EE

Internal Report
DESY M-94-05
August 1994

Spin Tune Shift due to Closed Orbit Distortions

by
D. P. Barber, K. Heinemann, G. Ripken

Spin Tune Shift due to Closed Orbit Distortions

D.P. Barber, K. Heinemann, G. Ripken

August 1994

Abstract

We present a perturbative formalism, up to second order, for calculating spin tune shift on the closed orbit of a storage ring due to misalignment. This is based on the familiar concepts of the SLIM[1] formalism and can treat rings of arbitrary geometry. The final formulae agree with those already given using another approach by Yokoya [2] and are valid for arbitrary particle velocity i.e. above and below transition.

Contents

1 Introduction

In a perfectly planar storage ring with no longitudinal fields, the frequency of spin precesssion, Q_{spin}, around the vertical dipole field along the design orbit is $a\gamma$ precessions per turn [3]. Q_{spin} is called the spin tune. The spin tune can be measured by resonantly depolarizing the beam with a radial sinusoidal magnetic field. In the case of the perfect planar ring it is then possible to determine $a\gamma$ and hence the beam energy with very high precision [4, 5]. However, in the presence of misalignments, the closed orbit is distorted. In the vertical plane the distortion is due to horizontal quadrupole fields, sextupole fields and correction fields. In this case the spin tune is no longer exactly given by $a\gamma$ and the precision on the measurement of the energy is determined by the spin tune shift δQ_{spin} caused by distortions. An estimate of δQ_{spin} is therefore an essential ingredient when measuring the beam energy by resonant depolarization.

In proton storage rings operating in the TeV range, the spin tune shift can become very large, and although the energy might have been chosen so that Q_{spin} is far from an integer, δQ_{spin} can be so big that one is in reality sitting close to integer spin tune. In that case the equilibrium polarization axis $\vec{n}_0$ could be tilted away from the design direction by a large amount. Since the imperfections are usually insufficiently well known, the direction of the equilibrium polarization would be unknown. Thus in this example we see again that we need to be able to estimate δQ_{spin} with sufficient precision. This is still the case if the ring contains solenoids or Siberian Snakes [2, 6]: the spin tune of the otherwise perfectly aligned ring, Q_{spin} is no longer given by $a\gamma$ [7] but the spin precession rate in transverse fields is still proportional to the energy and at very high energy the shift δQ_{spin} could be large in this case too.

Methods for estimating δQ_{spin} have been presented on several occasions. For example Assmann and Koutchouk have treated the special case of LEP, by studying the trace of the one turn 3×3 spin transfer matrix on the closed orbit [8] and Yokoya has used a canonical perturbation theory to study the spin tune shift in very high energy proton storage rings with and without snakes [2].

Spin tune can also be calculated from the trace of the one turn spinor transfer matrix [7]. In this paper we use methods familiar from the SLIM formalism [1] and obtain the same result as Yokoya up to second order and for arbitrary storage ring geometry. The calculations are similar in spirit to those used for calculating energy level shifts in quantum mechanics.

2 The Equations of Spin Motion.

2.1 The Unperturbed Eigenvalue Spectrum.

Using the notation of Refs. [3, 9], the motion of a spin vector $\vec{\xi}$ on the design orbit is described by the Thomas-BMT equation [10, 11]:

$$\begin{aligned}\frac{d}{ds}\vec{\xi} &= \vec{\Omega} \times \vec{\xi} \\ &= \underline{\Omega} \cdot \vec{\xi} \end{aligned} \qquad (2.1)$$

where we write the spin as

$$\vec{\xi} = \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix}$$

and where the precession vector $\vec{\Omega}$

$$\vec{\Omega} = \begin{pmatrix} \Omega_s \\ \Omega_x \\ \Omega_z \end{pmatrix} \; ; \quad \underline{\Omega} = \begin{pmatrix} 0 & -\Omega_z & \Omega_x \\ \Omega_z & 0 & -\Omega_s \\ -\Omega_x & \Omega_s & 0 \end{pmatrix}$$

depends on the magnetic and electric fields, the velocity and the energy. We calculate in the machine coordinate system. In the presence of misalignments the closed orbit deviates from the design orbit and we write:

$$\begin{aligned} \frac{d}{ds}\vec{\xi} &= \left[\vec{\Omega} + \vec{\omega}\right] \times \vec{\xi} \\ &= \left[\underline{\Omega} + \underline{\omega}\right] \cdot \vec{\xi} \end{aligned} \tag{2.2}$$

where

$$\vec{\omega} = \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \; ; \quad \underline{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_x \\ \omega_z & 0 & -\omega_s \\ -\omega_x & \omega_s & 0 \end{pmatrix}$$

is the contribution to the precession vector due to closed orbit distortions and is assumed to be small compared with $\vec{\Omega}$. The detailed form can be found in Refs. [3, 9].

Denoting the 3x3 orthogonal rotation transfer matrix solving (2.1) by $M_0(s+L,s)$, the corresponding matrix in the presence of distortions is written as:

$$\underline{M}(s+L,s) = \underline{M}_0(s+L,s) + \delta\underline{M}(s+L,s) \; .$$

The spin tune is extracted from the eigenvalues of the one turn eigen problem. For a perfectly aligned machine of arbitrary geometry this takes the form (see Refs. [3, 9, 12]):

$$\underline{M}_0(s_0+L,s_0)\;\vec{v}_\mu(s_0) = \alpha_\mu \cdot \vec{v}_\mu(s_0) \; ; \qquad (\mu = 1,\; 2,\; 3) \tag{2.3}$$

with

$$\begin{aligned} \alpha_1 &= 1 \; ; \\ \alpha_2 &= e^{+i\,\cdot\,2\pi\,Q_{spin}} \; ; \\ \alpha_3 &= e^{-i\,\cdot\,2\pi\,Q_{spin}} \end{aligned} \tag{2.4}$$

and

$$\vec{v}_1(s_0) = \vec{n}_0(s_0) \; ; \tag{2.5a}$$

$$\vec{v}_2(s_0) = \frac{1}{\sqrt{2}} \cdot \left[\vec{m}_0(s_0) + i \cdot \vec{l}_0(s_0)\right] \; ; \tag{2.5b}$$

$$\vec{v}_3(s_0) = \frac{1}{\sqrt{2}} \cdot \left[\vec{m}_0(s_0) - i \cdot \vec{l}_0(s_0)\right] \; ; \tag{2.5c}$$

The $(\vec{n}_0,\ \vec{m}_0,\ \vec{l}_0)$ are real vectors and the spin tune is the real number Q_{spin}.

The vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$ form an orthonormal system:

$$|\vec{n}_0(s_0)| \;=\; |\vec{m}_0(s_0)| \;=\; |\vec{l}_0(s_0)| \;=\; 1\ ; \tag{2.6a}$$

$$\vec{n}_0(s_0) \;\perp\; \vec{m}_0(s_0) \;\perp\; \vec{l}_0(s_0)\ . \tag{2.6b}$$

Choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) \;=\; \vec{m}_0(s_0) \times \vec{l}_0(s_0) \tag{2.6c}$$

these vectors represent a right-handed coordinate system.

$\vec{n}_0$ is by definition periodic, having eigenvalue equal to 1. In one turn around the ring, $\vec{m}_0$ and $\vec{l}_0$ effectively rotate around $\vec{n}_0$ by the angle $2\pi Q_{spin}$.

The orthonormal system of vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_0(s, s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$:

$$\vec{n}_0(s) \;=\; \underline{M}_0(s, s_0)\ \vec{n}_0(s_0)\ ; \tag{2.7a}$$
$$\vec{m}_0(s) \;=\; \underline{M}_0(s, s_0)\ \vec{m}_0(s_0)\ ; \tag{2.7b}$$
$$\vec{l}_0(s) \;=\; \underline{M}_0(s, s_0)\ \vec{l}_0(s_0) \tag{2.7b}$$

whereby the orthonormality relations remain unchanged:

$$\vec{n}_0(s) \;=\; \vec{m}_0(s) \times \vec{l}_0(s) \tag{2.8a}$$
$$\vec{m}_0(s) \;\perp\; \vec{l}_0(s)\ ; \tag{2.8b}$$
$$|\vec{n}_0(s)| \;=\; |\vec{m}_0(s)| \;=\; |\vec{l}_0(s)| \;=\; 1\ . \tag{2.8c}$$

The eigenvectors $\vec{v}_\mu(s)$

$$\vec{v}_\mu(s) \;=\; \underline{M}_0(s, s_0)\ \vec{v}_\mu(s_0)$$

obey the orthonormality relations:

$$\vec{v}_\mu^{\,+}(s) \cdot \vec{v}_\nu(s) \;=\; \delta_{\mu\nu}\ . \tag{2.9}$$

In a perfectly aligned planar machine $\vec{n}_0$ is vertical and $\vec{m}_0$, $\vec{l}_0$ are horizontal.

Remark:

For

$$Q_{spin} \;=\; \frac{1}{2}$$

we obtain from (2.4):

$$\alpha_2 \;=\; \alpha_3 \;=\; -1\ ,$$

i.e. the eigenvalue spectrum is degenerate. Then the eigenvectors $\vec{v}_2$ and $\vec{v}_3$ become real since the eigenvalues α_2 and α_3 are real. This means that the vector $\vec{l}_0$ in (2.5b, c) vanishes and $\vec{v}_2$ and $\vec{v}_3$ are arbitrary vectors perpendicular to $\vec{n}_0$.

Thus as is usually the case with degeneracy, the eigenvectors are no longer orthogonal and eqn. (2.9) is no longer valid. However, given $\vec{n}_0$ and an arbitrary unit vector perpendicular to $\vec{n}_0$ which we will call $\vec{m}_0$ we can always construct an $\vec{l}_0$ as $\vec{n}_0(s_0) \times \vec{m}_0(s_0)$. We then have a perfectly valid orthonormal dreibein even in the case of degeneracy. This is in any case the procedure to follow if one wants to construct a dreibein at s_0 which varies smoothly with energy and which is not susceptible to the arbitrary rotations around $\vec{n}_0$, (i.e. arbitrary phase factors) which subroutines in computer calculations introduce.

2.2 The Perturbed Part of the Revolution Matrix

In order to determine the perturbation $\delta \underline{M}(s+L, s)$ of the revolution matrix due to distortions we first note that according to eqn. (2.2) the transfer matrix

$$\underline{M}_0(s, s_0) + \delta \underline{M}(s, s_0)$$

obeys the equation:

$$\frac{d}{ds}\left[\underline{M}_0(s, s_0) + \delta \underline{M}(s, s_0)\right] = \left[\underline{\Omega} + \underline{\omega}\right] \cdot \left[\underline{M}_0(s, s_0) + \delta \underline{M}(s, s_0)\right] ; \tag{2.10a}$$

$$\underline{M}_0(s_0, s_0) + \delta \underline{M}(s_0, s_0) = \underline{1} . \tag{2.10b}$$

Furthermore we write:

$$\delta \underline{M} = \delta \underline{M}^{(1)} + \delta \underline{M}^{(2)} + \delta \underline{M}^{(3)} + \cdots , \tag{2.11}$$

whereby $\delta \underline{M}^{(\nu)}$ denotes the ν^{th} order in $\vec{\omega}$ of $\delta \underline{M}$.

a) Calculation of $\delta \underline{M}^{(1)}$.

Taking into account the corresponding equations for the unperturbed transfer matrix:

$$\frac{d}{ds}\underline{M}_0(s, s_0) = \underline{\Omega} \cdot \underline{M}_0(s, s_0) ;$$

$$\underline{M}_0(s_0, s_0) = \underline{1}$$

we obtain from (2.10) the differential equation for $\delta \underline{M}^{(1)}(s, s_0)$ in the form:

$$\frac{d}{ds}\delta \underline{M}^{(1)}(s, s_0) = \underline{\Omega} \cdot \delta \underline{M}^{(1)}(s, s_0) + \underline{\omega} \cdot \underline{M}_0(s, s_0)$$

with the initial condition:

$$\delta \underline{M}^{(1)}(s_0, s_0) = \underline{0} .$$

The solution of this equation (and thus the first order solution of eqn. (2.2)) reads as:

$$\begin{aligned} \delta \underline{M}^{(1)}(s,s_0) &= \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0(s,\tilde{s}) \cdot \underline{\omega}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0) \\ &= \underline{M}_0(s,s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \underline{\omega}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0) \ . \end{aligned} \tag{2.12}$$

The perturbative part $\delta \underline{M}^{(1)}(s_0+L,s_0)$ of the one turn matrix is therefore:

$$\begin{aligned} \delta \underline{M}^{(1)}(s_0+L,s_0) &= \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}_0(s_0+L,\tilde{s}) \cdot \underline{\omega}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0) \\ &= \underline{M}_0(s_0+L,s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \underline{\omega}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0) \ . \end{aligned} \tag{2.13a}$$

In general we have:

$$\delta \underline{M}^{(1)}(s+L,s) = \underline{M}_0(s+L,s) \cdot \int_{s}^{s+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s) \cdot \underline{\omega}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s) \ . \tag{2.13b}$$

b) Calculation of $\delta \underline{M}^{(2)}$.

