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Symmetries and Solutions of Colored Yang-Baxter Equation for Six-Vertex Model†

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Abstract: In this paper, symmetries and spectral-dependent solutions of colored Yang-Baxter equation with six-vertex are discussed. It is shown that each six-vertex type solution of colored Yang-Baxter equation is equivalent to one of the six basic solutions up to five solution transformations and all solutions can be classified into two types called Baxter type and Free-Fermion type. The unitary condition of the solutions is also discussed.

§1. Colored Yang-Baxter equation and its symmetries

It is well known that the Yang-Baxter equation

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u). \tag{1.1}$$

plays an important role in the theory of two-dimensional integral system of quantum field theory and quantum statistics on two-dimensional lattice.[1-4] It ensures commutativity of the transfer matrices in the systems and models.

As one of generalization of the Yang-Baxter equation, the colored Yang-Baxter equation is defined as:[5]

$$\check{R}_{12}(u,\xi,\eta)\check{R}_{23}(u+v,\xi,\lambda)\check{R}_{12}(v,\eta,\lambda) = \check{R}_{23}(v,\eta,\lambda)\check{R}_{12}(u+v,\xi,\lambda)\check{R}_{23}(u,\xi,\eta), \tag{1.2}$$

where ξ, η and λ are color parameters, u, v and $u+v$ are spectral parameters and

$$\check{R}_{12}(u,\xi,\eta) = \check{R}(u,\xi,\eta)\otimes E, \qquad \check{R}_{23}(u,\xi,\eta) = E\otimes \check{R}(u,\xi,\eta),$$

here E is unit matrix of order n, $\otimes$ means the tensor product of two matrices. Recently, much attention has been attracted to the colored Yang-Baxter equation (1.2) and its

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solutions. Some subjects, such as the close relation of the colored Yang-Baxter equation with the free fermionic model in magnetic field, the connection between solutions of the equation and the multivariable invariants of links as well as representations of quantum algebras and so on are also discussed.[6-11]

In this paper, we will focus on the discussion of the symmetries and non-degenerate six-vertex type solutions of the colored Yang-Baxter equation (1.2). The case of the eight-vertex type solutions will be discussed in a coming paper. The first section of this paper is applied to introduce the symmetries of six-vertex type solutions of (1.2). In the second section we give the differential equations that solutions of the colored Yang-Baxter equation satisfy. In the third section we will construct all six-vertex type solutions of Yang-Baxter equation only with colored parameters. Based on the second and third sections we present six basic solutions of the equation (1.2) in section four and classify the obtained solutions into two type: the Baxter type and Free-Fermion type. These basic solutions, in fact, give all non-degenerate six-vertex type solutions of the colored Yang-Baxter equation (1.2) up to five solution transformations.

The six-vertex type solution of colored Y-B equation is the solution of (1.2) in the following form:

$$\check{R}(u,\xi,\eta) = \begin{pmatrix} a_1(u,\xi,\eta) & 0 & 0 & 0 \\ 0 & a_2(u,\xi,\eta) & a_5(u,\xi,\eta) & 0 \\ 0 & a_6(u,\xi,\eta) & a_3(u,\xi,\eta) & 0 \\ 0 & 0 & 0 & a_4(u,\xi,\eta) \end{pmatrix}.$$

If the weights functions $a_i(u,\xi,\eta) \not\equiv 0$ $(i = 1,2,\cdots,6)$ are satisfied additionally, the solution is called the non-degenerate six-vertex type solution of colored Y-B equation. Otherwise, it is called the degenerate six-vertex type solution. In the case of six-vertex type solutions, the Free-Fermion condition can be expressed as[6]

$$a_2(u,\xi,\eta)a_3(u,\xi,\eta) - a_1(u,\xi,\eta)a_4(u,\xi,\eta) - a_5(u,\xi,\eta)a_6(u,\xi,\eta) = 0. \tag{1.3}$$

For the six-vertex type solutions of Y-B equation, the matrix equation (1.2) is equivalent to the following 13 equations:

$$P_0: \quad a_2^{(1)}a_2^{(2)}a_3^{(3)} - a_3^{(1)}a_3^{(2)}a_2^{(3)} = 0, \tag{1.4a}$$

$$\left.\begin{aligned} P_1: &\quad a_1^{(1)}a_1^{(2)}a_2^{(3)} - a_2^{(1)}a_2^{(2)}a_1^{(3)} - a_6^{(1)}a_5^{(2)}a_2^{(3)} = 0, \\ P_2: &\quad a_1^{(1)}a_2^{(2)}a_5^{(3)} - a_5^{(1)}a_2^{(2)}a_1^{(3)} - a_3^{(1)}a_5^{(2)}a_2^{(3)} = 0, \\ P_3: &\quad a_2^{(1)}a_1^{(2)}a_6^{(3)} - a_2^{(1)}a_6^{(2)}a_1^{(3)} - a_6^{(1)}a_3^{(2)}a_2^{(3)} = 0, \end{aligned}\right\} \tag{1.4b}$$

$$\left.\begin{aligned} P_4: &\quad a_4^{(1)}a_4^{(2)}a_2^{(3)} - a_2^{(1)}a_2^{(2)}a_4^{(3)} - a_5^{(1)}a_6^{(2)}a_2^{(3)} = 0, \\ P_5: &\quad a_4^{(1)}a_2^{(2)}a_6^{(3)} - a_6^{(1)}a_2^{(2)}a_4^{(3)} - a_3^{(1)}a_6^{(2)}a_2^{(3)} = 0, \\ P_6: &\quad a_2^{(1)}a_4^{(2)}a_5^{(3)} - a_2^{(1)}a_5^{(2)}a_4^{(3)} - a_5^{(1)}a_3^{(2)}a_2^{(3)} = 0, \end{aligned}\right\} \tag{1.4c}$$

$$\left.\begin{array}{ll}
P_7: & a_1^{(1)}a_1^{(2)}a_3^{(3)} - a_3^{(1)}a_3^{(2)}a_1^{(3)} - a_6^{(1)}a_5^{(2)}a_3^{(3)} = 0, \\
P_8: & a_1^{(1)}a_3^{(2)}a_5^{(3)} - a_5^{(1)}a_3^{(2)}a_1^{(3)} - a_2^{(1)}a_5^{(2)}a_3^{(3)} = 0, \\
P_9: & a_3^{(1)}a_1^{(2)}a_6^{(3)} - a_3^{(1)}a_6^{(2)}a_1^{(3)} - a_6^{(1)}a_2^{(2)}a_3^{(3)} = 0, \\
P_{10}: & a_4^{(1)}a_4^{(2)}a_3^{(3)} - a_3^{(1)}a_3^{(2)}a_4^{(3)} - a_5^{(1)}a_6^{(2)}a_3^{(3)} = 0, \\
P_{11}: & a_4^{(1)}a_3^{(2)}a_6^{(3)} - a_6^{(1)}a_3^{(2)}a_4^{(3)} - a_2^{(1)}a_6^{(2)}a_3^{(3)} = 0, \\
P_{12}: & a_3^{(1)}a_4^{(2)}a_5^{(3)} - a_3^{(1)}a_5^{(2)}a_4^{(3)} - a_5^{(1)}a_2^{(2)}a_3^{(3)} = 0,
\end{array}\right\} \qquad (1.4d)$$

where for simplicity we denote

$$a_i^{(1)} = a_i(u,\xi,\eta), \quad a_i^{(2)} = a_i(v,\eta,\lambda), \quad a_i^{(3)} = a_i(u+v,\xi,\lambda), \qquad i = 1,2,\cdots,6. \quad (1.5)$$

Assume $\check{R}(u,\xi,\eta)$ is a solution of (1.2). Detailed study of the system of equations (1.4) shows that there are five symmetries of six-vertex type solution of Y-B equation as follows.

(A) Symmetry of exchanging indices.

The system of equations (1.4) is invariant if we exchange the two sub-indices 2 and 3 or 1, 5 and 4, 6 respectively.

(B) The scaling symmetry.

$f(u,\xi,\eta)\check{R}(u,\xi,\eta)$ is also a solution of Y-B equation (1.2), where $f(u,\xi,\eta)$ is an arbitrary function.

(C) Symmetry of weights $a_5(u,\xi,\eta)$, and $a_6(u,\xi,\eta)$

If the weights $a_5(u,\xi,\eta)$ and $a_6(u,\xi,\eta)$ are replaced by the new weights $\bar{a}_5(u,\xi,\eta) = \mu^{-1}a_5(u,\xi,\eta)$ and $\bar{a}_6(u,\xi,\eta) = \mu a_6(u,\xi,\eta)$ respectively, where μ is a non-zero constant, the new $\check{R}(u,\xi,\eta)$ is also a solution of (1.2).

(D) Symmetry of weights $a_2(u,\xi,\eta)$, and $a_3(u,\xi,\eta)$

If the weights $a_2(u,\xi,\eta)$ and $a_3(u,\xi,\eta)$ are replaced by the new weights $\bar{a}_2(u,\xi,\eta) = e^{\mu u}\frac{f(\xi)}{f(\eta)}a_2(u,\xi,\eta)$ and $\bar{a}_3(u,\xi,\eta) = e^{-\mu u}\frac{f(\eta)}{f(\xi)}a_3(u,\xi,\eta)$ respectively, where $f(\xi)$ is an arbitrary function and μ is a constant, the new $\check{R}(u,\xi,\eta)$ is also a solution of (1.2).

(E) Symmetry of spectral and color parameters.

If we take the new spectral parameter $\bar{u} = \mu u$ and new color parameters $\alpha = f(\xi)$, $\beta = f(\eta)$, where $f(\xi)$ is an arbitrary function and μ is a constant, the new matrix $\check{R}(u',\alpha,\beta)$ is also a solution of (1.2).

