Racah Coefficients and $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$

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Abstract

The tensor product of three irreducible representations of the quantum superalgebra $U_q(osp(1|2))$ is studied and the super q-analogs of Racah coefficients and $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$ are defined. Racah coefficients and $6-j$ symbols depend on the superspin l and the parity λ which characterize the irreducible representations of $U_q(osp(1|2))$ but the dependence on parities can be factored out so that one can define parity independent $6-j$ symbols. It is shown that the $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$ satisfy symmetry properties and orthogonality relations similar to those of the $6-j$ symbols for quantum algebra $U_q(su(2))$.

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I Introduction.

This paper is a continuation of the study, begun in earlier papers [1], [2], of the Racah-Wigner calculus for quantum superalgebra $U_q(osp(1|2))$. The quantum superalgebra $U_q(osp(1|2))$ can be considered either as the super-analog of the $U_q(su(2))$ quantum algebra, for which the Racah-Wigner calculus has been developed in several papers [3], [4], [5], [6] or as the quantum deformation of the super algebra $osp(1|2)$, the Racah-Wigner calculus of which has also been constructed in Refs.[7], [8], [9], [10].

The quantum superalgebra $U_q(osp(1|2))$ has been defined and studied by Kulish and Reshetikhin [11], [12]. Its Clebsch-Gordan coefficients were derived by Kulish [13] using a recursion relation and in Ref.[1] using the projection operator method. A particular case of Clebsch-Gordan coefficients was also given by Saleur [14].
The general case of reduction of tensor product of two arbitrary irreducible representations of the quantum superalgebra $U_q(osp(1|2))$, where the bilinear Hermitean form is not necessarily positive definite, has been studied in Ref.[2]. It has been shown that the Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$ do not depend either on the signatures of the bilinear Hermitean form or on the class of the represntations but they depend on the parities of the representation spaces. Several properties of Clebsch-Gordan coefficients for $U_q(osp(1|2))$, such as orthogonality relations and symmetry properties of these coefficients have been derived in Ref.[2]. In particular, it was shown that under some condition, Clebsch-Gordan coefficients satisfy Regge symmetry. The study of the symmetry properties of the Clebsch-Gordan coefficients allowed one to define $3-j\lambda$ symbols which possess good symmetry properties but which depend on the parity of bases in graded representation spaces. It has been shown that the dependence on parities in this symbol can be factored out, so that one can define parity independent $3-j$ symbols, which are super-analogs of $3-j$ symbols for the quantum algebra $U_q(su(2))$ [6]. In this article, Clebsch-Gordan coefficients and $3-j$ symbols will be used to define Racah coefficients and $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$.

The paper has the following structure : Section II contains the definition of the quantum superalgebra $U_q(osp(1|2))$ and the basic properties of its irreducible representations. In section III, we recall the definition and some properties of Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$. Section IV is devoted to the definition and basic properties of parity dependent and parity independent $3-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. In section V, we consider the tensor product of three irreducible representations of $U_q(osp(1|2))$ with arbitrary Hermitean forms. We define Racah coefficients for the quantum superalgebra $U_q(osp(1|2))$ and we derive the basic properties of these coefficients: symmetry properties, pseudo-orthogonality relations and some particular values. In Section VI, we consider $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. In the first subsection, we define in a conventional way using an invariant metric and parity dependent $3-j$ symbols, parity dependent $6-j$ symbols and we derive their basic properties. In the second subsection, it is shown that that it is possible to factor out the dependence on parities in the parity dependent $6-j$ symbols, so one can define parity independent $6-j$ symbols whose properties are quite similar to those of the $6-j$ symbols for the quantum algebra $U_q(su(2))$.

II The irreducible representations of the quantum superalgebra $U_q(osp(1|2))$.

The quantum superalgebra $U_q(osp(1|2))$ is generated by three elements: H (even) and $v_\pm$ (odd) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\tfrac{1}{2}v_\pm, \quad [v_+, v_-]_+ = -\frac{sh(\eta H)}{sh(2\eta)}, \tag{2.1}$$

where the deformation parameter η is real and $q = e^{-\frac{\eta}{2}}$. We choose $\eta > 0$ so that $q < 1$. The following formulae for coproduct Δ, antipode S and counit ϵ define on $U_q(osp(1|2))$ the structure of a Hopf algebra

$$\Delta(v_\pm) = v_\pm \otimes q^H + q^{-H} \otimes v_\pm, \tag{2.2}$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(1) = 1 \otimes 1, \tag{2.3}$$

$$S(H) = -H, \quad S(v_\pm) = -q^{\pm\frac{1}{2}}v_\pm, \tag{2.4}$$

$$\epsilon(H) = \epsilon(v_\pm) = 0, \quad \epsilon(1) = 1, \tag{2.5}$$

where the coproduct Δ is the homomorphism

$$\Delta : U_q(osp(1|2)) \to U_q(osp(1|2)) \otimes U_q(osp(1|2)). \tag{2.6}$$

A representation of a quantum superalgebra $U_q(osp(1|2))$ in a finite dimensional graded space V is a homomorphism T

$$T: \; U_q(osp(1|2)) \;\to\; L(V, V), \tag{2.7}$$

of the associative graded algebra $U_q(osp(1|2))$ into the associative graded algebra $L(V, V)$ of linear operators in V, such that

$$[T(H), T(v_\pm)] = \pm\tfrac{1}{2}T(v_\pm), \quad [T(v_+), T(v_-)]_+ = -\frac{sh(\eta T(H))}{sh(2\eta)}, \tag{2.8}$$

It has been shown in Ref.[2], that any finite dimensional grade star representation of $U_q(osp(1|2))$ is characterized by four parameters: the superspin l (a non negative integer), the parity $\lambda = 0, 1$ of the highest weight vector in the representation space and by $\varphi, \psi = 0, 1$ the signature parameters of the Hermitean form in the representation space V. The parity λ and the signature φ define the class $\epsilon = 0, 1$ of the grade star representation by

$$\epsilon = \lambda + \varphi + 1 \;\; mod(2). \tag{2.9}$$

The representation space $V = V^l(\lambda)$ is a graded vector space of dimension $2l + 1$ with basis $e^{lq}_m(\lambda)$, where $-l \le m \le l$. The parity of the basis vectors $e^{lq}_m(\lambda)$ is determined by the values of l, m and λ,

$$deg(e^l_m(\lambda)) = l - m + \lambda \;\; mod(2). \tag{2.10}$$

The vectors $e^{lq}_m(\lambda)$ are orthogonal with respect to the Hermitean form and their normalization is determined by the signature parameters φ, ψ

$$(e^{lq}_m(\lambda), e^{lq}_{m'}(\lambda)) = (-1)^{\varphi(l-m)+\psi}\delta_{mm'}, \tag{2.11}$$

where (,) denotes the Hermitean form in the representation space. The operators $T(v_\pm)$ and $T(H)$ act on the basis $e^{lq}_m(\lambda)$ in the following way :

