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CALCULATION OF THE α–PARTICLE GROUND STATE

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The Correlated Hyperspherical Harmonic expansion method is used to calculate α-particle properties with a realistic hamiltonian consisting of the Argonne V14 two nucleon and Urbana model VIII three nucleon potentials. The calculated binding energy, mass radius and wave percentages are close to the corresponding quantities obtained with Green's Function Monte Carlo and Faddeev-Yakubovsky techniques.

I. INTRODUCTION

The bound and scattering states of the four-nucleon system have been the object of a large number of theoretical and experimental studies. Among the ground state properties experimentally accessible we can mention the binding energy, the asymptotic normalization constants for the separation in 2+2 and 3+1 clusters and the structure functions. Scattering states are also of interest, for istance in studying the reactions p+^{3}H $\rightarrow$ n+^{3}He or d+d $\rightarrow$ α + γ.

The four-nucleon problem with realistic interactions is quite involved and until a few years ago only the Green's Function Monte Carlo (GFMC) method had been successfully applied [1,2]. More recently, the Faddeev-Yakubovsky (FY) technique has reached a good numerical accuracy [3-5], too. The GFMC method is based on the use of the evolution propagator in imaginary time $\exp[-\tau H]$. The ground state expectation value of an operator O (for example the hamiltonian H) is obtained from $\langle \Psi_T | \exp[-\tau H] O \exp[-\tau H] | \Psi_T \rangle$ in the limit $\tau \rightarrow \infty$. In principle, Ψ_T is an arbitrary trial wave function (w.f.). In numerical applications

it must represent a good variational choice in the description of the system in order to ensure convergence. The FY technique, used in refs. [3,4] consists in solving directly the four-body Schrödinger equation in momentum space after the decomposition of the w.f. in partial waves. To compare the two methods, for the Argonne V14 (AV14) potential [6] (including the Coulomb interactions), the GFMC technique [2] provides a g.s. ^{4}He energy of -24.2 ± 0.2 MeV and FY gives -23.87 MeV (obtained by adding the mean Coulomb energy, assumed to be 0.75 MeV, to the value of ref. [4]). The reason of the discrepancy could be due *i)* to the difficulties of GFMC in treating the L^2 and $(LS)^2$ components of the two-body potential and *ii)* to an incomplete convergence of the partial wave expansion in FY.

A sophisticated variational technique to describe the bound and scattering states of three-nucleon systems has been developed by the authors in refs. [7,8]. The w.f. is expanded in channels, as in the Faddeev technique, and the radial amplitude of each channel is expanded in terms of correlated functions. Two different types of correlations have been investigated. The Pair correlated Hyperspherical Harmonic (PHH) basis includes a pair correlation function per channel, while the Correlated Hyperspherical Harmonic (CHH) basis uses a correlation factor of the product (Jastrow) form. Such bases result to be well suited for taking into account the correlations induced by the large repulsive terms of the nucleon-nucleon (NN) potential. The technique was used in ref. [7] to calculate the bound state w.f. of the triton, with the AV14 model NN interaction. In ref. [8] the method has been extended also to include Three Nucleon Interaction (TNI) terms, and to study the N-d scattering process below the break-up threshold. The obtained results are in complete agreement with those given by the best available methods [9-11].

The generalization of the CHH expansion method to the four-nucleon system with "realistic" interactions is presented in this paper. The w.f. is expressed as a sum over a number of channels having different angular-spin-isospin quantum numbers. Each channel is then expanded by means of the CHH functions. Due to the presence of the correlation factors, the number of expansion functions per channel can be kept low and it has been possible, therefore, to perform accurate calculations. The results obtained for the ground-state of the α-particle will be given for the AV14 potential without and with the inclusion of the Urbana VIII (UR) model of the TNI [12].

The paper is organized as follows. The expansion of the w.f. is outlined in sec. II, while the choice of the correlation functions is discussed is sec. III. The numerical results obtained for the ground state of the α-particle are reported and discussed in section IV. The last section is devoted to the conclusions.

II. THE CHH EXPANSION

The wave function of a four–nucleon system with total angular momentum J, J_z and total isospin T, T_z can be written as a sum of amplitudes

$$\Psi = \sum_{p=1}^{12}\left[\psi_A(x_{1p}, x_{2p}, x_{3p}) + \psi_B(y_{1p}, y_{2p}, y_{3p})\right], \qquad (1)$$

where p denotes an even permutation of the particles. Through the present work the permutation p corresponds to the order i, j, k, m of the particles. The dependence of ψ_A and ψ_B on the nucleon spin–isospin variables is understood. The vectors $x_{1,2,3}$ and $y_{1,2,3}$ are the two possible sets of Jacobi vectors which can be constructed for a system of four particles of equal masses, and they are defined as

set A

$$x_{1p} = \sqrt{\tfrac{3}{2}}(r_m - \frac{r_i + r_j + r_k}{3}),$$

$$x_{2p} = \sqrt{\tfrac{4}{3}}(r_k - \frac{r_i + r_j}{2}),$$

$$x_{3p} = r_j - r_i,$$

set B

$$y_{1p} = r_m - r_k,$$

$$y_{2p} = \sqrt{2}(\frac{r_m + r_k}{2} - \frac{r_i + r_j}{2}),$$

$$y_{3p} = r_j - r_i.$$

$$(2)$$

In the L–S coupling scheme the amplitudes ψ_A and ψ_B are written as

$$\psi_A(x_{1p}, x_{2p}, x_{3p}) = \sum_\alpha F_{\alpha p}\Phi_{A\alpha}(x_{1p}, x_{2p}, x_{3p})\mathcal{Y}^A_{\alpha p}, \qquad (3)$$

