III \; . \tag{9.8b}$$

Thus we have for W:

$$W^{(stat)} = \frac{1}{(2\pi)^3} \cdot \frac{1}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}} \cdot \exp\left[-\left(\frac{J_I}{\hat{J}_I} + \frac{J_{II}}{\hat{J}_{II}}\right)\right] \cdot \exp\left[-J \cdot \frac{2a}{M} - J^2 \cdot \frac{d}{4M}\right] \; . \tag{9.9}$$

In Appendix E it is shown that this solution is unique.

From eqn. (9.9) we obtain for the average $< J_k >$ of J_k $(k = I,\ II,\ III)$:

$$\begin{aligned} < J_k > \quad &\equiv \quad \int_0^{2\pi} d\Phi_k \int_0^\infty dJ_k \cdot w_k(J_k, \Phi_k) \cdot J_k \\ &= \begin{cases} \hat{J}_k \quad \text{for } k = I,\ II \; ; \\ \frac{1}{2\pi} \cdot \frac{1}{\hat{J}} \int_0^\infty dJ \cdot J \cdot \exp\left[-J \cdot \frac{2a}{M} - J^2 \cdot \frac{d}{4M}\right] \quad \text{for } k = III \end{cases} \end{aligned} \tag{9.10}$$

and using this result and eqn. (5.12):

$$\begin{aligned} < \tilde{y}_m \tilde{y}_n > \quad &\equiv \quad \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^\infty dJ_I \int_0^\infty dJ_{II} \int_0^\infty dJ_{III} \\ &\quad \times W^{(stat)}(J_k, \Phi_k) \cdot \tilde{y}_m(s)\, \tilde{y}_n(s) \\ &= \frac{1}{(2\pi)^3} \cdot \frac{1}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}} \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^\infty dJ_I \int_0^\infty dJ_{II} \int_0^\infty dJ_{III} \\ &\quad \times \exp\left[-\left(\frac{J_I}{\hat{J}_I} + \frac{J_{II}}{\hat{J}_{II}}\right)\right] \cdot \exp\left[-J \cdot \frac{2a}{M} - J^2 \cdot \frac{d}{4M}\right] \\ &\quad \times \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{\hat{v}_{km}(s) \cdot e^{-i\Phi_k} + \hat{v}^*_{km}(s) \cdot e^{+i\Phi_k}\right\} \\ &\quad \times \sum_{l=I,II,III} \sqrt{J_l} \cdot \left\{\hat{v}_{ln}(s) \cdot e^{-i\Phi_l} + \hat{v}^*_{ln}(s) \cdot e^{+i\Phi_l}\right\} \\ &= 2 \cdot \sum_{k=I,II,III} < J_k > \cdot \Re e \left\{\hat{v}_{km} \cdot \hat{v}^*_{kn}\right\} \\ &= 2 \cdot \sum_{k=I,II,III} < J_k > \cdot \Re e \left\{v_{km} \cdot v^*_{kn}\right\} \end{aligned} \tag{9.11}$$

where the term $< J_k >$ is given by eqn. (9.10).

Furthermore, the density of the particle distribution in the $(\tilde{x} - \tilde{p}_x - \tilde{z} - \tilde{p}_z - \tilde{\sigma} - \tilde{p}_\sigma)$ - phase space is given by:

$$\rho(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{\sigma}, \tilde{p}_\sigma) = W^{(stat)} \cdot |\det(\underline{\mathcal{J}})|^{-1} \tag{9.12}$$

where $\underline{\mathcal{J}}$ denotes the Jacobian matrix (5.16).

But from eqn. (5.15) it follows that:

$$|\det(\underline{\mathcal{J}})| = 1 \ .$$

Using then the relationship

$$J_k = |\vec{v}_k^{\,+} \underline{S}\, \vec{\tilde{y}}|^2$$

which may be derived from eqn. (5.12) and (5.9), eqn. (9.12) finally takes the form:

$$\begin{aligned} \rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma) &= \frac{1}{(2\pi)^3}\cdot\frac{1}{\hat{J}_I\cdot\hat{J}_{II}\cdot\hat{J}}\exp\left[-\left(\frac{J_I}{\hat{J}_I}+\frac{J_{II}}{\hat{J}_{II}}\right)\right]\cdot\exp\left[-J\cdot\frac{2a}{M}-J^2\cdot\frac{d}{4M}\right] \\ &= \frac{1}{(2\pi)^2}\cdot\frac{1}{\hat{J}_I\cdot\hat{J}_{II}}\exp\left[-\left(|\vec{v}_I^{\,+}\underline{S}\,\vec{\tilde{y}}|^2\cdot\frac{1}{\hat{J}_I}+|\vec{v}_{II}^{\,+}\underline{S}\,\vec{\tilde{y}}|^2\cdot\frac{1}{\hat{J}_{II}}\right)\right] \\ &\quad\times\frac{1}{2\pi}\cdot\frac{1}{\hat{J}}\exp\left[-\left(|\vec{v}_{III}^{\,+}\underline{S}\,\vec{\tilde{y}}|^2\cdot\frac{2a}{M}+|\vec{v}_{III}^{\,+}\underline{S}\,\vec{\tilde{y}}|^4\cdot\frac{d}{4M}\right)\right]\ . \end{aligned} \tag{9.13}$$

For further investigations it is useful to adopt the special form of the eigenvectors $\vec{v}_k$ in eqn. (5.23) (—no synchro-betatron coupling) to decompose the orbit vector $\vec{\tilde{y}}$ into two subvectors $\vec{\tilde{y}}^{(\beta)}$ and $\vec{\tilde{y}}^{(\sigma)}$:

$$\vec{\tilde{y}} = \begin{pmatrix} \vec{\tilde{y}}^{(\beta)} \\ \vec{\tilde{y}}^{(\sigma)} \end{pmatrix} , \tag{9.14}$$

a transverse part (betatron oscillations):

$$\vec{\tilde{y}}^{(\beta)} = \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \end{pmatrix} \tag{9.15}$$

and a longitudinal part (synchrotron oscillations):

$$\vec{\tilde{y}}^{(\sigma)} = \begin{pmatrix} \tilde{\sigma} \\ \tilde{p}_\sigma \end{pmatrix} . \tag{9.16}$$

Then one has:

$$J_k = \begin{cases} [\vec{v}_k^{(\beta)}]^+ \underline{S}_4 \cdot \vec{\tilde{y}}^{(\beta)} & \text{for } k = I,\ II\ ; \\ \vec{w}_\sigma^{\,+} \underline{S}_2 \cdot \vec{\tilde{y}}^{(\sigma)} & \text{for } k = III \end{cases} \tag{9.17}$$

and the density $\rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma)$ may be factorized as:

$$\rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma) = \rho_\beta(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z)\cdot\rho_\sigma(\tilde{\sigma},\tilde{p}_\sigma) \tag{9.18}$$

with

$$\begin{aligned}
\rho_\beta(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z) &= w_I(J_I,\Phi_I)\cdot w_{II}(J_{II},\Phi_{II}) \\
&= \frac{1}{(2\pi)^2}\frac{1}{\hat{J}_I\cdot\hat{J}_{II}}\exp\left[-\left(\frac{J_I}{\hat{J}_I}+\frac{J_{II}}{\hat{J}_{II}}\right)\right] \\
&= \frac{1}{(2\pi)^2}\frac{1}{\hat{J}_I\cdot\hat{J}_{II}}\exp\left[-\frac{|(\vec{v}_I^{\,(\beta)})^+\underline{S}_4\;\vec{\tilde{y}}^{\,(\beta)}|^2}{\hat{J}_I}-\frac{|(\vec{v}_{II}^{\,(\beta)})^+\underline{S}_4\;\vec{\tilde{y}}^{\,(\beta)}|^2}{\hat{J}_{II}}\right]; \qquad (9.19a) \\
\rho_\sigma(\tilde{\sigma},\tilde{p}_\sigma) &= w(J,\Phi) \\
&= \frac{1}{2\pi\hat{J}}\exp\left[-\left(J\cdot\frac{2a}{M}+J^2\cdot\frac{d}{4M}\right)\right] \\
&= \frac{1}{2\pi\hat{J}}\exp\left[-\frac{2a}{M}\cdot|\vec{w}_\sigma^{\,+}\underline{S}_2\;\vec{\tilde{y}}^{\,(\sigma)}|^2-\frac{d}{4M}\cdot|\vec{w}_\sigma^{\,+}\underline{S}_2\;\vec{\tilde{y}}^{\,(\sigma)}|^4\right]. \qquad (9.19b)
\end{aligned}$$

Here $\rho_\beta(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z)$ denotes the density in the transverse $(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z)$ - phase space (betatron motion) and $\rho_\sigma(\tilde{\sigma},\tilde{p}_\sigma)$ the density in the longitudinal $(\tilde{\sigma},\tilde{p}_\sigma)$ - phase space (synchrotron oscillation).

For the term $|\vec{w}_\sigma^{\,+}\underline{S}_2\;\vec{\tilde{y}}^{\,(\sigma)}|^2$ appearing in (9.19b) one has from (5.23b):

$$|\vec{w}_\sigma^{\,+}\underline{S}_2\;\vec{\tilde{y}}^{\,(\sigma)}|^2 = \frac{1}{2\beta_\sigma}\cdot\left\{[\alpha_\sigma\cdot\tilde{\sigma}+\beta_\sigma\cdot\tilde{p}_\sigma]^2+\tilde{\sigma}^2\right\}\;. \qquad (9.20a)$$

Using the oscillator model, this equation simplifies to:

$$|\vec{w}_\sigma^{\,+}\underline{S}_2\;\vec{\tilde{y}}^{\,(\sigma)}|^2 = \frac{\Omega}{2\kappa}\cdot\left\{\frac{\kappa^2}{\Omega^2}\cdot\tilde{p}_\sigma^2+\tilde{\sigma}^2\right\} \qquad (9.20b)$$

where κ and Ω are given by (5.30a) and (5.31).

It follows from (9.19b) and (9.20) that the lines of constant density in the longitudinal $(\tilde{\sigma},\tilde{p}_\sigma)$ - phase space are represented by the ellipse equation:

$$\frac{1}{2\beta_\sigma}\cdot\left\{[\alpha_\sigma\cdot\tilde{\sigma}+\beta_\sigma\cdot\tilde{p}_\sigma]^2+\tilde{\sigma}^2\right\} \;=\; const. \qquad (9.21)$$

The areas of constant density in the transverse $(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z)$ - space are investigated in Appendix F.

Finally we remark that from eqn. (9.19b) one obtains the energy distribution by integrating $\rho_\sigma(\tilde{\sigma},\tilde{p}_\sigma)$ over $\tilde{\sigma}$:

$$\hat{\rho}_\sigma(\tilde{p}_\sigma) = \int_{-\infty}^{+\infty} d\tilde{\sigma}\cdot\rho_\sigma(\tilde{\sigma},\tilde{p}_\sigma)\;.$$

Using eqn. (9.20b) for the oscillator model we get:

$$\begin{aligned}
\hat{\rho}_\sigma(\tilde{p}_\sigma) \;&=\; \frac{1}{2\pi\hat{J}}\int_{-\infty}^{+\infty} d\tilde{\sigma}\cdot\exp\left\{-\frac{2a}{M}\cdot\frac{\Omega}{2\kappa}\cdot\left[\frac{\kappa^2}{\Omega^2}\cdot\tilde{p}_\sigma^2+\tilde{\sigma}^2\right]\right\} \\
&\quad\times\exp\left\{-\frac{d}{4M}\cdot\frac{\Omega^2}{4\kappa^2}\cdot\left[\frac{\kappa^2}{\Omega^2}\cdot\tilde{p}_\sigma^2+\tilde{\sigma}^2\right]^2\right\}\;. \qquad (9.22)
\end{aligned}$$

10 Summary

As in Ref. [2] we have investigated the influence of linear and nonlinear radiation damping and quantum fluctuations on the motion of charged particles in storage rings by using the Fokker-Planck equation. In this treatment we have shown how to include closed orbit distortion, pointlike rf cavities, transverse coupling due to skew quadrupoles and solenoids, all within a 6 x 6 symplectic dispersion formalism.

A number of approximations have been used which were mentioned explicitly in the text. However, it is clear that quadrupole-sextupole and octupole-dipole wigglers could in principle, and if used in sufficient quantities, modify the tails of the longitudinal phase space distribution so that these would no longer be Gaussian. Clearly, in a realistic evaluation of the utility of nonlinear wigglers in a real storage ring the effect of the various approximations would need to be considered in more detail and the effect of closed orbit deviations on the damping should be included. Naturally, it would then be necessary to consider the nonlinear Hamiltonian motion and the stochastic radiation effects in the nonlinear fields.