The five symmetries (A), (B), (C), (D) and (E) are called solution transformations A, B, C, D and E of six-vertex type solutions of colored Y-B equation (1.2) respectively.

§2. The differential equation related to the Colored Yang-Baxter equation

Since we only consider non-degenerate solutions of (1.2), from the equation (1.4a) and the assumption of non-degeneration we have

$$\frac{a_3}{a_2}(u,\xi,\eta) = \frac{a_3}{a_2}(u+v,\xi,\lambda)\Big/\frac{a_3}{a_2}(v,\eta,\lambda) \qquad (2.1)$$

due to $a_2(u,\xi,\eta) \not\equiv 0$. Let $\xi = \eta = \lambda$ in (2.1), we have

$$\frac{a_3}{a_2}(u,\xi,\xi) = exp(C(\xi)u). \qquad (2.2)$$

By letting $\eta = \lambda$ in (2.1), we find

$$\frac{a_3}{a_2}(u,\xi,\eta) = \frac{a_3}{a_2}(u+v,\xi,\eta)exp(-C(\eta)v). \qquad (2.3)$$

Therefore, the combination of (2.1), (2.2) and (2.3) gives

$$\frac{a_3}{a_2}(u,\xi,\eta) = exp(C(\lambda)u)\,\frac{a_3}{a_2}(0,\xi,\lambda)\Big/\frac{a_3}{a_2}(0,\eta,\lambda). \qquad (2.4)$$

Noticing that left hand side of (2.4) is independent of λ and considering the solution transformations B and D, we can assume $a_2(u,\xi,\eta) = a_3(u,\xi,\eta) = 1$ in the following discussion of non-degenerate solutions without losing generality. Therefore equations (1.4) can be reduced to following six equations:

$$\begin{array}{l}
a_1(u,\xi,\eta)a_1(v,\eta,\lambda) - a_1(u+v,\xi,\lambda) - a_6(u,\xi,\eta)a_5(v,\eta,\lambda) = 0 \\
a_1(u,\xi,\eta)a_5(u+v,\xi,\lambda) - a_5(u,\xi,\eta)a_1(u+v,\xi,\lambda) - a_5(v,\eta,\lambda) = 0 \\
a_1(v,\eta,\lambda)a_6(u+v,\xi,\lambda) - a_6(v,\eta,\lambda)a_1(u+v,\xi,\lambda) - a_6(u,\xi,\eta) = 0
\end{array} \qquad (2.5a)$$

$$\begin{array}{l}
a_4(u,\xi,\eta)a_4(v,\eta,\lambda) - a_4(u+v,\xi,\lambda) - a_5(u,\xi,\eta)a_6(v,\eta,\lambda) = 0 \\
a_4(u,\xi,\eta)a_6(u+v,\xi,\lambda) - a_6(u,\xi,\eta)a_4(u+v,\xi,\lambda) - a_6(v,\eta,\lambda) = 0 \\
a_4(v,\eta,\lambda)a_5(u+v,\xi,\lambda) - a_5(v,\eta,\lambda)a_4(u+v,\xi,\lambda) - a_5(u,\xi,\eta) = 0
\end{array} \qquad (2.5b)$$

In order to solve (2.5), we will use $a_i(u+v,\xi,\lambda)$ and $a_i(v,\eta,\lambda)$ $(i = 1,2,\cdots,6)$ to represent $a_i(u,\xi,\eta)$ $(i = 1,2,\cdots,6)$. from (2.5a) we have

$$\begin{array}{rl}
a_1(u,\xi,\eta) = & \dfrac{a_1(u+v,\xi,\lambda)}{a_1(v,\eta,\lambda)}(1 - a_5(v,\eta,\lambda)a_6(v,\eta,\lambda)) \\
& +a_5(v,\eta,\lambda)a_6(u+v,\xi,\lambda), \\
a_5(u,\xi,\eta) = & \dfrac{a_5(u+v,\xi,\lambda)}{a_1(v,\eta,\lambda)}(1 - a_5(v,\eta,\lambda)a_6(v,\eta,\lambda)) \\
& -\dfrac{a_5(v,\eta,\lambda)}{a_1(u+v,\xi,\lambda)}(1 - a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)), \\
a_6(u,\xi,\eta) = & a_1(v,\eta,\lambda)a_6(u+v,\xi,\lambda) - a_1(u+v,\xi,\lambda)a_6(v,\eta,\lambda),
\end{array} \qquad (2.6)$$

and from (2.5b) we have

$$\begin{array}{rl}
a_4(u,\xi,\eta) = & \dfrac{a_4(u+v,\xi,\lambda)}{a_4(v,\eta,\lambda)}(1 - a_5(v,\eta,\lambda)a_6(v,\eta,\lambda)) \\
& +a_5(u+v,\xi,\lambda)a_6(v,\eta,\lambda), \\
a_5(u,\xi,\eta) = & a_4(v,\eta,\lambda)a_5(u+v,\xi,\lambda) - a_4(u+v,\xi,\lambda)a_5(v,\eta,\lambda), \\
a_6(u,\xi,\eta) = & \dfrac{a_6(u+v,\xi,\lambda)}{a_4(v,\eta,\lambda)}(1 - a_5(v,\eta,\lambda)a_6(v,\eta,\lambda)) \\
& -\dfrac{a_6(v,\eta,\lambda)}{a_4(u+v,\xi,\lambda)}(1 - a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)).
\end{array} \qquad (2.7)$$

By comparing the expressions of $a_5(u,\xi,\eta)$ and $a_6(u,\xi,\eta)$ in (2.6) and (2.7), we have

$$\begin{aligned}
& a_1(u+v,\xi,\lambda)a_5(u+v,\xi,\lambda)(1-a_1(v,\eta,\lambda)a_4(v,\eta,\lambda)-a_5(v,\eta,\lambda)a_6(v,\eta,\lambda))\\
=\ & a_1(v,\eta,\lambda)a_5(v,\eta,\lambda)(1-a_1(u+v,\xi,\lambda)a_4(u+v,\xi,\lambda)\\
& -a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)),\\
& a_4(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)(1-a_1(v,\eta,\lambda)a_4(v,\eta,\lambda)-a_5(v,\eta,\lambda)a_6(v,\eta,\lambda))\\
=\ & a_4(v,\eta,\lambda)a_6(v,\eta,\lambda)(1-a_1(u+v,\xi,\lambda)a_4(u+v,\xi,\lambda)\\
& -a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)),
\end{aligned}$$

i.e.

$$\begin{aligned}
\frac{(1-a_1(u+v,\xi,\lambda)a_4(u+v,\xi,\lambda)-a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda))}{a_1(u+v,\xi,\lambda)a_5(u+v,\xi,\lambda)} &= C_1(\lambda),\\
\frac{(1-a_1(u+v,\xi,\lambda)a_4(u+v,\xi,\lambda)-a_5(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda))}{a_4(u+v,\xi,\lambda)a_6(u+v,\xi,\lambda)} &= C_2(\lambda),
\end{aligned} \tag{2.8}$$

where $C_1(\lambda), C_2(\lambda)$ are functions of λ. Therefore, from the combination of (2.6), (2.7) and (2.8), the expressions of $a_i(u,\xi,\eta)$ $(i=1,4,5,6)$ by $a_i(u+v,\xi,\lambda), a_i(v,\eta,\lambda)$ $(i=1,4,5,6)$ are as follows

$$\begin{aligned}
a_1(u,\xi,\eta) =\ & a_1(u+v,\xi,\lambda)(a_4(v,\eta,\lambda)+C_1(\lambda)a_5(v,\eta,\lambda))\\
& +a_5(v,\eta,\lambda)a_6(u+v,\xi,\lambda),\\
a_4(u,\xi,\eta) =\ & a_4(u+v,\xi,\lambda)(a_1(v,\eta,\lambda)+C_2(\lambda)a_6(v,\eta,\lambda))\\
& +a_5(u+v,\xi,\lambda)a_6(v,\eta,\lambda),\\
a_5(u,\xi,\eta) =\ & a_4(v,\eta,\lambda)a_5(u+v,\xi,\lambda)-a_4(u+v,\xi,\lambda)a_5(v,\eta,\lambda),\\
a_6(u,\xi,\eta) =\ & a_1(v,\eta,\lambda)a_6(u+v,\xi,\lambda)-a_1(u+v,\xi,\lambda)a_6(v,\eta,\lambda),
\end{aligned} \tag{2.9}$$

From the expression of $a_5(u,\xi,\eta)$ and $a_6(u,\xi,\eta)$ in (2.6) and (2.7), it is easy to show that

$$a_5(u,\xi,\eta) = -a_5(-u,\eta,\xi), \qquad a_6(u,\xi,\eta) = -a_6(-u,\eta,\xi). \tag{2.10}$$

Letting $u=0$, $\eta=\xi$ in (2.5) and then solving the equation with respect to $\{a_1(0,\xi,\xi), a_4(0,\xi,\xi), a_5(0,\xi,\xi), a_5(0,\xi,\xi)\}$ we obtain

$$a_1(0,\xi,\xi) = a_4(0,\xi,\xi) = 1, \qquad a_5(0,\xi,\xi) = a_6(0,\xi,\xi) = 0. \tag{2.11}$$

(2.11) can be regarded as initial condition of Y-B equation (1.2).