$$\mathcal{Y}^A_{\alpha p} = \left\{\left[\left[Y_{\ell_{1\alpha}}(\hat{x}_{1p})Y_{\ell_{2\alpha}}(\hat{x}_{2p})\right]_{\ell_{12\alpha}}Y_{\ell_{3\alpha}}(\hat{x}_{3p})\right]_{L_\alpha} \times \right.$$

$$\left.\left[\left[[s_i s_j]_{S_{a\alpha}} s_k\right]_{S_{b\alpha}} s_m\right]_{S_\alpha}\right\}_{JJ_z}\left[\left[[t_i t_j]_{T_{a\alpha}} t_k\right]_{T_{b\alpha}} t_m\right]_{TT_z}, \quad (4)$$

and

$$\psi_B(y_{1p}, y_{2p}, y_{3p}) = \sum_\alpha F_{\alpha p}\Phi_{B\alpha}(y_{1p}, y_{2p}, y_{3p})\mathcal{Y}^B_{\alpha p}, \qquad (5)$$

$$\mathcal{Y}^B_{\alpha p} = \left\{\left[\left[Y_{\ell_{1\alpha}}(\hat{y}_{1p})Y_{\ell_{2\alpha}}(\hat{y}_{2p})\right]_{\ell_{12\alpha}}Y_{\ell_{3\alpha}}(\hat{y}_{3p})\right]_{L_\alpha} \times \right.$$

$$\left.\left[[s_i s_j]_{S_{a\alpha}}[s_k s_m]_{S_{b\alpha}}\right]_{S_\alpha}\right\}_{JJ_z}\left[[t_i t_j]_{T_{a\alpha}}[t_k t_m]_{T_{b\alpha}}\right]_{TT_z}, \quad (6)$$

respectively. Here, x_p, y_p denote the magnitudes of the Jacobi vectors, and s_i (t_i) denotes the spin (isospin) function of particle i. Each α–channel is specified by the angle–spin–isospin quantum numbers $\ell_{1\alpha}, \ell_{2\alpha}, \ell_{3\alpha}, \ell_{12\alpha} L_\alpha, S_{a\alpha}, S_{b\alpha}, S_\alpha, T_{a\alpha}$ and $T_{b\alpha}$. In eqs. (4) and (6) L_α and S_α are coupled to give J, J_z and the functions $F_{\alpha p}$ are correlation factors. In order to ensure the antisymmetry of the w.f. the amplitudes $\psi_{A,B}$ must change sign under the exchange of particles i and j. Therefore, if the function $F_{\alpha p}$ is even (odd) when $i \leftrightarrow j$, then the integer $l_{3\alpha} + S_{a\alpha} + T_{a\alpha}$ must be odd (even); moreover, $\ell_{1\alpha} + \ell_{2\alpha} + \ell_{3\alpha}$ must be even or odd number depending on whether the parity of the state considered is either even or odd.

The functions $F_{\alpha p}$ have been chosen to depend only on the interparticle distances, i.e. $F_{\alpha p} = F_{\alpha p}(r_{ij}, r_{ik}, r_{jk}, r_{im}, r_{jm}, r_{km})$. In the case of purely central potentials, a simple product form for each radial function $F_{\alpha p}$ allows for quite an accurate description of the $A = 4$ ground state [13]. As an example, the calculated binding energy with the Malfliet–Tjon V interaction (MTV) [14] and an optimized trial w.f. of the Jastrow form $\Psi = \prod_{i<j} f(r_{ij})$ is found to be $B = 31.32$ MeV [13], very close the (presumably) exact value $B = 31.36$ MeV obtained with a variety of different methods [3,9,16]. The inclusion of (non-optimized) Jastrow factors significantly improves the convergence rate of the HH expansion, as it has been shown in ref. [7] for the $A = 3$ system with realistic interactions, and in ref. [15] for the $A = 4$ system with the MTV interaction. It is therefore useful to incorporate such correlation factors in the present study of the $A = 4$ ground state with realistic interactions. The correlation functions $F_{\alpha p}$ in eqs. (3) and (5) are taken of the form

$$F_{\alpha p} = f_{a\alpha}(r_{ij})f_{b\alpha}(r_{ik})f_{b\alpha}(r_{jk})f_{c\alpha}(r_{im})f_{c\alpha}(r_{jm})f_{d\alpha}(r_{km}), \qquad (7)$$

where $f_{a\alpha}$, $f_{b\alpha}$, $f_{c\alpha}$ and $f_{d\alpha}$ are one–dimensional functions of the interparticle distances. The choice of these functions will be discussed in section III. The α–channels selected in our calculation are specified in table I together with the corresponding choice of the radial functions f_a, f_b, f_c and f_d.

One can replace the magnitudes of the Jacobi variables with the hyperspherical coordinates, given by the hyperradius [17]

$$\rho = \sqrt{(x_{1p})^2 + (x_{2p})^2 + (x_{3p})^2} = \sqrt{(y_{1p})^2 + (y_{2p})^2 + (y_{3p})^2}, \qquad (8)$$

which turns out to be independent on p, and the "hyperangular" variables,

$$\cos\phi_{3p} = x_{3p}/\rho = y_{3p}/\rho,$$

$$\cos\phi^A_{2p} = x_{2p}/(\rho\sin\phi_{3p}),$$

$$\cos\phi^B_{2p} = y_{2p}/(\rho\sin\phi_{3p}).$$

$$(9)$$

Each Φ_α function in eqs. (3) and (5) is then expanded in terms of the HH basis as

$$\Phi_{A\alpha}(x_{1p}, x_{2p}, x_{3p}) = \sum_{n,m} \frac{u_{nm}^\alpha(\rho)}{\rho^4} x_{1p}^{\ell_{1\alpha}} x_{2p}^{\ell_{2\alpha}} x_{3p}^{\ell_{3\alpha}} Y_{nm}^\alpha(\phi_{2p}^A, \phi_{3p}) , \qquad (10)$$