The differential equation for $\delta \underline{M}^{(2)}(s,s_0)$ reads as:

$$\frac{d}{ds}\, \delta \underline{M}^{(2)}(s,s_0) = \underline{\Omega} \cdot \delta \underline{M}^{(2)}(s,s_0) + \underline{\omega} \cdot \delta \underline{M}^{(1)}(s,s_0)$$

with the initial condition:

$$\delta \underline{M}^{(2)}(s_0,s_0) = \underline{0} \ .$$

The solution of this equation (and thus the second order solution of eqn. (2.2)) is given by:

$$\delta \underline{M}^{(2}(s,s_0) = \underline{M}_0(s,s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \underline{\omega}(\tilde{s}) \cdot \delta \underline{M}^{(1)}(\tilde{s},s_0) \ .$$

The perturbative part $\delta \underline{M}^{(2)}(s_0+L,s_0)$ of the revolution matrix is therefore:

$$\delta \underline{M}^{(2)}(s_0+L,s_0) = \underline{M}_0(s_0+L,s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \underline{\omega}(\tilde{s}) \cdot \delta \underline{M}^{(1)}(\tilde{s},s_0) \tag{2.14a}$$

and for $\delta \underline{M}^{(2)}(s+L,s)$ one may thus write:

$$\delta \underline{M}^{(2)}(s+L,s) = \underline{M}_0(s+L,s) \cdot \int_{s}^{s+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s) \cdot \underline{\omega}(\tilde{s}) \cdot \delta \underline{M}^{(1)}(\tilde{s},s) \tag{2.14b}$$

with $\delta \underline{M}^{(1)}$ given by eqn. (2.12).

Remark:

In general we can write for $n \geq 0$:

$$\begin{aligned} \frac{d}{ds}\, \delta \underline{M}^{(n+1)}(s,s_0) &= \underline{\Omega} \cdot \delta \underline{M}^{(n+1)}(s,s_0) + \underline{\omega} \cdot \delta \underline{M}^{(n)}(s,s_0) \ ; \\ \delta \underline{M}^{(n+1)}(s_0,s_0) &= \underline{0} \end{aligned}$$

with the solution:

$$\delta \underline{M}^{(n+1}(s,s_0) = \underline{M}_0(s,s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \underline{\omega}(\tilde{s}) \cdot \delta \underline{M}^{(n)}(\tilde{s},s_0) \ .$$

In this paper we will only consider $\delta \underline{M}^{(1)}$ and $\delta \underline{M}^{(2)}$.

3 Perturbation Theory for the Spin Tune Shift

Since the sum of the eigenvalues of a diagonalisable matrix is given by its trace we have:

$$Sp\,[\delta \underline{M}(s_0 + L, s_0)] \;=\; \delta\alpha_1 + \delta\alpha_2 + \delta\alpha_3 \tag{3.1}$$

with [1]

$$\delta\alpha_1 \;=\; 0\,; \tag{3.2a}$$

$$\begin{aligned}\delta\alpha_2 &= e^{+i\,\cdot\,2\pi\,[Q_{spin} + \delta Q_{spin}]} - e^{+i\,\cdot\,2\pi\,Q_{spin}} \\ &= \alpha_2 \cdot \left[e^{i\,\cdot\,2\pi\,\delta Q_{spin}} - 1\right]; \end{aligned}\tag{3.2b}$$

$$\begin{aligned}\delta\alpha_3 &= e^{-i\,\cdot\,2\pi\,[Q_{spin} + \delta Q_{spin}]} - e^{-i\,\cdot\,2\pi\,Q_{spin}} \\ &= \delta\alpha_2^* \,. \end{aligned}\tag{3.2c}$$

In the spirit of the series expansion (2.11) we write:

$$\delta Q_{spin} \;=\; \delta Q_{spin}^{(1)} + \delta Q_{spin}^{(2)} + \cdots \,. \tag{3.3}$$

From (3.2b) and (3.3) we then obtain:

$$\delta\alpha_2 \;=\; \alpha_2 \cdot \left[e^{i\,\cdot\,2\pi\,\left(\delta Q_{spin}^{(1)} + \delta Q_{spin}^{(2)} + \cdots\right)} - 1\right]$$

with

$$\begin{aligned} e^{i\,\cdot\,2\pi\,\delta Q_{spin}^{(1)}} &= 1 + 2\pi i \cdot \delta Q_{spin}^{(1)} + \frac{1}{2}\cdot\left(2\pi i \cdot \delta Q_{spin}^{(1)}\right)^2 + \cdots\,; \\ e^{i\,\cdot\,2\pi\,\delta Q_{spin}^{(2)}} &= 1 + 2\pi i \cdot \delta Q_{spin}^{(2)} + \cdots\,. \end{aligned}$$

Thus:

$$\begin{aligned} e^{i\,\cdot\,2\pi\,\left[\delta Q_{spin}^{(1)} + \delta Q_{spin}^{(2)}\right]} &= \left[1 + 2\pi i \cdot \delta Q_{spin}^{(1)} + \frac{1}{2}\cdot\left(2\pi i \cdot \delta Q_{spin}^{(1)}\right)^2 + \cdots\right] \\ &\qquad \times \left[1 + 2\pi i \cdot \delta Q_{spin}^{(2)} + \cdots\right] \\ &= 1 + 2\pi i \cdot \delta Q_{spin}^{(1)} + 2\pi i \cdot \delta Q_{spin}^{(2)} - 2\pi^2 \cdot \left(\delta Q_{spin}^{(1)}\right)^2 + \cdots \end{aligned}$$

and

$$\delta\alpha_2 \;=\; \delta\alpha_2^{(1)} + \delta\alpha_2^{(2)} + \cdots$$

with

$$\delta\alpha_2^{(1)} \;=\; \alpha_2 \cdot 2\pi i \cdot \delta Q_{spin}^{(1)}\,; \tag{3.4a}$$

$$\delta\alpha_2^{(2)} \;=\; \alpha_2 \cdot \left[2\pi i \cdot \delta Q_{spin}^{(2)} - 2\pi^2 \cdot \left(\delta Q_{spin}^{(1)}\right)^2\right]\,. \tag{3.4b}$$

[1] $\underline{M}$ is orthogonal, so it still has at least one unit eigenvalue [12].