$$T(H)e^{lq}_m(\lambda) = \frac{m}{2}\, e^{lq}_m(\lambda), \tag{2.12}$$

$$T(v_+)e^{lq}_m(\lambda) = (-1)^{l-m}\sqrt{[l-m][l+m+1]}\gamma\, e^{lq}_{m+1}(\lambda), \tag{2.13}$$

$$T(v_-)e^{lq}_m(\lambda) = \sqrt{[l+m][l-m+1]}\gamma\, e^{lq}_{m-1}(\lambda), \tag{2.14}$$

where the symbol $[n]$ is the Kulish graded quantum symbol [13] defined by

$$[n] = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} \tag{2.15}$$

Note that the action of the operators $T(v_\pm)$ and $T(H)$ does not depend on the parameters λ, φ, ψ. The representation T of class ϵ which acts in the representation space $V^l(\lambda)$ with an Hermitean form characterized by the signature parameters φ and ψ is denoted by $T^{l\epsilon}_{\varphi\psi}$. However, for simplicity, the indices ϵ, φ, ψ will sometimes be omited in the following. Note that the parity λ is expressed in terms of the other parameters by relation (2.9), i.e.,

$$\lambda = \epsilon + \varphi + 1, \quad mod(2). \tag{2.16}$$

In the limit $q \to 1$, the grade star representation $T^{l\epsilon}_{\varphi\psi}$ becomes a grade star representation of superalgebra $osp(1|2)$. For more details on the irreducible grade star representations of $U_q(osp(1|2))$, see Ref.[2].

III Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$.

A Tensor product of two irreducible representations.

In this section, we will recall basic properties of the tensor product of two irreducible representations $T^{l_1\epsilon}_{\varphi_1\psi_1}$ and $T^{l_2\epsilon}_{\varphi_2\psi_2}$ of the same class ϵ. The bilinear Hermitean form in the tensor product space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ is defined in the following way :

$$((X_1 \otimes X_2), (Y_1 \otimes Y_2)) = (-1)^{deg(X_2)deg(Y_1)}(X_1, Y_1)(X_2, Y_2), \tag{3.1}$$

where $X_1, Y_1 \in V^{l_1}(\lambda_1)$, $X_2, Y_2 \in V^{l_2}(\lambda_2)$. The space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ is a representation space for the tensor product of two representations $T^{l_1\epsilon}_{\varphi_1\psi_1} \otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$. The generators $v_\pm$ and H are represented in this case by the following operators acting in the space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$

$$v^{\otimes}_\pm(1,2) = (T^{l_1} \otimes T^{l_2})(\Delta(v_\pm)) = T^{l_1}(v_\pm) \otimes q^{T^{l_2}(H)} + q^{-T^{l_1}(H)} \otimes T^{l_2}(v_\pm), \tag{3.2}$$

$$H^{\otimes}(1,2) = (T^{l_1} \otimes T^{l_2})\Delta(H) = T^{l_1}(H) \otimes T^{l_2}(1) + T^{l_1}(1) \otimes T^{l_2}(H). \tag{3.3}$$

It has been shown in [2] and [1] that the tensor product of representations $T^{l_1\epsilon}_{\varphi_1\psi_1} \otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$ is a representation of class ϵ with respect to the Hermitean form (3.1) and that the tensor product of representation spaces $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ can be decomposed into a direct sum of subspaces $V^l(\lambda)$

$$V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2) = \oplus_l V^l(\lambda), \tag{3.4}$$

where l is an integer satisfying the conditions

$$|l_1 - l_2| \le l \le l_1 + l_2 . \tag{3.5}$$

In this decomposition, each subspace appears only once, i.e., the tensor product of two representations of the same class is simply reducible.

B $U_q(osp(1|2))$ Clebsch-Gordan coefficients.

By definition, the Clebsch-Gordan coefficients $(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q$ relate the pseudo-normalized basis $e^{l_1 q}_{m_1}(\lambda_1) \otimes e^{l_2 q}_{m_2}(\lambda_2)$ to the reduced pseudo-normalized basis $e^{lq}_{m}(l_1, l_2, \lambda)$ in the following way :

$$e^{lq}_{m}(l_1, l_2, \lambda) = \sum_{m_1, m_2} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q \, e^{l_1 q}_{m_1}(\lambda_1) \otimes e^{l_2 q}_{m_2}(\lambda_2) . \tag{3.6}$$

where $m = m_1 + m_2$. The bases $e^{lq}_{m}(l_1, l_2, \lambda)$ in the reduced tensor product space are orthogonal and satisfy the normalization condition

$$(e^{lq}_{m}(l_1, l_2, \lambda) \, , \, e^{l'q}_{m'}(l_1, l_2, \lambda)) = (-1)^{\varphi(l-m)+\psi} \, \delta_{ll'} \, \delta_{mm'} , \tag{3.7}$$

where

$$\varphi = l_1 + l_2 + l + \lambda_1 + \varphi_2 \quad mod(2), \tag{3.8}$$

$$\psi = (l_1 + l_2 + l + \lambda_2)\lambda_1 + \varphi_2(l_1 + l_2 + l) + \psi_1 + \psi_2 \quad mod(2), \tag{3.9}$$

$$\lambda = l_1 + l_2 + l + \lambda_1 + \lambda_2 \quad mod(2) . \tag{3.10}$$

It should be noticed that even if the bases $e^{l_1 q}_{m_1}(\lambda_1)$ and $e^{l_2 q}_{m_2}(\lambda_2)$ are positive definite, i.e., $\varphi_i = \psi_i = 0$, $i = 1, 2$, the reduced basis $e^{lq}_{m}(l_1, l_2, \lambda)$ need not be positive definite.
The operators $H^{\otimes}(1,2)$ and $v^{\otimes}_{\pm}(1,2)$ act on the vectors $e^{lq}_{m}(l_1, l_2, \lambda)$ in a standard way given by formulae (2.12), (2.14). Recently, Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2)$ have been derived by use of various methods [13], [1], [2]. In the following we will use the results obtained in Refs.[1], [2] by application of the projection the operator method. In the limit $q \to 1$, the quantum Clebsch-Gordan coefficients become $osp(1|2)$ Clebsch-Gordan coefficients

$$\lim_{q \to 1} (l_1 m_1 \lambda_1, \; l_2 m_2 \lambda_2 | l m \lambda)_q = (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda) , \tag{3.11}$$