In following context, the differential equations of $a_i(u,\xi,\eta)$ is developed and we will

use the following notations for simplicity:

$$\begin{aligned}
X &= a_1(u,\xi,\lambda), & X' &= \frac{\partial a_1(u,\xi,\lambda)}{\partial u}, & X'' &= \frac{\partial^2 a_1(u,\xi,\lambda)}{\partial u^2},\\
W &= a_4(u,\xi,\lambda), & W' &= \frac{\partial a_4(u,\xi,\lambda)}{\partial u}, & W'' &= \frac{\partial^2 a_4(u,\xi,\lambda)}{\partial u^2},\\
Y &= a_5(u,\xi,\lambda), & Y' &= \frac{\partial a_5(u,\xi,\lambda)}{\partial u}, & Y'' &= \frac{\partial^2 a_5(u,\xi,\lambda)}{\partial u^2},\\
Z &= a_6(u,\xi,\lambda), & Z' &= \frac{\partial a_6(u,\xi,\lambda)}{\partial u}, & Z'' &= \frac{\partial^2 a_6(u,\xi,\lambda)}{\partial u^2},\\
c_i(\xi) &= \left.\frac{\partial a_i(u,\xi,\lambda)}{\partial u}\right|_{u=0,\lambda=\xi}, & & i=1,4,5,6.
\end{aligned} \tag{2.12}$$

If we differentiate the system of equations (2.5) with respect to the variable u and then letting $u=0$, $\eta=\xi$. We can obtain following group of differential equations with appropriate arrangement of the arguments

$$\begin{aligned}
X' &= Xc_1(\xi) - Yc_6(\xi), & Y' &= -Yc_1(\xi) + Xc_5(\xi),\\
W' &= Wc_4(\xi) - Zc_5(\xi), & Z' &= -Zc_4(\xi) + Wc_6(\xi),\\
c_6(\xi) &= -(X'Z) + XZ', & c_5(\xi) &= -W'Y + WY'.
\end{aligned} \tag{2.13}$$

Differentiating once more (2.13) with respect to the spectral variable u, we have four differential equations in the same form,

$$\begin{aligned}
X'' &= (c_1(\xi)^2 - c_5(\xi)c_6(\xi))X, & Y'' &= (c_1(\xi)^2 - c_5(\xi)c_6(\xi))Y,\\
W'' &= (c_4(\xi)^2 - c_5(\xi)c_6(\xi))W, & Z'' &= (c_4(\xi)^2 - c_5(\xi)c_6(\xi))Z.
\end{aligned} \tag{2.14}$$

With the similar process, doing the differential with respect to the variable v, we can also have

$$\begin{aligned}
X' &= Xc_1(\lambda) - Zc_5(\lambda), & Z' &= -Zc_1(\lambda) + Xc_6(\lambda),\\
W' &= Wc_4(\lambda) - Yc_6(\lambda), & Y' &= -Yc_4(\lambda) + Wc_5(\lambda),\\
c_5(\lambda) &= -(X'Y) + XY', & c_6(\lambda) &= -W'Z + WZ'.
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
X'' &= (c_1(\lambda)^2 - c_5(\lambda)c_6(\lambda))X, & Y'' &= (c_4(\lambda)^2 - c_5(\lambda)c_6(\lambda))Y,\\
W'' &= (c_4(\lambda)^2 - c_5(\lambda)c_6(\lambda))W, & Z'' &= (c_1(\lambda)^2 - c_5(\lambda)c_6(\lambda))Z.
\end{aligned} \tag{2.16}$$

From the equations (2.15), we can obtain a system of compatibility relations for the Y-B equation (1.4) as follows:

$$\begin{aligned}
XY(c_1(\lambda)+c_4(\lambda)) &= c_5(\lambda)(-1+WX+YZ),\\
WZ(c_1(\lambda)+c_4(\lambda)) &= c_6(\lambda)(-1+WX+YZ).
\end{aligned} \tag{2.17}$$

Comparing (2.17) with (2.8), we have

$$C_1(\lambda) = \frac{c_1(\lambda)+c_4(\lambda)}{c_5(\lambda)}, \quad C_2(\lambda) = \frac{c_1(\lambda)+c_4(\lambda)}{c_6(\lambda)} \tag{2.18}$$

Hence by comparing (2.14) with (2.16) we see

$$c_1(\xi)^2 - c_5(\xi)c_6(\xi) = c_1(\lambda)^2 - c_5(\lambda)c_6(\lambda)$$

is a constant independent of colored parameters ξ and λ and

$$c_1(\xi)^2 = c_4(\xi)^2. \tag{2.19}$$

So What we only need to do is to solve the differential equations of degree 2

$$U''(u,\xi,\eta) = k^2 U(u,\xi,\eta), \tag{2.20}$$

where $k^2 = (c_1(\xi)^2 - c_5(\xi)c_6(\xi))$. If $k \neq 0$, by solution transformation E to the spectral parameter u, we can assume $k = 1$. Therefore the general solutions of the equation are

$$a_i(u,\xi,\eta) = A_i(\xi,\eta)\cosh(u) + B_i(\xi,\eta)\sinh(u), \tag{2.21a}$$

for the case $k \neq 0$ and

$$a_i(u,\xi,\eta) = A_i(\xi,\eta) + B_i(\xi,\eta)u, \quad (i = 1,2,5,6) \tag{2.21b}$$

for the case $k = 0$, where

$$A_i(\xi,\eta) = a_i(0,\xi,\eta), \qquad B_i(\xi,\eta) = \left.\frac{da_i(u,\xi,\eta)}{du}\right|_{u=0}, \quad (i = 1,2,5,6) \tag{2.21c}$$

§3 The solutions of the Y-B equation only with colored parameters

In this section, we consider the Yang-Baxter equation only with colored parameters as follows.

$$\check{R}_{12}(\xi,\eta)\check{R}_{23}(\xi,\lambda)\check{R}_{12}(\eta,\lambda) = \check{R}_{23}(\eta,\lambda)\check{R}_{12}(\xi,\lambda)\check{R}_{23}(\xi,\eta), \tag{3.1}$$

This equation can be reduced from the Yang-Baxter equation of colored parameters (1.2) if we take the spectral parameter $u = v = 0$. Clearly, if $R(u,\xi,\eta)$ is solution of (1.2), then $R(0,\xi,\eta)$ is a solution of (3.1). So in roughly speaking, a solution of (3.1) gives the initial values of a solutions of (1.2). This implies $A_i(\xi,\eta), (i = 1,2,5,6)$ in (2.21a) and (2.21b) must be a solution of the Y-B equation only with colored parameters. For the Y-B equation only with colored parameter definition of non-degenerate and degenerate six-vertex type solutions are similar to that of (1.2). Generally, five solution transformations and the formulas given in the first section are also true for the six-vertex solutions of Y-B equation (3.1) except for the spectral parameters being replaced by 0.

We first consider the non-degenerate solutions. Similarly, by considering with solution transformation B and D, we can assume $a_2(\xi,\eta) = a_3(\xi,\eta) = 1$ in the following

discussion of non-degenerate solutions without losing generality. Therefore 13 equations in (1.4) for spectral parameters $u = v = 0$ are equivalent to following six equations up to solution transformations B and D:

$$\left.\begin{aligned} a_1(\xi,\eta)a_1(\eta,\lambda) - a_1(\xi,\lambda) - a_6(\xi,\eta)a_5(\eta,\lambda) = 0 \\ a_1(\xi,\eta)a_5(\xi,\lambda) - a_5(\xi,\eta)a_1(\xi,\lambda) - a_5(\eta,\lambda) = 0 \\ a_1(\eta,\lambda)a_6(\xi,\lambda) - a_6(\eta,\lambda)a_1(\xi,\lambda) - a_6(\xi,\eta) = 0 \end{aligned}\right\} \tag{3.2a}$$

$$\left.\begin{aligned} a_4(\xi,\eta)a_4(\eta,\lambda) - a_4(\xi,\lambda) - a_5(\xi,\eta)a_6(\eta,\lambda) = 0 \\ a_4(\xi,\eta)a_6(\xi,\lambda) - a_6(\xi,\eta)a_4(\xi,\lambda) - a_6(\eta,\lambda) = 0 \\ a_4(\eta,\lambda)a_5(\xi,\lambda) - a_5(\eta,\lambda)a_4(\xi,\lambda) - a_5(\xi,\eta) = 0 \end{aligned}\right\} \tag{3.2b}$$

Similar to the discussion in section 2, from equations (3.2) we have the expressions of $a_i(\xi,\eta)$ $(i = 1,2,\cdots,6)$ in forms of $a_i(\xi,\lambda)$ and $a_i(\eta,\lambda)$ $(i = 1,2,\cdots,6)$ as follows

$$\begin{aligned} a_1(\xi,\eta) &= \frac{a_1(\xi,\lambda)}{a_1(\eta,\lambda)}(1 - a_5(\eta,\lambda)a_6(\eta,\lambda)) + a_5(\eta,\lambda)a_6(\xi,\lambda), \\ a_4(\xi,\eta) &= \frac{a_4(\xi,\lambda)}{a_4(\eta,\lambda)}(1 - a_5(\eta,\lambda)a_6(\eta,\lambda)) + a_5(\xi,\lambda)a_6(\eta,\lambda), \\ a_5(\xi,\eta) &= a_4(\eta,\lambda)a_5(\xi,\lambda) - a_4(\xi,\lambda)a_5(\eta,\lambda), \\ a_6(\xi,\eta) &= a_1(\eta,\lambda)a_6(\xi,\lambda) - a_1(\xi,\lambda)a_6(\eta,\lambda). \end{aligned} \tag{3.3}$$

