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Non-Commutative Geometry on Discrete Space

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Abstract

A new formulation is given for the non-commutative geometry on Z_2-discrete space. This is useful for deriving the Brans -Dicke theory for gravity on two-sheeted space-time.

§ 1. Introduction

Recently, in order to build the standard model of particle physics, the non-commutative geometry (NCG)[1] has been applied to a two-sheeted space-time $M_4 \times Z_2$.[2] A nice picture of these studies is that the Higgs scalar field is introduced as an extended gauge field related to the Z_2 symmetry.

Besides particle model building there are some works on gravity in terms of NCG on $M_4 \times Z_2$,[3] where a scalar field is coupled to gravity. However, the scalar fields in these works are different with each other. The reason of this is partially owing to different torsion-free conditions used in their papers, but mostly owing to lack of proper formalism of gravity in terms of NCG.

In a previous work[4] we gave one of such a formalism. We found a proper definition of affine connection and then obtained the precise Brans-Dicke theory on $M_4 \times Z_2$. In this paper we would like to give another formalism, the vielbein formalism.

§ 2. Differential caluculus on discrete group Z_2

1. Derivative

Let A be the algebra of complex-valued function on $M_4 \times Z_2$, where M_4 is a manifold of four-dimensional space-time and Z_2 is a discrete space with two points e and r. Let $x^N = (x^n, x^\bullet)$, n = 1,2,3,4, be coordinates on $M_4 \times Z_2$, where $x^\bullet = x^\bullet(g)$, $g \in Z_2$, is a coordinate on Z_2. We then define a derivative ∂_{Z_2} on Z_2 for any element $f(x, x^\bullet(g)) \in A$ by

$$\partial_{Z_2} f(x^\bullet(g)) \equiv \frac{\Delta f(x^\bullet(g))}{\Delta x^\bullet(g)} = \frac{f(x^\bullet(g)) - f(x^\bullet(g'))}{x^\bullet(g) - x^\bullet(g')} = \frac{f-f'}{x^\bullet - x^{\bullet\prime}}, \tag{2.1}$$

where

$$\Delta x^\bullet(g) = x^\bullet(g) - x^\bullet(g') = x^\bullet - x^{\bullet\prime}, \quad (g \neq g' \in Z_2) \tag{2.2}$$

is taken to be an infinitesimally small quantity, but it does not have a zero limit. From this definition we see that the second-order derivative always vanishes

$$\partial_{Z_2}\partial_{Z_2} f = \partial_{Z_2}\left(\frac{f-f'}{x^\bullet - x^{\bullet\prime}}\right) = \frac{1}{\Delta x^\bullet}\left(\frac{f-f'}{x^\bullet - x^{\bullet\prime}} - \frac{f'-f}{x^{\bullet\prime} - x^\bullet}\right) = 0. \tag{2.3}$$

2. Exterior derivative

Let d_{Z_2} be the exterior derivative on Z_2. By introducing a one-form χ dual to ∂_{Z_2}, $\chi(\partial_{Z_2}) = 1$, the exterior derivative on f is defined by

$$d_{Z_2} f = (\partial_{Z_2} f)\chi = \frac{f-f'}{\Delta x^\bullet}\chi. \tag{2.4}$$

In order that the Leibnitz rule is valid for this exterior derivative, we should add a condition

$$\chi f(g) = f(g')\chi = f'\chi. \tag{2.5}$$

<u>Proof</u>

Let us calculate the exterior derivative for products $f_1 f_2$:

$$\begin{aligned} d_{Z_2}(f_1 f_2) &= \partial_{Z_2}(f_1 f_2)\chi = (f_1 f_2 - f_1' f_2')\frac{\chi}{\Delta x^\bullet} \\ &= (f_1 f_2 - f_1 f_2' + f_1 f_2' - f_1' f_2')\frac{\chi}{\Delta x^\bullet} \\ &= f_1(\partial_{Z_2} f_2)\chi + (\partial_{Z_2} f_1) f_2' \chi. \end{aligned} \tag{2.6}$$

e second term violates the Leibnitz rule. However, if we set the condition .5), then the Leibnitz rule will be recovered

$$d_{Z_2}(f_1 f_2) = f_1\, d_{Z_2} f_2 + d_{Z_2} f_1\, f_2 .$$

The nilpotency $d^2_{Z_2} = 0$ will be realized if

$$d_{Z_2} \chi = 0. \tag{2.7}$$

his can be seen as follows:

$$d^2_{Z_2} f = d_{Z_2}(\partial_{Z_2} f \chi) = \partial_{Z_2}\partial_{Z_2} f\, \chi\chi + \partial_{Z_2} f\, d_{Z_2}\chi. \tag{2.8}$$

Since the second derivative vanishes by (2.3), then we have $d_{Z_2}\chi = 0$. The equation $d_{Z_2}\chi = 0$ is different from the usual condition $d_{Z_2}\chi = -2\chi\chi$ in NCG. In (2.8) the product $\chi\chi$ does not always vanish. In the following we assume

$$\chi\chi \neq 0. \tag{2.9}$$

Both equations (2.5) and (2.9) are characteristic in NCG.