Let us recall some properties of the Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$ [2]. They satisfy the pseudo-orthogonality relations

$$\sum_{m_1 m_2} (-1)^{(l_1 - m_1)(l_2 - m_2)} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda)_q = (-1)^{(l-m)L} \delta_{ll'} \delta_{mm'}, \tag{3.12}$$

$$\sum_{lm} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q (l_1 m'_1 \lambda_1, l_2 m'_2 \lambda_2 | l m \lambda)_q = (-1)^{(l_1 - m_1)(l_2 - m_2)} \delta_{m_1 m'_1} \delta_{m_2 m'_2} , \tag{3.13}$$

where $L = l_1 + l_2 + l$. These pseudo-orthogonality relations do not depend on the parameters φ_i, ψ_i, λ_i and ϵ.
Considering the action of operators $v^{\otimes}_{\pm}(1,2)$ on the defining relations for Clebsch-Gordan coefficients, one can derive the following recursion relations for Clebsch-Gordan coefficients, cf. Ref.[13]

$$\begin{aligned} &\sqrt{[l+m][l-m+1]\gamma} \, (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m-1 \; \lambda)_q \\ &= q^{\frac{m_2}{2}} \sqrt{[l_1 - m_1][l_1 + m_1 + 1]\gamma} \, (l_1 m_1 + 1 \; \lambda_1, l_2 m_2 \lambda_2 | l m \; \lambda)_q \\ &\quad + (-1)^{l_1 - m_1 + \lambda_1} q^{\frac{-m_1}{2}} \sqrt{[l_2 - m_2][l_2 + m_2 + 1]\gamma} \, (l_1 m_1 \; \lambda_1, l_2 m_2 + 1 \; \lambda_2 | l m \; \lambda)_q , \end{aligned} \tag{3.14}$$

$$\begin{aligned} &(-1)^{l_1 + l_2 + l + \lambda_1} \sqrt{[l-m][l+m+1]\gamma} \, (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m+1 \; \lambda)_q \\ &= (-1)^{l_2 - m_2 + \lambda_1} q^{\frac{m_2}{2}} \sqrt{[l_1 + m_1][l_1 - m_1 + 1]\gamma} \, (l_1 m_1 - 1 \; \lambda_1, l_2 m_2 \lambda_2 | l m \; \lambda)_q \\ &\quad + q^{\frac{-m_1}{2}} \sqrt{[l_2 + m_2][l_2 - m_2 + 1]\gamma} \, (l_1 m_1 \; \lambda_1, l_2 m_2 - 1 \; \lambda_2 | l m \; \lambda)_q . \end{aligned} \tag{3.15}$$

In the classical theory of Racah-Wigner calculus, a very important role is played by the Clebsch-Gordan coefficient $(jm, jn|00)$, which defines an invariant metric. In the case of the quantum superalgebra $U_q(osp(1|2)$, the corresponding coefficient also defines an invariant metric. It has the form [1]

$$C^{lq}_{mn}(\lambda) = \sqrt{[2l+1]} (l m \lambda, l n \lambda | 000)_q = (-1)^{(\lambda)(l-m)} (-1)^{\frac{(l-m)(l-m-1)}{2}} q^{\frac{m}{2}} \delta_{m,-n} , \tag{3.16}$$

and it satisfies the properties

$$C^{lq}_{mn}(\lambda) \;=\; (-1)^m \, C^{lq^{-1}}_{nm}(\lambda), \qquad C^{lq}_{mn}(\lambda) \, C^{lq^{-1}}_{pn}(\lambda) \;=\; \delta_{mp} . \tag{3.17}$$

In the following, we will use this invariant metric to construct parity dependent $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2)$.
One easily checks that in the limit $q = 1$, the invariant metric (3.16) becomes the invariant metric for the classical superalgebra $osp(1|2)$ defined in Ref.[10].
It has been shown in Ref.[2], that Clebsch-Gordan coefficients have the following symmetries

$$\begin{aligned} (l_1 \, m_1 \, \lambda_1, \; l_2 \, m_2 \, \lambda_2 | l_3 \, m_3 \, \lambda_3)_q \;&=\; (-1)^{(l_1 - m_1 + \lambda_1)(l_2 - m_2 + \lambda_2)} \, (-1)^{(\lambda_1 + \lambda_2)(l_1 + l_2 + l_3) + \lambda_1 \lambda_2} \\ &\times (-1)^{\frac{(l_1 + l_2 - l_3)(l_1 + l_2 - l_3 + 1)}{2}} (l_2 \, m_2 \, \lambda_2, \; l_1 \, m_1 \lambda_1 | l_3 \, m_3 \, \lambda_3)_{q^{-1}} , \end{aligned} \tag{3.18}$$

$$\begin{aligned} (l_1 \, m_1 \, \lambda_1, \; l_2 \, m_2 \, \lambda_2 | l_3 \, m_3 \, \lambda_3)_q \;&=\; (-1)^{\frac{(l_2 + m_2)(l_2 + m_2 - 1)}{2}} (-1)^{\lambda_1 (l_1 + l_3 - m_2)} \\ &\times (-1)^{(\lambda_2 + L)(l_1 + l_2 - m_3)} q^{-\frac{m_2}{2}} \left(\frac{[2l_3 + 1]}{[2l_1 + 1]} \right)^{\frac{1}{2}} (l_2 \; -m_2 \, \lambda_2, \; l_3 \, m_3 \lambda_3 | l_1 \, m_1 \, \lambda_1)_{q^{-1}} , \end{aligned} \tag{3.19}$$

$$\begin{aligned} (l_1 \, m_1 \, \lambda_1, \; l_2 \, m_2 \, \lambda_2 | l_3 \, m_3 \, \lambda_3)_q \;&=\; (-1)^{\frac{(l_1 - m_1)(l_1 - m_1 - 1)}{2}} (-1)^{\lambda_3 (l_2 + l_3 - m_1)} \\ &\times (-1)^{(\lambda_1 + L)(l_1 + l_3 - m_2)} q^{\frac{m_1}{2}} \left(\frac{[2l_3 + 1]}{[2l_2 + 1]} \right)^{\frac{1}{2}} (l_3 \, m_3 \, \lambda_3, \; l_1 \; -m_1 \lambda_1 | l_2 \, m_2 \, \lambda_2)_{q^{-1}} . \end{aligned} \tag{3.20}$$

The Clebsch-Gordan coefficients satisfy also the "mirror" symmetry

$$\begin{aligned}
&(l_1\, m_1\, \lambda_1,\, l_2\, m_2\, \lambda_2 | l_3\, m_3\, \lambda_3)_q \\
&= (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i)(l_i-m_i-1)}{2}} (l_1\, -m_1\, \lambda_1,\, l_2\, -m_2\, \lambda_2 | l_3\, -m_3\, \lambda_3)_{q^{-1}}\,, \qquad (3.21)
\end{aligned}$$

where $L = l_1 + l_2 + l_3$. Al these symmetries of Clebsch-Gordan coefficients present the same structure as the symmetries of Clebsch-Gordan coefficients for quantum algebra $U_q(su(2))$ [6], excepted that the phases are non linear in l_i, m_i and that they depend on the parities λ_i, $(i = 1, 2, 3)$.
Besides, it has been shown that these Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$ satisfy also Regge symmetry. However in this case, Regge symmetry is realized only under some condition on the arguments, cf. Ref.[15].

IV Symmetric $3-j$ symbols for quantum superalgebra $U_q(osp(1|2))$

A Parity dependent $3 - j\lambda$ symbols.