One of the original reasons for considering [2] these wigglers was that they could perhaps reduce spin depolarization effects by reducing the energy spread. They could have other uses [2].

In this paper we have only considered the case of ultrarelativistic particles. In order to study the case of arbitrary velocity one would introduce the variable $\sigma = s - v_0 \cdot t$ (v_0 =average speed of the particles) as described in Ref. [25, 9]. But for nonrelativistic particles the radiation effects could be neglected.

Finally we remark that the classical spin motion in linear approximation could be easily incorporated in our dispersion treatment in analogy to the investigations in Ref. [1].

Appendix A: Description of the Electromagnetic Field

Using the freedom to select a gauge, we can choose any vector potential which leads (via eqns. (2.8) and (2.9)) to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity [6].

A.1 Bending Magnet

If the curvatures K_x and K_z of the design orbit are fixed, the magnetic bending fields on the design orbit, $\mathcal{B}_x^{(0)}(s)$ and $\mathcal{B}_z^{(0)}(s)$:

$$\begin{aligned} \mathcal{B}_x^{(0)}(s) &= \mathcal{B}_x(0,0,s)\,; && \text{(A.1a)} \\ \mathcal{B}_z^{(0)}(s) &= \mathcal{B}_z(0,0,s) && \text{(A.1b)} \end{aligned}$$

are given by (with $\dot{s} = c$) :

$$\begin{aligned} \frac{e}{E_0} \cdot \mathcal{B}_x^{(0)} &= -K_z\;; && \text{(A.2a)} \\ \frac{e}{E_0} \cdot \mathcal{B}_z^{(0)} &= +K_x\;. && \text{(A.2b)} \end{aligned}$$

The corresponding vector potential can be written as

$$\frac{e}{E_0} \cdot A_s = -\frac{1}{2}(1 + K_x \cdot x + K_z \cdot z) \; ; \tag{A.3a}$$

$$A_x = A_z = 0 \; . \tag{A.3b}$$

A.2 Quadrupole

The quadrupole fields are

$$\mathcal{B}_x = z \cdot \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0} \; ; \tag{A.4a}$$

$$\mathcal{B}_z = x \cdot \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0} \; , \tag{A.4b}$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}\left(z^2 - x^2\right) \; ; \tag{A.5a}$$

$$A_x = A_z = 0 \; . \tag{A.5b}$$

In the following we rewrite the term $(e/E_0) \cdot A_s$ in (2.2) as:

$$\frac{e}{E_0} A_s = \frac{1}{2} g \cdot (z^2 - x^2) \; ; \tag{A.6a}$$

$$g = \frac{e}{E_0} \cdot \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0} \; . \tag{A.6b}$$

A.3 Skew Quadrupole

The fields are

$$\mathcal{B}_x = -\frac{1}{2} \cdot \left(\frac{\partial \mathcal{B}_z}{\partial z} - \frac{\partial \mathcal{B}_x}{\partial x}\right)_{x=z=0} \cdot x \; ; \tag{A.7a}$$

$$\mathcal{B}_z = +\frac{1}{2} \cdot \left(\frac{\partial \mathcal{B}_z}{\partial z} - \frac{\partial \mathcal{B}_x}{\partial x}\right)_{x=z=0} \cdot z \; . \tag{A.7b}$$

Thus we may use

$$A_s = -\frac{1}{2}\left(\frac{\partial \mathcal{B}_z}{\partial z} - \frac{\partial \mathcal{B}_x}{\partial x}\right)_{x=z=0} \cdot xz \; ; \tag{A.8a}$$

$$A_x = A_z = 0 \; , \tag{A.8b}$$

and we write

$$\frac{e}{E_0} A_s = N \cdot xz \; ; \tag{A.9a}$$

$$N = \frac{1}{2} \cdot \frac{e}{E_0}\left(\frac{\partial \mathcal{B}_x}{\partial x} - \frac{\partial \mathcal{B}_z}{\partial z}\right)_{x=z=0} \; . \tag{A.9b}$$

A.4 Sextupole

$$\mathcal{B}_x = \left(\frac{\partial^2 \mathcal{B}_z}{\partial x^2}\right)_{x=z=0} \cdot xz \; ; \tag{A.10a}$$

$$\mathcal{B}_z = \left(\frac{\partial^2 \mathcal{B}_z}{\partial x^2}\right)_{x=z=0} \cdot \frac{1}{2}(x^2 - z^2) \tag{A.10b}$$

so that

$$\frac{e}{E_0} A_s = -\lambda \cdot \frac{1}{6}(x^3 - 3\,xz^2) \tag{A.11a}$$

with

$$\lambda = \frac{e}{E_0} \cdot \left(\frac{\partial^2 \mathcal{B}_z}{\partial x^2}\right)_{x=z=0} . \tag{A.11b}$$

A.5 Octupole

$$\mathcal{B}_x = \frac{1}{6}\left(\frac{\partial^3 \mathcal{B}_x}{\partial z^3}\right)_{x=z=0} \cdot (z^3 - 3\,x^2 z) \; ; \tag{A.12a}$$

$$\mathcal{B}_z = \frac{1}{6}\left(\frac{\partial^3 \mathcal{B}_x}{\partial z^3}\right)_{x=z=0} \cdot (3\,xz^2 - x^3) \tag{A.12b}$$

so that

$$\frac{e}{E_0} A_s = \mu \cdot \frac{1}{24}(z^4 - 6\,x^2 z^2 + x^4) \tag{A.13a}$$

with

$$\mu = \frac{e}{E_0} \cdot \left(\frac{\partial^3 \mathcal{B}_x}{\partial z^3}\right)_{x=z=0} . \tag{A.13b}$$

A.6 Solenoid Fields

The field components in the current free region are given by [6, 26]:

$$\mathcal{B}_x(x,z,s) = x \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + z^2)^\nu \; ; \tag{A.14a}$$

$$\mathcal{B}_z(x,z,s) = z \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + z^2)^\nu \; ; \tag{A.14b}$$

$$\mathcal{B}_s(x,z,s) = \sum_{\nu=0}^{\infty} b_{2\nu} \cdot (x^2 + z^2)^\nu \tag{A.14c}$$

where for consistency with Maxwell's equations the coefficients b_μ obey the recursion equations:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \; ; \tag{A.15a}$$

$$b_{2\nu+2}(s) = +\frac{1}{(2\nu+2)} \cdot b'_{2\nu+1}(s) \; ; \tag{A.15b}$$

$$(\nu = 0, 1, 2, ...)$$

and where

$$b_0(s) \equiv \mathcal{B}_s(0,0,s) \ . \qquad (A.16)$$

The vector potential leading to the solenoid field of eqn. (A.14) is then:

$$\begin{aligned}
A_x(x,z,s) &= -z \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \ ; && (A.17a) \\
A_z(x,z,s) &= +x \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \ ; && (A.17b) \\
A_s(x,z,s) &= 0 && (A.17c)
\end{aligned}$$

with

$$r^2 = x^2 + z^2 \ .$$

Thus we can write:

$$\begin{aligned}
\frac{e}{E_0} A_x &= -H(s) \cdot z + \frac{1}{8} H''(s) \cdot (x^2 + z^2) \cdot z + \cdots \ ; && (A.18a) \\
\frac{e}{E_0} A_z &= +H(s) \cdot x - \frac{1}{8} H''(s) \cdot (x^2 + z^2) \cdot x + \cdots && (A.18b)
\end{aligned}$$

with

$$\begin{aligned}
H(s) &= \frac{1}{2} \cdot \frac{e}{E_0} \cdot b_0(s) && (A.19) \\
&\equiv \frac{1}{2} \cdot \frac{e}{E_0} \cdot \mathcal{B}_s(0,0,s) \ .
\end{aligned}$$

Note that the cyclotron radius for the longitudinal field (A.16) is given by

$$R = \frac{1}{2 \cdot H} \ .$$

A.7 Correction Coil

$$\left\{ \begin{aligned} \mathcal{B}_x &= \Delta\hat{\mathcal{B}}_x \cdot \delta(s - s_0) \ ; \\ \mathcal{B}_z &= \Delta\hat{\mathcal{B}}_z \cdot \delta(s - s_0) \end{aligned} \right. \qquad (A.20)$$

so that

$$\frac{e}{E_0} A_s = \frac{e}{E_0} \cdot \delta(s - s_0) \cdot \left[\Delta\hat{\mathcal{B}}_x \cdot z - \Delta\hat{\mathcal{B}}_z \cdot x \right] \ . \qquad (A.21)$$

A.8 Cavity Field

For a longitudinal electric field

$$\begin{aligned} \mathcal{E}_x &= 0 ; \\ \mathcal{E}_z &= 0 ; \\ \mathcal{E}_s &= \mathcal{E}(s,\sigma) \end{aligned} \tag{A.22}$$

we write:

$$\begin{aligned} A_x &= 0 ; \\ A_z &= 0 ; \\ A_s &= \int_{\sigma_0}^{\sigma} d\tilde{\sigma} \cdot \mathcal{E}(s,\tilde{\sigma}) , \end{aligned} \tag{A.23}$$

which by (2.8) immediately gives $\mathcal{E}_s$.

Now the cavity field may be represented by

$$\mathcal{E}(s,\sigma) = V(s)\sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \tag{A.24}$$

and using (A.23) we obtain:

$$A_s = -\frac{L}{2\pi \cdot h} \cdot V(s) \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] , \tag{A.25}$$

in which the phase φ is defined so that the average energy radiated away in the magnets is replaced by the cavities and h is the harmonic number.

Appendix B: Solution of the Equations of Motion; Thin Lens Approximation

B.1 Using the Variables x, p_x, z, p_z, σ, p_σ (Oscillations Around the Design-Orbit)

When written in thin lens approximation (excluding the solenoid case) the transfer matrices of eqn. (2.48), $\underline{M} + \delta\underline{M}$, defined by:

$$\begin{pmatrix} \vec{y}(s+\Delta s/2) \\ 1 \end{pmatrix} = (\underline{M} + \delta\underline{M}) \begin{pmatrix} \vec{y}(s-\Delta s/2) \\ 1 \end{pmatrix} \tag{B.1}$$

may be written in the form:

$$\begin{aligned} &\underline{M}\left(s+\frac{\Delta s}{2}, s-\frac{\Delta s}{2}\right) + \delta\underline{M}\left(s+\frac{\Delta s}{2}, s-\frac{\Delta s}{2}\right) = \\ &\underline{M}_D\left(s+\frac{\Delta s}{2}, s\right) \cdot \{\underline{1} + ((m_{ik})) + ((\delta m_{ik}))\} \cdot \underline{M}_D\left(s, s-\frac{\Delta s}{2}\right) ; \\ &\qquad (i,k = 1, 2 \cdots 7) ; \end{aligned}$$

δm_{ik} = are the non-symplectic elements due to the matrix $\delta \underline{A}$ which contains dissipative terms

and

$$\underline{M}_D = \text{transfer matrix for a drift space .}$$

The thin lens matrices $\underline{m}$, $\delta \underline{m}$ are then given by:

1) Quadrupole:

$$\begin{aligned} m_{21} &= -g \cdot \Delta s \; ; \\ m_{43} &= -m_{21} \; ; \\ m_{67} &= -C_1 \cdot g^2 \cdot \left(x^2 + z^2\right) \cdot (1 + 2\eta) \cdot \Delta s \; . \end{aligned} \tag{B.2a}$$

2) Skew quadrupole:

$$\begin{aligned} m_{23} &= N \cdot \Delta s \; ; \\ m_{41} &= m_{23} \; ; \\ m_{67} &= -C_1 \cdot g^2 \cdot \left(x^2 + z^2\right) \cdot (1 + 2\eta) \cdot \Delta s \; . \end{aligned} \tag{B.2b}$$

3) Bending magnet:

$$\begin{aligned} m_{21} &= -K_x^2 \cdot \Delta s \; ; \\ m_{26} &= K_x \cdot \Delta s \; ; \\ m_{43} &= -K_z^2 \cdot \Delta s \; ; \\ m_{46} &= K_z \cdot \Delta s \; ; \\ m_{51} &= -K_x \cdot \Delta s \; ; \\ m_{53} &= -K_z \cdot \Delta s \; ; \\ m_{67} &= -C_1 \cdot (K_x^2 + K_z^2) \cdot \Delta s \; ; \\ \delta m_{61} &= -C_1 \cdot K_x^3 \cdot \Delta s \; ; \\ \delta m_{63} &= -C_1 \cdot K_z^3 \cdot \Delta s \; ; \\ \delta m_{66} &= -2C_1 \cdot (K_x^2 + K_z^2) \cdot \Delta s \; . \end{aligned} \tag{B.2c}$$