And we also have the algebraic relations

$$\begin{aligned} \frac{(1 - a_1(\xi,\lambda)a_4(\xi,\lambda) - a_5(\xi,\lambda)a_6(\xi,\lambda))}{a_1(\xi,\lambda)a_5(\xi,\lambda)} &= C_1(\lambda), \\ \frac{(1 - a_1(\xi,\lambda)a_4(\xi,\lambda) - a_5(\xi,\lambda)a_6(\xi,\lambda))}{a_4(\xi,\lambda)a_6(\xi,\lambda)} &= C_2(\lambda), \end{aligned} \tag{3.4}$$

where C_1, C_2 are functions of λ. Now we fix λ to a value and take 0 as the fixed value without losing gerality. If we denote $f_i(\xi) = a_i(\xi,\lambda)$ $(i = 1,4,5,6)$, then $f_i(\xi)$ $(i = 1,4,5,6)$ satisfy

$$\begin{aligned} 1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) &= C_1 f_1(\xi)f_5(\xi), \\ 1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) &= C_2 f_4(\xi)f_6(\xi), \end{aligned} \tag{3.5}$$

where C_1, C_2 are constants. Therefore, from the combination of (3.3) and (3.5), we can write $a_i(\xi,\eta)$ $(i = 1,4,5,6)$ by $f_i(\xi), f_i(\eta)$ $(i = 1,4,5,6)$ as

$$\begin{aligned} a_1(\xi,\eta) &= f_1(\xi)(f_4(\eta) + C_1 f_5(\eta)) + f_5(\eta)f_6(\xi), \\ a_4(\xi,\eta) &= f_4(\xi)(f_1(\eta) + C_2 f_6(\eta)) + f_5(\xi)f_6(\eta), \\ a_5(\xi,\eta) &= f_4(\eta)f_5(\xi) - f_4(\xi)f_5(\eta), \\ a_6(\xi,\eta) &= f_1(\eta)f_6(\xi) - f_1(\xi)f_6(\eta). \end{aligned} \tag{3.6}$$

Since we consider non-degenerate solutions, there are only two cases for C_1 and C_2, one is $C_1 = C_2 = 0$, the other is $C_1 \neq 0$ and $C_2 \neq 0$. And by solution transformation C, we can assume $C_1 = C_2 = -2\cos(C)$.

1. The case $C_1 = C_2 = 0$

For this case, we have

$$1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) = 0.$$

Equivalently, $f_i(\xi)$ $(i = 1,4,5,6)$ can be parameterized as

$$\begin{aligned} f_1(\xi) &= (F(\xi)+1)G(\xi),\\ f_4(\xi) &= (-F(\xi)+1)/G(\xi),\\ f_5(\xi) &= F(\xi)H(\xi),\\ f_6(\xi) &= F(\xi)/H(\xi), \end{aligned} \tag{3.7}$$

where F, G, H are arbitrary functions of one variable. And corresponding expressions of $a_i(\xi,\eta)$ $(i = 1,2,\cdots,6)$ are

$$\begin{aligned} a_1(\xi,\eta) &= (F(\xi)+1)(-F(\eta)+1)\frac{G(\xi)}{G(\eta)} + F(\xi)F(\eta)\frac{H(\eta)}{H(\xi)},\\ a_2(\xi,\eta) &= a_3(\xi,\eta) = 1,\\ a_4(\xi,\eta) &= (F(\eta)+1)(-F(\xi)+1)\frac{G(\eta)}{G(\xi)} + F(\xi)F(\eta)\frac{H(\xi)}{H(\eta)},\\ a_5(\xi,\eta) &= F(\xi)(-F(\eta)+1)\frac{H(\xi)}{G(\eta)} - F(\eta)(-F(\xi)+1)\frac{H(\eta)}{G(\xi)},\\ a_6(\xi,\eta) &= F(\xi)(F(\eta)+1)\frac{G(\eta)}{H(\xi)} - F(\eta)(F(\xi)+1)\frac{G(\xi)}{H(\eta)}. \end{aligned} \tag{3.8}$$

This solution is called Free-Fermion type solution because it satisfies the Free-Fermion condition (1.3).

2. The case $C_1 \neq 0$ and $C_2 \neq 0$

For this case, by solution transformation C, we can assume $C_1 = C_2 = -2\cos(C) \neq 0$,

$$1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) = -2\cos(C)f_1(\xi)f_5(\xi) = -2\cos(C)f_4(\xi)f_6(\xi). \tag{3.9}$$

Since $-2\cos(C) \neq 0$,

$$f_1(\xi)f_5(\xi) = f_4(\xi)f_6(\xi). \tag{3.10}$$

By (17), $f_i(\xi)$ $(i = 1,4,5,6)$ can expressed as follows

$$\begin{aligned} f_1(\xi) &= H(\xi)/G(\xi), & f_4(\xi) &= H(\xi)G(\xi),\\ f_5(\xi) &= F(\xi)G(\xi), & f_6(\xi) &= F(\xi)/G(\xi). \end{aligned} \tag{3.11}$$

Then we can rewrite (3.9) as

$$H^2(\xi) - 2\cos(C)H(\xi)F(\xi) + F^2(\xi) - 1 = 0. \tag{3.12}$$

Considering (3.12) as a second order equation of $H(\xi)$, we have the solutions as

$$H(\xi) = \cos(C)F(\xi) \pm \sqrt{1 - \sin^2(C)F^2(\xi)}. \tag{3.13}$$

So we have following two sub-cases.

Subcase 1. If $\cos(C) = \pm 1$, then $\sin(C) = 0$,

$$H(\xi) = \pm F(\xi) \pm 1.$$

After considering $f_i(\xi)$ $(i = 1,4,5,6)$ and $H(\xi), F(\xi)$ with the influence of solution transformations B,C,E and other equivalent conditions, we find that we need only assume

$$\cos(C) = 1, \qquad H(\xi) = F(\xi) + 1.$$

Thus

$$\begin{aligned} f_1(\xi) &= (F(\xi)+1)/G(\xi), & f_5(\xi) &= F(\xi)G(\xi),\\ f_4(\xi) &= (F(\xi)+1)G(\xi), & f_6(\xi) &= F(\xi)/G(\xi), \end{aligned} \tag{3.14}$$

The solution of (3.1) is

$$\begin{aligned} a_1(\xi,\eta) &= (F(\xi) - F(\eta) + 1)\frac{G(\eta)}{G(\xi)},\\ a_2(\xi,\eta) &= a_3(\xi,\eta) = 1,\\ a_4(\xi,\eta) &= (F(\xi) - F(\eta) + 1)\frac{G(\xi)}{G(\eta)},\\ a_5(\xi,\eta) &= (F(\xi) - F(\eta))G(\xi)G(\eta),\\ a_6(\xi,\eta) &= (F(\xi) - F(\eta))\frac{1}{G(\xi)G(\eta)}. \end{aligned} \tag{3.15}$$

Subcase 2. If $\cos(C) \neq \pm 1$, then $\sin(C) \neq 0$. Up to solution transformation E, we can replace $F(\xi)$ by $\sin(F(\xi))/\sin(C)$. Thus

$$H(\xi) = \frac{\cos(C)\sin(F(\xi))}{\sin(C)} \pm \cos(\xi) = \sin(F(\xi) \pm C)/\sin(C).$$

After considering the influence of solution transformations B,C,E and other equivalent conditions, we find that we need only assume

$$H(\xi) = \sin(F(\xi) + C)/\sin(C).$$

Therefore

$$\begin{aligned} f_1(\xi) &= \frac{\sin(F(\xi)+C)}{\sin(C)G(\xi)}, & f_5(\xi) &= \frac{\sin(F(\xi))G(\xi)}{\sin(C)},\\ f_4(\xi) &= \frac{\sin(F(\xi)+C)G(\xi)}{\sin(C)}, & f_6(\xi) &= \frac{\sin(F(\xi))}{\sin(C)G(\xi)}, \end{aligned} \tag{3.16}$$

The solution of (3.1) for this case is

$$
\begin{aligned}
a_1(\xi,\eta) &= \frac{\sin(F(\xi)-F(\eta)+C)}{\sin(C)}\frac{G(\eta)}{G(\xi)},\\
a_2(\xi,\eta) &= a_3(\xi,\eta) = 1,\\
a_4(\xi,\eta) &= \frac{\sin(F(\xi)-F(\eta)+C)}{\sin(C)}\frac{G(\xi)}{G(\eta)},\\
a_5(\xi,\eta) &= \frac{\sin(F(\xi)-F(\eta))}{\sin(C)}G(\xi)G(\eta),\\
a_6(\xi,\eta) &= \frac{\sin(F(\xi)-F(\eta))}{\sin(C)}\frac{1}{G(\xi)G(\eta)},
\end{aligned}
\tag{3.17}
$$

where C is arbitrary constant.

Therefore, up to solution transformations A,B,C,D,E, any non-degenerate six-vertex type solution of YBE (3.1) with color parameters is equivalent to one of the three sets of basic solutions: (3.8), (3.15) and (3.17).