For the quantum superalgebra $U_q(osp(1|2))$, the parity dependent $3 - j$ symbols, denoted $sq3 - j\lambda$, have been defined in Ref.[2] as follows

$$\begin{aligned}
\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= (-1)^{(l_1+l_2-m_3)\lambda_3}(-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}}(-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \\
&\times \frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l_3 - m_3 \lambda_3)_q\,. \qquad (4.1)
\end{aligned}$$

Their arguments satisfy the same constraints as for Clebsch-Gordan coefficients

$$|l_1 - l_2| < l_3 < l_1 + l_2\,, \qquad (4.2)$$

$$m_1 + m_2 + m_3 = 0\,, \qquad (4.3)$$

$$l_1 + l_2 + l_3 = \lambda_3 + \lambda_1 + \lambda_2 \quad mod(2)\,, \qquad (4.4)$$

and the symbols have symmetry properties similar to the symmetry properties of $3 - j$ symbols for the quantum algebra $U_q(su(2))$. Under an even permutation of columns one has

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \begin{pmatrix} l_3\lambda_3 & l_1\lambda_1 & l_2\lambda_2 \\ m_3 & m_1 & m_2 \end{pmatrix}_q = \begin{pmatrix} l_2\lambda_2 & l_3\lambda_3 & l_1\lambda_1 \\ m_2 & m_3 & m_1 \end{pmatrix}_q \qquad (4.5)$$

and under an odd permutation they tranform as

$$\begin{aligned}
\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= \beta \begin{pmatrix} l_2\lambda_2 & l_1\lambda_1 & l_3\lambda_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_{q^{-1}} \\
&= \beta \begin{pmatrix} l_1\lambda_1 & l_3\lambda_3 & l_2\lambda_2 \\ m_1 & m_3 & m_2 \end{pmatrix}_{q^{-1}} = \beta \begin{pmatrix} l_3\lambda_3 & l_2\lambda_2 & l_1\lambda_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q^{-1}}, \qquad (4.6)
\end{aligned}$$

where β is the phase factor

$$\beta = (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i-\lambda_i)(l_i-m_i-\lambda_i-1)}{2}}\,. \qquad (4.7)$$

Under "mirror" symmetry, the $sq3 - j\lambda$ symbol transform in a slightly different way

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}_q = (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i)(l_i-m_i-1)}{2}} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_{q^{-1}}. \qquad (4.8)$$

Note also that the $sq3 - j$ symbols, as well as the Clebsch-Gordan coefficients, satisfy a conditional Regge symmetry, cf. Ref.[2], and that they satisfy modified pseudo-orthogonality relations, namely

$$\begin{aligned}
\sum_{m_1 m_2} (-1)^{(l_1-m_1)(l_2-m_2)} q^{\frac{1}{3}(m_1-m_2)} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l'_3\lambda_3 \\ m_1 & m_2 & m'_3 \end{pmatrix}_q \\
= (-1)^{(l_3-m_3)L} \frac{\delta_{l_3 l'_3}\, \delta_{m_3 m'_3}}{[2l_3+1]}\,, \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
\sum_{l_3 m_3} (-1)^{(l_3-m_3)L}[2l_3+1] \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix}_q \\
= (-1)^{(l_1-m_1)(l_2-m_2)}\, q^{-\frac{1}{3}(m_1-m_2)}\, \delta_{m_1 m'_1}\, \delta_{m_2 m'_2}\,. \qquad (4.10)
\end{aligned}$$

In the limit $q \rightarrow 1$, the $sq3 - j\lambda$ symbols become identical to the $s3 - j\lambda$ symbols for superalgebra $osp(1|2)$ defined in Ref.[10].

B Parity independent $3 - j$ symbols.

In Ref.[2], it has been shown that for $sq3 - j\lambda$ symbols, the dependence on λ_i can be factored out, so that for the quantum superalgebra $U_q(osp(1|2))$ one can define parity independent $3 - j$ symbols that will be called $sq3 - j$ symbols

$$\begin{aligned}
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= (-1)^{\lambda_1(l_1+l_3-m_2)}(-1)^{l_1(l_2+l_3-m_1)}\,(-1)^{l_2(l_1+l_2-m_3)} \\
&\times (-1)^{L(l_1+l_2-m_3)}\,(-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}}(-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \\
&\times \frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}}(l_1\, m_1\, \lambda_1, l_2\, m_2\, \lambda_2 | l_3\, -m_3\, \lambda_3)_q. \qquad (4.11)
\end{aligned}$$

The parity independent $sq3 - j$ symbols satisfy the constraints (4.2) and (4.3) and they have symmetry properties similar to those of the $sq3 - j\lambda$ symbols: for an even permutation of columns

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix}_q = \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix}_q, \qquad (4.12)$$

and for an odd permutation of columns

$$\begin{aligned}
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q &= \alpha \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_{q^{-1}} \\
&= \alpha \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix}_{q^{-1}} = \alpha \begin{pmatrix} l_3 & l_2 & l_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q^{-1}}, \qquad (4.13)
\end{aligned}$$

where α is the phase factor

$$\alpha = (-1)^{\sum_{i=1}^{3}\frac{(l_i-m_i)(l_i-m_i-1)}{2}}(-1)^{\sum_{i=1}^{3}l_i m_i}. \qquad (4.14)$$

As $sq3-j\lambda$ symbols, these $sq3-j$ symbols satisfy also a "mirror" symmetry and a conditional Regge symmetry. But for the latter property, the $sq3-j$ symbols are not invariant, they are multiplied by a phase factor, cf. Ref.[2]. Moreover, the $sq3-j$ symbols satisfy the same pseudo-orthogonality relations (4.9) (4.10) as $sq3-j\lambda$ symbols.
Using the parity independent $sq3-j$ symbol, one can define a parity independent invariant metric

$$C^{lq}_{mn} = q^{\frac{1}{2}m}\sqrt{[2l+1]}\begin{pmatrix} l & l & 0 \\ m & n & 0 \end{pmatrix}_q = (-1)^{l(l-m)}(-1)^{\frac{(l-m)(l-m-1)}{2}}q^{\frac{m}{2}}\delta_{m,-n}. \qquad (4.15)$$

It is related to the parity dependent invariant metric $C^{lq}_{mn}(\lambda)$ defined in relation (3.16) by

$$C^{lq}_{mn} = (-1)^{(l-m)(\lambda+l)}C^{lq}_{mn}(\lambda). \qquad (4.16)$$

The parity independent $sq3-j$ symbols and invariant metric will be used in the following to construct the parity independent symmetric $6-j$ symbols for the quantum superalgebra $U_s(osp(1|2))$.
In the limit $q \to 1$, $sq3-j$ symbols become identical to the parity independent $s3-j$ symbols for the superalgebra $osp(1|2)$ defined in Ref.[10].

V Racah Coefficients for the quantum superalgebra $U_q(osp(1|2))$.