4) Combined function magnet:

$$\begin{aligned} m_{21} &= -\left[K_x^2 + g\right] \cdot \Delta s \; ; \\ m_{26} &= K_x \cdot \Delta s \; ; \\ m_{43} &= -\left[K_z^2 - g\right] \cdot \Delta s \; ; \\ m_{46} &= K_z \cdot \Delta s \; ; \\ m_{51} &= -K_x \cdot \Delta s \; ; \\ m_{53} &= -K_z \cdot \Delta s \; ; \\ m_{67} &= -C_1 \cdot \left\{(K_x^2 + K_z^2) + g^2 \cdot [x^2 + z^2] \cdot (1 + 2\eta)\right\} \cdot \Delta s \; ; \\ \delta m_{61} &= -C_1 \cdot \left[K_x^3 + 2\, G_{CF}^{(x)}\right] \cdot \Delta s \; ; \\ \delta m_{63} &= -C_1 \cdot \left[K_z^3 - 2\, G_{CF}^{(z)}\right] \cdot \Delta s \; ; \\ \delta m_{66} &= -2\, C_1 \cdot (K_x^2 + K_z^2) \cdot \Delta s \; . \end{aligned} \tag{B.2d}$$

5) Cavity:

$$\begin{aligned} V &= \hat{V} \cdot \delta(s - s_0) ; \\ m_{65} &= \frac{e\hat{V}}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi ; \\ m_{67} &= \frac{e\hat{V}}{E_0} \cdot \sin\varphi ; \\ \delta m_{22} &= -\frac{e\hat{V}}{E_0} \cdot \sin\varphi ; \\ \delta m_{44} &= -\frac{e\hat{V}}{E_0} \cdot \sin\varphi . \end{aligned} \tag{B.2e}$$

6) Sextupole:

$$\begin{aligned} m_{27} &= \frac{1}{2} \lambda(s) \cdot \left(z^2 - x^2\right) \cdot \Delta s ; \\ m_{47} &= \lambda(s) \cdot xz \cdot \Delta s . \end{aligned} \tag{B.2f}$$

7) Dipole correction magnet:

$$\begin{aligned} \delta m_{66} &= -2\, C_1 \cdot \left(\frac{e}{E_0}\right)^2 \left[(\Delta \mathcal{B}_x)^2 + (\Delta \mathcal{B}_z)^2\right] \cdot \Delta s ; \\ m_{67} &= -C_1 \cdot \left(\frac{e}{E_0}\right)^2 \left[(\Delta \mathcal{B}_x)^2 + (\Delta \mathcal{B}_z)^2\right] \cdot \Delta s . \end{aligned} \tag{B.2g}$$

8) Quadrupole - sextupole wiggler:

$$\begin{aligned} m_{21} &= -g \cdot \Delta s ; \\ m_{43} &= -m_{21} ; \\ m_{27} &= \frac{1}{2} \lambda(s) \cdot \left(z^2 - x^2\right) \cdot \Delta s ; \\ m_{47} &= \lambda(s) \cdot xz \cdot \Delta s ; \\ m_{67} &= -C_1 \cdot g^2 \cdot \left(x^2 + z^2\right) \cdot (1 + 2\eta) \cdot \Delta s \\ &\quad -C_1 \cdot G_{QS} \cdot \left[x^3 + xz^2\right] \cdot \Delta s . \end{aligned} \tag{B.2h}$$

9) Octupole - dipole wiggler:

$$\begin{aligned} m_{27} &= -\frac{e}{E_0} \Delta \mathcal{B}_z \cdot \Delta s ; \\ m_{47} &= +\frac{e}{E_0} \Delta \mathcal{B}_x \cdot \Delta s ; \\ m_{67} &= -C_1 \cdot \left(\frac{e}{E_0}\right)^2 (\Delta \mathcal{B}_x)^2 \cdot (1 + 2\eta) \cdot \Delta s \\ &\quad -C_1 \cdot G_{DO} \cdot \frac{1}{3} \left[3\, xz^2 - x^3\right] \cdot \Delta s . \end{aligned} \tag{B.2i}$$

10) Solenoid (for which we don't use a thin lens approximation but instead consider the transfer matrix for thin slices over which the field H remains constant):

$$\begin{aligned}
M_{11} &= \frac{1}{2}(1+\cos 2\Theta) ; \\
M_{12} &= \frac{1}{2H}\sin 2\Theta ; \\
M_{13} &= \frac{1}{2}\sin 2\Theta ; \\
M_{14} &= \frac{1}{2H}(1-\cos 2\Theta) ; \\
M_{21} &= -H\cdot\frac{1}{2}\sin 2\Theta ; \\
M_{22} &= M_{11} ; \\
M_{23} &= H\cdot\frac{1}{2}(1-\cos 2\Theta) ; \\
M_{24} &= M_{13} ; \\
M_{31} &= -M_{13} ; \\
M_{32} &= -M_{14} ; \\
M_{33} &= M_{11} ; \\
M_{34} &= M_{12} ; \\
M_{41} &= -M_{23} ; \\
M_{42} &= -M_{13} ; \\
M_{43} &= M_{21} ; \\
M_{44} &= M_{11} ; \\
M_{55} &= 1 ; \\
M_{66} &= 1 ; \\
M_{77} &= 1 ; \\
M_{ik} &= 0 \quad \text{otherwise}
\end{aligned} \tag{B.2j}$$

with $\Theta = H\cdot\Delta s$ and H constant over the intervall Δs .

(Equation (B.2j) is valid also for a solenoid field of arbitrary length if H is constant across the whole length.)

The determining equations of the closed orbit $\vec{y}_0(s)$ (see eqn. (3.2a, b)) now read as:

$$\vec{y}_0(s_0+L) \equiv \{\underline{M}(s_0+L,s_0)+\delta\underline{M}(s_0+L,s_0)\}\,\vec{y}_0(s_0) = \vec{y}_0(s_0) ; \tag{B.3a}$$

(periodicity condition);

$$\vec{y}_0(s) = \{\underline{M}(s,s_0)+\delta\underline{M}(s,s_0)\}\,\vec{y}_0(s_0) . \tag{B.3b}$$

Since eqn. (3.2a) is nonlinear, it must be solved iteratively.

B.2 Using the Variables $\bar{x}$, $\bar{p}_x$, $\bar{z}$, $\bar{p}_z$, $\bar{\sigma}$, $\bar{p}_\sigma$ (Oscillations Around the Closed Orbit)

The linear part of eqn. (3.3) reads as:

$$\frac{d}{ds}\,\vec{\bar{y}} \;=\; \underline{\bar{A}} \cdot \vec{\bar{y}} + \delta\underline{\bar{A}} \cdot \vec{\bar{y}}\;. \tag{B.4}$$

For the resulting transfer matrix

$$\begin{aligned}
&\underline{\bar{M}}\left(s+\frac{\Delta s}{2}, s-\frac{\Delta s}{2}\right) + \delta\underline{\bar{M}}\left(s+\frac{\Delta s}{2}, s-\frac{\Delta s}{2}\right) \;=\\
&\underline{\bar{M}}_D\left(s+\frac{\Delta s}{2}, s\right)\cdot\{\underline{1} + ((\bar{m}_{ik})) + ((\delta\bar{m}_{ik}))\}\cdot\underline{\bar{M}}_D\left(s, s-\frac{\Delta s}{2}\right)\;;\\
&\qquad (i, k = 1, 2 \cdots 6)
\end{aligned}$$

(where the quantities $\delta\bar{m}_{ik}$ are the non-symplectic elements due to the matrix $\delta\underline{A}$ which contains dissipative terms and $\underline{\bar{M}}_D$ = transfer matrix for a drift space), one obtains:

1) Quadrupole:

$$\begin{aligned}
\bar{m}_{ik} &= m_{ik}\;;\\
\delta\bar{m}_{61} &= -2\,C_1\cdot g^2\cdot x_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{63} &= -2\,C_1\cdot g^2\cdot z_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{66} &= -2\,C_1\cdot g^2\cdot(x_0^2+z_0^2)\cdot\Delta s\;.
\end{aligned}\tag{B.5a}$$

Thus a closed orbit deviation in a quadrupole produces damping effects similar to those in a combined function magnet (eqn. (B.2d)).

2) Skew quadrupole:

$$\begin{aligned}
\bar{m}_{ik} &= m_{ik}\;;\\
\delta\bar{m}_{61} &= -2\,C_1\cdot N^2\cdot x_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{63} &= -2\,C_1\cdot N^2\cdot z_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{66} &= -2\,C_1\cdot N^2\cdot(x_0^2+z_0^2)\cdot\Delta s\;.
\end{aligned}\tag{B.5b}$$

3) Bending magnet:

$$\begin{aligned}
\bar{m}_{ik} &= m_{ik}\;;\\
\delta\bar{m}_{ik} &= \delta m_{ik}\;.
\end{aligned}\tag{B.5c}$$

4) Combined function magnet:

$$\begin{aligned}
\bar{m}_{ik} &= m_{ik}\;;\\
\delta\bar{m}_{61} &= \delta m_{61} - 2\,C_1\cdot g^2\cdot x_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{63} &= \delta m_{63} - 2\,C_1\cdot g^2\cdot z_0\cdot(1+2\eta_0)\cdot\Delta s\;;\\
\delta\bar{m}_{66} &= \delta m_{66} - 2\,C_1\cdot g^2\cdot(x_0^2+z_0^2)\cdot\Delta s\;.
\end{aligned}\tag{B.5d}$$

5) Cavity :

$$\begin{aligned} \bar{m}_{ik} &= m_{ik} \ ; \\ \delta\bar{m}_{ik} &= \delta m_{ik} \ . \end{aligned} \tag{B.5e}$$

6) Sextupole :

$$\begin{aligned} \bar{m}_{21} &= -\lambda \cdot x_0 \cdot \Delta s \ ; \\ \bar{m}_{23} &= +\lambda \cdot z_0 \cdot \Delta s \ ; \\ \bar{m}_{41} &= +\bar{m}_{23} \ ; \\ \bar{m}_{43} &= -\bar{m}_{21} \ . \end{aligned} \tag{B.5f}$$

7) Dipole correction magnet :

$$\delta\bar{m}_{66} = -2\,C_1 \cdot \left(\frac{e}{E_0}\right)^2 \left[(\Delta\mathcal{B}_x)^2 + (\Delta\mathcal{B}_z)^2\right] \cdot \Delta s \ . \tag{B.5g}$$

8) Quadrupole - sextupole wiggler :

$$\begin{aligned} \bar{m}_{21} &= -[g + \lambda \cdot x_0] \cdot \Delta s \ ; \\ \bar{m}_{23} &= \lambda \cdot z_0 \cdot \Delta s \ ; \\ \bar{m}_{41} &= +\bar{m}_{23} \ ; \\ \bar{m}_{43} &= -\bar{m}_{21} \ ; \\ \delta\bar{m}_{61} &= -2\,C_1 \cdot g^2 \cdot x_0 \cdot (1 + 2\eta_0) \cdot \Delta s \\ &\quad -C_1 \cdot G_{QS} \cdot [3\,x_0^2 + z_0^3] \cdot \Delta s \ ; \\ \delta\bar{m}_{63} &= -2\,C_1 \cdot g^2 \cdot z_0 \cdot (1 + 2\eta_0) \cdot \Delta s \\ &\quad -C_1 \cdot G_{QS} \cdot 2\,x_0 z_0 \cdot \Delta s \ ; \\ \delta\bar{m}_{66} &= -2\,C_1 \cdot g^2 \cdot (x_0^2 + z_0^2) \cdot \Delta s \ . \end{aligned} \tag{B.5h}$$

9) Octupole - dipole wiggler :

$$\begin{aligned} \delta\bar{m}_{61} &= -C_1 \cdot G_{DO} \cdot [z_0^2 - x_0^2] \cdot \Delta s \ ; \\ \delta\bar{m}_{63} &= -C_1 \cdot G_{DO} \cdot 2\,x_0 z_0 \cdot \Delta s \ ; \\ \delta\bar{m}_{66} &= -2\,C_1 \cdot \left(\frac{e}{E_0}\right)^2 (\Delta\mathcal{B}_x)^2 \cdot \Delta s \ . \end{aligned} \tag{B.5i}$$

11) Solenoid :

$$\bar{m}_{ik} = m_{ik} \ . \tag{B.5k}$$

These matrices are those used in the standard thin lens version of SLIM.