Equations (2.11), (3.1) and (3.4) finally lead to:

$$\begin{aligned} Sp\,[\delta\underline{M}^{(1)}(s_0+L,s_0)] &= \delta\alpha_2^{(1)} + \left[\delta\alpha_2^{(1)}\right]^* \\ &= 2\pi i\cdot\delta Q_{spin}^{(1)}\cdot(\alpha_2-\alpha_3)\ ; \end{aligned}$$

$$\begin{aligned} Sp\,[\delta\underline{M}^{(2)}(s_0+L,s_0)] &= \delta\alpha_2^{(2)} + \left[\delta\alpha_2^{(2)}\right]^* \\ &= 2\pi i\cdot\delta Q_{spin}^{(2)}\cdot(\alpha_2-\alpha_3) - 2\pi^2\cdot\left(\delta Q_{spin}^{(1)}\right)^2\cdot(\alpha_2+\alpha_3) \end{aligned}$$

or:

$$\delta Q_{spin}^{(1)} = \frac{1}{2\pi i\cdot(\alpha_2-\alpha_3)}\cdot Sp\,[\delta\underline{M}^{(1)}(s_0+L,s_0)]\ ; \tag{3.5a}$$

$$\delta Q_{spin}^{(2)} = \frac{1}{2\pi i\cdot(\alpha_2-\alpha_3)}\cdot\left\{Sp\,[\delta\underline{M}^{(2)}(s_0+L,s_0)] + 2\pi^2\cdot\left(\delta Q_{spin}^{(1)}\right)^2\cdot(\alpha_2+\alpha_3)\right\}\ . \tag{3.5b}$$

We now write the orthogonality relation of the eigenvectors (2.9) in the form

$$\underline{V}^+\cdot\underline{V} = \underline{1}_{(3\times3)} \quad\Longrightarrow\quad \underline{V}\cdot\underline{V}^+ = \underline{1}_{(3\times3)}$$

by introducing the 3×3 matrix

$$\underline{V} = (\vec{v}_1,\,\vec{v}_2,\,\vec{v}_3) \quad\Longrightarrow\quad \underline{V}^+ = \begin{pmatrix}\vec{v}_1^{\,+}\\ \vec{v}_2^{\,+}\\ \vec{v}_3^{\,+}\end{pmatrix}$$

where the columns of $\underline{V}$ are the eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$. Recalling the relation for the trace of a matrix

$$Sp\,(\underline{A}\cdot\underline{B}) = Sp\,(\underline{B}\cdot\underline{A})$$

we may now write the trace of $Sp\,[\delta\underline{M}^{(n)}]$ appearing in (3.5a,b) as:

$$\begin{aligned} Sp\,[\delta\underline{M}^{(n)}] &= Sp\left\{(\vec{v}_1,\,\vec{v}_2,\,\vec{v}_3)\cdot\begin{pmatrix}\vec{v}_1^{\,+}\\ \vec{v}_2^{\,+}\\ \vec{v}_3^{\,+}\end{pmatrix}\cdot\delta\underline{M}^{(n)}\right\} \\ &= Sp\left\{\begin{pmatrix}\vec{v}_1^{\,+}\\ \vec{v}_2^{\,+}\\ \vec{v}_3^{\,+}\end{pmatrix}\cdot\delta\underline{M}^{(n)}\cdot(\vec{v}_1,\,\vec{v}_2,\,\vec{v}_3)\right\} \\ &= Sp\begin{pmatrix} \vec{v}_1^{\,+}\delta\underline{M}^{(n)}\vec{v}_1 & \vec{v}_1^{\,+}\delta\underline{M}^{(n)}\vec{v}_2 & \vec{v}_1^{\,+}\delta\underline{M}^{(n)}\vec{v}_3 \\ \vec{v}_2^{\,+}\delta\underline{M}^{(n)}\vec{v}_1 & \vec{v}_2^{\,+}\delta\underline{M}^{(n)}\vec{v}_2 & \vec{v}_2^{\,+}\delta\underline{M}^{(n)}\vec{v}_3 \\ \vec{v}_3^{\,+}\delta\underline{M}^{(n)}\vec{v}_1 & \vec{v}_3^{\,+}\delta\underline{M}^{(n)}\vec{v}_2 & \vec{v}_3^{\,+}\delta\underline{M}^{(n)}\vec{v}_3 \end{pmatrix} \\ &= \sum_{\nu=1}^{3}\vec{v}_\nu^{\,+}\cdot\delta\underline{M}^{(n)}\cdot\vec{v}_\nu\ . \end{aligned} \tag{3.6}$$

Furthermore, taking into account the orthogonality relation of the transfer matrix $\underline{M}_0(s_1,s_2)$:

$$\underline{M}_0^T(s_1,s_2)\cdot\underline{M}_0(s_1,s_2) = \underline{1}\ ;$$

$$\Longrightarrow \quad \vec{v}_\nu^{\,+}(s)\cdot \underline{M}_0(s+L,s) \;=\; \vec{v}_\nu^{\,+}(s)\cdot \left[\underline{M}_0^{-1}(s+L,s)\right]^T$$
$$= \left[\underline{M}_0^{-1}(s+L,s)\cdot \vec{v}_\nu(s)\right]^+$$
$$= \left[\alpha_\nu^{-1}\cdot \vec{v}_\nu(s)\right]^+$$
$$= \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s)$$
$$\left(\text{since } \; (\alpha_\nu^{-1})^* = \alpha_\nu\right)$$

we finally have:

$$Sp\,[\delta\underline{M}^{(n)}] = \sum_{\nu=1}^{3} \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s_0)\cdot \underline{M}_0^{-1}(s_0+L,s_0)\cdot \delta\underline{M}^{(n)}(s_0+L,s_0)\cdot \vec{v}_\nu(s_0)$$

and thus:

$$Sp\,[\delta\underline{M}^{(1)}] = \sum_{\nu=1}^{3} \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s_0)\cdot \underline{M}_0^{-1}(s_0+L,s_0)\cdot \delta\underline{M}^{(1)}(s_0+L,s_0)\cdot \vec{v}_\nu(s_0); \tag{3.7a}$$

$$Sp\,[\delta\underline{M}^{(2)}] = \sum_{\nu=1}^{3} \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s_0)\cdot \underline{M}_0^{-1}(s_0+L,s_0)\cdot \delta\underline{M}^{(2)}(s_0+L,s_0)\cdot \vec{v}_\nu(s_0) \tag{3.7b}$$

with α_ν given by eqn. (2.4).