A Tensor product of three irreducible representations

Let us consider the tensor product of three irreducible representations of the quantum superalgebra $U_q(osp(1|2))$ of the same class ϵ

$$T^{l_1\epsilon}_{\varphi_1\psi_1} \otimes T^{l_2\epsilon}_{\varphi_2\psi_2} \otimes T^{l_3\epsilon}_{\varphi_3\psi_3}. \qquad (5.1)$$

The representation space for such a tensor product is the tensor product of the representation spaces

$$V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2) \otimes V^{l_3}(\lambda_3), \qquad (5.2)$$

with a basis spanned by the vectors

$$e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) \otimes e^{l_3}_{m_3}(\lambda_3). \qquad (5.3)$$

The bilinear Hermitean form in the graded space (5.2) is defined with the bilinear Hermitean forms in each space $V^{l_i}(\lambda_i)$, $i = 1,2,3$, in the following way

$$\begin{aligned}(e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) \otimes e^{l_3}_{m_3}(\lambda_3)\ ,\ e^{l_1}_{n_1}(\lambda_1) \otimes e^{l_2}_{n_2}(\lambda_2) \otimes e^{l_3}_{n_3}(\lambda_3))\\ = (-1)^{(\sum_{i<j}(l_i-m_i+\lambda_i)(l_j-m_j+\lambda_j))}(-1)^{(\sum_{i=1}^{3}\varphi_i(l_i-m_i)+\psi_i)}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}. \qquad (5.4)\end{aligned}$$

The reduction of the tensor product of representations (5.1) can be done, as in the classical case, in two different schemes. In the first scheme, one couples first the representations T^{l_1} and T^{l_2} and then the result is coupled to the representation T^{l_3} in order to give as a final result the representation T^l. This scheme can expressed in the short way

$$T^l \subset ((T^{l_1} \otimes T^{l_2})_q \otimes T^{l_3})_q. \qquad (5.5)$$

The reduced basis corresponding to this scheme is given by the expression

$$\begin{aligned}e^{lq}_m(l_{12}, l_3, \lambda) &= \sum_{m_i}(l_1m_1\lambda_1, l_2m_2\lambda_2|l_{12}m_{12}\lambda_{12})_q \\ &\times (l_{12}m_{12}\lambda_{12}, l_3m_3\lambda_3|lm\lambda)_q\, e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) \otimes e^{l_3}_{m_3}(\lambda_3), \qquad (5.6)\end{aligned}$$

where $i = 1,2,3,12$ and we have

$$m = m_1 + m_2 + m_3, \qquad (5.7)$$

$$\lambda = \sum_{i=1}^{3}(\lambda_i + l_i) + l. \qquad (5.8)$$

The operators $H, v_\pm$ are represented in this reduced representation space by

$$v^{\otimes}_{\pm}(12,3) = (T^{l_1} \otimes T^{l_2} \otimes T^{l_3})((\Delta \otimes id)\Delta(v_\pm)), \qquad (5.9)$$

$$H^{\otimes}(12,3) = (T^{l_1} \otimes T^{l_2} \otimes T^{l_3})((\Delta \otimes id)\Delta(H)). \qquad (5.10)$$

In the second scheme, one couples T^{l_1} with result $T^{l_{23}}$ of the coupling of representations T^{l_2} and T^{l_3} in order to yield T^l. That is we consider

$$T^l \subset (T^{l_1} \otimes (T^{l_2} \otimes T^{l_3})_q)_q. \qquad (5.11)$$

In this case, the reduced basis vectors are of the form

$$\begin{aligned}e^{lq}_m(l_1, l_{23}, \lambda) &= \sum_{m_i}(l_2m_2\lambda_2, l_3m_3\lambda_3|l_{23}m_{23}\lambda_{23})_q \\ &\times (l_1m_1\lambda_1, l_{23}m_{23}\lambda_{23}|lm\lambda)_q\, e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) \otimes e^{l_3}_{m_3}(\lambda_3), \qquad (5.12)\end{aligned}$$

here $i = 1,2,3,23$ and m and λ satisfy the same condition as in the previous reduction scheme.
The operators H and $v_\pm$ are represented in this case by

$$v^{\otimes}_{\pm}(1,23) = (T^{l_1} \otimes T^{l_2} \otimes T^{l_3})((id \otimes \Delta)\Delta(v_\pm)), \qquad (5.13)$$

$$H^{\otimes}(12,3) = (T^{l_1} \otimes T^{l_2} \otimes T^{l_3})((id \otimes \Delta)\Delta(H)). \qquad (5.14)$$

Due to the coassociativity of the coproduct, we have

$$v^{\otimes}_{\pm}(1,23) = v^{\otimes}_{\pm}(12,3) = v^{\otimes}_{\pm}(123), \qquad (5.15)$$

$$H^{\otimes}(12,3) = H^{\otimes}(1,23) = H^{\otimes}(123) \qquad (5.16)$$

and

$$\begin{aligned}v^{\otimes}_{\pm}(123) &= T^{l_1}(v_\pm) \otimes q^{T^{l_2}(H)} \otimes q^{T^{l_3}(H)} + q^{-T^{l_1}(H)} \otimes T^{l_2}(v_\pm) \otimes q^{T^{l_3}}(H) \\ &+ q^{-T^{l_1}(H)} \otimes q^{-T^{l_2}(H)} \otimes T^{l_3}(v_\pm), \qquad (5.17)\end{aligned}$$

$$H^{\otimes}(123) = T^{l_1}(H)\otimes T^{l_2}(1)\otimes T^{l_3}(1) + T^{l_1}(1)\otimes T^{l_2}(H)\otimes T^{l_3}(1) + T^{l_1}(1)\otimes T^{l_2}(1)\otimes T^{l_3}(H). \quad (5.18)$$

Using the recursion formulae (3.14) and (3.15), one can check that the operators $v_{\pm}^{\otimes}(123)$ and $H^{\otimes}(123)$ act on the bases $e_m^{lq}(l_1,l_{23},\lambda)$ and $e_m^{lq}(l_{12},l_3,\lambda)$ in the standard way given by formulae (2.12), (2.14). The bases $e_m^{lq}(l_1,l_{23},\lambda)$ and $e_m^{lq}(l_{12},l_3,\lambda)$ are orthogonal and normalized in the following way

$$(e_m^{lq}(l_{12},l_3,\lambda)\ ,\ e_{m'}^{l'q}(l'_{12},l_3,\lambda)) = (-1)^{\varphi_{12,3}(l-m)+\psi_{12,3}}\ \delta_{ll'}\ \delta_{mm'}\ \delta_{l_{12}l'_{12}}\ , \quad (5.19)$$

$$(e_m^{lq}(l_1,l_{23},\lambda)\ ,\ e_{m'}^{l'q}(l_1,l'_{23},\lambda)) = (-1)^{\varphi_{1,23}(l-m)+\psi_{1,23}}\ \delta_{ll'}\ \delta_{mm'}\ \delta_{l_{23}l'_{23}}\ , \quad (5.20)$$

where

$$\varphi_{1,23} = \varphi_{12,3} = \mathcal{L} + \lambda_1 + \lambda_2 + \varphi_3 \quad mod(2), \quad (5.21)$$

$$\psi_{12,3} = (l_1+l_2+l_{12})(\mathcal{L}+1) + (\lambda_1+\lambda_3+\varphi_2)\mathcal{L} + \sum_{i<j}\lambda_i\lambda_j + \sum_{i=1}^{3}\psi_i \quad mod(2), \quad (5.22)$$

$$\psi_{1,23} = (l_2+l_3+l_{23})(\mathcal{L}+1) + (\lambda_1+\lambda_3+\varphi_2)\mathcal{L} + \sum_{i<j}\lambda_i\lambda_j + \sum_{i=1}^{3}\psi_i \quad mod(2), \quad (5.23)$$

and

$$\mathcal{L} = l_1 + l_2 + l_3 + l. \quad (5.24)$$

B Racah Coefficients

Racah Coefficients for the quantum superalgebra $U_q(osp(1|2))$, called sqRacah coefficients and denoted by $U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)$, are defined in the standard way as the coefficients that relate two reduced bases in two different reduction schemes

$$e_m^{lq}(l_{12},l_3,\lambda) = \sum_{l_{23}}(-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}\ U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ e_m^{lq}(l_1,l_{23},\lambda). \quad (5.25)$$