3. Exterior derivative on $M_4 \times Z_2$

Let d_4 be the exterior derivative on M_4. Then the exterior derivative on A is given by

$$\begin{aligned} df &= d_4 f + d_{Z_2} f \\ &= \partial_m f\, dx^m + \partial_{Z_2} f\, \chi . \qquad (m = 1,2,3,4) \end{aligned} \tag{2.10}$$

In the following sometimes we use the notation as

$$df = \partial_N f\, dx^N, \qquad N = (m, \bullet), \quad dx^\bullet = \chi. \tag{2.11}$$

From the requirement of nilpotency of $d^2 = 0$, $d_4^2 = 0$, $d^2_{Z_2} = 0$ and hence $d_4 d_{Z_2} + d_{Z_2} d_4 = 0$, we get

$$dx^n \wedge dx^m = - dx^m \wedge dx^n,$$
$$dx^m \wedge \chi = - \chi \wedge dx^n.$$

However, the one-form χ on Z_2 with itself is not anticommutative by (2.9).

§ 3. A vielbein formalism

1. Bases

Let us consider the tangent space T_p at a point $p = (x^n, x^\bullet) \in M_4 \times Z_2$. We denote by $\{\mathbb{E}_A(p),\ A = 1,2,3,4,5\}$ a set of orthonormal bases of T_p and by $\{\mathbb{E}_N(p),\ N = (n, \bullet),\ n = 1,2,3,4\}$ a set of general coordinate bases of T_p. The vielbein $e^A{}_N(p)$ is a transformation between them

$$\mathbb{E}_N = \mathbb{E}_A\, e^A{}_N, \qquad \mathbb{E}_A = \mathbb{E}_N\, e^N{}_A, \qquad \langle \mathbb{E}_A, \mathbb{E}_B \rangle = \eta_{AB}. \tag{3.1}$$

A metric G_{MN} on $M_4 \times Z_2$ is then given by the inner product

$$G_{MN} = \langle \mathbb{E}_M, \mathbb{E}_N \rangle = \eta_{AB}\, e^A{}_M e^B{}_N. \tag{3.2}$$

2. Spin connection

A spin connection $\omega_L{}^A{}_B$ is introduced by mapping

$$\mathbb{E}_A(x+\Delta x, x^\bullet) \sim \mathbb{E}_A(x, x^\bullet) + \mathbb{E}_B(x, x^\bullet)\, \omega_m{}^B{}_A\, \Delta x^m, \quad (m = 1,2,3,4) \tag{3.3}$$

$$\mathbb{E}_A(x, x^\bullet + \Delta x^\bullet) \sim \mathbb{E}_A(x, x^\bullet) - \mathbb{E}_B(x, x^\bullet)\, \omega_\bullet{}^B{}_A\, \Delta x^\bullet, \tag{3.4}$$

where $\Delta x^\bullet = x^\bullet - x^{\bullet\prime}$ is an infinitesimally small quantity as well as Δx^n. Differential forms of them are given by

$$d_4 \mathbb{E}_A = \mathbb{E}_B\, \omega_m{}^B{}_A\, dx^m, \tag{3.5}$$

$$d_{Z_2} \mathbb{E}_A = \mathbb{E}_B\, \omega_\bullet{}^B{}_A\, \chi. \tag{3.6}$$

Sometimes they will be used in a single form

$$d\mathbb{E}_A = \mathbb{E}_B\,\omega_N{}^B{}_A\,dx^N, \qquad dx^\bullet = \chi. \tag{3.7}$$

3. Affine connection

An affine connection $\Gamma^L{}_{MN}$ is also introduced by mapping

$$\mathbb{E}_M(x+\Delta x, x^\bullet) \sim \mathbb{E}_M(x,x^\bullet) + \mathbb{E}_L(x,x^\bullet)\Gamma^L{}_{nM}\,\Delta x^n, \tag{3.8}$$

$$\mathbb{E}_M(x, x^\bullet+\Delta x^\bullet) \sim \mathbb{E}_M(x,x^\bullet) - \mathbb{E}_L(x,x^\bullet)\Gamma^L{}_{\bullet M}\,\Delta x^\bullet . \tag{3.9}$$

Differential forms of them are given by

$$d_4\mathbb{E}_M = \mathbb{E}_L\Gamma^L{}_{nM}\,dx^n, \tag{3.10}$$

$$d_{Z_2}\mathbb{E}_M = \mathbb{E}_L\,\Gamma^L{}_{\bullet M}\,\chi \tag{3.11}$$

or simply

$$d\mathbb{E}_M = \mathbb{E}_L\,\Gamma^L{}_{NM}\,dx^N, \qquad dx^\bullet = \chi . \tag{3.12}$$

