Properties of Clebsch-Gordan Coefficients and $3-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$

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Abstract

The structure of finite dimensional representations of the quantum superalgebra $U_q(osp(1|2))$ is studied. Tensor product of two representation spaces with arbitrary (not necessarily positive definite) Hermitean forms is considered. An explicit analytical formula for Clebsch-Gordan coefficients is derived using the projection operator method. Pseudo-orthogonality relations are given and symmetry properties, including Regge symmetry, are discussed. The quantum analogue of super $3-j$ symbols are defined and their symmetry properties are analyzed.

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1 Introduction.

Recently, quantum algebras [1] have raised considerable interest among theoretical physicists. This wide interest may be explained by the fact that quantum algebras are continuous deformations of well known Lie algebras and that their representation theory is very similar to that of non deformed Lie algebras. In particular, the irreducible representations of quantum algebra $U_q(su(2))$ have the same structure as those of $su(2)$ algebra. It has been shown in several papers [2], [3], [4], [5], that for the quantum algebra $U_q(su(2))$, the Racah-Wigner calculus can be fully developed following the same lines as in the classical case. It is quite remarkable that all topics that are relevant for the classical Racah-Wigner calculus have their direct quantum analogue in the representation theory of quantum algebra $U_q(su(2))$.
A very efficient method for the analysis of the properties of irreducible representations is the projection operator method, first introduced to derive the $su(2)$ Clebsch-Gordan coefficients (C-Gc) by Shapiro [6]. Recently, Smirnov, Tolstoy and Kharitonov [7], [8] have used the projection operator method to derive an analytical formula for the C-Gc of the quantum algebra $U_q(su(2))$ and to study the corresponding Racah-Wigner calculus .

The superalgebra $osp(1|2)$ was first introduced by Pais and Rittenberg [9]. Using the inclusion $sl(2) \subset osp(1|2)$, Scheunert, Nahm and Rittenberg [10] and Berezin and Tolstoy [11] shew that any $osp(1|2)$ Clebsch-Gordan coefficient can be factorized into the product of a usual $su(2)$ Clebsch-Gordan coefficient and a so-called scalar factor. In Refs.[12], [13], [14] (see also references therein), it has been shown that the Racah-Wigner calculus can also be constructed for this superalgebra. In particular, the super $s3$-j and super $s6$-j symbols have been defined and expressed in terms of the classical 3-j and 6-j symbols.

The quantum superalgebra $U_q(osp(1|2))$, which is the subject of this article, can be considered either as the quantum analogue of $osp(1|2)$ superalgebra or as the super-analogue of $U_q(su(2))$ quantum algebra. However, it is to be noticed that there is no inclusion $U_q(sl(2)) \not\subset U_q osp(1|2))$. This quantum superalgebra has been defined and studied by Kulish and Reshetikhin [15], [16]. Its Clebsch-Gordan coefficients were derived by Kulish [17], using a recursion relation, and in Ref.[18] using the projection operator method. A particular case of C-Gc was also given by Saleur [19].

In Ref.[18], it has been shown that in the reduction of the tensor product of two irreducible representation spaces of $U_q(osp(1|2))$ with positive definite bilinear Hermitean forms, it appears representation spaces the Hermitean forms of which are not positive definite and where the highest weight vector is normalized to -1. In this paper, in order to study the most general case, we consider the reduction of tensor product of representation spaces whose bilinear Hermitean forms are not necessarily positive definite, and using the projection operator method, we derive an analytical formula for the Clebsch-Gordan coefficients. This analytical formula does not differ from the analytical formula obtained in [18], which proves that Clebsch-Gordan coefficients do not depend on the signatures of the bilinear Hermitean forms defined in the representation spaces. Besides, we study several properties of Clebsch-Gordan coefficients: orthogonality relations, symmetry properties, particular values and it is also shown that Clebsch-Gordan coefficients satisfy a conditional Regge symmetry.
Similarly to the case of $U_q(su(2))$, the study of the symmetry properties of the Clebsch-Gordan coefficients allows one to define for $U_q(osp(1|2))$, $sq3-j$ symbols that possess good

symmetry properties. We first define $sq3-j\lambda$ symbols which depend on the parities λ of the graded representation space bases. Then, we show that the dependence on parities can be factorized out, so that one can define for quantum superalgebra $U_q(osp(1|2))$ parity independent $3-j$ symbols that are superanalogues of $3-j$ symbols for quantum algebra $U_q(su(2))$ and have symmetry properties very similar to those of these $3-j$ symbols [7], [8].

This paper has the following structure: Section II contains the definition of the quantum superalgebra $U_q(osp(1|2))$ and the basic properties of its irreducible representations, and we recall the explicit expression of the projection operator for the quantum superalgebra $U_q(osp(1|2))$. In section III, we consider the tensor product of two irreducible representations of $U_q(osp(1|2))$ with arbitrary Hermitean forms and the projection operator is used to derive an analytical formula for the Clebsch-Gordan coefficients. Pseudo-orthogonality relations, recursion relations and symmetry properties of the Clebsch-Gordan coefficients are given in the following subsections of section III. Section IV is devoted to $3-j$ symbols: parity dependent and parity independent $3-j$ symbols for $U_q(osp(1|2))$ are defined and their properties are discussed.