The sympletic part $\underline{\bar{M}}$ of the whole transfer matrix $(\underline{\bar{M}} + \delta\underline{\bar{M}})$ is needed for a calculation of dispersion.

The nonsymplectic part $\delta\underline{\bar{M}}$ can be used to calculate the linear damping constants within the framework of the fully coupled 6-dimensional formalism [1].

B.3 Using the Variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ (Oscillations Around the Dispersion-Orbit)

In order to calculate the eigenvectors $\vec{v}_k$ (see eqn. (5.9)) we need the transfer matrices due to eqn. (5.1):

$$\frac{d}{ds}\vec{\tilde{y}} = \underline{\tilde{A}} \cdot \vec{\tilde{y}} \tag{B.6}$$

with $\underline{\tilde{A}}$ given by eqn. (5.18) (oscillations around the dispersion-orbit).

Writing:

$$\underline{\tilde{M}}\left(s + \frac{\Delta s}{2}, s - \frac{\Delta s}{2}\right) =$$

$$\underline{\tilde{M}}_D\left(s + \frac{\Delta s}{2}, s\right) \cdot \{\underline{1} + ((\tilde{m}_{ik}))\} \cdot \underline{\tilde{M}}_D\left(s, s - \frac{\Delta s}{2}\right) ;$$

$$(i, k = 1, 2 \cdots 6)$$

we get:

$$\begin{aligned} \tilde{m}_{ik} &= \bar{m}_{ik} \text{ for } (i, k = 1, 2 \cdots 4) ; \\ \tilde{m}_{56} &= -[K_x \cdot D_1 + K_z \cdot D_3] \cdot \Delta s ; \\ \tilde{m}_{65} &= \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \end{aligned} \tag{B.7}$$

with $\hat{V}$ defined in (B.2e).

The nonsymplectic component of the whole transfer matrix resulting from the term $\delta\underline{\tilde{A}}$ in the equation of motion (4.19) reads as:

$$\delta\underline{\tilde{M}}\left(s + \frac{\Delta s}{2}, s - \frac{\Delta s}{2}\right) = \delta\underline{\tilde{A}} \cdot \Delta s \tag{B.8}$$

with $\delta\underline{\tilde{A}}$ given by (4.23). This matrix can be used to determine the linear damping constants (see Appendix C).

Appendix C: Perturbation Theory and Damping Constants

C.1 Introductory Remark: Two Methods for Calculating the Linear Damping Constants

In chapter 8 we have derived an analytic expression for the damping constants α_k of the (coupled) synchro-betatron oscillations (see eqn. (8.19)). This expression allows the calculation of α_k if one knows the eigenvectors $\vec{v}_k(s)$ of the unperturbed problem (5.1) and the matrix elements $\delta\tilde{A}_{ik}$ of eqn. (4.23).

On the other hand, A. Chao [12] calculates the damping constants by using the eigenvalue spectrum of the revolution matrix

$$\underline{\tilde{M}}(s_0 + L, s_0) + \delta\underline{\tilde{M}}(s_0 + L, s_0)$$

of the perturbed problem

$$\frac{d}{ds}\vec{\tilde{y}} \;=\; \left(\underline{\tilde{A}} + \delta\underline{\tilde{A}}\right) \cdot \vec{\tilde{y}} \,. \tag{C.1}$$

This matrix with the perturbation part $\delta\underline{\tilde{M}}(s_0 + L, s_0)$ is not symplectic in contrast to

$$\underline{\tilde{M}}(s_0 + L, s_0) \,.$$

Therefore, writing the perturbed eigenvalues $(\lambda_k + \delta\lambda_k)$ in the form (see eqn. (6.16)):

$$\lambda_k + \delta\lambda_k = e^{-i\cdot 2\pi(Q_k + \delta Q_k)}$$

one will generally obtain complex values for the Q-shift δQ_k caused by the perturbation $\delta\underline{A}$. According to A. Chao [12] we put

$$\begin{aligned} \alpha_k &= -2\pi \cdot \Im m\{Q_k + \delta Q_k\} \\ &= -2\pi \cdot \Im m\{\delta Q_k\} \,. \end{aligned} \tag{C.2}$$

The purpose of this appendix is to show the equivalence of (8.19) and (C.2).

C.2 Equivalence of the two Methods

C.2.1 Calculation for the Perturbed Part of the Revolution Matrix

In order to prove the equivalence of the two methods mentioned above we determine the perturbation part $\delta\underline{\tilde{M}}(s_0 + L, s_0)$ of the revolution matrix (of the perturbed problem).

According to eqn. (C.1) the transfer matrix

$$\underline{\tilde{M}}(s, s_0) + \delta\underline{\tilde{M}}(s, s_0)$$

obeys the equation:

$$\frac{d}{ds}\left[\underline{\tilde{M}}(s, s_0) + \delta\underline{\tilde{M}}(s, s_0)\right] \;=\; \left[\underline{\tilde{A}}(s) + \delta\underline{\tilde{A}}(s)\right] \cdot \left[\underline{\tilde{M}}(s, s_0) + \delta\underline{\tilde{M}}(s, s_0)\right] \,; \tag{C.3a}$$

$$\underline{\tilde{M}}(s_0, s_0) + \delta\underline{\tilde{M}}(s_0, s_0) \;=\; \underline{1} \,. \tag{C.3b}$$

Taking into account the corresponding equations for the unperturbed transfer matrix $\underline{\tilde{M}}(s,s_0)$:

$$\begin{aligned}\frac{d}{ds}\underline{\tilde{M}}(s,s_0) &= \underline{\tilde{A}}(s)\cdot\underline{\tilde{M}}(s,s_0)\ ;\\ \underline{\tilde{M}}(s_0,s_0) &= \underline{1}\end{aligned}$$

we obtain from (C.3), to first order, the differential equation for $\delta\underline{\tilde{M}}(s,s_0)$:

$$\frac{d}{ds}\delta\underline{\tilde{M}}(s,s_0) = \underline{\tilde{A}}(s)\cdot\delta\underline{\tilde{M}}(s,s_0)+\delta\underline{\tilde{A}}(s)\cdot\underline{\tilde{M}}(s,s_0)$$

with the initial condition:

$$\delta\underline{\tilde{M}}(s_0,s_0) = \underline{0}\ .$$

The solution of this equation (and thus the first order solution of eqn. (C.3)) reads as:

$$\begin{aligned}\delta\underline{\tilde{M}}(s,s_0) &= \int_{s_0}^{s} d\tilde{s}\cdot\underline{\tilde{M}}(s,\tilde{s})\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s_0)\\ &= \underline{\tilde{M}}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s_0)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s_0)\ .\end{aligned}$$

For the perturbative part $\delta\underline{\tilde{M}}(s_0+L,s_0)$ of the revolution matrix one therefore gets, to first order, the expression:

$$\begin{aligned}\delta\underline{\tilde{M}}(s_0+L,s_0) &= \int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{\tilde{M}}(s_0+L,\tilde{s})\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s_0)\\ &= \underline{\tilde{M}}(s_0+L,s_0)\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s_0)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s_0) \qquad \text{(C.4a)}\end{aligned}$$

and for $\delta\underline{\tilde{M}}(s+L,s)$ one thus may write:

$$\delta\underline{\tilde{M}}(s+L,s) = \underline{\tilde{M}}(s+L,s)\cdot\int_{s}^{s+L} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s)\ . \qquad \text{(C.4b)}$$

Remark:

Writing:

$$\delta\underline{\tilde{M}} = \delta\underline{\tilde{M}}^{(1)}+\delta\underline{\tilde{M}}^{(2)}+\delta\underline{\tilde{M}}^{(3)}+\cdots$$

and

$$\delta\underline{\tilde{M}}^{(0)} \equiv \underline{\tilde{M}}\ ,$$

whereby $\delta\underline{\tilde{M}}^{(\nu)}$ denotes the ν^{th} order in $\delta\underline{\tilde{A}}$ of $\delta\underline{\tilde{M}}$, we obtain from (C.3a, b) for $n\geq 0$ [27]:

$$\begin{aligned}\frac{d}{ds}\delta\underline{\tilde{M}}^{(n+1)}(s,s_0) &= \underline{\tilde{A}}(s)\cdot\delta\underline{M}^{(n+1)}(s,s_0)+\delta\underline{\tilde{A}}(s)\cdot\delta\underline{M}^{(n)}(s,s_0)\ ;\\ \delta\underline{M}^{(n+1)}(s_0,s_0) &= \underline{0}\end{aligned}$$

with the solution :

$$\delta\underline{\tilde{M}}^{(n+1)}(s,s_0) \;=\; \underline{\tilde{M}}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s_0)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\delta\underline{\tilde{M}}^{(n)}(\tilde{s},s_0)\;.$$

In particular we have :

$$\delta\underline{\tilde{M}}^{(1)}(s,s_0) \;=\; \underline{\tilde{M}}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s_0)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\delta\underline{\tilde{M}}(\tilde{s},s_0)\;.$$

The matrix $\delta\underline{\tilde{M}}$ used in (C.4) is identical to $\delta\underline{\tilde{M}}^{(1)}$.

C.2.2 Perturbation Theory

Equation (C.4b) determines the perturbed part $\delta\underline{\tilde{M}}(s+L,s)$ of the revolution matrix if the (unperturbed) transfer matrix $\underline{\tilde{M}}(\tilde{s},s)$ and the perturbation $\delta\underline{\tilde{A}}(\tilde{s})$ are known. Using the eigenvalue equation

$$\begin{aligned}(\underline{\tilde{M}}+\delta\underline{\tilde{M}})\cdot(\vec{v}_\mu+\delta\vec{v}_\mu) \;&=\; (\lambda_\mu+\delta\lambda_\mu)\cdot(\vec{v}_\mu+\delta\vec{v}_\mu)\;;\\ (\mu=\pm I,\ \pm II,\ \pm III)&\end{aligned}$$

or (since $\underline{\tilde{M}}\,\vec{v}_\mu=\lambda_\mu\vec{v}_\mu$)

$$\underline{\tilde{M}}\cdot\delta\vec{v}_\mu+\delta\underline{\tilde{M}}\cdot\vec{v}_\mu=\lambda_\mu\cdot\delta\vec{v}_\mu+\delta\lambda_\mu\cdot\vec{v}_\mu \tag{C.5}$$

we can calculate the Q-shift

$$\delta Q_\kappa=\frac{i}{2\pi\cdot\lambda_\kappa}\cdot\delta\lambda_\kappa \tag{C.6}$$

caused by $\delta\underline{\tilde{M}}$ [22, 28].