$$\Phi_{B\alpha}(y_{1p}, y_{2p}, y_{3p}) = \sum_{n,m} \frac{w_{nm}^\alpha(\rho)}{\rho^4} y_{1p}^{\ell_{1\alpha}} y_{2p}^{\ell_{2\alpha}} y_{3p}^{\ell_{3\alpha}} Y_{nm}^\alpha(\phi_{2p}^B, \phi_{3p}) , \qquad (11)$$

where

$$Y_{nm}^\alpha(\beta, \gamma) = N_{nm}^\alpha (\sin\beta)^m P_n^{K_{2\alpha}, \ell_{3\alpha}+\frac{1}{2}}(\cos 2\beta) P_m^{\ell_{1\alpha}+\frac{1}{2}, \ell_{2\alpha}+\frac{1}{2}}(\cos 2\gamma) . \qquad (12)$$

In the last expression, $P^{a,b}$ are Jacobi polynomials, the integers n and m range from zero to infinity, $K_{2\alpha} = \ell_{1\alpha} + \ell_{2\alpha} + 2m + 2$ and N_{nm}^α are normalization factors. When the functions Y_{nm}^α are multiplied by the product of the spherical harmonics $Y_{\ell_1} Y_{\ell_2} Y_{\ell_3}$, as in eqs. (3) and (5), the standard expression [17] of the Hyperspherical Harmonic functions is recovered. It is convenient to rewrite the w.f. given by eq. (1) as:

$$\Psi = \sum_{\alpha,n,m} \frac{U_{nm}^\alpha(\rho)}{\rho^4} \mathcal{H}_{nm}^\alpha(\rho, \Omega) , \qquad (13)$$

where U stands for u or w, depending on whether the channel α is constructed with either set A or B of the Jacobi variables, and Ω denotes the hyperangular and angular variables (the dependence of $\mathcal{H}$ on the variable ρ is due to the correlation factor). The functions $\mathcal{H}_{nm}^\alpha(\rho, \Omega)$ are explicitly given by

$$\mathcal{H}_{nm}^\alpha(\rho, \Omega) = \sum_{p=1}^{12} F_{\alpha p} \, x_{1p}^{\ell_{1\alpha}} \, x_{2p}^{\ell_{2\alpha}} \, x_{3p}^{\ell_{3\alpha}} \, Y_{nm}^\alpha(\phi_{2p}^A, \phi_{3p}) \mathcal{Y}_{\alpha p}^A , \qquad (14)$$

if channel α is constructed with the set A of Jacobi vectors, and

$$\mathcal{H}_{nm}^\alpha(\rho, \Omega) = \sum_{p=1}^{12} F_{\alpha p} \, y_{1p}^{\ell_{1\alpha}} \, y_{2p}^{\ell_{2\alpha}} \, y_{3p}^{\ell_{3\alpha}} \, Y_{nm}^\alpha(\phi_{2p}^B, \phi_{3p}) \mathcal{Y}_{\alpha p}^B , \qquad (15)$$

if it is constructed with the set B of Jacobi vectors. The summation over n and m in eq. (13) will be limited to include the CHH functions $\mathcal{H}_{nm}^\alpha$ with $0 \leq 2(n+m) \leq K_\alpha^M$, K_α^M being a positive even integer. For a given K_α^M, the number M_α of functions included in the expansion of channel α is given by

$$M_\alpha = (K_\alpha^M + 2)(K_\alpha^M + 4)/8 . \qquad (16)$$

However, we note that some of the functions $\mathcal{H}_{nm}^\alpha$ are not linearly independent from the other ones, and therefore should be removed from the expansion. In general, it may happen that CHH functions constructed in terms of different sets of Jacobi vectors but having the same quantum numbers are equal. As an example, it can be verified that $\mathcal{H}_{n,0}^{\alpha \equiv A} = \mathcal{H}_{n,0}^{\alpha \equiv B}$ for all n. The values of M_α, given by eq. (16), will not be modified but the components in the w.f., involving linearly depending functions $\mathcal{H}_{nm}^\alpha$ in the following will be taken with zero amplitude.

The Rayleigh–Ritz variational principle is used to determine the functions $U_{nm}^\alpha(\rho)$. The functional derivative of the hamiltonian expectation value with respect to each function $U(\rho)$ is set zero,

$$< \delta_U \Psi | H - E | \Psi > = 0 , \qquad (17)$$

$\delta_U \Psi$ denoting the change in the w.f. due to an infinitesimal variation of any of the functions $U_{nm}^\alpha(\rho)$. The volume element $d\tau$ in the nine–dimensional space is written as $d\tau = \rho^8 d\rho \, d\Omega$. After performing an integration over $d\Omega$ in eq. (17) and a summation over the spin–isospin variables, we obtain a set of coupled second–order differential equations of the form

$$\sum_{q'=1}^{M} \left(A_{qq'}(\rho) \frac{d^2}{d\rho^2} + B_{qq'}(\rho) \frac{d}{d\rho} + C_{qq'}(\rho) + \frac{m}{\hbar^2} E \, N_{qq'}(\rho) \right) U_{q'}(\rho) = 0 . \qquad (18)$$