Remark 1. If we take $F(\xi)=\xi$, $G(\xi)=H(\xi)=1$ in (3.8), solution (3.8) becomes

$$
\begin{aligned}
a_1(\xi,\eta) &= \xi-\eta+1,\\
a_2(\xi,\eta) &= a_3(\xi,\eta)=1,\\
a_4(\xi,\eta) &= \eta-\xi+1,\\
a_5(\xi,\eta) &= \xi-\eta,\\
a_6(\xi,\eta) &= \xi-\eta.
\end{aligned}
\tag{3.18}
$$

If we take $F(\xi)=\sin(\xi)/\sin(C)$, $G(\xi)=\sin(\xi+C)/(\sin(xi)+\sin(C))$ and $H(\xi)=1$ in (3.8), solution (3.8) becomes

$$
\begin{aligned}
a_1(\xi,\eta) &= \frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_2(\xi,\eta) &= a_3(\xi,\eta)=1,\\
a_4(\xi,\eta) &= \frac{\sin(\eta-\xi+C)}{\sin(C)},\\
a_5(\xi,\eta) &= \frac{\sin(\xi-\eta)}{\sin(C)},\\
a_6(\xi,\eta) &= \frac{\sin(\xi-\eta)}{\sin(C)}.
\end{aligned}
\tag{3.19}
$$

If we take $F(\xi)=\xi$, $G(\xi)=1$ in (3.15), solution (3.15) becomes

$$
\begin{aligned}
a_1(\xi,\eta) &= \xi-\eta+1,\\
a_2(\xi,\eta) &= a_3(\xi,\eta)=1,\\
a_4(\xi,\eta) &= \xi-\eta+1,\\
a_5(\xi,\eta) &= \xi-\eta,\\
a_6(\xi,\eta) &= \xi-\eta.
\end{aligned}
\tag{3.20}
$$

If we take $F(\xi)=\xi$, $G(\xi)=1$ in (3.17), solution (3.17) becomes

$$
\begin{aligned}
a_1(\xi,\eta) &= \frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_2(\xi,\eta) &= a_3(\xi,\eta)=1,\\
a_4(\xi,\eta) &= \frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_5(\xi,\eta) &= \frac{\sin(\xi-\eta)}{\sin(C)},\\
a_6(\xi,\eta) &= \frac{\sin(\xi-\eta)}{\sin(C)}.
\end{aligned}
\tag{3.21}
$$

In (3.18), (3.19), (3.20) and (3.21), $a_i(\xi,\eta)$ $(i=1,2,\cdots,6)$ are in forms of functions of two variables, but they can also be regarded as one-variable functions of $\xi-\eta$. Therefore from three basic solutions (3.8), (3.15) and (3.17) of (3.1), we obtain the four basic solutions of (1.1). In fact, (3.20) is the solution given by Yang[1], (3.21) is just the trigonometric one for ice model [2], (3.21) and (3.19) are just the type-I and type-II six-vertex solutions given by Sogo et al[12], (3.18) can be obtained by taking limit in (3.19).

Now that we have discussed the non-degenerate solution, we come to the degenerate six-vertex type solutions of Y-B equation (3.1).

Case 1. If $a_2(\xi,\eta)=a_3(\xi,\eta)\equiv 0$, then $a_1(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta)$ can be arbitrary functions.

Case 2. If $a_2(\xi,\eta)\equiv 0$ or $a_3(\xi,\eta)\equiv 0$, then up to solution transformation A, we assume $a_2(\xi,\eta)\not\equiv 0$, $a_3(\xi,\eta)\equiv 0$ without losing generality, equation (2) is equivalent to following six equations:

$$
\begin{aligned}
&a_1(\xi,\eta)a_1(\eta,\lambda)-a_1(\xi,\lambda)-a_6(\xi,\eta)a_5(\eta,\lambda)=0,\\
&a_1(\xi,\eta)a_5(\xi,\lambda)-a_5(\xi,\eta)a_1(\xi,\lambda)=0,\\
&a_1(\eta,\lambda)a_6(\xi,\lambda)-a_6(\eta,\lambda)a_1(\xi,\lambda)=0,\\
&a_4(\xi,\eta)a_4(\eta,\lambda)-a_4(\xi,\lambda)-a_5(\xi,\eta)a_6(\eta,\lambda)=0,\\
&a_4(\xi,\eta)a_6(\xi,\lambda)-a_6(\xi,\eta)a_4(\xi,\lambda)=0,\\
&a_4(\eta,\lambda)a_5(\xi,\lambda)-a_5(\eta,\lambda)a_4(\xi,\lambda)=0.
\end{aligned}
\tag{3.22}
$$

Subcase 2.1 If we have $a_5(\xi,\eta)=a_6(\xi,\eta)\equiv 0$ additionally, then up to solution transformations, the basic solutions for this case are

$$
\begin{cases}
a_1(\xi,\eta)=\varepsilon_1 f_1(\xi)/f_1(\eta),\\
a_2(\xi,\eta)=1,\\
a_4(\xi,\eta)=\varepsilon_4 f_4(\xi)/f_4(\eta),\\
a_3(\xi,\eta)=a_5(\xi,\eta)=a_6(\xi,\eta)=0,
\end{cases}
\tag{3.23}
$$

where f_1, f_4 are arbitrary functions, $\varepsilon_1,\varepsilon_4$ are 1 or 0.

Subcase 2.2 If we have $a_5(\xi,\eta)\equiv 0$ or $a_6(\xi,\eta)\equiv 0$ additionally, then up to solution transformation A, we assume $a_5(\xi,\eta)\not\equiv 0$, $a_6(\xi,\eta)\equiv 0$ without losing generality, therefore for this case there are four sets of basic solutions up to the solution transformations:

i) $a_1(\xi,\eta)=a_3(\xi,\eta)=a_4(\xi,\eta)=a_6(\xi,\eta)\equiv 0$, $a_2(\xi,\eta), a_5(\xi,\eta)$ are arbitrary functions,

ii) $a_1(\xi,\eta)=a_3(\xi,\eta)=a_6(\xi,\eta)\equiv 0$, $a_4(\xi,\eta)=f_4(\xi)/f_4(\eta)$, $a_2(\xi,\eta)=1$, $a_5(\xi,\eta)=f_4(\xi)f_5(\eta)$,

iii) $a_3(\xi,\eta)=a_4(\xi,\eta)=a_6(\xi,\eta)\equiv 0$, $a_1(\xi,\eta)=f_1(\xi)/f_1(\eta)$, $a_2(\xi,\eta)=1$, $a_5(\xi,\eta)=f_5(\xi)/f_1(\eta)$,

iv) $a_3(\xi,\eta)=a_6(\xi,\eta)\equiv 0$, $a_1(\xi,\eta)=f_1(\xi)/f_1(\eta)$, $a_4(\xi,\eta)=f_4(\xi)/f_4(\eta)$, $a_2(\xi,\eta)=1$, $a_5(\xi,\eta)=f_4(\xi)/f_1(\eta)$,

where f_1, f_4, f_5 are arbitrary functions.

Subcase 2.3. If we have $a_5(\xi,\eta)\not\equiv 0$ and $a_6(\xi,\eta)\not\equiv 0$ additionally, up to solution transformations, the basic solutions for this case are

$$\left\{\begin{aligned} a_1(\xi,\eta)&=\frac{f_6(\xi)}{f_6(\eta)}(g(\xi,\eta)+1/2),\\ a_2(\xi,\eta)&=1,\\ a_3(\xi,\eta)&=0,\\ a_4(\xi,\eta)&=\frac{f_5(\xi)}{f_5(\eta)}(-g(\xi,\eta)+1/2),\\ a_5(\xi,\eta)&=\frac{f_5(\xi)}{Cf_6(\eta)}(g(\xi,\eta)+1/2),\\ a_6(\xi,\eta)&=-\frac{Cf_6(\xi)}{f_5(\eta)}(-g(\xi,\eta)+1/2),\end{aligned}\right. \tag{3.24}$$

and

$$\left\{\begin{aligned} a_1(\xi,\eta)&=\frac{C_1C_4}{1+C_1C_4}\frac{f_6(\xi)}{f_6(\eta)},\\ a_2(\xi,\eta)&=1,\\ a_3(\xi,\eta)&=0,\\ a_4(\xi,\eta)&=\frac{C_1C_4}{1+C_1C_4}\frac{f_5(\xi)}{f_5(\eta)},\\ a_5(\xi,\eta)&=\frac{C_4}{1+C_1C_4}\frac{f_5(\xi)}{f_6(\eta)},\\ a_6(\xi,\eta)&=\frac{C_1}{1+C_1C_4}\frac{f_6(\xi)}{f_5(\eta)},\end{aligned}\right. \tag{3.25}$$

where f_1, f_4, f_5 are arbitrary one-variable functions, g is arbitrary two-variable function, C_1, C_4 are non-zero constants. And solution (31) satisfies Free-Fermion condition (1.3).

Case 3. If $a_2(\xi,\eta)\not\equiv 0$ and $a_3(\xi,\eta)\not\equiv 0$, we also have following cases,

Subcase 3.1 If we have $a_5(\xi,\eta)=a_6(\xi,\eta)\equiv 0$ additionally, then up to solution

transformations, the basic solution for this case is

$$\left\{\begin{aligned} a_1(\xi,\eta)&=f_1(\xi)/f_1(\eta),\\ a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\ a_4(\xi,\eta)&=f_4(\xi)/f_4(\eta),\\ a_5(\xi,\eta)&=a_6(\xi,\eta)=0,\end{aligned}\right. \tag{3.26}$$

where f_1, f_4 are arbitrary functions.

Subcase 3.2 If we have $a_5(\xi,\eta)\equiv 0$ or $a_6(\xi,\eta)\equiv 0$ additionally, then up to solution transformation A, we assume $a_5(\xi,\eta)\not\equiv 0$, $a_6(\xi,\eta)\equiv 0$ without losing generality, equation (2) is equivalent to following six equations:

$$\begin{aligned} &a_1(\xi,\eta)a_1(\eta,\lambda)-a_1(\xi,\lambda)=0,\\ &a_1(\xi,\eta)a_5(\xi,\lambda)-a_5(\xi,\eta)a_1(\xi,\lambda)-a_5(\eta,\lambda)=0,\\ &a_4(\xi,\eta)a_4(\eta,\lambda)-a_4(\xi,\lambda)=0,\\ &a_4(\eta,\lambda)a_5(\xi,\lambda)-a_5(\eta,\lambda)a_4(\xi,\lambda)-a_5(\xi,\eta)=0.\end{aligned}$$

Then up to solution transformations, the basic solutions for this case are

$$\left\{\begin{aligned} a_1(\xi,\eta)&=\frac{f_1(\xi)}{f_1(\eta)},\\ a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\ a_4(\xi,\eta)&=\frac{f_1(\eta)}{f_1(\xi)},\\ a_5(\xi,\eta)&=\frac{f_5(\xi)}{f_1(\eta)}-\frac{f_5(\eta)}{f_1(\xi)},\\ a_6(\xi,\eta)&=0,\end{aligned}\right. \tag{3.27}$$

and

$$\left\{\begin{aligned} a_1(\xi,\eta)&=\frac{f_1(\xi)}{f_1(\eta)},\\ a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\ a_4(\xi,\eta)&=\frac{f_4(\xi)}{f_4(\eta)},\\ a_5(\xi,\eta)&=\frac{f_4(\eta)(1-f_1(\xi)f_4(\xi))}{Cf_1(\xi)}-\frac{f_4(\xi)(1-f_1(\eta)f_4(\eta))}{Cf_1(\eta)},\\ a_6(\xi,\eta)&=0,\end{aligned}\right. \tag{3.28}$$

where f_1, f_4, f_5 are arbitrary functions, C is non-zero constant. And solution (3.27) satisfied Free-Fermion condition (1.3).