As we shall see, these forms allow the traces to be calculated directly in terms of the unperturbed vectors $\vec{n}_0(s), \vec{m}_0(s), \vec{l}_0(s)$ and the perturbation $\vec{\omega}$ instead of by evaluating $Sp\,[\delta\underline{M}]$ from scratch.

a) Calculation of $\delta Q_{spin}^{(1)}$:

From (3.7a) and (2.13) we obtain:

$$\begin{aligned}
Sp\,[\delta\underline{M}^{(1)}] &= \sum_{\nu=1}^{3} \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s_0)\cdot \underline{M}_0^{-1}(s_0+L,s_0)\cdot \delta\underline{M}^{(1)}(s_0+L,s_0)\cdot \vec{v}_\nu(s_0) \\
&= \sum_{\nu=1}^{3} \alpha_\nu \cdot \vec{v}_\nu^{\,+}(s_0)\cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot \underline{M}_0^{-1}(\tilde{s},s_0)\cdot \underline{\omega}(\tilde{s})\cdot \underline{M}_0(\tilde{s},s_0)\cdot \vec{v}_\nu(s_0) \\
&= \sum_{\nu=1}^{3} \alpha_\nu \cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot [\underline{M}_0(\tilde{s},s_0)\cdot \vec{v}_\nu(s_0)]^+ \cdot \underline{\omega}(\tilde{s})\cdot \vec{v}_\nu(\tilde{s}) \\
&= \sum_{\nu=1}^{3} \alpha_\nu \cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot \vec{v}_\nu^{\,+}(\tilde{s})\cdot \underline{\omega}(\tilde{s})\cdot \vec{v}_\nu(\tilde{s}) \\
&= \sum_{\nu=1}^{3} \alpha_\nu \cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot \vec{v}_\nu^{\,+}(\tilde{s})\cdot [\vec{\omega}(\tilde{s})\times \vec{v}_\nu(\tilde{s})] \\
&= \sum_{\nu=1}^{3} \alpha_\nu \cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot \vec{\omega}^{\,T}(\tilde{s})\cdot [\vec{v}_\nu(\tilde{s})\times \vec{v}_\nu^{\,*}(\tilde{s})] \\
&= \int_{s_0}^{s_0+L} d\tilde{s}\cdot \vec{\omega}^{\,T}(\tilde{s})\cdot \{\alpha_2\cdot [\vec{v}_2(\tilde{s})\times \vec{v}_2^{\,*}(\tilde{s})] + \alpha_3\cdot [\vec{v}_3(\tilde{s})\times \vec{v}_3^{\,*}(\tilde{s})]\} \\
&= \frac{1}{2}\cdot \int_{s_0}^{s_0+L} d\tilde{s}\cdot \vec{\omega}^{\,T}(\tilde{s})\cdot \Big\{\alpha_2\cdot \left[\vec{m}_0(\tilde{s}) + i\cdot \vec{l}_0(\tilde{s})\right]\times \left[\vec{m}_0(\tilde{s}) - i\cdot \vec{l}_0(\tilde{s})\right]
\end{aligned}$$

$$+\alpha_3 \cdot \left[\vec{m}_0(\tilde{s}) - i \cdot \vec{l}_0(\tilde{s})\right] \times \left[\vec{m}_0(\tilde{s}) + i \cdot \vec{l}_0(\tilde{s})\right]\Big\}$$

$$= \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{\omega}^T(\tilde{s}) \cdot i \left\{\vec{m}_0(\tilde{s}) \times \vec{l}_0(\tilde{s})\right\} \cdot (-\alpha_2 + \alpha_3)$$

$$= -i \cdot (\alpha_2 - \alpha_3) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{\omega}^T(\tilde{s}) \cdot \vec{n}_0(\tilde{s}) \tag{3.8}$$

leading to:

$$\delta Q_{spin}^{(1)} = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{\omega}^T(\tilde{s}) \cdot \vec{n}_0(\tilde{s}) \ . \tag{3.9}$$

(see eqn. (3.5a)) [2]

This expression agrees with that given by Yokoya [2]. .

Remark:

In the case of a coasting beam, i.e. in the absence of accelerating cavities and radiation, the energy along the closed orbit is constant. Eqn. (3.9) can then be used to calculate at first order the spin chromaticity defined as the derivative of the spin tune with respect to a constant fractional energy deviation from the design energy [13].

In a nominally planar machine, vertical closed orbit distortions cause no first order spin tune shift since $\vec{n}_0$ and the $\vec{\omega}$ due to radial fields on the closed orbit are orthogonal. Uncompensated solenoids cause no first order tune shift for the same reason.

For horizontal closed orbit distortions, $\vec{\omega}$ due to vertical fields on the closed orbit is parallel to $\vec{n}_0$ but since the horizontal closed orbit is periodic the one turn integral vanishes and the first order tune shift is also zero.

b) Calculation of $\delta Q_{spin}^{(2)}$:

From (3.7b) and (2.14) we obtain:

$$\begin{aligned} Sp\,[\delta \underline{M}^{(2)}] &= \sum_{\mu=1}^{3} \alpha_\mu \cdot \vec{v}_\mu^{\,+}(s_0) \cdot \underline{M}_0^{-1}(s_0+L, s_0) \cdot \delta \underline{M}^{(2)}(s_0+L, s_0) \cdot \vec{v}_\mu(s_0) \\ &= \sum_{\mu=1}^{3} \alpha_\mu \cdot \vec{v}_\mu^{\,+}(s_0) \cdot \int_{s_0}^{s_0+L} ds' \cdot \underline{M}_0^{-1}(s', s_0) \cdot \underline{\omega}(s') \\ &\qquad \times \delta \underline{M}^{(1)}(s', s_0) \cdot \vec{v}_\mu(s_0) \\ &= \sum_{\mu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot [\underline{M}_0(s', s_0) \cdot \vec{v}_\mu(s_0)]^+ \cdot \underline{\omega}(s') \\ &\qquad \times \int_{s_0}^{s'} ds'' \cdot \underline{M}_0(s', s'') \cdot \underline{\omega}(s'') \cdot \underline{M}_0(s'', s_0) \cdot \vec{v}_\mu(s_0) \end{aligned}$$