E is the total energy of the system and the indices q and q' run over the channels α and the quantum numbers n, m. It is convenient to perform all the numerical integrations with respect to the variables corresponding to a given set of Jacobi vectors and a given permutation. In the following we will refer to the Jacobi set A and the permutation $p = 1$, corresponding to the order 1234 of the particles (hereafter, the index p will be omitted). The quantities to be calculated, as functions of ρ, are the coefficients A, B, C and N in eq. (18) and this requires the numerical integration over the hyperangular and angular variables. As a first step, let us introduce the variables $z = \cos\phi_3$, $x = \cos\phi_2^A$, $\mu_1 = \hat{x}_1 \cdot \hat{x}_3$, $\mu_2 = \hat{x}_2 \cdot \hat{x}_3$ and φ, the angle between the projection of the vectors $\hat{x}_1$ and $\hat{x}_2$ in the plane perpendicular to $\hat{x}_3$. The integration over Ω can be replaced by

$$\int d\Omega \rightarrow 8\pi^2 \int_0^1 dx \, x^2 \sqrt{1-x^2} \int_0^1 dz \, z^2 (1-z^2)^2 \int_{-1}^{+1} d\mu_1 \int_{-1}^{+1} d\mu_2 \int_0^{2\pi} d\varphi . \qquad (19)$$

We now define a new set of variables ξ_i, $i = 1, 5$, all ranging in the interval $(0,1)$, by means of the relations

$$\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^1 d\xi_1\,, \qquad \int_0^1 z^2(1-z^2)^2\,dz = \int_0^1 d\xi_2\,, \qquad (20)$$

$$\xi_3 = \frac{\mu_1 + 1}{2} \qquad \xi_4 = \frac{\mu_2 + 1}{2}\,, \qquad \xi_4 = \frac{\varphi}{2\pi}\,. \qquad (21)$$

The integration over Ω is reduced to the integration over the variables ξ_i and the corresponding numerical calculations have been done by means of the Quasi Random Number (QRN) technique [18]. A good precision has been obtained by using $50,000 \div 100,000$ QRN integration points. The solution of eq. (18) is carried out by standard numerical methods. The procedure we have adopted consists in replacing the derivatives with finite differences and in solving the corresponding eigenvalue problem with the Lanczos algorithm [19].

III. THE CORRELATION FACTOR

It is well known that the rate of convergence of the (uncorrelated) HH expansion results is very slow when the interparticle interaction is strongly repulsive at short distances. As already pointed out in the Introduction, the role of the correlation factors is to accelerate the convergence of the expansion by improving the description of the system when two particles are close to each other. In such configurations, there are large cancellations between the contributions from kinetic and potential energy terms, and therefore the w.f. must be very precisely constructed. It is therefore convenient to include in the w.f. appropriate terms, in order to describe these configurations. This has the advantage of reducing the number of basis functions necessary to obtain a converged result.

The correlation factors in eq. (7) are of the Jastrow form, namely they are product of correlation functions depending only on the interparticle distances. In principle, these functions could be determined variationally, as was done in ref. [13] for a simplified problem (central interaction). However, such an approach would be numerically involved in the present case, and we have used the following simpler procedure based on the observation that, in a generic nuclear system, when a given pair is far away from all other particles, the dependence of the total w.f. on the coordinates of these two particles is mainly determined by their mutual interaction. Therefore, the radial w.f. $\varphi_\beta(r_{ij})$ for the relative motion of pair i,j in the angle–spin–isospin state β ($\equiv j_\beta, \ell_\beta, S_\beta, T_\beta$), can be approximately described by the solution of an equation of the form

$$\sum_{\beta'}\left\{-\frac{\hbar^2}{m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell_\beta(\ell_\beta + 1)}{r^2}\right]\delta_{\beta\beta'} + V_{\beta\beta'}(r) + \lambda_{\beta\beta'}(r)\right\}\varphi_{\beta'}(r) = 0\,, \qquad (22)$$

where $V_{\beta\beta'}(r) = <\beta'|V(i,j)|\beta>$ ($V(i,j)$ is the interparticle potential). Depending on the state β, eq. (22) can be a single equation or two coupled equations. The additional term $\lambda_{\beta\beta'}(r)$ in eq. (22) takes into account the effect of the other particles on the interacting pair. There is a large arbitrariness in the choice of $\lambda_{\beta\beta'}(r)$, since the relevant condition to be satisfied is $|\lambda_{\beta\beta'}(r)| \ll |V_{\beta\beta'}(r)|$ at small r values. As in our previous study of the three–nucleon system, we parametrize $\lambda_{\beta\beta'}(r)$ as $\lambda_{\beta\beta'}(r) = \Lambda_\beta \exp(-\gamma r)\delta_{\beta\beta'}$. The $1/\gamma$ value should be greater than the range R of the potential $V_{\beta\beta'}(r)$, but its precise value is found to be unimportant ($1/\gamma$ is taken to be 2.0 fm). The depth Λ_β is determined by requiring that $\varphi_\beta(r)$ satisfies some appropriate healing condition. In this paper we impose the condition $\varphi_\beta(r) = 1$ when $r \gg R$.

It should be noted from eq. (7) that the function $f_{a\alpha}(r)$ is related to the reference pair (i, j), characterized by definite values of angular momentum, spin and isospin for each channel. Therefore, these functions can be taken as solutions of an equation of the form (22). Since the total w.f. has been constructed in the LS coupling, in general the total angular momentum j_β of the reference pair does not have a definite value. However, for the first three channels ($\alpha = 1$–3) $\ell_1 = \ell_2 = \ell_{12} = 0$ and, since $J = 0$, one easily obtains $j_\beta = S_{a\alpha}$. As a consequence, the correlation functions $f_{a\alpha} \equiv f_\alpha$, $\alpha = 1$–3, correspond to the states $^3S_1$, $^1S_0$ and $^3D_1$, respectively (of course, the functions f_1 and f_3 satisfy two coupled equations).