From the defining relations relations (5.6), (5.12) and from the orthogonality relations for Clebsch-Gordan coefficients (3.12), (3.13) the sqRacah coefficients are related to the Clebsch-Gordan coefficients as follows

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) &= \sum_{m_i}(-1)^{(l-m)\mathcal{L}}(-1)^{\sum_{k<j}(l_k-m_k)(l_j-m_j)} \\ &\times (l_1m_1\lambda_1, l_2m_2\lambda_2|l_{12}m_{12}\lambda_{12})_q(l_{12}m_{12}\lambda_{12}, l_3m_3\lambda_3|lm\lambda)_q \\ &\times (l_2m_2\lambda_2, l_3m_3\lambda_3|l_{23}m_{23}\lambda_{23})_q(l_1m_1\lambda_1, l_{23}m_{23}\lambda_{23}|lm\lambda)_q, \end{aligned} \quad (5.26)$$

where $i = 1,2,3,12,23$ and $k,j = 1,2,3$. Similarly to Racah coefficients for $U_q(osp(1|2))$, sqRacah coefficients depend on the parities λ_i but they depend neither on class ϵ nor on the signature parameters φ_i, ψ_i i=1,2,3.
Considering the action of operators $v_{\pm}^{\otimes}(123)$ and $H^{\otimes}(123)$ on the defining relation (5.25) for sqRacah coefficients and using the relation $\varphi_{12,3} = \varphi_{1,23}$, one can show that the coefficients $U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)$ do not depend on m. Besides, sqRacah coefficients satisfy the following orthogonality relations

$$\sum_{l_{12}}(-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ U^s(l_1,l_2,l_3,l,l_{12},l'_{23},q) = (-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}\delta_{l_{23}l'_{23}}, \quad (5.27)$$

$$\sum_{l_{23}}(-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ U^s(l_1,l_2,l_3,l,l'_{12},l_{23},q) = (-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}\delta_{l_{12}l'_{12}}. \quad (5.28)$$

Using these properties, one can easily inverse relation (5.25) :

$$e_m^{lq}(l_1,l_{23},\lambda) = \sum_{l_{12}}(-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)e_m^{lq}(l_{12},l_3,\lambda). \quad (5.29)$$

Let us now derive some special values of sqRacah coefficients. If any one of the superspins l_1,l_2,l_3 is zero, then by definition the sqRacah coefficient is equal to unity

$$U^s(0,l_2,l_3,l,l_2,l,q) = U^s(l_1,0,l_3,l,l_1,l_3,q) = U^s(l_1,l_2,0,l,l,l_2,q) = 1. \quad (5.30)$$

The case $l=0$ can be calculated directly by means of expression (5.26), which gives

$$U^s(l_1,l_2,l_3,0,l_3,l_1,q) = (-1)^{(\lambda_1+\lambda_3)(l_1+l_2+l_3)}. \quad (5.31)$$

Similarly, it can be proved that

$$U^s(l_1,l_1,l_3,l_3,0,l_{13},q) = (-1)^{\lambda_1(l_1+l_3+l_{13})}(-1)^{\frac{(l_1+l_3-l_{13})(l_1+l_3-l_{13}+1)}{2}}\left(\frac{[2l_{13}+1]}{[2l_1+1][2l_3+1]}\right)^{\frac{1}{2}}, \quad (5.32)$$

$$U^s(l_1,l_2,l_2,l_3,l_{12},0,q) = (-1)^{\lambda_2(l_1+l_2+l_{12})}(-1)^{\frac{(l_1+l_2-l_{12})(l_1+l_2-l_{12}+1)}{2}}\left(\frac{[2l_{12}+1]}{[2l_1+1][2l_2+1]}\right)^{\frac{1}{2}}. \quad (5.33)$$

These formulae are very similar to the corresponding ones for quantum algebra $U_q(su(2))$ cf. Ref.[6].

Taking into account the fact that the coefficients $U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)$ do not depend on m and making use of the identity

$$\sum_{m=-l}^{l}(-1)^{(l-m)}q^{-m} = [2l+1], \quad (5.34)$$

one can express the coefficient $U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)$ in terms of Clebsch-Gordan coefficients in a more symmetric way

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) &= \frac{1}{[2l+1]}\sum_{all\ m}(-1)^{(l-m)(\mathcal{L}+1)}(-1)^{\sum_{k<j}^{kj=1,2,3}(l_k-m_k)(l_j-m_j)}q^{-m} \\ &\times (l_1m_1\lambda_1, l_2m_2\lambda_2|l_{12}m_{12}\lambda_{12})_q(l_{12}m_{12}\lambda_{12}, l_3m_3\lambda_3|lm\lambda)_q \\ &\times (l_2m_2\lambda_2, l_3m_3\lambda_3|l_{23}m_{23}\lambda_{23})_q(l_1m_1\lambda_1, l_{23}m_{23}\lambda_{23}|lm\lambda)_q. \end{aligned} \quad (5.35)$$

Using this formula and the "mirror" symmetry of Clebsch-Gordan coefficients (3.21), one can prove the following property of the sqRacah coefficients

$$U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) = U^s(l_1,l_2,l_3,l,l_{12},l_{23},q^{-1}). \quad (5.36)$$

Comparison of the two coupling schemes

$$((T^{l_3}\otimes T^{l_2})_{q^{-1}}\otimes T^{l_1})_{q^{-1}} \quad \text{and} \quad (T^{l_3}\otimes(T^{l_2}\otimes T^{l_1})_{q^{-1}})_{q^{-1}}, \quad (5.37)$$

and use of the first symmetry (3.18) of Clebsch-Gordan coefficients yield the following symmetry of sqRacah coefficients

$$U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \;=\; U^s(l_3,l_2,l_1,l,l_{23},l_{12},q). \tag{5.38}$$

Using symmetry properties of the Clebsch-Gordan coefficients (3.18-3.20), one can write the expression (5.26) in the form