[2]Equation (3.9) can be rewritten as

$$\delta Q_{spin}^{(1)} = -\frac{i}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_2^{\,+}(\tilde{s}) \cdot \underline{\omega}(\tilde{s}) \cdot \vec{v}_2(\tilde{s}) \ ,$$

revealing the similarity with the expression for an energy shift in quantum mechanics.

$$= \sum_{\mu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \underline{\omega}(s') \times \int_{s_0}^{s'} ds'' \cdot \underline{M}_0(s', s'') \cdot \underline{\omega}(s'') \cdot \vec{v}_\mu(s'') \ . \tag{3.10}$$

Using the completeness relation

$$\vec{x} \;=\; \sum_{\nu=1}^{3} \vec{v}_\nu \cdot \left(\vec{v}_\nu^{\,+} \cdot \vec{x}\right)$$

valid for an arbitrary vector $\vec{x}$ for the case when

$$\vec{x} \;=\; \int_{s_0}^{s'} ds'' \cdot \underline{M}_0(s', s'') \cdot \underline{\omega}(s'') \cdot \vec{v}_\mu(s'') \ ,$$

we may write [3] :

$$\begin{aligned}
Sp\,[\delta \underline{M}^{(2)}] \;&=\; \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \underline{\omega}(s') \cdot \vec{v}_\nu(s') \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \vec{v}_\nu^{\,+}(s') \cdot \underline{M}_0(s', s'') \cdot \underline{\omega}(s'') \cdot \vec{v}_\mu(s'') \\
&=\; \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \underline{\omega}(s') \cdot \vec{v}_\nu(s') \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \left[\underline{M}_0^{-1}(s', s'') \cdot \vec{v}_\nu(s')\right]^{+} \underline{\omega}(s'') \cdot \vec{v}_\mu(s'') \\
&=\; \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \underline{\omega}(s') \cdot \vec{v}_\nu(s') \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \vec{v}_\nu^{\,+}(s'') \cdot \underline{\omega}(s'') \cdot \vec{v}_\mu(s'') \\
&=\; -\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \underline{\omega}(s') \cdot \vec{v}_\nu(s') \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \left[\vec{v}_\mu^{\,+}(s'') \cdot \underline{\omega}(s'') \cdot \vec{v}_\nu(s'')\right]^{+} \\
&\qquad \left(\text{since } \ \underline{\omega}^{+} \;=\; -\underline{\omega}\right) \\
&=\; -\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{v}_\mu^{\,+}(s') \cdot \left[\vec{\omega}(s') \times \vec{v}_\nu(s')\right] \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \left\{\vec{v}_\mu^{\,+}(s'') \cdot \left[\vec{\omega}(s'') \times \vec{v}_\nu(s'')\right]\right\}^{+} \\
&=\; -\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} \alpha_\mu \cdot \int_{s_0}^{s_0+L} ds' \cdot \vec{\omega}^{\,T}(s') \cdot \left[\vec{v}_\nu(s') \times \vec{v}_\mu^{\,*}(s')\right] \\
&\qquad \times \int_{s_0}^{s'} ds'' \cdot \left\{\vec{\omega}^{\,T}(s'') \cdot \left[\vec{v}_\nu(s'') \times \vec{v}_\mu^{\,*}(s'')\right]\right\}^{+}
\end{aligned} \tag{3.11}$$

[3]This is equivalent to inserting the relation $\underline{V}^{+}\underline{V} \;=\; \underline{1}$ in (3.10) before the second integrand.

with

$$\begin{aligned} \vec{v}_1 \times \vec{v}_1^* &= \vec{n}_0 \times \vec{n}_0 \\ &= 0 \; ; \end{aligned} \tag{3.12a}$$

$$\begin{aligned} \vec{v}_1 \times \vec{v}_2^* &= \vec{n}_0 \times \frac{1}{\sqrt{2}} \left[\vec{m}_0 - i \cdot \vec{l}_0\right] \\ &= \frac{1}{\sqrt{2}} \left[\vec{l}_0 + i \cdot \vec{m}_0\right] \\ &= +i \cdot \frac{1}{\sqrt{2}} \left[\vec{m}_0 - i \cdot \vec{l}_0\right] \\ &= +i \cdot \vec{v}_3 \; ; \end{aligned} \tag{3.12b}$$

$$\begin{aligned} \vec{v}_1 \times \vec{v}_3^* &= \vec{n}_0 \times \frac{1}{\sqrt{2}} \left[\vec{m}_0 + i \cdot \vec{l}_0\right] \\ &= \frac{1}{\sqrt{2}} \left[\vec{l}_0 - i \cdot \vec{m}_0\right] \\ &= -i \cdot \frac{1}{\sqrt{2}} \left[\vec{m}_0 + i \cdot \vec{l}_0\right] \\ &= -i \cdot \vec{v}_2 \; ; \end{aligned} \tag{3.12c}$$

$$\begin{aligned} \vec{v}_2 \times \vec{v}_1^* &= -\left[\vec{v}_1 \times \vec{v}_2^*\right]^* \\ &= +i \cdot \vec{v}_3^* \\ &= +i \cdot \vec{v}_2 \; ; \end{aligned} \tag{3.12d}$$

$$\begin{aligned} \vec{v}_2 \times \vec{v}_2^* &= \frac{1}{\sqrt{2}} \left[\vec{m}_0 + i \cdot \vec{l}_0\right] \times \frac{1}{\sqrt{2}} \left[\vec{m}_0 - i \cdot \vec{l}_0\right] \\ &= \frac{i}{2} \left[\vec{l}_0 \times \vec{m}_0 - \vec{m}_0 \times \vec{l}_0\right] \\ &= -i \cdot \vec{n}_0 \; ; \end{aligned} \tag{3.12e}$$