The $\alpha > 3$ channels are less important than the first three ones. Therefore, the functions $f_{a\alpha}(r)$, with $\alpha > 3$, have been determined by a simplified procedure: we have used the function f_1 for the channels with $\ell_3 = 0$, $S_a = 1$, $T_a = 0$, f_2 for those with $\ell_3 = 0$, $S_a = 0$, $T_a = 1$ and f_3 for those with $\ell_3 = 2$, $S_a = 1$, $T_a = 0$. Finally, for the channels with $\ell_3 = 1$, the function f_4, calculated by considering in eq. (22) only the central part of the pair potential in the state $^3P_1$, has been adopted.

Let us now discuss the choice of the functions $f_{b\alpha}(r_{ik})$, $f_{c\alpha}(r_{im})$ and $f_{d\alpha}(r_{km})$ with k, m different from the reference pair indices i, j (see eq. (7)). These functions correlate pairs which are not in a definite angle–spin–isospin state. For the correlations of the first channel listed in table I the following procedure has been adopted: first of all, let us suppose in eq. (3) $\Phi_{A\alpha=1}(x_{1p}, x_{2p}, x_{3p}) = 1$ and moreover, let us consider a configuration where the particles 1 and 2 are very close to each other and the others interparticle distances are very large, so that the corresponding correlation functions have reached the asymptotic value $f(r_{ij\neq12}) \approx 1$. In this case, the expression of channel $\alpha = 1$ reduces to:

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$$\sum_p F_{1,p} \mathcal{Y}_{1,p}^A \to \sum_p \mathcal{F}_p(r_{12}) \left[\left[[s_i s_j]_1 s_k \right]_{\frac{1}{2}} s_m \right]_0 \left[\left[[t_i t_j]_0 t_k \right]_{\frac{1}{2}} t_m \right]_0 , \qquad (23)$$

where $\mathcal{F}_p$ stands for one of the f_{a1}, f_{b1}, f_{c1} or f_{d1} functions, depending on the permutation p. The spin–isospin states in the last equation can be expressed in terms of the states constructed with the 1234 ordering of the particles, which are given by

$$|S,T\rangle = \left[\left[[s_1 s_2]_S s_3 \right]_{\frac{1}{2}} s_4 \right]_0 \left[\left[[t_1 t_2]_T t_3 \right]_{\frac{1}{2}} t_4 \right]_0 . \qquad (24)$$

Eq. (23) can be then explicitely written as

$$\sum_p F_{1,p} \mathcal{Y}_{1,p}^A \to f_{a1}(r_{12})|1,0\rangle + f_{b1}(r_{12}) \left[\sqrt{\frac{1}{10}} |1,0\rangle - \sqrt{\frac{9}{10}} |0,1\rangle \right]$$

$$+ f_{c1}(r_{12}) \left[\sqrt{\frac{1}{10}} |1,0\rangle - \sqrt{\frac{9}{10}} |0,1\rangle \right] + f_{d1}(r_{12}) |1,0\rangle . \qquad (25)$$

From this expression, we see that the function f_{d1} is associated to the same spin–isospin state as the function f_{a1}, so that $f_{d1} = f_{a1} = f_1$ (we have previously discussed how to determine f_1). Analogously, the functions f_{b1} and f_{c1} are taken to be equal ($\equiv g_1$). They have been calculated by using the potential $V_{\beta\beta'}(r) = 1/2[V_{10}^{(c)}(r_{12}) + V_{01}^{(c)}(r_{12})]$ in eq. (22), where $V_{ST}^{(c)}(r)$ stands for the projection of the central part of the two–body potential on the spin state S and isospin state T ($= 0$ or 1). We note that different averages of the spin–triplet and spin–singlet parts of the central potential lead to negligible differences in the calculated binding energies, if a sufficient number of HH expansion terms are included in the w.f.. These considerations remain valid also in the determination of the second channel functions.

For the third channel any pair of particle is in a triplet spin state since the total spin is 2, therefore the function $f_{d4} \equiv g_2$ has been calculated with the potential $V_{10}^{(c)}(r)$. On the other hand, the functions f_{b3} and f_{c3} correlate particles with a non definite angular momentum and isospin. For the sake of simplicity, we take $f_{b3} = f_{c3} = g_1$. We have however checked that other choices of f_{b3} and f_{c3} (for example, different mixing of S–state and D–state functions) do not influence the final results in a significant way.

A simpler procedure has been followed for the correlation functions of the other channels constructed with the set A of the Jacobi vectors ($\alpha = 4 \div 8$ and $12 \div 19$, as listed in Table I). We have taken $f_{b\alpha} = f_{c\alpha} = f_{d\alpha} = g_1$. The correlation functions of the channels constructed with the set B of the Jacobi vectors ($\alpha = 9 \div 11$ and $20 \div 22$) have been chosen in a similar way. In this case, the functions correlating pairs with definite angle–spin–isospin quantum numbers are $f_{a\alpha}$ and $f_{d\alpha}$ and so, they have been chosen to be f_1 or f_3 depending on value of ℓ_3 and ℓ_1, respectively. The functions $f_{b\alpha}$ and $f_{c\alpha}$ have been chosen to be g_1, as in the other channels. A summary of the correlation factors used in the various channels is given in table I.

IV. RESULTS AND DISCUSSION

The list of the channels included in the present calculation with the corresponding orbital angular momentum, spin and isospin quantum numbers is reported in table I. The convergence of the correlated expansion for the four–nucleon ground state has been studied by the authors in ref. [15] for the case of semirealistic, purely central NN interactions, such as those given by Malfliet and Tjon [14] and Afnan and Tang [20]. Since such interactions are spin–dependent, the first two channels listed in table I were considered. The conclusion was that inclusion of these two channels leads to a very good convergence of the expansion with a few (5 − 10) hyperradial components per channel.