Subcase 3.3 If we have $a_5(\xi,\eta)\not\equiv 0$ and $a_6(\xi,\eta)\not\equiv 0$ additionally, there is no degenerate solution for this case. Otherwise, let $a_4\equiv 0$ for example, then from equations (1.4c) one can see at least one of $a_2(\xi,\eta), a_3(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta)$ must be zero, that gives the contradiction, i.e. there are only non-degenerate solutions for this case.

In fact, from above discussion, we give all the six-vertex type degenerate solution only with color parameters up to solution transformations A,B,C,D,E.

Remark 2. In both references [8,9], a solution of equations (2) is mentioned and can be expressed in the notations of this paper as:

$$\check{R} = \begin{pmatrix} p(\xi) & 0 & 0 & 0 \\ 0 & \dfrac{p^2(\xi)-1}{p(\eta)} & 1 & 0 \\ 0 & \dfrac{p(\xi)}{p(\eta)} & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{p(\eta)} \end{pmatrix}, \tag{3.29}$$

If we let

$$g(\xi,\eta) = \frac{p^2(\eta)+1}{2(p^2(\eta)-1)}, \quad f_5(\xi) = \frac{1}{p^2(\xi)-1}, \quad f_6(\xi) = \frac{p(\xi)}{C(p^2(\xi)-1)},$$

in the degenerate basic solution (3.24) discussed in this paper, the above solution (3.29) is obtained.

§4. Solutions of colored Yang-Baxter equation

Based on the results of first three sections, we are going to give all non-degenerate six-vertex type solutions of colored Yang-Baxter equation(1.2). In section 2, we have already obtained the general form of solutions for the Yang-Baxter equation (1.2) in (2.21a) and (2.21b), $a_2(u,\xi,\eta) = a_3(u,\xi,\eta) = 1$. And in section 3, we present $A_i(\xi,\eta)$ in (2.21a) and (2.21b) as solutions of Y-B equation only with colored parameters (3.1). So in order to construct a non-degenerate six-vertex type solutions of colored Yang-Baxter equation, what we remain to do is to find $B_i(x,y)(i=1,4,5,6)$.

Based on the discussion in section 2, from (2.9) and (2.18), we have the expressions of $a_i(u,\xi,\eta)$ $(i=1,2,\cdots,6)$ as

$$\begin{aligned} a_1(u,\xi,\eta) = & \; a_1(u+v,\xi,\lambda)\left(a_4(v,\eta,\lambda) + \frac{c_1(\lambda)+c_4(\lambda)}{c_5(\lambda)}a_5(v,\eta,\lambda)\right) \\ & + a_5(v,\eta,\lambda)a_6(u+v,\xi,\lambda), \\ a_4(u,\xi,\eta) = & \; a_4(u+v,\xi,\lambda)\left(a_1(v,\eta,\lambda) + \frac{c_1(\lambda)+c_4(\lambda)}{c_6(\lambda)}a_6(v,\eta,\lambda)\right) \\ & + a_5(u+v,\xi,\lambda)a_6(v,\eta,\lambda), \\ a_5(u,\xi,\eta) = & \; a_4(v,\eta,\lambda)a_5(u+v,\xi,\lambda) - a_4(u+v,\xi,\lambda)a_5(v,\eta,\lambda), \\ a_6(u,\xi,\eta) = & \; a_1(v,\eta,\lambda)a_6(u+v,\xi,\lambda) - a_1(u+v,\xi,\lambda)a_6(v,\eta,\lambda), \end{aligned} \tag{4.1}$$

Since $A_i(\xi,\eta) = a_i(0,\xi,\eta)$ $(i=1,4,5,6)$, from the initial values of $a_i(u,\xi,\eta)$, we have

$$A_1(\xi,\xi) = A_4(\xi,\xi) = 1, \qquad A_5(\xi,\xi) = A_6(\xi,\xi) = 0.$$

Of all the solutions, only solutions (3.8), (3.15), (3.17), (3.26), (3.27) and (3.28) satisfy these conditions. And $A_i(\xi,\eta)$ should also satisfy (4.1) with $u=v=0$ and $a_i(0,\xi,\eta)$ replaced by $A_i(\xi,\eta)$ $(i=1,4,5,6)$. Therefore, only four basic solutions (3.8), (3.15), (3.17) and (3.27) are suitable to be chose as $A_i(\xi,\eta)$ $(i=1,4,5,6)$.

Next, we are going to give $B_i(\xi,\eta)$ $(i=1,4,5,6)$. Firstly, from the differential equations (2.15) and the general form of solutions (2.21a), (2.21b), $B_i(\xi,\eta)$ $(i=1,4,5,6)$ can be expressed as

$$\begin{aligned} B_1(\xi,\eta) &= c_1(\eta)A_1(\xi,\eta) - c_5(\eta)A_6(\xi,\eta), \\ B_4(\xi,\eta) &= c_4(\eta)A_4(\xi,\eta) - c_6(\eta)A_5(\xi,\eta), \\ B_5(\xi,\eta) &= -c_4(\eta)A_5(\xi,\eta) + c_5(\eta)A_4(\xi,\eta), \\ B_6(\xi,\eta) &= -c_1(\eta)A_6(\xi,\eta) + c_6(\eta)A_1(\xi,\eta). \end{aligned} \tag{4.2}$$

Secondly, by letting $v=0$ in (4.1) and from the general form of solutions, we have

$$\begin{aligned} B_1(\xi,\eta) &= B_1(\xi,\lambda)\left(A_4(\eta,\lambda) + \frac{c_1(\lambda)+c_4(\lambda)}{c_5(\lambda)}A_5(\eta,\lambda)\right) + B_6(\xi,\lambda)A_5(\eta,\lambda), \\ B_4(\xi,\eta) &= B_4(\xi,\lambda)\left(A_1(\eta,\lambda) + \frac{c_1(\lambda)+c_4(\lambda)}{c_6(\lambda)}A_6(\eta,\lambda)\right) + B_5(\xi,\lambda)A_6(\eta,\lambda), \\ B_5(\xi,\eta) &= B_5(\xi,\lambda)A_4(\eta,\lambda) - B_4(\xi,\lambda)A_5(\eta,\lambda), \\ B_6(\xi,\eta) &= B_6(\xi,\lambda)A_1(\eta,\lambda) - B_1(\xi,\lambda)A_6(\eta,\lambda), \end{aligned} \tag{4.3}$$

We will first use (4.2) to obtain $B_i(\xi,0)$ from $A_i(\xi,0)$, then use (4.3) to obtain $B_i(\xi,\eta)$ from $A_i(\xi,0)$ and $B_i(\xi,0)$ $(i=1,4,5,6)$. For simplicity, we denote c_i for $c_i(0)$ $(i=1,4,5,6)$. All the solutions can be classified into two type. Solutions satisfying Free-Fermion condition (1.3) are called the Free-Fermion type solutions, and others are called the Baxter type solutions.

Free-Fermion type solutions

From (3.8) and (3.27), we can obtain four basic solutions satisfying (1.3). Since $A_i(\xi,\eta)$ of (3.8) and (3.27) satisfy the Free-Fermion condition, $c_1 = -c_4$. By solution transformation C, we can assume $c_5 = c_6$. If the solutions are in the form as (2.21a), i.e.

$$\begin{aligned} a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1, \\ a_i(u,\xi,\eta) &= A_i(\xi,\eta)\cosh(u) + B_i(\xi,\eta)\sinh(u), \quad i=1,4,5,6, \end{aligned} \tag{4.4}$$

$c_1^2 - c_5^2 = 1$, we denote

$$c_1 = \cosh(C), \quad c_4 = -\cosh(C), \quad c_5 = c_6 = \sinh(C). \tag{4.5}$$

If the solutions are in the form as (2.21b), i.e.

$$a_2(u,\xi,\eta) = a_3(u,\xi,\eta) = 1, \quad a_i(u,\xi,\eta) = A_i(\xi,\eta) + B_i(\xi,\eta)u, i=1,4,5,6, \tag{4.6}$$