$$\begin{aligned} \vec{v}_2 \times \vec{v}_3^* &= \frac{1}{\sqrt{2}} \left[\vec{m}_0 + i \cdot \vec{l}_0\right] \times \frac{1}{\sqrt{2}} \left[\vec{m}_0 + i \cdot \vec{l}_0\right] \\ &= \frac{i}{2} \left[\vec{l}_0 \times \vec{m}_0 + \vec{m}_0 \times \vec{l}_0\right] \\ &= 0 \; ; \end{aligned} \tag{3.12f}$$

$$\begin{aligned} \vec{v}_3 \times \vec{v}_1^* &= -\left[\vec{v}_1 \times \vec{v}_3^*\right]^* \\ &= -i \cdot \vec{v}_2^* \\ &= -i \cdot \vec{v}_3 \; ; \end{aligned} \tag{3.12g}$$

$$\vec{v}_3 \times \vec{v}_2^* = -\left[\vec{v}_2 \times \vec{v}_3^*\right]^*$$

$$= \quad 0\ ; \tag{3.12h}$$

$$\begin{aligned}
\vec{v}_3 \times \vec{v}_3^* &= \frac{1}{\sqrt{2}}\left[\vec{m}_0 - i\cdot\vec{l}_0\right] \times \frac{1}{\sqrt{2}}\left[\vec{m}_0 + i\cdot\vec{l}_0\right] \\
&= \frac{i}{2}\left[-\vec{l}_0 \times \vec{m}_0 + \vec{m}_0 \times \vec{l}_0\right] \\
&= +i\cdot\vec{n}_0\ .
\end{aligned} \tag{3.12i}$$

Thus:

$$\begin{aligned}
Sp\,[\delta\underline{M}^{(2)}] &= -\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\alpha_\mu\cdot\int_{s_0}^{s_0+L} ds'\cdot\vec{\omega}^{\,T}(s')\cdot\left[\vec{v}_\nu(s')\times\vec{v}_\mu^*(s')\right] \\
&\qquad\times\int_{s_0}^{s'} ds''\cdot\left\{\vec{\omega}^{\,T}(s'')\cdot\left[\vec{v}_\nu(s'')\times\vec{v}_\mu^*(s'')\right]\right\}^+ \\
&= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\qquad\times\Big\{\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_2(s')\times\vec{v}_1^*(s')]\right)\cdot\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_2(s'')\times\vec{v}_1^*(s'')]\right)^* \\
&\qquad+\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_3(s')\times\vec{v}_1^*(s')]\right)\cdot\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_3(s'')\times\vec{v}_1^*(s'')]\right)^* \\
&\qquad+\alpha_2\cdot\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_2(s')\times\vec{v}_2^*(s')]\right)\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_2(s'')\times\vec{v}_2^*(s'')]\right)^* \\
&\qquad+\alpha_3\cdot\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_3(s')\times\vec{v}_3^*(s')]\right)\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_3(s'')\times\vec{v}_3^*(s'')]\right)^* \\
&\qquad+\alpha_2\cdot\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_1(s')\times\vec{v}_2^*(s')]\right)\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_1(s'')\times\vec{v}_2^*(s'')]\right)^* \\
&\qquad+\alpha_3\cdot\left(\vec{\omega}^{\,T}(s')\cdot[\vec{v}_1(s')\times\vec{v}_3^*(s')]\right)\left(\vec{\omega}^{\,T}(s'')\cdot[\vec{v}_1(s'')\times\vec{v}_3^*(s'')]\right)^*\Big\}\,.
\end{aligned} \tag{3.13}$$

Putting (3.12) into (3.13), we obtain:

$$\begin{aligned}
Sp\,[\delta\underline{M}^{(2)}] &= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\qquad\times\Big\{\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_2(s'')\right]^* \\
&\qquad+\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_3(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_3(s'')\right]^* \\
&\qquad+\alpha_2\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{n}_0(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{n}_0(s'')\right]^* \\
&\qquad+\alpha_3\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{n}_0(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{n}_0(s'')\right]^* \\
&\qquad+\alpha_2\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_3(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_3(s'')\right]^* \\
&\qquad+\alpha_3\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_2(s'')\right]^*\Big\} \\
&= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\qquad\times\Big\{(1+\alpha_3)\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_3(s'')\right] \\
&\qquad+(1+\alpha_2)\cdot\left[\vec{\omega}^{\,T}(s')\cdot\vec{v}_3(s')\right]\cdot\left[\vec{\omega}^{\,T}(s'')\cdot\vec{v}_2(s'')\right]
\end{aligned}$$

$$+(\alpha_2+\alpha_3)\cdot\left[\vec{\omega}^T(s')\cdot\vec{n}_0(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{n}_0(s'')\right]\Big\}$$

$$= T_{21}+T_{22}$$

with (3.14)

$$\begin{aligned}
T_{21} &= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\quad \times(\alpha_2+\alpha_3)\cdot\left[\vec{\omega}^T(s')\cdot\vec{n}_0(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{n}_0(s'')\right] \\
&= -2\cos\left[2\pi Q_{spin}\right]\cdot\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds''\cdot\left[\vec{\omega}^T(s')\cdot\vec{n}_0(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{n}_0(s'')\right] \\
&= -\cos\left[2\pi Q_{spin}\right]\cdot\left[\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{\omega}^T(\tilde{s})\cdot\vec{n}_0(\tilde{s})\right]^2 \\
&= -\cos\left[2\pi Q_{spin}\right]\cdot 4\pi^2\cdot\left(\delta Q_{spin}^{(1)}\right)^2
\end{aligned}$$

(where we use the fact that the integrand is symmetric under interchange of s' and s'') and (3.15a)

$$\begin{aligned}
T_{22} &= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\quad \times\Big\{(1+\alpha_3)\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_3(s'')\right] \\
&\quad +(1+\alpha_2)\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_3(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_2(s'')\right]\Big\} \\
&= -\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\quad \times\Big\{2\cdot e^{-i\pi Q_{spin}}\cdot\cos[\pi Q_{spin}]\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_3(s'')\right] \\
&\quad +2\cdot e^{+i\pi Q_{spin}}\cdot\cos[\pi Q_{spin}]\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_3(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_2(s'')\right]\Big\} \\
&= -2\cos\left[\pi Q_{spin}\right]\cdot\int_{s_0}^{s_0+L} ds'\cdot\int_{s_0}^{s'} ds'' \\
&\quad \times\Big\{e^{-i\pi Q_{spin}}\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_2(s')\right]\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_2(s'')\right]^* \\
&\quad +e^{+i\pi Q_{spin}}\cdot\left[\vec{\omega}^T(s')\cdot\vec{v}_2(s')\right]^*\cdot\left[\vec{\omega}^T(s'')\cdot\vec{v}_2(s'')\right]\Big\}\,.
\end{aligned}$$