However, for the case of realistic NN potentials, it is more difficult to solve the four–body problem with a satisfactory accuracy. In this section the results obtained for the AV14 potential will be presented. All the 22 channels listed in table I have been considered in the variational w.f. These turn out to be the most important in describing both the p+^{3}H (neutron+^{3}He) and d+d configurations. It is well known that realistic local interactions, which accurately reproduce NN scattering data, underbind the $A = 3, 4$ nuclei. This discrepancy can be eliminated by including three–nucleon interactions in the hamiltonian. We use the Urbana VIII (UR) model of TNI [12] in order to compare with results obtained with different approaches.

The values obtained for the binding energy B and the mean value of the kinetic energy T are reported in table II. A few of the cases we have investigated are presented in the table to show the convergence properties with respect to the numbers M_α of the HH expansion terms included in the various channels, and the number N_c of channels taken into account. As an example, when $M_{1\div3}$ is increased from 10 to 21 (and $M_{\alpha>3} = 0$), the gain ΔB_3 in the binding energy of the system is 0.22 MeV for the AV14 potential and 0.50 MeV for the full hamiltonian including the TNI. This slow convergence is a consequence of the non–orthogonality of the channels due to *i)* to the presence of the correlation factors, which produce a mixing of the orbital angular momenta of the channels and *ii)*

to the sum over the permutations of the Faddeev amplitudes. As a consequence, when only very few channels are considered, the increase in the number of HH components allows for a partial inclusion of higher order channels. This point may be made clearer by comparing ΔB_3 with ΔB_{22} (the change in the binding energy corresponding to $N_c = 22$ when $M_{1 \div 3}$ is increased from 10 to 21). The last three rows of table II can be used and the values $\Delta B_{22} = 0.06$ MeV for the AV14 and 0.11 for the AV14+UR models are obtained. They are appreciably smaller than ΔB_3. Therefore, the conclusion is that the number of HH functions per channel can be kept sufficiently small (≈ 10) when all the relevant channels are taken into account.

As far as the channels with $\alpha > 4$ are concerned, we see from inspection of table II that a reasonable convergence is reached with a rather small numbers of the CHH components ($M_{\alpha > 3} = 3 \div 10$). Moreover, the values of M_α can be taken smaller for increasing values of α. This behavior should be a consequence of the relative minor importance of these channels with respect to the first three ones. This interpretation is also confirmed by the mean values of the kinetic operator T given in table II. Indeed, for the AV14 + UR case, they increase by 10%, 5% and 1% when the w.f. contains 8, 11 and 22 channels, respectively.

We will now try to estimate the contribution to the binding energy that is not included in the present calculation. As it has been mentioned before, it is difficult to estimate the convergence of the CHH expansion relative to channels, due to the non–orthogonality among them. We have therefore ordered the HH functions and their energy contributions according to their grand–angular quantum number value $G = \ell_1 + \ell_2 + \ell_3 + 2(n + m)$. The motivation lies in the fact that the kinetic operator acting on a HH function with grand–angular number G produces a "centrifugal" term equal to $G(G+4)/\rho^2$. This term is more and more repulsive as G increases, therefore it is useful to examine the convergence of the expansion as function of G. In ref. [17] it has been shown that in the case of a central Yukawa–type potential, the missing energy δB could be estimated as

$$\delta B \approx \text{const} \sum_{G=G_{max}+2}^{\infty} \frac{1}{(G/2 + 1)^4} , \qquad (26)$$

after the inclusion of the (uncorrelated) HH functions with $G = 0, 2, \ldots G_{max}$. If the same formula for δB is applied to the realistic interactions studied in this paper, the corrections $\delta B(\text{AV}14) = 0.09$ MeV and $\delta B(\text{AV}14 + \text{UR}) = 0.37$ MeV are obtained by means of the results shown in table III. However, it should be observed that in the present calculations not all the HH functions with a given G value have been included, but only those belonging to the channels specified in table I. Nevertheless, we are sufficiently confident of the validity of the previous

estimates for δB. We therefore conclude that the truncated CHH expansion has reached a good convergence for the AV14 potential, whereas (a few) more channels must be added for the AV14+UR potential.

The results obtained with the CHH technique are compared in table IV with those from other methods. In the case of the AV14 interaction, the Faddeev–Yakubovsky (FY) and the CHH estimates for the binding energy are rather close. At present, also in the FY approach there are problems concerning the full convergence in the channel expansion. The missing binding energy is evaluated [4] to be about 0.1 MeV and the extrapolated B value is in agreement with the corresponding one given in the last row of table III. The binding energy estimates from the previous two methods are consistent with that from the GFMC [2] once the statistical error in the GFMC value is considered.

We have considered the same CHH channels as for the AV14 potential if the TNI terms are included. However, in this case the B value obtained differs from the GFMC one by approximately 0.8 MeV (reduced to 0.45 MeV for the extrapolated CHH binding energy). A larger number of channels (perhaps also of HH components per channel) should be presumably considered in order to resolve this discrepancy.

In table IV the values obtained for the average kinetic and potential energies, mass radius and percentages of various waves are shown. It is interesting to compare the percentages with the corresponding ones for triton [7]; namely $P_{S'} = 1.27$, $P_P = 0.076$ and $P_D = 9.962$, in the case of the AV14 potential. From table IV we see that the percentage of the mixed simmetry S–wave is noticeably reduced. This should be due to the increase in binding energy, and, in fact, the relation $P_{S'} \propto 1/B^2$ is essentially verified. On the other hand, the P– and D–wave percentages are greater for the α particle, probably since there is a larger number of channels with total orbital angular momentum $L = 1$ or 2. For example, when A= 3 the most important contribution to the P wave comes from the channel with two angular momenta $\ell = 2$ coupled to give $L = 1$. For A= 4, an important contribution comes from the 16–th channel of table I. The corresponding state contains only angular momenta values $\ell = 1$, and therefore it should be preferred due to the smaller associated kinetic energy. The component with a fully antisymmetric radial dependence gives negligible contributions for both A=3 and 4.