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \;=\; & \frac{\rho}{[2l_1+1]} \sum_{all\ m} (-1)^{(l_1-m_1)(\mathcal{L}+1)} (-1)^{\sum_{k<j}^{kj=1,2,3}(l_k-m_k)(l_j-m_j)} q^{-m_1} \\ & \times (l_{12}m_{12}\lambda_{12}, l_2m_2\lambda_2 | l_1m_1\lambda_1)_q (lm\lambda, l_3m_3\lambda_3|l_{12}m_{12}\lambda_{12})_q \\ & \times (l_3m_3\lambda_3, l_2m_2\lambda_2|l_{23}m_{23}\lambda_{23})_q (lm\lambda, l_{23}m_{23}\lambda_{23}|l_1m_1\lambda_1)_q , \end{aligned} \tag{5.39}$$

where the phase ρ is

$$\rho \;=\; (-1)^{\mathcal{L}(l_2+l_3+l_{23})}(-1)^{\lambda_2(l_2+l+l_{23}+l_{12})+\lambda_3(l_1+l_3+l_{23}+l_{12})}. \tag{5.40}$$

This relation, together with relation (5.35), leads to the symmetry property

$$U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \;=\; \rho\, U^s(l,l_3,l_2,l_1,l_{12},l_{23},q). \tag{5.41}$$

In the next section, we will show that similarly as in the non deformed case of $osp(1|2)$ [10], sqRacah coefficients are related to $6-j$ symbols that have better symmetry properties.

VI The $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$

A Parity dependent $6-j$ symbols.

By analogy with the corresponding construction of $su(2)$ $6-j$ symbols, the parity dependent $sq6-j\lambda$ symbols for the quantum superalgebra $U_q(osp(1|2))$ are defined by contraction of a product of four $sq3-j\lambda$ symbols with use of the parity dependent invariant metric $C^{lq}_{mn}(\lambda)$:

$$\begin{aligned} \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{matrix} \right\}_q \;=\; & \sum_{all\ m,m'} \left(\prod_{i=1}^{6} C^{l_i q}_{m_i m_i'}(\lambda_i) \right) \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \\ & \times \begin{pmatrix} l_1\lambda_1 & l_5\lambda_5 & l_6\lambda_6 \\ m_1' & m_5' & m_6 \end{pmatrix}_q \begin{pmatrix} l_4\lambda_4 & l_2\lambda_2 & l_6\lambda_6 \\ m_4 & m_2' & m_6' \end{pmatrix}_q \begin{pmatrix} l_4\lambda_4 & l_5\lambda_5 & l_3\lambda_3 \\ m_4' & m_5 & m_3' \end{pmatrix}_q . \end{aligned} \tag{6.1}$$

Using the explicit form (3.16) of the invariant metric, the symbol may be written

$$\begin{aligned} \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{matrix} \right\}_q \;=\; & \sum_{all\ m} (-1)^{(\sum_{i=1}^{6}\lambda_i(l_i-m_i))} (-1)^{(\sum_{i=1}^{6}\frac{(l_i-m_i)(l_i-m_i-1)}{2}} q^{\frac{1}{2}(m_4+m_5+m_6)} \\ & \times \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \begin{pmatrix} l_1\lambda_1 & l_5\lambda_5 & l_6\lambda_6 \\ -m_1 & -m_5 & m_6 \end{pmatrix}_q \\ & \times \begin{pmatrix} l_4\lambda_4 & l_2\lambda_2 & l_6\lambda_6 \\ m_4 & -m_2 & -m_6 \end{pmatrix}_q \begin{pmatrix} l_4\lambda_4 & l_5\lambda_5 & l_3\lambda_3 \\ -m_4 & m_5 & -m_3 \end{pmatrix}_q . \end{aligned} \tag{6.2}$$

In order for the $sq6-j\lambda$ to exist, the four superspin-parity triplets

$$\{(l_1,\lambda_1),(l_2,\lambda_2),(l_3,\lambda_3)\}\,, \quad \{(l_3,\lambda_3),(l_4,\lambda_4),(l_5,\lambda_5)\}\,, \tag{6.3}$$

$$\{(l_1,\lambda_1),(l_5,\lambda_5),(l_6,\lambda_6)\}\,, \quad \{(l_2,\lambda_2),(l_4,\lambda_4),(l_6,\lambda_6)\}\,, \tag{6.4}$$

must satisfy triangular constraints of the form (4.2), (4.3) and relation (4.4) for the parities. Comparison of definition (6.2) with the expression (5.35) for the sqRacah coefficients shows that both quantities are proportional. Explicitly, taking into account a change of notation, this relation reads

$$\begin{aligned} U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \;=\; & (-1)^{\lambda_1(l_2+l_3+l_{23})+\lambda_3(l_1+l_2+l_{12})}(-1)^{(l_1+l_3+l_{12}+l_{23})(\mathcal{L}+1)} \\ & \times (-1)^{\frac{\mathcal{L}(\mathcal{L}+1)}{2}} \sqrt{[2l_{12}+1][2l_{23}+1]} \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{matrix} \right\}_q . \end{aligned} \tag{6.5}$$

The symmetry properties of $sq3-j\lambda$ symbols and sqRacah coefficients imply that the $sq6-j\lambda$ symbols satisfy the same symmetry properties as $su(2)$ and $osp(1|2)$ $6-j$ symbols. Namely, they are invariant under any permutation of columns and they are invariant under interchange of upper and lower arguments in each pair of columns.
Besides, the orthogonality relations (5.27), (5.28) for sqRacah coefficients imply the following orthogonality relations for the $sq6-j\lambda$ symbols :

$$\begin{aligned} \sum_{l_{23}} (-1)^{(\mathcal{L}+1)(l_2+l_3+l_{23})} [2l_{12}+1][2l_{23}+1] \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{matrix} \right\}_q \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l'_{12}\lambda'_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{matrix} \right\}_q \\ = (-1)^{(\mathcal{L}+1)(l_1+l_2+l_{12})}\delta_{l_{12}l'_{12}} . \end{aligned} \tag{6.6}$$

$$\begin{aligned} \sum_{l_{12}} (-1)^{(\mathcal{L}+1)(l_1+l_2+l_{12})} [2l_{12}+1][2l_{23}+1] \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l_{23}\lambda_{23} \end{matrix} \right\}_q \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_{12}\lambda_{12} \\ l_3\lambda_3 & l\lambda & l'_{23}\lambda'_{23} \end{matrix} \right\}_q \\ = (-1)^{(\mathcal{L}+1)(l_2+l_3+l_{23})}\delta_{l_{23}l'_{23}} . \end{aligned} \tag{6.7}$$

By means of relations (5.30-5.33), one can readily obtain particular values for $sq6-j\lambda$ symbols with one zero superspin, for instance

$$\left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_3\lambda_3 & 0\;0 & l_1\lambda_1 \end{matrix} \right\}_q = \frac{(-1)^{\frac{(l_1+l_2+l_3)(l_1+l_2+l_3+1)}{2}}}{\sqrt{[2l_1+1][2l_3+1]}}. \tag{6.8}$$

In the limit $q\to 1$, the $sq6-j\lambda$ symbols are identical, up to a phase factor, to the $s6-j\lambda$ symbols for the superalgebra $osp(1|2)$ defined in Ref.[10].