For that purpose we expand $\delta\vec{v}_\mu$ in terms of the eigenvectors $\vec{v}_\mu$ of the unperturbed problem:

$$\delta\vec{v}_\mu=\sum_\nu a_{\mu\nu}\cdot\vec{v}_\nu \tag{C.7}$$

and by inserting (C.7) into (C.5) we get :

$$\sum_\nu a_{\mu\nu}\cdot\lambda_\nu\vec{v}_\nu+\delta\underline{\tilde{M}}\cdot\vec{v}_\mu \;=\; \lambda_\mu\cdot\sum_\nu a_{\mu\nu}\vec{v}_\nu+\delta\lambda_\mu\cdot\vec{v}_\mu\;. \tag{C.8}$$

Multiplying this equation from the left hand side with

$$\frac{1}{i}\cdot\vec{v}_\kappa^{+}\,\underline{S}$$

and taking into account eqn. (6.28) we obtain

$$a_{\mu\kappa}\cdot\lambda_\kappa\cdot\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa+\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\cdot\delta\underline{\tilde{M}}\cdot\vec{v}_\mu \;=\; \lambda_\mu\cdot a_{\mu\kappa}\cdot\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa+\delta\lambda_\kappa\cdot\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\cdot\delta_{\mu\kappa} \tag{C.9}$$

with

$$\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa=\begin{cases}+1 & \text{for } \kappa=I,\ II,\ III\;;\\ -1 & \text{for } \kappa=-I,-II,-III\;.\end{cases} \tag{C.10}$$

For $\kappa \neq \mu$ the expansion coefficients are given by (see eqn. (C.4)):

$$\begin{aligned} a_{\mu\kappa} &= \left(\frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right) \cdot \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \cdot \delta \underline{\tilde{M}} \cdot \vec{v}_\mu(s) \\ &= \left(\frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right) \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \cdot \underline{\tilde{M}}(s+L, s) \\ &\quad \times \int_s^{s+L} d\tilde{s} \cdot \underline{\tilde{M}}^{-1}(\tilde{s}, s) \cdot \delta \underline{\tilde{A}}(\tilde{s}) \cdot \underline{\tilde{M}}(\tilde{s}, s) \cdot \vec{v}_\mu(s) \,. \end{aligned}$$

Using the symplectic condition of the transfer matrix $\underline{\tilde{M}}(s_1, s_2)$:

$$\underline{\tilde{M}}^T(s_1, s_2) \cdot \underline{S} \cdot \underline{\tilde{M}}(s_1, s_2) = \underline{S}$$

and the equation

$$\begin{aligned} \vec{v}_\kappa^+(s) \cdot \underline{S} \cdot \underline{\tilde{M}}(s+L, s) &= \vec{v}_\kappa^+(s) \cdot \left[\underline{\tilde{M}}^{-1}(s+L, s)\right]^T \cdot \underline{S} \\ &= \left[\underline{\tilde{M}}^{-1}(s+L, s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\ &= \left[\lambda_\kappa^{-1} \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\ &= \lambda_\kappa \cdot \vec{v}_\kappa^+(s) \cdot \underline{\tilde{M}}^T(\tilde{s}, s) \cdot \underline{S} \cdot \underline{\tilde{M}}(\tilde{s}, s) \\ &\quad \left(\text{since } \ (\lambda_\kappa^{-1})^* = \lambda_\kappa \ \text{ and } \ \underline{S} = \underline{\tilde{M}}^T \cdot \underline{S} \cdot \underline{\tilde{M}}\right) \\ &= \lambda_\kappa \cdot \left[\underline{\tilde{M}}(\tilde{s}, s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \cdot \underline{\tilde{M}}(\tilde{s}, s) \\ &= \lambda_\kappa \cdot \vec{v}_\kappa^+(\tilde{s}) \cdot \underline{S} \cdot \underline{\tilde{M}}(\tilde{s}, s) \end{aligned} \qquad \text{(C.11)}$$

$a_{\mu\kappa}$ can be rewritten as:

$$\begin{aligned} a_{\mu\kappa} &= \left(\frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right) \frac{\lambda_\kappa}{\lambda_\mu - \lambda_\kappa} \\ &\quad \times \frac{1}{i} \cdot \int_s^{s+L} d\tilde{s} \cdot \vec{v}_\kappa^+(\tilde{s}) \cdot \underline{S} \cdot \delta \underline{\tilde{A}}(\tilde{s}) \cdot \vec{v}_\mu(\tilde{s}) \end{aligned} \qquad \text{(C.12)}$$

so that the perturbation $\delta\vec{v}_\mu$ of $\vec{v}_\mu$ is given by (see eqns. (C.7) and (C.12)):

$$\begin{aligned} \delta\vec{v}_\mu(s) &= \sum_{\kappa\neq\mu} \left(\frac{1}{i}\,\vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right) \frac{\lambda_\kappa}{\lambda_\mu - \lambda_\kappa} \\ &\quad \times \frac{1}{i} \cdot \left[\int_s^{s+L} d\tilde{s} \cdot \vec{v}_\kappa^+(\tilde{s}) \cdot \underline{S} \cdot \delta \underline{\tilde{A}}(\tilde{s}) \cdot \vec{v}_\mu(\tilde{s})\right] \cdot \vec{v}_\kappa(s) \\ &\quad + a_{\mu\mu} \cdot \vec{v}_\mu(s) \,. \end{aligned} \qquad \text{(C.13)}$$

Here the coefficient $a_{\mu\mu}$ remains undetermined but can be determined to first order by using the normalization condition (6.28) applied to the perturbed eigenvector $\vec{v}_\mu + \delta\vec{v}_\mu$:

$$[\vec{v}_\mu(s) + \delta\vec{v}_\mu(s)]^+ \cdot \underline{S} \cdot [\vec{v}_\mu(s) + \delta\vec{v}_\mu(s)] \;=\; \vec{v}_\mu^+(s) \cdot \underline{S} \cdot \vec{v}_\mu(s)$$

with $\delta\vec{v}_\mu$ given by (C.13) which leads to:

$$\begin{aligned} 0 &= \delta\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\mu(s)+\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\delta\vec{v}_\mu(s) \\ &= (a_{\mu\mu}+a_{\mu\mu}^{*})\cdot\left[\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\mu(s)\right] \end{aligned}$$

$$\Longrightarrow \quad a_{\mu\mu} = i\cdot\varphi_\mu$$

where φ_μ is an arbitrary real number. This is consistent with the fact that one can multiply an eigenvector $\vec{v}_\mu$ with an arbitrary phase factor $e^{i\varphi_\mu}$ without disturbing the normalization. Without loss of generality we may set:

$$\varphi_\mu = 0 \quad\Longrightarrow\quad a_{\mu\mu} = 0 \ .$$

For $\mu = \kappa$ the first terms on both sides of eqn. (C.9) cancel and one obtains with (C.4), (C.6) and (C.11) the following approximate expression for the Q-shift δQ_κ in linear order:

$$\begin{aligned} \delta Q_\kappa &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\,\underline{S}\cdot\delta\underline{\tilde{M}}(s+L,s)\cdot\vec{v}_\kappa(s) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\,\underline{S}\cdot\underline{\tilde{M}}(s+L,s) \\ &\qquad\times\int_s^{s+L} d\tilde{s}\cdot\underline{\tilde{M}}^{-1}(\tilde{s},s)\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\underline{\tilde{M}}(\tilde{s},s)\vec{v}_\kappa(s) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s}) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s}) \end{aligned}$$

(in the last step we have used the fact that the integrand is a periodic function of period L; see eqn. (6.22)).
or for $\kappa = k$ and $\kappa = -k$ $\quad (k = \pm I, \pm II, \pm III)$:

$$\delta Q_k = \frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_k^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_k(\tilde{s})\ ; \tag{C.14a}$$

$$\delta Q_{-k} = -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_{-k}^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_{-k}(\tilde{s})\ . \tag{C.14b}$$

Using the facts that:

$$\begin{aligned} \delta Q_\kappa^{*} &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)^{+}\cdot\frac{1}{2\pi}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\left[\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s})\right]^{+} \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\left[-\vec{v}_\kappa^{+}(\tilde{s})\cdot\delta\underline{\tilde{A}}^{T}(\tilde{s})\cdot\underline{S}\cdot\vec{v}_\kappa(\tilde{s})\right] \end{aligned}$$

as well as

$$\vec{v}_{-\kappa} = (\vec{v}_\kappa)^{*}$$

the following relations can be derived from (C.14a,b):

$$\begin{aligned}\Re e\{\delta Q_k\} &= \frac{1}{4\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_k^{+}(\tilde{s})\cdot\left[\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s}) - \delta\underline{\tilde{A}}^T(\tilde{s})\cdot\underline{S}\right]\cdot\vec{v}_k(\tilde{s}) \\ &= -\Re e\{\delta Q_{-k}\}\ ; \end{aligned} \tag{C.15a}$$

$$\begin{aligned}\Im m\{\delta Q_k\} &= -\frac{i}{4\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_k^{+}(\tilde{s})\cdot\left[\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s}) + \delta\underline{\tilde{A}}^T(\tilde{s})\cdot\underline{S}\right]\cdot\vec{v}_k(\tilde{s}) \\ &= +\Im m\{\delta Q_{-k}\}\ . \end{aligned} \tag{C.15b}$$

This means that in addition to a real Q-shift, there is also a complex Q-shift, and comparing (9.18) with (C.15) we find the desired result that the two methods mentioned at the beginning of the appendix are equivalent. Equation (9.18) allows the calculation of the damping constants simply by a numerical integration if one knows the eigenvectors of the unperturbed problem instead of calculating the eigenvalue spectrum of the perturbed revolution matrix necessary for evaluating (C.2).

Finally it is worth mentioning that the applied perturbation theory is only valid if $\delta\lambda_\mu$ and $\delta\vec{v}_\mu$ are small compared with the unperturbed quantities λ_μ and $\vec{v}_\mu$:

$$\begin{aligned}|\delta\lambda_\mu| &\ll |\lambda_\mu|\ ; \\ \|\delta\vec{v}_\mu\| &\ll \|\vec{v}_\mu\|\ .\end{aligned}$$

Therefore, in order to apply this kind of perturbation theory the following condition must hold (see eqn. (C.13)):

$$\left|\int_{s}^{s+L} d\tilde{s}\cdot\left[\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{\tilde{A}}(\tilde{s})\cdot\vec{v}_\mu(\tilde{s})\right]\right| \ll |\lambda_\mu - \lambda_\kappa|\ .$$

This condition is well satisfied if the values for different $\lambda_\mu, \lambda_\kappa$ are far apart. However the calculation breaks down if two eigenvalues coincide:

$$\lambda_k = e^{-i\cdot 2\pi Q_k} \approx \lambda_{k'} = e^{-i\cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k - Q_{k'} = n$$

or

$$\lambda_k = e^{-i\cdot 2\pi Q_k} \approx \lambda_{-k'} = e^{+i\cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k + Q_{k'} = n$$

(n=integer).

Since these Q-resonances can lead to instabilities of the particle motion we do not investigate these effects in this report.

Remark:

Replacing in (C.1) the perturbation matrix $\delta\underline{\tilde{A}}$ by the matrix $\Delta\underline{\tilde{A}}$ (as defined in eqns. (4.15b) and (4.17b)) we obtain the result that the Q-shift induced by $\Delta\underline{\tilde{A}}$, i.e. by dispersion in the cavities, is given by:

$$\Delta Q_k = \frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} ds\cdot\vec{v}_k^{+}(s)\cdot\underline{S}\cdot\Delta\underline{\tilde{A}}(s)\cdot\vec{v}_k(s)\ ; \tag{C.16}$$

($k = I,\ II,\ III$).

The term ΔQ_k is a real number because of eqn. (4.26).

Appendix D: The Dependence of Damping Distributions on R.F. Frequency

Using the relation

$$\sigma \;=\; s - ct\ ,$$

the cavity field in eqn. (A.24):

$$\mathcal{E} \;=\; V(s)\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \tag{D.1}$$

can be written in an alternative form as:

$$\mathcal{E} \;=\; V(s)\sin\left[h\cdot 2\pi\frac{s}{L}-\omega_{rf}\,t+\varphi\right] \tag{D.2}$$

with

$$\omega_{rf} \;=\; h\cdot 2\pi\frac{c}{L} \tag{D.3}$$

defining the design frequency of the cavities in a storage ring of length L.

It has long been recognized that the damping behaviour in electron storage rings can be modified by making slight changes in the frequency of the accelerating fields [29, 30], and the aim of this appendix is to show how frequency shifts can be incorporated in the equations of motion (2.48) (see also Ref. [31]) when each magnet is represented by a set of thin lenses separated by straight drift sections.

If the r.f. frequency is slightly changed from ω_{rf} to $\omega_{rf}+\delta\omega_{rf}$ but the cavities remain in their usual positions, particles moving in the straight sections become progressively out of synchronisation with the common phase. In this case the voltage seen by the particle may be written as:

$$\mathcal{E} \;=\; V(s)\sin\left[h\cdot 2\pi\frac{s}{L}-(\omega_{rf}+\delta_{rf})\,t+\varphi\right]\ . \tag{D.4}$$

Defining

$$\hat{\sigma} \;=\; \sigma-\frac{\delta\omega_{rf}}{\omega_{rf}}\cdot(1-\sigma) \tag{D.5}$$

we obtain:

$$\mathcal{E} \;=\; V(s)\sin\left[h\cdot\frac{2\pi}{L}\cdot\hat{\sigma}+\varphi\right]\ . \tag{D.6}$$

Thus the r.f. phase at a cavity is now given in terms of the new quantity $\hat{\sigma}$ differing from the simple bunch length variable by the term $\frac{\delta\omega_{rf}}{\omega_{rf}}\,[s-\sigma]$.