In order to get an approximate estimate of the contribution due to higher channels not included in the present calculation, we have included in the expansion other channels to those listed in table I. In all cases, the gain in binding energy obtained by including one more channel at a time was less than 0.02 MeV for the AV14 potential. When the TNI terms are taken into consideration, the

gains are slightly larger. The TNI strongly depends on the particle state, particularly on the interparticle distances, so the channels with higher angular momenta values can give appreciable contribution to the structure of the w.f. Moreover, the correlation factors have been determined by using only the two-body NN potential and no attempt has been made to improve them in the presence of TNI.

It should be pointed out that when the CHH expansion is applied to the study of the triton ground state, there are convergence problems similar to those encountered for the alpha particle. For A=3, the CHH expansion gives satisfactory results just with a small number of channels. However, after the inclusion of the 12 channels, there is a missing binding energy of approximately 0.02 MeV with respect to the (presumably) exact value obtained by a variety of different methods [7,9-11] that results very difficult to be recovered by adding more channels. Once again, the problem is due to the mixing of orbital angular momenta caused by the Jastrow correlation factor. In fact, this convergence problem has been solved [7] by replacing the Jastrow factor with a simpler pair correlation function $f(r_{ij}) = f(\rho^{(3)} \cos \phi^{(3)})$, where $\rho^{(3)}$ and $\phi^{(3)}$ are the corresponding hypervariables for A=3. Probably, similar considerations hold for the four-body system too, and again a PHH expansion should allow for better results. Such an expansion is obtained by simplifying the correlation factors discussed in section III, by taking $f_b = f_c = f_d = 1$ in eq. (7). However the numbers of channels to be included becomes larger, of the order $N_c \approx 100$, a number comparable to the one used in the FY approach. All these points require further investigation.

V. CONCLUSIONS

The use of suitable expansion bases has been shown to be a powerful tool in the development of theoretical investigations of nuclear structure. However, such an approach has been proven to be really successful only in the case of effective interactions without large repulsions. On the contrary, the phenomenological NN potentials are characterized by a strong state dependence and large repulsion at small distances. When these potentials are used to study the structure of few-nucleon systems, it is necessary to set-up new ad hoc devised expansion bases. An expansion over a set of correlated Hyperspherical Harmonic functions was proposed in ref. [7] and applied to the A=3 nucleus ground state. The conclusion was that only a few correlated HH funtions per channel were sufficient to obtain at least four correct figures for the quantities of interest. The correlation factors were obtained in a simple way. However, if the correlation functions were taken

to be equal to one, with the corresponding number of HH considered in ref. [7], no binding at all would be found.

In the present paper the generalization of the CHH method to the four-nucleon system has been discussed. For the ground state the convergence of the expansion results to be more critical than in the A=3 case; this can be due to the larger number i) of expansion terms necessary for an accurate description of each channel, and ii) of channels to be included. However, with a rather small number (22) of channels and a limited total number (= 164) of expansion functions, it is possible to obtain a satisfactory description of the ground state, comparable to the one given by the GFMC or FY methods. It turns out that, in the case of the AV14 potential, there are small differences between the results of the three methods, whereas larger differences are present when also TNI terms are taken into account. Such differences may be due to an incomplete convergence of the partial wave expansion present in the CHH and FY calculations, or to an underestimate of the statistical error in the GFMC technique.

It should be pointed out that the present Jastrow correlation factors have not been optimized and that this could slow down the rate of convergence in the expansion. Other types of correlation functions, as for example in the PHH approach, could be perhaps more appropriate. However, to clarify these apects of the problem further investigations are necessary.

In conclusion, it appears that at the present time the description of the α-particle ground state, within the framework of the standard non relativistic theory with realistic potentials, has reached a good accuracy.

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Table Caption

Table I. Quantum numbers and correlation functions for the channels $\alpha = 1 \div 22$ included in the partial wave decomposition of the w.f. given by eq. 3 (set A) or eq. 5 (set B). The choice of the correlation factors f_k and g_k is discussed in section III.

Table II. α–particle binding energy B and average kinetic energy T (in MeV) for the AV14 and AV14 + UR interaction models, with Coulomb interaction included, in correspondence to different values of M_α, the number of CHH functions in the channel α.

Table III. α–particle binding energy B and average kinetic energy T as function of the grandangular quantum number G. The last row reports the extrapolated binding energy calculated with eq. (26) by assuming $G_{\text{MAX}} = 8$. The missing energy δB is then calculated by subtracting the binding energy reported in the last row of table II from the extrapolated binding energy.

Table IV. α–particle energy breakdown for the AV14 and AV14 + UR potentials. Energies are in MeV and radii in fm. V_{NN} is the mean value of the two–nucleon potential energy and the various terms V^X give the contributions of the corresponding parts of the potential. U_{TNI} is the mean value of the TNI potential energy and $U^{2\pi}$ (U^R) gives the contribution of the two–pion–exchange (repulsive) term of the Urbana VIII model TNI. Finally, R is the mass radius and P_S, P_P, P_D and $P_{S'}$ are the wave percentages. For the sake of comparison also the results obtained with the GFMC [2], VMC [12] and FY [4] techniques have been reported. The numbers in parentheses give the statistical errors in the last digit. The result denoted by (*) has been obtained by adding to the one reported in ref. [4] the mean Coulomb potential, assumed to be 0.75 MeV.

TABLE I.