Substituting (3.1) into (3.12) we get an expression

$$d\mathbb{E}_M = \mathbb{E}_A e^A{}_L\,\Gamma^L{}_{NM}\,dx^N = \mathbb{E}_A\,\Gamma^A{}_{NM}\,dx^N, \tag{3.13}$$

where

$$\Gamma^A{}_{NM} = e^A{}_L\,\Gamma^L{}_{NM}. \tag{3.14}$$

4. Covariant-free equation

We derive the covariant-free equation

$$\nabla_N e^A{}_M = \partial_N e^A{}_M + \omega_N{}^A{}_B\,e'^B{}_M - e^A{}_L\,\Gamma^L{}_{NM} = 0. \tag{3.15}$$

Proof

Substituting (3.1) into the left-hand side of (3.13) we get

$$d\mathbb{E}_M = d(\mathbb{E}_A\,e^A{}_M) = d\mathbb{E}_A\,e^A{}_M + \mathbb{E}_A\,de^A{}_M$$
$$= \mathbb{E}_B\,\omega_N{}^B{}_A\,dx^N e^A{}_M + \mathbb{E}_A\,\partial_N e^A{}_M\,dx^N$$

$$= (\mathbb{E}_B\,\omega_N{}^B{}_A\,e^{A'}_M + \mathbb{E}_A\,\partial_N e^A{}_M)\,dx^N$$
$$= \mathbb{E}_A\,(\partial_N e^A{}_M + \omega_N{}^A{}_B\,e^{B'}_M)\,dx^N, \tag{3.16}$$

where we have used the non-commutativity

$$dx^N e^A{}_M(g) = e^A{}_M(g')\,dx^N = e^{A'}_M\,dx^N \quad \text{for } N=\bullet . \tag{3.17}$$

Comparing (3.16) and (3.13) one obtains

$$\Gamma^A{}_{NM} = e^A{}_L\,\Gamma^L{}_{NM} = \partial_N e^A{}_M + \omega^A_{NB}\,e^{B'}_M . \tag{3.18}$$

Note that this equation reduces to the ordinary form

$$\nabla_N e^A{}_M = \partial_N e^A{}_M + \omega_N{}^A{}_B\,e^B_M - e^A{}_L\,\Gamma^L{}_{NM}, \tag{3.19}$$

if $e^B{}_M{}' = e^B{}_M$, i.e., the vielbein is independent of $x^\bullet$.

5. Proof of $\omega_N{}^{AB} = -\omega_N{}^{BA}$

First we prove $\omega_\bullet^{AB} = -\ \omega_\bullet^{BA}$. Let us assume that the inner product of basis $\mathbb{E}_A\cdot\mathbb{E}_B$ be invariant under a change $g \to g'$. Then one gets

$$0 = \mathbb{E}_A\cdot\mathbb{E}_B - \mathbb{E}'_A\cdot\mathbb{E}'_B$$
$$= \mathbb{E}_A\cdot\mathbb{E}_B - (\mathbb{E}_A - \mathbb{E}_C\,\omega_\bullet{}^C{}_A\,\Delta x^\bullet)\cdot(\mathbb{E}_B - \mathbb{E}_C\,\omega_\bullet{}^C{}_B\,\Delta x^\bullet)$$
$$= (\mathbb{E}_A\cdot\mathbb{E}_C\,\omega_\bullet{}^C{}_B + \mathbb{E}_B\cdot\mathbb{E}_C\,\omega_\bullet{}^C{}_A)\,\Delta x^\bullet$$
$$= (\omega_{\bullet AB} + \omega_{\bullet BA})\,\Delta x^\bullet \implies \omega_{\bullet AB} = -\omega_{\bullet BA} . \tag{3.20}$$

Here we have neglected the second-order term $0(\Delta x^\bullet)^2$. In the same way one gets $\omega_{nAB} + \omega_{nBA} = 0$. This completes the proof.

6. Covariant-free equation for G_{MN}

Both equations of $\nabla_M e^A{}_N = 0$ and $\omega_N{}^{AB} = -\omega_N{}^{BA}$ are necessary to prove the equation

$$\nabla_L G_{MN} = \partial_L G_{MN} - \Gamma_{MLN} - \Gamma_{NLM} = 0, \tag{3.21}$$

where

$$\Gamma_{MLN} = G_{MK}\,\Gamma^K{}_{LN}\,. \tag{3.22}$$

If the affine connection $\Gamma^L{}_{MN}$ is symmetric with respect to M and N, then it is well known that $\Gamma^L{}_{MN}$ is expressed in terms of the metric G_{MN} as follows:

$$\Gamma^L{}_{MN} = \frac{1}{2} G^{LK}(\partial_M G_{KN} + \partial_N G_{KM} - \partial_K G_{MN})\,. \tag{3.23}$$

The equation (3.21) is also derived directly by using (3.2), (3.8) and (3.9).