In addition we may write:

$$\frac{d\hat{\sigma}}{ds} = \frac{d\sigma}{ds} - \frac{\delta\omega_{rf}}{\omega_{rf}}\left[1 - \frac{d\sigma}{ds}\right] \tag{D.7}$$

so that in first order the equation of motion for $\hat{\sigma}$ reads as:

$$\frac{d\hat{\sigma}}{ds} = \frac{d\sigma}{ds} - \frac{\delta\omega_{rf}}{\omega_{rf}}\,. \tag{D.8}$$

The complete equation of motion (2.48) then has to be replaced by:

$$\frac{d}{ds}\,\vec{y} = \underline{A}\cdot\vec{y} + \delta\underline{A}\cdot\vec{y} + \vec{c}_0 + \vec{\hat{c}}_0 + \vec{c}_1 + \vec{c}_{qua} + \vec{c}_{sqd} + \vec{c}_{sex} + \vec{c}_w + \delta\vec{c} \tag{D.9}$$

where we have replaced $\hat{\sigma}$ by σ.

This equation contains an additional inhomogeneous term:

$$\vec{\hat{c}}_0^{\;T} = \left(0,\,0,\,0,\,0,\,-\frac{\delta\omega_{rf}}{\omega_{rf}},\,0\right) \tag{D.10}$$

resulting from eqn. (D.8) and it induces a closed orbit shift.

As described in chapter B1 of Appendix B, the new closed orbit may be found by using the thin lens approximation. In this study, where all optical elements where divided into a sufficient number of thin lenses, the only modification to the 7×7 matrices given in chapter B1 is to put

$$m_{57} = -\frac{\delta\omega_{rf}}{\omega_{rf}}\cdot l \tag{D.11}$$

for drift spaces of length l. Since all other elements (thin lenses) have zero length, their m_{57} elements remain zero. This merely expresses the fact mentioned above that in this thin lens approximation it is in the straight sections that the particles loose synchronism with the r.f. phase.

The oscillations around the new closed orbit can be calculated by the analysis described in sections B.2 and B.3 of Appendix B. These chapters remain valid and may be taken over completely.

Appendix E: The Essential Uniqueness of the Fokker-Planck Solutions for Large Times

The following considerations are based mainly on a method outlined in Ref. [15]. The difference between our treatment and that of Ref. [15] lies in the use of different boundary conditions.

In order to investigate the asymptotic time behaviour of the Fokker-Planck solutions we first introduce the abbreviations

$$(x_1, x_2, x_3, x_4, x_5, x_6) \equiv (J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}) \tag{E.1}$$

so that the Fokker-Planck equation (8.16) can be written in the form:

$$\frac{\partial W}{\partial s} = \left\{ -\sum_{i=1}^{6} \frac{\partial}{\partial x_i} L_i + \sum_{i,j=1}^{6} \frac{\partial^2}{\partial x_i \partial x_j} L_{ij} \right\} W \tag{E.2}$$

where

$$\begin{aligned} &(L_1,\ L_2,\ L_3,\ L_4,\ L_5,\ L_6) \\ = &(-2a_I J_I + M_I,\ -2a_{II} J_{II} + M_{II},\ -2a_{III} J_{III} + M_{III} - \frac{1}{2} d_{III} \cdot J_{III}^2,\ b_I,\ b_{II},\ b_{III}) \end{aligned} \tag{E.3}$$

and where

$$L_{ij} = \delta_{ij} \cdot \tilde{L}_i \tag{E.4a}$$

with

$$\begin{aligned} &(\tilde{L}_1,\ \tilde{L}_2,\ \tilde{L}_3,\ \tilde{L}_4,\ \tilde{L}_5,\ \tilde{L}_6) \\ \equiv &\left(J_I \cdot M_I,\ J_{II} \cdot M_{II},\ J_{III} \cdot M_{III},\ \frac{M_I}{4J_I},\ \frac{M_{II}}{4J_{II}},\ \frac{M_{III}}{4J_{III}} \right) . \end{aligned} \tag{E.4b}$$

From eqns. (2.22), (2.47), and (8.14b) it is clear that the diffusion matrix $((D_{ij}))$ is positive definite which is connected with the positivity of the radiative energy loss of the electron. We will need this property later.

We now introduce the Lyapunov functional [14] of W with respect to another special physical solution W_0 [15] (W_0 is assumed to have no zeros):

$$\begin{aligned} \hat{H}(s) &= \int_V d^6x \cdot W \cdot \ln\left(\frac{W}{W_0}\right) \\ &= \int_V d^6x \cdot W \cdot [\ln W - \ln W_0] \end{aligned} \tag{E.5}$$

(V denotes the action - angle phase space).

Defining the quantity

$$R = \frac{W}{W_0} \tag{E.6}$$

and using the relation ($R \geq 0$)

$$R \cdot \ln R - R + 1 = \int_1^R dx \cdot \ln x \geq 0 \tag{E.7}$$

[14] sometimes called the 'relative entropy'

[15] A solution W of the Fokker-Planck equation (8.16) is called physical if it is normalized and nonnegative and if the moments of J_I, J_{II}, J_{III} with respect to W are finite.

as well as the normalization condition for W and W_0:

$$\int_V d^6x \cdot W = 1 ; \tag{E.8a}$$

$$\int_V d^6x \cdot W_0 = 1 \tag{E.8b}$$

we obtain the inequality:

$$\begin{aligned}
\hat{H}(s) &= \int_V d^6x \cdot W \cdot \ln R \\
&= \int_V d^6x \cdot [W \cdot \ln R - W + W_0] \\
&= \int_V d^6x \cdot W_0 \cdot [R \cdot \ln R - R + 1] \geq 0 ,
\end{aligned} \tag{E.9}$$

i.e. $H(s)$ cannot have negative values.

For the derivative

$$\frac{\partial}{\partial s} \hat{H}(s)$$

of the Lyapunov functional we get:

$$\begin{aligned}
\frac{\partial}{\partial s} \hat{H}(s) &= \int_V d^6x \cdot \left\{ \left(\frac{\partial}{\partial s} W \right) \cdot \ln R + W \cdot \left[\frac{1}{W} \cdot \frac{\partial}{\partial s} W - \frac{1}{W_0} \cdot \frac{\partial}{\partial s} W_0 \right] \right\} \\
&= \int_V d^6x \cdot \left\{ \left(\frac{\partial}{\partial s} W \right) \cdot \ln R - R \cdot \left(\frac{\partial}{\partial s} W_0 \right) \right\} + \frac{\partial}{\partial s} \int_V d^6x \cdot W \\
&= \int_V d^6x \cdot \left\{ \left(\frac{\partial}{\partial s} W \right) \cdot \ln R - R \cdot \left(\frac{\partial}{\partial s} W_0 \right) \right\} .
\end{aligned} \tag{E.10}$$

Using eqn. (E.2), the first term on the r.h.s. of eqn. (E.10) can be written as:

$$\begin{aligned}
\int_V d^6x \cdot \left(\frac{\partial}{\partial s} W \right) \cdot \ln R &= \int_V d^6x \cdot \ln R \cdot \left\{ -\sum_{i=1}^{6} \frac{\partial}{\partial x_i} L_i + \sum_{i,j=1}^{6} \frac{\partial^2}{\partial x_i \partial x_j} L_{ij} \right\} W \\
&= \int_V d^6x \cdot W \cdot \left\{ \sum_{i=1}^{6} L_i + \sum_{i,j=1}^{6} L_{ij} \frac{\partial}{\partial x_j} \right\} \frac{\partial}{\partial x_i} \ln R \\
&= \int_V d^6x \cdot W \cdot \left\{ \sum_{i=1}^{6} L_i + \sum_{i,j=1}^{6} L_{ij} \frac{\partial}{\partial x_j} \right\} \left(\frac{1}{R} \frac{\partial R}{\partial x_i} \right) \\
&= \int_V d^6x \cdot \frac{W}{R} \cdot \left\{ \sum_{i=1}^{6} L_i + \sum_{i,j=1}^{6} L_{ij} \frac{\partial}{\partial x_j} \right\} \frac{\partial R}{\partial x_i} \\
&\quad - \int_V d^6x \cdot W \cdot \sum_{i,j=1}^{6} L_{ij} \cdot \frac{1}{R^2} \cdot \frac{\partial R}{\partial x_j} \cdot \frac{\partial R}{\partial x_i} \\
&= \int_V d^6x \cdot R \cdot \left\{ -\sum_{i=1}^{6} \frac{\partial}{\partial x_i} L_i + \sum_{i,j=1}^{6} \frac{\partial^2}{\partial x_i \partial x_j} L_{ij} \right\} W_0
\end{aligned}$$

$$- \int_V d^6x \cdot W \cdot \sum_{i,j=1}^{6} L_{ij} \cdot \frac{1}{R}\frac{\partial R}{\partial x_i} \cdot \frac{1}{R}\frac{\partial R}{\partial x_j}$$

$$= \int_V d^6x \cdot R \cdot \left(\frac{\partial}{\partial s} W_0\right) - \int_V d^6x \cdot W \cdot \sum_{i,j=1}^{6} L_{ij} \cdot \frac{1}{R}\frac{\partial R}{\partial x_i} \cdot \frac{1}{R}\frac{\partial R}{\partial x_j} \ . \qquad \text{(E.11)}$$

To carry out the partial integration we have used the fact that W and W_0 are periodic functions in Φ_k ($k = I,\ II,\ III$) with period 2π and that the probability currents (see eqn. (8.27b)) of W, i.e.

$$\Im_i \equiv L_i W - \sum_{j=1}^{6} \frac{\partial}{\partial x_j} [L_{ij} W]; \quad (i = 1,\ 2,\ 3) \qquad \text{(E.12)}$$

and of W_0 associated with the action variables J_k ($k = I,\ II,\ III$) vanish for $J_k = 0$ and $J_k \to \infty$.

Equations (E.10) and (E.11) lead to

$$\frac{\partial}{\partial s} \hat{H}(s) = - \int_V d^6x \cdot W \cdot \sum_{i,j=1}^{6} D_{ij} \cdot \frac{\partial}{\partial x_i}\left[\ln\left(\frac{W}{W_0}\right)\right] \cdot \frac{\partial}{\partial x_j}\left[\ln\left(\frac{W}{W_0}\right)\right] \ . \qquad \text{(E.13)}$$

Now, since the diffusion matrix $((D_{ij}))$ is positive definite we have

$$\frac{\partial}{\partial s} \hat{H}(s) < 0 \qquad \text{(E.14a)}$$

if

$$\sum_{i=1}^{6} \left\{\frac{\partial}{\partial x_i}\left[\frac{W}{W_0}\right]\right\}^2 \neq 0 \ . \qquad \text{(E.14b)}$$

Hence from eqns. (E.9) and (E.14) we have:

$$\lim_{s \to \infty} \left[\frac{\partial}{\partial x_i}\left(\frac{W}{W_0}\right)\right] = 0 \qquad \text{(E.15)}$$

(otherwise $\hat{H}(s)$ would decrease below 0).

From (E.15) and using (E.9) we can finally see that the two solutions W and W_0 must coincide for long times.

Choosing for W_0 the solution (9.9), we obtain the result that the stationary distribution (9.9) is unique.

Note that W_0 given by (9.9) is in fact nowhere vanishing so that the assumption made after (E.4) is allowed.