α	set	ℓ_1	ℓ_2	ℓ_3	ℓ_{12}	L	S_a	S_b	S	T_a	T_b	T	f_a	f_b	f_c	f_d
1	A	0	0	0	0	0	1	1/2	0	0	1/2	0	f_1	g_1	g_1	f_1
2	A	0	0	0	0	0	0	1/2	0	1	1/2	0	f_2	g_1	g_1	f_2
3	A	0	0	2	0	2	1	3/2	2	0	1/2	0	f_3	g_1	g_1	g_2
4	A	0	2	0	2	2	1	3/2	2	0	1/2	0	f_1	g_1	g_1	g_1
5	A	0	2	2	2	0	1	1/2	0	0	1/2	0	f_3	g_1	g_1	g_1
6	A	0	2	2	2	1	1	1/2	1	0	1/2	0	f_3	g_1	g_1	g_1
7	A	0	2	2	2	1	1	3/2	1	0	1/2	0	f_3	g_1	g_1	g_1
8	A	0	2	2	2	2	1	3/2	2	0	1/2	0	f_3	g_1	g_1	g_1
9	B	2	0	2	2	0	1	1	0	0	0	0	f_3	g_1	g_1	f_3
10	B	2	0	2	2	1	1	1	1	0	0	0	f_3	g_1	g_1	f_3
11	B	2	0	2	2	2	1	1	2	0	0	0	f_3	g_1	g_1	f_3
12	A	0	1	1	1	0	1	1/2	0	1	1/2	0	f_4	g_1	g_1	g_1
13	A	0	1	1	1	1	1	1/2	1	1	1/2	0	f_4	g_1	g_1	g_1
14	A	0	1	1	1	1	1	3/2	1	1	1/2	0	f_4	g_1	g_1	g_1
15	A	0	1	1	1	2	1	3/2	2	1	1/2	0	f_4	g_1	g_1	g_1
16	A	1	1	0	0	0	1	1/2	0	0	1/2	0	f_1	g_1	g_1	g_1
17	A	1	1	0	1	1	1	1/2	1	0	1/2	0	f_1	g_1	g_1	g_1
18	A	1	1	0	1	1	1	3/2	1	0	1/2	0	f_1	g_1	g_1	g_1
19	A	1	1	0	2	2	1	3/2	2	0	1/2	0	f_1	g_1	g_1	g_1
20	B	0	0	0	0	0	1	1	0	0	0	0	f_1	g_1	g_1	f_1
21	B	0	0	2	0	2	1	1	2	0	0	0	f_3	g_1	g_1	f_1
22	B	2	0	0	2	2	1	1	2	0	0	0	f_1	g_1	g_1	f_3

TABLE II.

Number of CHH functions					AV14		AV14 + UR	
$M_{1\div3}$	$M_{4\div8}$	$M_{8\div11}$	$M_{12\div19}$	$M_{20\div22}$	B	T	B	T
1	0	0	0	0	20.10	83.04	20.92	85.16
3	0	0	0	0	20.47	83.91	21.20	85.60
6	0	0	0	0	20.84	84.35	21.72	87.26
10	0	0	0	0	21.20	85.47	22.34	89.52
15	0	0	0	0	21.35	85.88	22.69	90.85
21	0	0	0	0	21.42	86.10	22.84	91.33
28	0	0	0	0	21.47	86.22	22.92	91.47
10	1	0	0	0	21.53	86.68	23.30	93.20
10	3	0	0	0	22.61	90.34	25.40	100.32
10	6	0	0	0	22.70	90.46	25.40	100.82
10	10	0	0	0	22.76	90.53	25.66	101.25
10	10	1	0	0	23.67	94.09	26.99	105.86
10	10	3	0	0	23.69	94.10	27.03	105.93
10	10	6	0	0	23.71	94.17	27.06	106.09
10	10	6	1	0	23.76	94.34	27.24	106.82
10	10	6	3	0	23.79	94.39	27.28	106.95
10	10	6	3	1	23.85	94.60	27.34	107.03
10	10	6	3	3	23.87	94.54	27.37	107.02
15	10	6	3	3	23.91	99.65	27.44	107.06
21	10	6	3	3	23.93	99.75	27.48	107.09

	AV14		AV14 + UR	
G	B	T	B	T
2	20.60	84.18	21.45	86.36
4	23.12	92.72	25.67	102.27
6	23.71	94.45	26.91	106.06
8	23.85	94.69	27.33	106.77
extr	24.02		27.85	

TABLE III.

Potential	AV14			AV14 + UR		
	CHH	FY(*)	GFMC	CHH	VMC	GFMC
B	23.93	23.87	24.2(2)	27.48	27.2(2)	28.3(2)
T	94.75			107.20	106.6(8)	113.3(20)
T_S	58.50			63.46		
T_P	1.39			2.66		
T_D	34.86			41.07		
V_{NN}	−118.68			−130.06	−129.7(7)	−136.5(20)
V^{CENT}	−41.95			−44.92		
V^{TENS}	−76.46			−85.35		
V^{LS}	1.28			1.75		
V^{L^2}	13.26			15.60		
V^{LS^2}	−15.56			−17.93		
V^{COUL}	0.75			0.78	0.74(1)	0.75(1)
U_{TNI}				−4.56	−4.84(9)	−5.8(3)
$U^{2\pi}$				−9.09	−9.48(11)	−10.8(3)
U^R				4.53	4.73(8)	5.0(2)
R	1.53			1.46	1.47(1)	1.45(1)
P_S (%)	85.44			82.88		
P_P (%)	0.35			0.64		
P_D (%)	14.20			16.24	15.5(1)	16.6(2)
$P_{S'}$ (%)	0.35			0.24		

TABLE IV.