7. Torsion

The torsions T^A and T^L are given by

$$\begin{aligned} T &= d(\boldsymbol{e}_N\, dx^N) = d(\boldsymbol{e}_A\, e^A{}_N\, dx^N) \equiv d(\boldsymbol{e}_A\, \mathbb{E}^A) \\ &= d\boldsymbol{e}_A \wedge \mathbb{E}^A + \boldsymbol{e}_A\, d\mathbb{E}^A \\ &= \boldsymbol{e}_B\, \omega^B{}_A \wedge \mathbb{E}^A + \boldsymbol{e}_A\, d\mathbb{E}^A \\ &= \boldsymbol{e}_A\,(d\mathbb{E}^A + \omega^A{}_B \wedge \mathbb{E}^B) = \boldsymbol{e}_A\, T^A \end{aligned}$$

hence

$$T^A = d\mathbb{E}^A + \omega^A{}_B \wedge \mathbb{E}^B\,, \quad \omega^A{}_B \equiv \omega_N{}^A{}_B\, dx^N,\ \mathbb{E}^A \equiv e^A{}_N\, dx^N \tag{3.24}$$

and

$$\begin{aligned} T &= d(\boldsymbol{e}_N\, dx^N) = d\boldsymbol{e}_N \wedge dx^N \\ &= \boldsymbol{e}_L\, \Gamma^L{}_{MN}\, dx^M \wedge dx^N = \boldsymbol{e}_L\, T^L \end{aligned}$$

i.e.,

$$\begin{aligned} T^L &= \Gamma^L{}_{MN}\, dx^M \wedge dx^N \\ &= \Gamma^L{}_{mn}\, dx^m \wedge dx^n + (\Gamma^L{}_{n\bullet} - \Gamma^L{}_{\bullet n})\, dx^n \wedge \chi + \Gamma^L{}_{\bullet\bullet}\, \chi\chi\,. \end{aligned} \tag{3.25}$$

Here we have used $d^2x^n = 0$ and $d_{Z_2}\chi = 0$. Since $\Gamma^L{}_{MN} = \Gamma^L{}_{NM}$ and $\chi\chi \neq 0$, we finally obtain

$$T^L = \Gamma^L{}_{\bullet\bullet}\, \chi\chi\,. \tag{3.26}$$

In the same way we have

$$T^A = \Gamma^A{}_{\bullet\bullet}\, \chi\chi\,. \tag{3.27}$$

8. Curvature formulas

By using (3.7) and (3.12) we have the curvature formulas

$$dd\boldsymbol{e}_B = \boldsymbol{e}_A\, \mathbb{R}^A{}_B\,, \tag{3.28}$$

$$dd\boldsymbol{e}_N = \boldsymbol{e}_M\, R^M{}_N\,, \tag{3.29}$$

$$\mathbb{R}^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B\,, \quad \omega^A{}_B \equiv \omega_L{}^A{}_B\, dx^L \tag{3.30}$$

$$R^M{}_N = d\Gamma^M{}_N + \Gamma^M{}_L \wedge \Gamma^L{}_N\,, \quad \Gamma^M{}_N \equiv \Gamma^M{}_{LN}\, dx^L \tag{3.31}$$

and

$$R^M{}_N = e^M{}_A\, \mathbb{R}^A{}_B\, e^B{}_N\,. \tag{3.32}$$

§ 4. Concluding remarks

We have given a new formulation of NCG on $M_4 \times Z_2$. This will be useful for driving the Brans-Dicke theory for gravity on the two-sheeted space-time. Namely , if we take the vielbein of the type

$$e^A{}_M = \begin{pmatrix} e^a_m & e^5_m \\ e^a_\bullet & e^5_\bullet \end{pmatrix} = \begin{pmatrix} e^a_m & 0 \\ 0 & \lambda \end{pmatrix}, \tag{4.1}$$

then we get the gravity action

$$\begin{aligned} I &= \int_{M_4}\int_{Z_2} \sqrt{-\det(G_{MN})}\, R \\ &= \int_{M_4} \sqrt{-g_4}\,\Big[\lambda R_4 - \frac{\partial_m\lambda\,\partial_n\lambda}{\lambda}\, g^{mn}\Big]\,, \end{aligned} \tag{4.2}$$

which describes exactly the Brans-Dicke scalar field λ coupled to Einstein gravity. In fact this has been already given in another previous work[5] by using the vielbein formalism, but without its proof. In this paper, therefore, we have given a foundation of the vielbein formalism of NCG.

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