$$(-1)^{\Psi} \left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{matrix} \right\}_{q=1} = \left\{ \begin{matrix} j_1\lambda_1 & j_2\lambda_2 & j_3\lambda_3 \\ j_4\lambda_4 & j_5\lambda_5 & j_6\lambda_6 \end{matrix} \right\}_{osp(1|2)} , \tag{6.9}$$

with $l_i = 2j_i$ and where the phase Ψ is

$$\Psi \;=\; \sum_{i=1}^{6} l_i + (l_1+l_4)(l_2+l_5) + (l_2+l_5)(l_3+l_6) + (l_3+l_6)(l_1+l_4)\,. \tag{6.10}$$

Note that the phase Ψ has the same symmetry as $sq6-j\lambda$ symbol. The difference between the two symbols in Eq.(6.9) follows from the fact that in Ref.[10] a different basis in the representation space has been used. For more details on bases in representation spaces, see Ref.[2].
In the next subsection it will be shown that the parity dependence of $sq6-j\lambda$ symbols can be factored out so that, exactly as in the case of $3-j$ symbols, it is possible to define parity independent $6-j$ symbols.

B Parity independent $6-j$ symbols

Along the same lines as in the previous subsection, we define parity independent $sq6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$ by contraction of four parity independent $sq3-j$ symbols with use of the parity independent metric C^{lq}_{mn}

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}_q = \sum_{all\ m} \left(\prod_{i=1}^{6} C^{l_i q}_{m_i m'_i} \right) \left(\begin{matrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{matrix} \right)_q \left(\begin{matrix} l_1 & l_5 & l_6 \\ m'_1 & m'_5 & m_6 \end{matrix} \right)_q \times \left(\begin{matrix} l_4 & l_2 & l_6 \\ m_4 & m'_2 & m'_6 \end{matrix} \right)_q \left(\begin{matrix} l_4 & l_5 & l_3 \\ m'_4 & m_5 & m'_3 \end{matrix} \right)_q . \quad (6.11)$$

Using the explicit form (4.16) of the invariant metric, the symbol may be written

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}_q = \sum_{all\ m} (-1)^{(\sum_{i=1}^{6} l_i(l_i-m_i))} (-1)^{(\sum_{i=1}^{6} \frac{(l_i-m_i)(l_i-m_i-1)}{2}} q^{\frac{1}{2}(m_4+m_5+m_6)} \times \left(\begin{matrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{matrix} \right)_q \left(\begin{matrix} l_1 & l_5 & l_6 \\ -m_1 & -m_5 & m_6 \end{matrix} \right)_q \times \left(\begin{matrix} l_4 & l_2 & l_6 \\ m_4 & -m_2 & -m_6 \end{matrix} \right)_q \left(\begin{matrix} l_4 & l_5 & l_3 \\ -m_4 & m_5 & -m_3 \end{matrix} \right)_q . \quad (6.12)$$

The $sq6-j$ symbols satisfy the same triangular conditions as the $sq6-j\lambda$ symbols.
Using the definition (6.12) of the $sq6-j\lambda$ symbol and the analytical formula of the $sq3-j$ symbols [2], one can derive the following proportionality relation between $sq6-j\lambda$ and $sq6-j$ symbols

$$\left\{ \begin{matrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ l_4\lambda_4 & l_5\lambda_5 & l_6\lambda_6 \end{matrix} \right\}_q = (-1)^{\Phi(\lambda_i)} \left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}_q , \quad (6.13)$$

where the phase $\Phi(\lambda_i)$ is given by

$$\Phi(\lambda_i) = (\sum_{i=1}^{6} \lambda_i)(\sum_{i=1}^{6} l_i) + (\sum_{i=1}^{6} l_i\lambda_i). \quad (6.14)$$

Thus, all the dependence on λ_i in $sq6-j\lambda$ symbols is contained in the phase factor. It is noticeable that $\Phi(\lambda_i)$ is a completely symmetric function of the six indices and hence, the $sq6-j$ symbols have all the symmetry properties of the $sq6-j\lambda$ symbols. Furthermore, it is easy to check that the $sq6-j$ symbols satisfy pseudo-orthogonality relations similar to relations (6.5), (6.7) satisfied by the $sq6-j\lambda$ symbols.
Let us remark that

$$\Phi(l_i) = 0 \quad mod(2), \quad (6.15)$$

therefore, the $sq6-j$ symbol coincides with the $sq6-j\lambda$ symbol for the fixed parity convention $\lambda_i = l_i \quad mod(2)$. Note also that when one argument vanishes, the value of the $sq6-j$ symbols is

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_3 & 0 & l_1 \end{matrix} \right\}_q = \frac{(-1)^{\frac{(l_1+l_2+l_3)(l_1+l_2+l_3+1)}{2}}}{\sqrt{[2l_1+1][2l_3+1]}} , \quad (6.16)$$

thus, in this particular case, parity dependent and parity independent $6-j$ symbols are equal.
In the limit $q \to 1$, the $sq6-j$ symbols become identical, up to the phase factor Ψ, to the $s6-j$ symbols for the superalgebra $osp(1|2)$ defined in Ref.[10]

$$(-1)^{\Psi} \left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}_{q=1} = \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}^{S}_{osp(1|2)} , \quad (6.17)$$

with $l_i = 2j_i$.

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References

[1] P. Minnaert, M. Mozrzymas, J. Math. Phys. **35** (6), (1994).

[2] P. Minnaert, M. Mozrzymas, LPTB 94-14 preprint, July 1994.

[3] A. N. Kirillov, N. YU. Reshetikhin, LOMI Preprint E-9-88, Leningrad, 1988.

[4] Zhong-Qi-MA, Preprint IC/89/162, ICTP, Trieste (1989).

[5] M. Nomura, J. Math. Phys. **30**(10), 2397 (1989).

[6] Yu. F. Smirnov, V. N. Tolstoy, Yu. I. Kharitonov, Sov. J. Nucl. Phys. **53**, 1068, (1991).

[7] M. Scheunert, W. Nahm, V. Rittenberg, J. Math. Phys. **18**, 155 (1977).

[8] F. A. Berezin, V. N. Tolstoy, Commun. Math. Phys. **8**, 409, (1981).

[9] P. Minnaert, M. Mozrzymas, J. Math. Phys. **33** (5), 1582, (1992).

[10] M. Daumens, P. Minnaert, M. Mozrzymas, S. Toshev, J. Math. Phys. **34** (6), 2475, (1993).

[11] P. P. Kulish, RIMS Preprint 615, Kyoto, 1988.

[12] P. P. Kulish, N. Yu. Reshetikhin, Lett. Math. Phys. **18**, 143, (1989).

[13] P. P. Kulish, LOMI Preprint published in Zapiski Nautchnovo Seminaria, LOMI, 1990.

[14] H. Saleur, Nucl. Phys.**B336**, 363, (1990).

[15] P. Minnaert, M. Mozrzymas, preprint LPTB 94-11, July 1994.