$c_1^2 - c_5^2 = 0$, by solution transformations we can assume

$$c_1 = c_5 = c_6 = 1, \quad c_4 = -1. \tag{4.7}$$

For solution (3.8),

$$\begin{array}{ll} A_1(\xi,0) = (F(\xi)+1)G(\xi), & A_4(\xi,0) = (F(\xi)+1)/G(\xi), \\ A_5(\xi,0) = F(\xi)H(\xi), & A_6(\xi,0) = F(\xi)/H(\xi), \end{array} \tag{4.8}$$

and

$$\begin{aligned} A_1(\xi,\eta) &= (F(\xi)+1)(-F(\eta)+1)\frac{G(\xi)}{G(\eta)} + F(\xi)F(\eta)\frac{H(\eta)}{H(\xi)}, \\ A_4(\xi,\eta) &= (F(\eta)+1)(-F(\xi)+1)\frac{G(\eta)}{G(\xi)} + F(\xi)F(\eta)\frac{H(\xi)}{H(\eta)}, \\ A_5(\xi,\eta) &= F(\xi)(-F(\eta)+1)\frac{H(\xi)}{G(\eta)} - F(\eta)(-F(\xi)+1)\frac{H(\eta)}{G(\xi)}, \\ A_6(\xi,\eta) &= F(\xi)(F(\eta)+1)\frac{G(\eta)}{H(\xi)} - F(\eta)(F(\xi)+1)\frac{G(\xi)}{H(\eta)}. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we have two basic solutions up to solution transformations. One is in the form (4.4) with $A_i(\xi,\eta)$ $(i = 1,4,5,6)$ expressed by (4.9) and $B_i(\xi,\eta)$ $(i = 1,4,5,6)$ by

$$\begin{aligned} B_1(\xi,\eta) = {} & \left((1+F(\xi))(1-F(\eta))\frac{G(\xi)}{G(\eta)} + F(\xi)F(\eta)\frac{H(\eta)}{H(\xi)}\right)\cosh(C) \\ & + \left(-F(\xi)(1-F(\eta))\frac{1}{G(\eta)H(\xi)} + (1+F(\xi))F(\eta)G(\xi)H(\eta)\right)\sinh(C), \\ B_4(\xi,\eta) = {} & \left(-(1-F(\xi))(1+F(\eta))\frac{G(\eta)}{G(\xi)} + F(\xi)F(\eta)\frac{H(\xi)}{H(\eta)}\right)\cosh(C) \\ & + \left(-F(\xi)(1+F(\eta))G(\eta)H(\xi) + (1-F(\xi))F(\eta)\frac{1}{G(\xi)H(\eta)}\right)\sinh(C), \\ B_5(\xi,\eta) = {} & \left(F(\xi)(1-F(\eta))\frac{H(\xi)}{G(\eta)} - F(\eta)(1-F(\xi))\frac{H(\eta)}{G(\xi)}\right)\cosh(C) \\ & + \left((1-F(\xi))(1-F(\eta))\frac{1}{G(\xi)G(\eta)} + F(\xi)F(\eta)H(\xi)H(\eta)\right)\sinh(C), \\ B_6(\xi,\eta) = {} & \left(-F(\xi)(1+F(\eta))\frac{G(\eta)}{H(\xi)} - F(\eta)(1+F(\xi))\frac{G(\xi)}{H(\eta)}\right)\cosh(C) \\ & + \left(\frac{F(\xi)F(\eta)}{H(\xi)H(\eta)} + (1+F(\xi))(1+F(\eta))G(\xi)G(\eta)\right)\sinh(C). \end{aligned} \tag{4.10}$$

The other is in the form (4.6) with $A_i(\xi,\eta)$ $(i = 1,4,5,6)$ expressed by (4.9) and

$B_i(\xi,\eta)$ $(i = 1,4,5,6)$ by

$$\begin{aligned} B_1(\xi,\eta) = {} & (1+F(\xi))(1-F(\eta))\frac{G(\xi)}{G(\eta)} + F(\xi)F(\eta)\frac{H(\eta)}{H(\xi)} \\ & + - F(\xi)(1-F(\eta))\frac{1}{G(\eta)H(\xi)} + (1+F(\xi))F(\eta)G(\xi)H(\eta), \\ B_4(\xi,\eta) = {} & -(1-F(\xi))(1+F(\eta))\frac{G(\eta)}{G(\xi)} + F(\xi)F(\eta)\frac{H(\xi)}{H(\eta)} \\ & + - F(\xi)(1+F(\eta))G(\eta)H(\xi) + (1-F(\xi))F(\eta)\frac{1}{G(\xi)H(\eta)}, \\ B_5(\xi,\eta) = {} & F(\xi)(1-F(\eta))\frac{H(\xi)}{G(\eta)} - F(\eta)(1-F(\xi))\frac{H(\eta}{G(\xi)} \\ & +(1-F(\xi))(1-F(\eta))\frac{1}{G(\xi)G(\eta)} + F(\xi)F(\eta)H(\xi)H(\eta), \\ B_6(\xi,\eta) = {} & -F(\xi)(1+F(\eta))\frac{G(\eta)}{H(\xi)} - (1+F(\xi))F(\eta)\frac{G(\xi)}{H(\eta)} \\ + {} & \frac{F(\xi)F(\eta)}{H(\xi)H(\eta)} + (1+F(\xi))(1+F(\eta))G(\xi)G(\eta). \end{aligned} \tag{4.11}$$

where $F(\xi), G(\xi)$ and $H(\xi)$ are arbitrary functions with $F(0) = 0, G(0) = 1$.

For solution (3.27)

$$A_1(\xi,0) = F(\xi), \quad A_4(\xi,0) = \frac{1}{F(\xi)}, \quad A_5(\xi,0) = G(\xi), \quad A_6(\xi,0) = 0, \tag{4.12}$$

and

$$\left\{ \begin{aligned} A_1(\xi,\eta) &= \frac{F(\xi)}{F(\eta)}, \\ A_4(\xi,\eta) &= \frac{F(\eta)}{F(\xi)}, \\ A_5(\xi,\eta) &= \frac{G(\xi)}{F(\eta)} - \frac{G(\eta)}{F(\xi)}, \\ A_6(\xi,\eta) &= 0, \end{aligned} \right. \tag{4.13}$$

From (4.12) and (4.13), we have two basic solutions up to solution transformations. One is in the form (4.4) with $A_i(\xi,\eta)$ $(i = 1,4,5,6)$ expressed by (4.13) and $B_i(\xi,\eta)$ $(i = 1,4,5,6)$ by

$$\begin{aligned} B_1(\xi,\eta) &= \frac{F(\xi)}{F(\eta)}\cosh(C) + F(\xi)G(\eta)\sinh(C), \\ B_4(\xi,\eta) &= -\frac{F(\eta)}{F(\xi)}\cosh(C) - F(\eta)G(\xi)\sinh(C), \\ B_5(\xi,\eta) &= \left(\frac{G(\xi)}{F(\eta)} + \frac{G(\eta)}{F(\xi)}\right)\cosh(C) + \left(\frac{1}{F(\xi)F(\eta)} + G(\xi)G(\eta)\right)\sinh(C), \\ B_6(\xi,\eta) &= F(\xi)F(\eta)\sinh(C). \end{aligned} \tag{4.14}$$

The other is in the form (4.6) with $A_i(\xi,\eta)$ $(i = 1,4,5,6)$ expressed by (4.13) and $B_i(\xi,\eta)$ $(i = 1,4,5,6)$ by

$$\begin{aligned} B_1(\xi,\eta) &= \frac{F(\xi)}{F(\eta)} + F(\xi)G(\eta), \\ B_4(\xi,\eta) &= -\frac{F(\eta)}{F(\xi)} - F(\eta)G(\xi), \\ B_5(\xi,\eta) &= \frac{G(\xi)}{F(\eta)} + \frac{G(\eta)}{F(\xi)} + \frac{1}{F(\xi)F(\eta)} + G(\xi)G(\eta), \\ B_6(\xi,\eta) &= F(\xi)F(\eta). \end{aligned} \tag{4.15}$$

where $F(\xi)$ and $G(\xi)$ are arbitrary functions with $F(0) = 1, G(0) = 0$.

The above four basic solutions satisfy the Free-Fermion condition (1.3).

Baxter type solutions

From (3.15) and (3.17), we can obtain two basic solutions. Since $A_i(\xi,\eta)$ of (3.15) and (3.17) do not satisfy the Free-Fermion condition and $c_1^2 = c_4^2$, $c_1 + c_4 \neq 0$ and therefore $c_1 = c_4$. By solution transformation C, we can assume $c_5 = c_6$.

For solution (3.15), $(c_1+c_4)/c_5 = 2\cos(C) = 2$, $c_1^2 - c_5^2 = 0$, therefore the six-vertex type solution corresponding to (3.15) must be in the form of (4.4). And

$$\begin{aligned} A_1(\xi,0) &= \frac{F(\xi)+1}{G(\xi)}, & A_4(\xi,0) &= (F(\xi)+1)G(\xi), \\ A_5(\xi,0) &= F(\xi)G(\xi), & A_6(\xi,0) &= \frac{F(\xi)}{G(\xi)}, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} A_1(\xi,\eta) &= (F(\xi) - F(\eta) + 1)\frac{G(\eta)}{G(\xi)}, \\ A_4(\xi,\eta) &= (F(\xi) - F(\eta) + 1)\frac{G(\xi)}{G(\eta)}, \\ A_5(\xi,\eta) &= (F(\xi) - F(\eta))G(\xi)G(\eta), \\ A_6(\xi,\eta) &= (F(\xi) - F(\eta))\frac{1}{G(\xi)G(\eta)}. \end{aligned} \tag{4.17}$$

From (4.16) and (4.17), we have one basic solutions up to solution transformations as

$$\begin{aligned} a_1(u,\xi,\eta) &= (1 + u + F(\xi) - F(\eta))\frac{G(\eta)}{G(\xi)}, \\ a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1, \\ a_4(u,\xi,\eta) &= (1 + u + F(\xi) - F(\eta))\frac{G(\xi)}{G(\eta)}, \\ a_5(u,\xi,\eta) &= (u + F(\xi) - F(\eta))G(\xi)G(\eta), \\ a_6(u,\xi,\eta) &= (u + F(\xi) - F(\eta))\frac{1}{G(\xi)G(\eta)}, \end{aligned} \tag{4.18}$$

where $F(\xi)$ and $G(\xi)$ are arbitrary functions with $F(0) = 0, G(0) = 1$.