Appendix F: The Areas of Constant Density in the Transverse $(\tilde{x} - \tilde{p}_x - \tilde{z} - \tilde{p}_z)$ - Phase Space

F.1 The Four - Dimensional Ellipsoid

We look for the areas of constant density in the transverse $(\tilde{x} - \tilde{p}_x - \tilde{z} - \tilde{p}_z)$ - phase space. From eqn. (9.19a) we obtain:

$$\begin{aligned} \rho_\beta(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z) &= \text{const} \\ \Longrightarrow \quad \frac{J_I}{\hat{J}_I} + \frac{J_{II}}{\hat{J}_{II}} &= \text{const} \, . \end{aligned} \tag{F.1}$$

The variables J_I and J_{II} may therefore be parameterised as:

$$\begin{aligned} \sqrt{J_I} &= a \cdot \cos\chi \cdot \sqrt{\hat{J}_I} \, ; \\ \sqrt{J_{II}} &= a \cdot \sin\chi \cdot \sqrt{\hat{J}_{II}} \end{aligned}$$

and it results from (5.16):

$$\begin{aligned} \vec{\tilde{y}}^{(\beta)}(s; \chi, \delta_I, \delta_{II}) &= a\sqrt{\hat{J}_I} \cdot \cos\chi \cdot \left[\vec{v}_I^{(\beta)}(s) \cdot e^{i\delta_I} + [\vec{v}_I^{(\beta)}(s)]^* \cdot e^{-i\delta_I} \right] \\ &+ a\sqrt{\hat{J}_{II}} \cdot \sin\chi \cdot \left[\vec{v}_{II}^{(\beta)}(s) \cdot e^{i\delta_{II}} + [\vec{v}_{II}^{(\beta)}(s)]^* \cdot e^{-i\delta_{II}} \right] \, . \end{aligned} \tag{F.2}$$

Equation (F.2) defines a four - dimensional ellipsoid in the $(\tilde{x} - \tilde{p}_x - \tilde{z} - \tilde{p}_z)$ - phase space [28, 32] which is periodic with period L [28] as may be seen by using eqns. (5.3) and (5.7):

$$\vec{\tilde{y}}^{(\beta)}(s + L; \chi, \delta_I, \delta_{II}) = \vec{\tilde{y}}^{(\beta)}(s; \chi, \delta_I - 2\pi Q_I, \delta_{II} - 2\pi Q_{II}) \, .$$

By a decomposition of the vectors

$$a \cdot \sqrt{J_k} \cdot \vec{v}_k^{(\beta)} \, ; \quad (k = I, \; II)$$

into a real and imaginary part:

$$\begin{aligned} a \cdot \sqrt{\hat{J}_I} \cdot \vec{v}_I^{(\beta)} &= \frac{1}{2} \cdot [\vec{y}_1 - i \cdot \vec{y}_2] \, ; \\ a \cdot \sqrt{\hat{J}_{II}} \cdot \vec{v}_{II}^{(\beta)} &= \frac{1}{2} \cdot [\vec{y}_3 - i \cdot \vec{y}_4] \end{aligned}$$

eqn. (F.2) takes the form:

$$\begin{aligned} \vec{\tilde{y}}^{(\beta)}(s; \chi, \delta_I, \delta_{II}) &= \cos\chi \cdot [\vec{y}_1(s) \cdot \cos\delta_I + \vec{y}_2(s) \cdot \sin\delta_I] \\ &+ \sin\chi \cdot [\vec{y}_3(s) \cdot \cos\delta_{II} + \vec{y}_4(s) \cdot \sin\delta_{II}] \, . \end{aligned} \tag{F.3}$$

It follows from eqn. (F.3) that the motion of this ellipsoid under the influence of the external fields can be described by four generating orbit vectors $\vec{y}_k$:

$$\vec{y}_\mu(s) = \underline{M}^{(\beta)}_{(4\times4)}(s, s_0) \; \vec{y}_\mu(s_0) \, ; \quad (\mu = 1, 2, 3, 4) \, . \tag{F.4}$$

Combining these vectors into a four-dimensional matrix $\underline{B}(s)$:

$$\underline{B}(s) = (\vec{y}_1(s),\ \vec{y}_2(s),\ \vec{y}_3(s),\ \vec{y}_4(s)) \tag{F.5}$$

one has [32]:

$$\underline{B}(s) = \underline{M}^{(\beta)}_{(4\times4)}(s, s_o)\ \underline{B}(s_o)\ . \tag{F.6}$$

This "bunch -shape matrix", $\underline{B}(s)$, now contains complete information about the transverse configuration of the bunch which can be obtained by projecting the ellipsoid (F.3) on the individuel phase planes [28, 32] as shall be discussed in the next chapter.

F.2 The Projections of the Four - Dimensional Ellipsoid. Beam Envelopes

In order to determine the projections of the four - dimensional ellipsoid which characterize the beam envelopes [28, 32] we first of all write eqn. (F.3) in component form:

$$\begin{aligned}\tilde{x}(s;\chi,\delta_I,\delta_{II}) &= \cos\chi\cdot[y_{11}(s)\cdot\cos\delta_I + y_{21}(s)\cdot\sin\delta_I] + \\ &\quad \sin\chi\cdot[y_{31}(s)\cdot\cos\delta_{II} + y_{41}(s)\cdot\sin\delta_{II}]\ ;\end{aligned} \tag{F.7a}$$

$$\begin{aligned}\tilde{p}_x(s;\chi,\delta_I,\delta_{II}) &= \cos\chi\cdot[y_{12}(s)\cdot\cos\delta_I + y_{22}(s)\cdot\sin\delta_I] + \\ &\quad \sin\chi\cdot[y_{32}(s)\cdot\cos\delta_{II} + y_{42}(s)\cdot\sin\delta_{II}]\ ;\end{aligned} \tag{F.7b}$$

$$\begin{aligned}\tilde{z}(s;\chi,\delta_I,\delta_{II}) &= \cos\chi\cdot[y_{13}(s)\cdot\cos\delta_I + y_{23}(s)\cdot\sin\delta_I] + \\ &\quad \sin\chi\cdot[y_{33}(s)\cdot\cos\delta_{II} + y_{43}(s)\cdot\sin\delta_{II}]\ ;\end{aligned} \tag{F.7c}$$

$$\begin{aligned}\tilde{p}_z(s;\chi,\delta_I,\delta_{II}) &= \cos\chi\cdot[y_{14}(s)\cdot\cos\delta_I + y_{24}(s)\cdot\sin\delta_I] + \\ &\quad \sin\chi\cdot[y_{34}(s)\cdot\cos\delta_{II} + y_{44}(s)\cdot\sin\delta_{II}]\ .\end{aligned} \tag{F.7d}$$

The computation of the single projections is then similar to that in Ref. [28] in which the functional relationship between pairs of components was investigated.

Since the details of the method have already been given in Refs. [28] and [32] only a summary will be needed here.

1) Projection on the $x - z$ plane.

We first investigate the projection on the $x - z$ plane. This describes the beam cross section. We will need the maximum amplitude in the x and z directions.

a) Maximum oscillation amplitude in the x direction:

Using the relation

$$Max_{(\delta)}\ \{A\cdot\cos\delta + B\cdot\sin\delta\} = \sqrt{A^2+B^2}$$

and eqn. (F.7a), the largest possible $\tilde{x}$ amplitude is

$$Max_{(\chi,\delta_I,\delta_{II})}\ \tilde{x}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) = \sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}$$

$$= E_x(s)\ . \tag{F.8}$$

This occurs for the values:

$$\cos\delta_I = \frac{y_{11}}{\sqrt{y_{11}^2 + y_{21}^2}}\ ; \quad \sin\delta_I = \frac{y_{21}}{\sqrt{y_{11}^2 + y_{21}^2}}\ ;$$

$$\cos\delta_{II} = \frac{y_{31}}{\sqrt{y_{31}^2 + y_{41}^2}}\ ; \quad \sin\delta_{II} = \frac{y_{41}}{\sqrt{y_{31}^2 + y_{41}^2}}\ ;$$

$$\cos\chi = \frac{\sqrt{y_{11}^2 + y_{21}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}}\ ;$$

$$\sin\chi = \frac{\sqrt{y_{31}^2 + y_{41}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}}\ . \tag{F.9}$$

The corresponding $\tilde{z}$ -coordinate is given by eqn. (F.7c) together with eqn. (F.9):

$$G_{xz} = \frac{1}{E_x(s)} \cdot \{y_{11}\cdot y_{13} + y_{21}\cdot y_{23} + y_{31}\cdot y_{33} + y_{41}\cdot y_{43}\}\ . \tag{F.10}$$

b) Maximum oscillation amplitude in the z direction:
Correspondingly, the maximum amplitude in the z - direction is obtained from (F.7c):

$$Max_{(\chi,\delta_I,\delta_{II})}\ \tilde{z}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) = \sqrt{y_{13}^2 + y_{23}^2 + y_{33}^2 + y_{43}^2}$$

$$= E_z(s)\ . \tag{F.11}$$

The accompanying $\tilde{x}$ -coordinate is then:

$$G_{zx} = \frac{1}{E_z(s)} \cdot \{y_{11}\cdot y_{13} + y_{21}\cdot y_{23} + y_{31}\cdot y_{33} + y_{41}\cdot y_{43}\}\ . \tag{F.12}$$

Thus

$$E_x \cdot G_{xz} = E_z \cdot G_{zx}\ . \tag{F.13}$$

c) The boundary curve of the beam cross section.

The projections of the ellipsoid (F.3) are ellipses, and these are described by the three independent quantities E_x, G_{xz}, E_z. The parameter G_{zx} depends on the other three (see eqn. (10.24)). In terms of E_x, G_{xz}, E_z, the ellipse can be written as:

$$E_z^2 \cdot \tilde{x}^2 - 2E_xG_{xz} \cdot \tilde{x}\tilde{z} + E_x^2 \cdot \tilde{z}^2 = \epsilon_{xz}^2 \tag{F.14a}$$

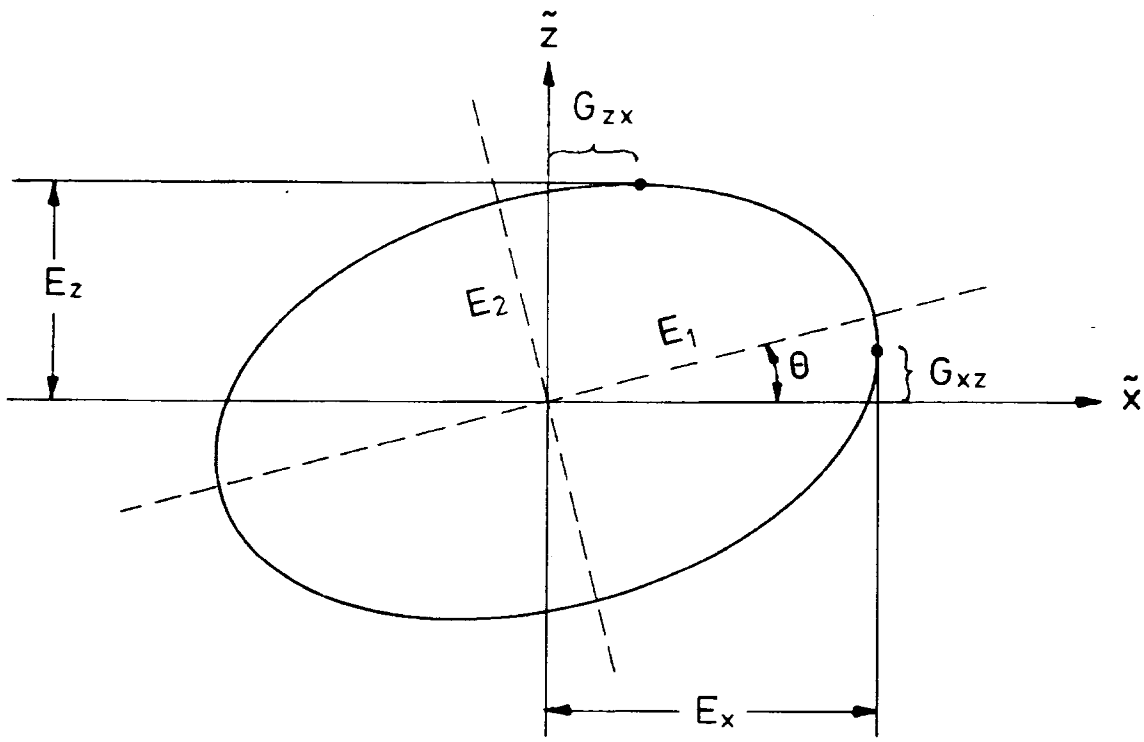

Figure 1: The beam cross section

with

$$\epsilon_{xz} = E_x \cdot \sqrt{E_z^2 - G_{xz}^2} \; . \tag{F.14b}$$

and where $\pi \, \epsilon_{xz}$ is the area of the ellipse.

The half axes E_1 and E_2 of the elliptical beam cross section are:

$$E_{1,2} = \frac{1}{2} \left\{ \left[E_x^2 + E_z^2 \right] \pm \sqrt{\left[E_x^2 - E_z^2 \right]^2 + 4 E_x^2 \cdot G_{xz}^2} \right\} \tag{F.15}$$

and the twist angle θ of the beam is given by:

$$\tan\, 2\theta = 2 \cdot \frac{E_x \cdot G_{xz}}{E_x^2 - E_z^2} \; . \tag{F.16}$$

The projections on the $(y,\ p_y)$ - plane ($y = x,\ z,\ \sigma$) can be found in a similar way [28, 32]. One obtains:

2) Projection on the $x - p_x$ plane.