(3.15b)

With

$$Sp\left[\delta\underline{M}^{(2)}(s_0+L,s_0)\right]+2\pi^2\cdot\left(\delta Q_{spin}^{(1)}\right)^2\cdot(\alpha_2+\alpha_3) = T_{22}$$

we finally obtain from (3.5b):

$$\delta Q_{spin}^{(2)} = \frac{1}{2\pi i\cdot(\alpha_2-\alpha_3)}\cdot T_{22}$$

$$
\begin{aligned}
&= \frac{1}{4\pi \cdot \sin 2\pi Q_{spin}} \cdot 2\cos\left[\pi Q_{spin}\right] \cdot \int_{s_0}^{s_0+L} ds' \cdot \int_{s_0}^{s'} ds'' \\
&\quad \times \left\{ e^{-i\pi Q_{spin}} \cdot \left[\vec{\omega}^T(s') \cdot \vec{v}_2(s')\right] \cdot \left[\vec{\omega}^T(s'') \cdot \vec{v}_2(s'')\right]^* \right. \\
&\quad \left. + e^{+i\pi Q_{spin}} \cdot \left[\vec{\omega}^T(s') \cdot \vec{v}_2(s')\right]^* \cdot \left[\vec{\omega}^T(s'') \cdot \vec{v}_2(s'')\right] \right\} \\
&= \frac{1}{4\pi \cdot \sin \pi Q_{spin}} \cdot \int_{s_0}^{s_0+L} ds' \cdot \int_{s_0}^{s'} ds'' \\
&\quad \times \left\{ e^{-i\pi Q_{spin}} \cdot \left[\vec{\omega}^T(s') \cdot \vec{v}_2(s')\right] \cdot \left[\vec{\omega}^T(s'') \cdot \vec{v}_2(s'')\right]^* \right. \\
&\quad \left. + e^{+i\pi Q_{spin}} \cdot \left[\vec{\omega}^T(s') \cdot \vec{v}_2(s')\right]^* \cdot \left[\vec{\omega}^T(s'') \cdot \vec{v}_2(s'')\right] \right\} \\
&= \frac{1}{4} \cdot \mathcal{I}m \left\{ \frac{1}{e^{-2\pi i Q_{spin}} - 1} \cdot \int_{s_0}^{s_0+L} ds' \cdot \left[\vec{\omega}^T(s') \cdot \vec{v}_2(s')\right]^* \right. \\
&\quad \left. \times \int_{s_0}^{s'} ds'' \left[\vec{\omega}^T(s'') \cdot \vec{v}_2(s'')\right] \right\} . \qquad (3.16)
\end{aligned}
$$

This is similar to the form given by K. Yokoya [2].

It is now clear that both vertical closed orbit distortions and solenoids can cause second order spin tune shifts. In the case of an isolated solenoid in an otherwise flat machine the tune shift can be calculated analytically in a very simple way using spinor algebra [7]. One sees there also that the lowest order term in the tune shift is of second order.

4 Summary

We have presented a perturbative formalism for the investigation of the spin tune shift on the closed orbit of a storage ring due to misalignment. The results are in agreement with those already obtained by Yokoya using another approach.

In this paper we have calculated δQ_{spin} only up to second order. But the method developed can be extended up to an arbitrary order in a straightforward manner.

Examples of the numerical application of these results may be found in [2].

Acknowledgements

We wish to thank K. Yokoya for checking the final formula (3.16) for the tune shift and for demonstrating the equivalence with his expression.

References

[1] A.W. Chao: Nucl. Inst. Meth., vol.180, p.29, (1981);

also in "Physics of High Energy Particle Accelerators", American Institute of Physics Proceedings 87, p.395, (1982).

[2] K. Yokoya: "On Multiple Siberian Snakes"; Superconducting Super Collider Report SSC-189, (1988).

[3] D.P. Barber, K. Heinemann, G. Ripken: "A Canonical Eight-Dimensional Formalism for Linear and Non-Linear Classical Spin-Orbit Motion in Storage Rings";
DESY Report 91-47, (1991)
and
"A Canonical 8-Dimensional Formalism for Classical Spin - Orbit Motion in Storage Rings. I: A New Pair of Canonical Spin Variables".
Accepted for publication in Zeitschrift für Physik C.

D.P. Barber, K. Heinemann, G. Ripken: "Nonlinear Canonical Spin - Orbit Dynamics in Storage Rings: Normal Forms and the $\vec{n}$ - Axis";
DESY Report 93-31, (1993).
and
"A Canonical 8-Dimensional Formalism for Classical Spin - Orbit Motion in Storage Rings. II: Normal Forms and the $\vec{n}$-Axis".
Accepted for publication in Zeitschrift für Physik C.

[4] D. P. Barber et al.: Phys. Lett., vol.135B, p.498, (1984).

[5] L. Arnaudon et al.: Phys. Lett., vol.284B, p.431, (1992).

[6] Ya.S. Derbenev, A.M. Kondratenko, A.N Shrinsky: Sov. Phys. JETP, vol.60, p.1216, (1971).

[7] B.W. Montague: "Polarized Beams in High Energy Storage Rings"; Physics Reports, vol.113 N^{0}1, (1983).

[8] G. Assmann and J.K. Koutchouk: "Spin Tune Shifts due to Optics Imperfections"; CERN SL/94-13 (AP), (1994).

[9] D.P. Barber, K. Heinemann, G. Ripken: "Notes on Spin Dynamics in Storage Rings"; DESY M-92-04, 1992.

[10] L. Thomas: Phil. Mag., vol.3, p.1, (1927).

[11] V. Bargmann, L. Michel and V.L. Telegdi: Phys. Rev. Lett., vol.2, p.435, (1959).

[12] Herbert Goldstein: "Classical Mechanics" ; Addison-Wesley, (1980).

[13] K. Yokoya: Part. Acc., vol.13, p.95, (1983).