For solution (3.17), $(c_1 + c_4)/c_5 = 2\cos(C) \neq \pm 2$, $c_1^2 - c_5^2 \neq 0$, therefore the six-vertex type solution corresponding to (3.17) must be in the form of (4.6). And

$$\begin{aligned} A_1(\xi,0) &= \frac{\sin(F(\xi)+C)}{\sin(C)G(\xi)}, & A_4(\xi,0) &= \frac{\sin(F(\xi)+C)G(\xi)}{\sin(C)}, \\ A_5(\xi,0) &= \frac{\sin(F(\xi))G(\xi)}{\sin(C)}, & A_6(\xi,0) &= \frac{\sin(F(\xi))}{\sin(C)G(\xi)}, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} A_1(\xi,\eta) &= \frac{\sin(F(\xi) - F(\eta) + C)}{\sin(C)}\frac{G(\eta)}{G(\xi)}, \\ A_4(\xi,\eta) &= \frac{\sin(F(\xi) - F(\eta) + C)}{\sin(C)}\frac{G(\xi)}{G(\eta)}, \\ A_5(\xi,\eta) &= \frac{\sin(F(\xi) - F(\eta))}{\sin(C)}G(\xi)G(\eta), \\ A_6(\xi,\eta) &= \frac{\sin(F(\xi) - F(\eta))}{\sin(C)}\frac{1}{G(\xi)G(\eta)}, \end{aligned} \tag{4.20}$$

where C satisfy $(c_1 + c_4)/c_5 = 2\cos(C) \neq \pm 2$.

From (4.19) and (4.20), we have one basic solutions up to solution transformations as

$$\begin{aligned} a_1(u,\xi,\eta) &= \frac{\sin(u + F(\xi) - F(\eta) + C)}{\sin(C)}\frac{G(\eta)}{G(\xi)}, \\ a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1, \\ a_4(u,\xi,\eta) &= \frac{\sin(u + F(\xi) - F(\eta) + C)}{\sin(C)}\frac{G(\xi)}{G(\eta)}, \\ a_5(u,\xi,\eta) &= \frac{\sin(u + F(\xi) - F(\eta))}{\sin(C)}G(\xi)G(\eta), \\ a_6(u,\xi,\eta) &= \frac{\sin(u + F(\xi) - F(\eta))}{\sin(C)}\frac{1}{G(\xi)G(\eta)} \end{aligned} \tag{4.21}$$

where $F(\xi), G(\xi)$ are arbitrary functions with $F(0) = 0, G(0) = 1$.

If we let $F(\xi) = \xi$, $G(\xi) = 1$, $v = \xi - \eta$ in (4.18) and (4.21), (4.18) and (4.21) become the solutions with two spectral parameters u and v. These solutions are essentially same as the solution in ice model. They are called Baxter type solutions.

Therefore, up to solution transformations A,B,C,D,E, any non-degenerate six-vertex type solution of colored YBE (1.2) is equivalent to one of the six sets of basic solutions: four free-fermion type solutions by (4.4), (4.6), (4.9), (4.10), (4.11), (4.13), (4.14) and (4.15) and two Baxter type solutions by (4.18) and (4.21).

Remark 3. In [8], a solution of colored Yang-Baxter equation is mentioned and can be written in the notation of this paper as

$$
\begin{aligned}
a_1(u,\xi,\eta) &= \cosh(\xi-\eta)\cos(\frac{\pi u}{2K}) + \sinh(\xi+\eta)\sin(\frac{\pi u}{2K}),\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1,\\
a_4(u,\xi,\eta) &= \cosh(\xi-\eta)\cos(\frac{\pi u}{2K}) - \sinh(\xi+\eta)\sin(\frac{\pi u}{2K}),\\
a_5(u,\xi,\eta) &= \sinh(\xi-\eta)\cos(\frac{\pi u}{2K}) + \cosh(\xi+\eta)\sin(\frac{\pi u}{2K}),\\
a_6(u,\xi,\eta) &= -\sinh(\xi-\eta)\cos(\frac{\pi u}{2K}) + \cosh(\xi+\eta)\sin(\frac{\pi u}{2K}),
\end{aligned} \tag{4.22}
$$

By taking solution transformation E to the coefficient of spectral parameter u and changing cos, sin to cosh, sinh, (4.22) is equivalent to

$$
\begin{aligned}
a_1(u,\xi,\eta) &= \cosh(\xi-\eta)\cosh(u) - i\sinh(\xi+\eta)\sinh(u),\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1,\\
a_4(u,\xi,\eta) &= \cosh(\xi-\eta)\cosh(u) + i\sinh(\xi+\eta)\sinh(u),\\
a_5(u,\xi,\eta) &= \sinh(\xi-\eta)\cosh(u) - i\cosh(\xi+\eta)\sinh(u),\\
a_6(u,\xi,\eta) &= -\sinh(\xi-\eta)\cos(\frac{\pi u}{2K}) - i\cosh(\xi+\eta)\sinh(u),
\end{aligned} \tag{4.23}
$$

By letting

$$
\begin{aligned}
&F(\xi) = i\sinh(\xi), \qquad G(\xi) = \frac{\cosh(\xi)}{1+i\sinh(\xi)} = \frac{1-i\sinh(\xi)}{\cosh(\xi)}, \qquad H(\xi) = -i,\\
&\cosh(C) = 0, \qquad \sinh(C) = -i,
\end{aligned}
$$

we can derive (4.23) from (4.4), (4.9) and (4.10).

Remark 4. In [11] Yang-Baxterization of colored $\check{R}$-matrix was discussed and some six-vertex type solutions were presented as examples. One of the solution can be expressed as follows in the notations of this paper with some correction to the misprints:

$$
\begin{aligned}
a_1(u,\xi,\eta) &= f(\xi,\eta)(q-q^{-1}e^u),\\
a_2(u,\xi,\eta) &= f(\xi,\eta)e^u w g^{-1}(\xi)g(\eta)\alpha(\xi)\alpha^{-1}(\eta),\\
a_3(u,\xi,\eta) &= f(\xi,\eta)w g(\xi)g^{-1}(\eta),\\
a_4(u,\xi,\eta) &= f(\xi,\eta)\alpha(\xi)\alpha^{-1}(\eta)(q-q^{-1}e^u),\\
a_5(u,\xi,\eta) &= f(\xi,\eta)\alpha(\xi)(1-e^u),\\
a_6(u,\xi,\eta) &= f(\xi,\eta)\alpha^{-1}(\eta)(1-e^u),
\end{aligned} \tag{4.24}
$$

where $w = q - q^{-1}$. By taking solution transformation D with $\mu = -1/2$ and $f(\xi) = g(\xi)\alpha^{-1/2}(\xi)$, solution transformation B with

$$
f(u,\xi,\eta) = f^{-1}(\xi,\eta)w^{-1}e^{-u/2}\alpha^{-1/2}(\xi)\alpha^{1/2}(\eta),
$$

and solution transfomration E with $\mu = -2$, $f(\xi) = \xi$, solution (4.24) is equivalent to

$$
\begin{aligned}
a_1(u,\xi,\eta) &= \frac{\sinh(u+C)}{\sinh(C)}\sqrt{\frac{\alpha(\eta)}{\alpha(\xi)}},\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1,\\
a_4(u,\xi,\eta) &= \frac{\sinh(u+C)}{\sinh(C)}\sqrt{\frac{\alpha(\xi)}{\alpha(\eta)}},\\
a_5(u,\xi,\eta) &= \frac{\sinh(u)}{\sinh(C)}\sqrt{\alpha(\xi)\alpha(\eta)},\\
a_6(u,\xi,\eta) &= \frac{\sinh(u)}{\sinh(C)}\sqrt{\frac{1}{\alpha(\xi)\alpha(\eta)}},
\end{aligned} \tag{4.25}
$$

where $q = e^C$. By letting

$$
F(\xi) = 1, \quad G(\xi) = \sqrt{\alpha(\xi)},
$$

we can derive (4.25) from (4.21). Similarly, other six-vertex type solutions in [11] can also be derived from six basic solutions.

References

[1] Yang, C.N., Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett., 19(1967)1312.

[2] Baxter, R.J., Exactly solved models in statistical mechanics, Academic Press, London, 1982.

[3] Lieb, E.H., Phys. Rev., **162** (1967)162-172.

[4] Sutherland, B., Phys. Rev. Lett. **18** (1967)103-104.

[5] P.P.Kulish and E.K.Sklyanin, Functional Anol. Appl. 16(1982) 1596.

[6] C. Fan and F.Y. Wu, Phys. Rev. B2(1970)723.

[7] Murakami, J., A state model for the multi-variable Alexander polynomial 1990, preprint, Osaka University;

[8] Murakami, J., The free-Fermion model in presence of field related to the quantum group $U_q(\widehat{sl_2})$ of affine type and the multi-variable Alexander polynomial of links, preprint RIMS-822, October 1991.

[9] R. Cuerno, C. Gómez, E. López and G. Sierra, The hidden quantum group of the 8-vertex free Fermion Model: q-Clifford algebras, preprint IMAFF-93-2 (1993), to appear in Phys. Lett. B.

[10] B. U. Felderhof, Physica 66(1973)279,723.

[11] M.L. Ge and K. Xue, Trigonometric Yang-Baxterization of coloured $\check{R}$-matrix, J. of Phys. A: Math. & Gen., Vol.26, No.2, 1993.

[12] K. Sogo, M. Uchinami, Y. Akutsu and M. Wadati, Prog. Theor. Phys., 68(1982)508.