For the projection of the ellipsoid (F.3) onto the $x - p_x$ plane the corresponding equations are (F.7a), (F.7b). Since these two relations have the same form as eqns. (F.7a) and (F.7c),

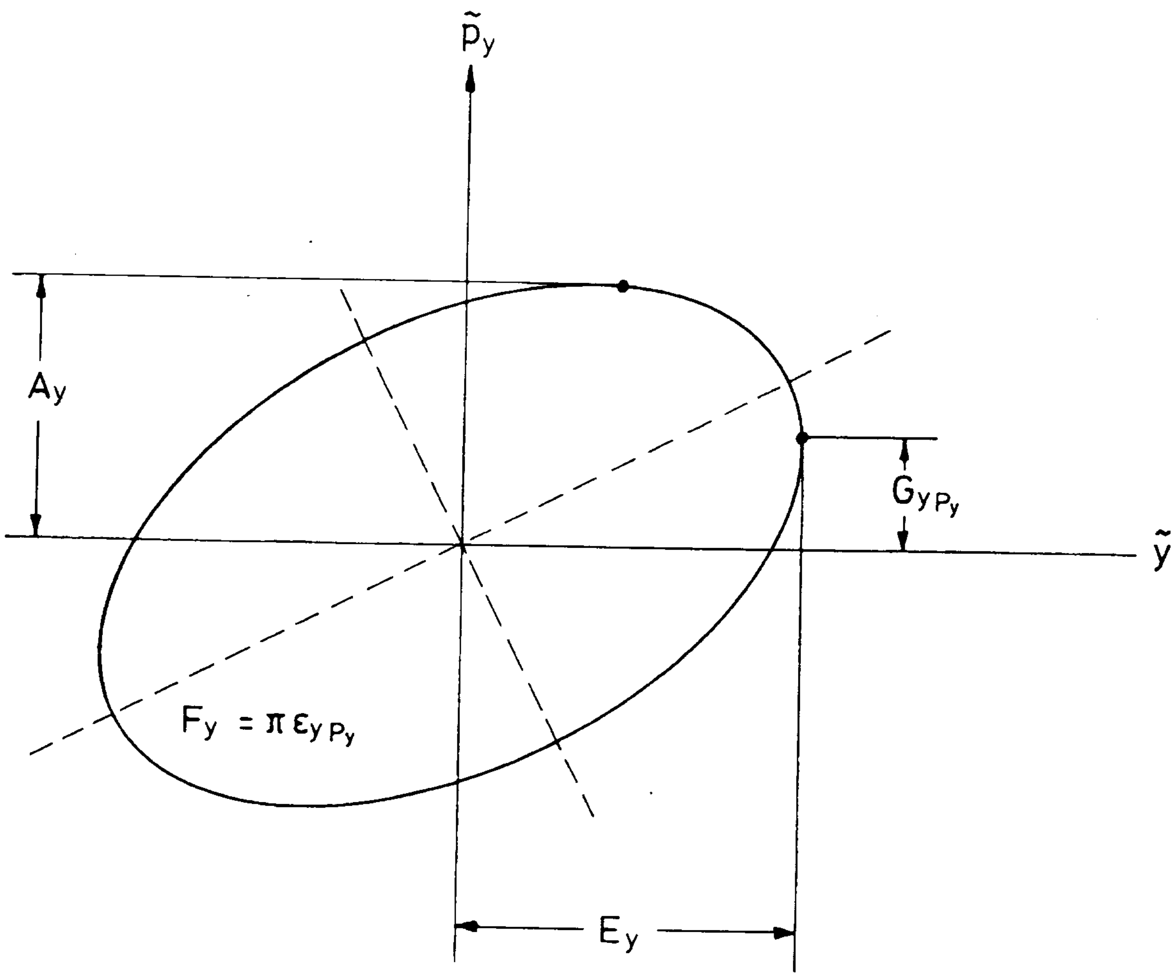

Figure 2: Projection on the $y - p_y$ plane; $(y \equiv x, z)$

we obtain an elliptical projection onto the $x - p_x$ plane by analogy with eqn. (F.14). We write the ellipse in the form:

$$A_x^2 \cdot \tilde{x}^2 - 2E_x G_{xp_x} \cdot \tilde{x}\tilde{p}_x + E_x^2 \cdot \tilde{p}_x^2 = \epsilon_{xp_x}^2 \tag{F.17}$$

with

$$A_x(s) = Max_{(\chi,\delta_I,\delta_{II})}\ \tilde{p}_x(s,\chi,\delta_I,\delta_{II})$$

$$= \sqrt{y_{12}^2 + y_{22}^2 + y_{32}^2 + y_{42}^2}\ ; \tag{F.18}$$

$$G_{xp_x}(s) = \frac{1}{E_x(s)} \cdot \{y_{11} \cdot y_{12} + y_{21} \cdot y_{22} + y_{31} \cdot y_{32} + y_{41} \cdot y_{42}\}\ ; \tag{F.19}$$

$$\pi\epsilon_{xp_x} = \pi \cdot E_x\sqrt{A_x^2 - E_{p_x}^2}\ ; \tag{F.20}$$

(area of the ellipse (F.17)).

Here, the function $A_x(s)$ represents the maximum amplitude of the momentum p_x and could be called the momentum envelope for the $x - p_x$ plane. $\pi\mathcal{E}_{xp_x}$ gives the area of the ellipse (F.17) and the meaning of E_{p_x} is indicated in Fig. 2.

3) Projection on the $z - p_z$ plane.

A similar treatment can be used to describe the projection on the $z - p_z$ plane. We write

$$A_z^2 \cdot \tilde{z}^2 - 2E_z G_{zp_z} \cdot \tilde{z}\tilde{p}_z + E_z^2 \cdot \tilde{p}_z^2 = \epsilon_{zp_z}^2 \tag{F.21}$$

where

$$\begin{aligned} A_z(s) &= Max_{(\chi,\delta_I,\delta_{II})}\ \tilde{p}_z(s,\chi,\delta_I,\delta_{II}) \\ &= \sqrt{y_{14}^2 + y_{24}^2 + y_{34}^2 + y_{44}^2}\ ; \end{aligned} \tag{F.22}$$

$$G_{zp_z}(s) = \frac{1}{E_z(s)} \cdot \{y_{13} \cdot y_{14} + y_{23} \cdot y_{24} + y_{33} \cdot y_{34} + y_{43} \cdot y_{44} + y_{53} \cdot y_{54} + y_{63} \cdot y_{64}\}\ ; \tag{F.23}$$

$$\pi\epsilon_{zp_z} = \pi \cdot E_z\sqrt{A_z^2 - E_{p_z}^2} \tag{F.24}$$

(area of the ellipse (F.21)).

References

[1] D.P. Barber, K. Heinemann, H. Mais, G. Ripken: "A Fokker-Planck-Treatment of Stochastic Particle Motion within the Framework of a fully coupled 6-dimensional Formalism for Electron-Positron Storage Rings including Classical Spin Motion in linear Approximation"; DESY 91-146.

[2] J.M. Jowett: "Electron Dynamics with Radiation and Nonlinear Wigglers"; CERN 87-03 and literature cited therein.

[3] G. Ripken and E. Karantzoulis: "A Fokker-Planck Treatment of Coupled Synchro-Betatron Motion in Proton Storage Rings under the Influence of Cavity Noise"; DESY 90-135, 1990.

[4] G. Ripken and E. Karantzoulis: "A Fokker-Planck Treatment of Phase Fluctuations for Proton Storage Rings within the Framework of a Fully Coupled 6 - dimensional Formalism with Applications to HERA"; DESY HERA 92-05, (1992).
Published in Part. Accel., vol.39, p.195, (1992).

[5] SLICK is the thick lens version (by D.P. Barber) of the program SLIM of A. Chao [12].

[6] G. Ripken: "Nonlinear canonical equations of coupled synchro- betatron motion and their solution within the framework of a nonlinear 6-dimensional (symplectic) tracking program for ultra-relativistic protons"; DESY 85-84.

[7] D.P. Barber, K. Heinemann, G. Ripken: "A Canonical Eight-Dimensional Formalism for Linear and Non-Linear Classical Spin-Orbit Motion in Storage Rings";
DESY Report 91-47, (1991)
and

"A Canonical 8-Dimensional Formalism for Classical Spin - Orbit Motion in Storage Rings. I: A New Pair of Canonical Spin Variables".
Z. Phys. C 64, 117-142 (1994).

D.P. Barber, K. Heinemann, G. Ripken: "Nonlinear Canonical Spin - Orbit Dynamics in Storage Rings: Normal Forms and the $\vec{n}$- Axis";
DESY Report 93-31, (1993).
and
"A Canonical 8-Dimensional Formalism for Classical Spin - Orbit Motion in Storage Rings. II: Normal Forms and the $\vec{n}$-Axis".
Z. Phys. C 64, 143-167 (1994).

[8] E. Karantzoulis: "Nonlinear covariant equations of transverse motion in arbitrary magnetic fields and with respect to arbitrary orbits"; Sincrotrone Trieste Report ST/M-89/21 (1989).

[9] D.P. Barber, K. Heinemann, G. Ripken, F. Willeke: "Notes on Synchro - Betatron Coupling for Particles of Arbitrary Energy Using a 6 - Dimensional Symplectic Dispersion Formalism"; DESY HERA 94-02.

[10] C. Bernardini, C. Pellegrini: "Linear theory of motion in electron storage rings", Ann. Phys. 46, 174 (1968).

[11] J.D. Jackson: "Classical Electrodynamics"; Second Edition; John Wiley and Sons Inc., New York (1975).

[12] A.Chao: "Evaluation of radiative spin polarization in an electron storage ring"; Nucl. Inst. Meth. 180,29 (1981).

[13] H. Mais and G. Ripken: "Spin-Orbit Motion in the Presence of Synchrotron Radiation using a Dispersion Formalism"; DESY 86-29.

[14] D.P. Barber, H. Mais, G. Ripken, F. Willeke: "Nonlinear Theory Of Coupled Synchro-Betatron Motion"; DESY 86-147 .

[15] H. Risken: "The Fokker-Planck Equation"; Second Edition; Springer (1984).

[16] J. Honerkamp: "Stochastische dynamische Systeme"; Weinheim: VCH (1990).

[17] Yu.P. Virchenko and Yu.N. Grigor'ev, Annals of Physics, vol 209, p 1 (1991).

[18] H. Mais and G. Ripken: "Spin-Orbit Motion in a Storage Ring in the Presence of Synchrotron Radiation using a Dispersion Formalism "; DESY 86-29 (1986).

[19] F. Ruggiero, E. Picasso and L.A. Radicati, Annals of Physics, vol 197, p 396 (1990).

[20] H. Mais and G. Ripken: "Theory of Spin-Orbit Motion in Electron-Positron Storage Rings. Summary of Results"; DESY 83-62.

[21] F. Willeke, G. Ripken: "Methods of Beam Optics"; DESY 88-114.

[22] G. Ripken, F. Willeke: "On the Impact of Linear Coupling on Nonlinear Dynamics"; DESY 90-001 (1990).

[23] N.G. van Kampen, Journal of Statistical Physics, vol 24, p 175 (1981).

[24] H. Mais and G. Ripken: "Influence of the synchrotron radiation on the spin orbit motion of a particle in a storage ring"; DESY M-82-20.

[25] D.P. Barber, G. Ripken, F. Schmidt: "A nonlinear canonical formalism for the coupled synchro-betatron motion of protons with arbitrary energy"; DESY 87-36.

[26] E. Freytag und G. Ripken: "Nichtlineare Störungen im Speicherring durch das longitudinale, rotationssymmetrische Magnetfeld einer Detektoranordnung"; DESY E3/R1-73/1 (1973).

[27] D.P. Barber, K. Heinemann, G. Ripken: "Spin Tune Shift due to Closed Orbit Distortions"; DESY M94-05.

[28] G. Ripken: "Untersuchungen zur Strahlführung und Stabilität der Teilchenbewegung in Beschleunigern und Storage-Ringen unter strenger Beruecksichtigung einer Kopplung der Betatronschwingungen"; DESY R1-70/04.

[29] R.D. Kohaupt and G.-A. Voss, Ann. Rev. Nucl. Part. Sci., vol 33, p 67 (1983).

[30] A. Piwinski, DESY 83-028 (1983).

[31] D.P. Barber, M. Mais, G. Ripken: "Calculation of the Dependence of Damping Distributions on R.F. Frequency in the Fully Coupled 6×6 Formalism"; DESY M-84-01.

[32] I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken: "Calculation of beam envelopes in storage rings and transport systems in the presence of transverse space charge effects and coupling"; DESY 87-161 (1987).
Published in Z.Phys., vol C39, p 339, (1988).