2 The irreducible representations of the quantum superalgebra $U_q(osp(1|2))$.

2.1 Representation spaces.

The quantum superalgebra $U_q(osp(1|2))$ is generated by three elements: H (even) and $v_\pm$ (odd) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\tfrac{1}{2}v_\pm, \quad [v_+, v_-]_+ = -\frac{sh\,(\eta H)}{sh\,(2\eta)}, \tag{2.1}$$

where the deformation parameter η is real and is related to the q-deformation parameter by $q = e^{-\frac{\eta}{4}}$. One can also define the quantum analogue of $L_\pm$ (which in non deformed case together with H span $sl(2)$ subalgebra of $osp(1|2)$) and their quantum commutation relations. Explicit formulae, worked out by Lukierski and Nowicki [20], show that for the quantum superalgebra $U_q(osp(1|2))$, the generators H and $L_\pm$ do not form a $U_q(sl(2))$ subalgebra. Thus, for $U_q(osp(1|2))$ there is no inclusion $U_q(sl(2)) \not\subset U_q(osp(1|2))$.
The following expressions for coproduct Δ, antipode S and counit ϵ define on $U_q(osp(1|2))$ the structure of a Hopf algebra

$$\Delta(v_\pm) = v_\pm \otimes q^H + q^{-H} \otimes v_\pm, \tag{2.2}$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(1) = 1 \otimes 1, \tag{2.3}$$

$$S(H) = -H, \quad S(v_\pm) = -q^{\pm\frac{1}{2}} v_\pm, \tag{2.4}$$

$$\epsilon(H) = \epsilon(v_\pm) = 0, \quad \epsilon(1) = 1. \tag{2.5}$$

For the homogenous elements of $U_q(osp(1|2))$, one can define a parity function deg such that

$$deg(H) = 0, \quad deg(v_\pm) = 1, \quad deg(1) = 0, \tag{2.6}$$

$$deg(\Delta(H)) = deg(H), \quad deg(\Delta(v_\pm)) = deg(v_\pm), \quad deg(\Delta(1)) = deg(1), \tag{2.7}$$

and furthermore one defines

$$deg(X_1 X_2 \ldots X_n) = \sum_{i=1}^{n} deg(X_i), \tag{2.8}$$

for any $X_i \in U_q(osp(1|2))$. Obviously, any product of generators $H, v_\pm$, involving an even (odd) number of generators $v_\pm$ is an even (odd) element of $U_q(osp(1|2))$.
A finite dimensional representation space V of $U_q(osp(1|2))$ is a graded vector space $V = V(0) \oplus V(1)$ where $V(0)$ is an even subspace and $V(1)$ is an odd subspace. We assume that there exists in V a bilinear Hermitean form (,), not necessarily positive definite, such that

$$(V(0), V(1)) = 0. \tag{2.9}$$

For the representation space, there exists also a parity function deg, defined on homogenous elements of V, such that

$$deg(y) = \begin{cases} 0 & \text{if} \quad y \in V(0), \\ 1 & \text{if} \quad y \in V(1). \end{cases} \tag{2.10}$$

A representation of the quantum superalgebra $U_q(osp(1|2))$ in the finite dimensional graded space V is a homomorphism T

$$T:\; U_q(osp(1|2)) \;\to\; L(V, V), \tag{2.11}$$

of the associative graded algebra $U_q(osp(1|2))$ in the associative graded algebra of linear operators in V, $L(V, V)$, such that for any $X \in U_q(osp(1|2))$

$$deg(X) = 0 \;\Rightarrow\; T(X)(V(0)) \subset V(0), \;\; T(X)(V(1)) \subset V(1), \tag{2.12}$$

$$deg(X) = 1 \;\Rightarrow\; T(X)(V(0)) \subset V(1), \;\; T(X)(V(1)) \subset V(0). \tag{2.13}$$

The operators $T(H), T(v_\pm)$, which represent the generators $H, v_\pm$ satisfy the defining relations

$$[T(H), T(v_\pm)] = \pm\tfrac{1}{2}T(v_\pm), \quad [T(v_+), T(v_-)]_+ = -\frac{sh\,(\eta T(H))}{sh\,(2\eta)}, \tag{2.14}$$

From the (anti)commutation relations (2.1), one can derive the following fundamental formula

$$(T(v_+))^m (T(v_-))^n = \sum_{i=0}^{min(m,n)} (-1)^{mn} (-1)^{\frac{i(i-1)}{2}} \frac{[m]![n]!}{[i]![m-i]![n-i]!} \times (T(v_-))^{n-i} (T(v_+))^{m-i} \frac{[4T(H) - n + m]!}{[4T(H) - n + m - i]!} \gamma^i, \tag{2.15}$$

where

$$\gamma = \frac{ch\,(\frac{\eta}{4})}{sh\,(2\eta)}, \tag{2.16}$$

and $[n]$ is the Kulish symbol defined as follows [17]

$$[n] = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}} = \begin{cases} \dfrac{sh\,(\frac{\eta}{4}(n))}{ch(\frac{\eta}{4})} & \text{if} \quad n \quad \text{is even} \\[2ex] \dfrac{ch\,(\frac{\eta}{4}(n))}{ch(\frac{\eta}{4})} & \text{if} \quad n \quad \text{is odd}. \end{cases} \tag{2.17}$$

We have set $q = e^{-\frac{\eta}{2}}$ therefore, the symbol $[n]$ is positive if $\eta > 0$. Note that the limit $q \to 1$ depends on the parity of the argument n of the symbol. As a particular case of relation (2.15), we have (see also Ref.[19])

$$T(v_+)(T(v_-))^n = (-1)^n((T(v_-))^n T(v_+) + (T(v_-))^{n-1}[4H - n + 1][n]\gamma) \tag{2.18}$$

Scheunert, Nahm and Rittenberg [21] have introduced the concept of grade star representations. In such a representation the operators satisfy the following relations

$$T(H)^* = T(H), \quad T(v_\pm)^* = \pm(-1)^\epsilon T(v_\mp), \quad T(1)^* = T(1), \tag{2.19}$$

where $(\star)$ is the grade adjoint operation defined in the following way

$$(T(X)^* f, g) = (-1)^{deg(X)deg(f)}(f, T(X)g), \tag{2.20}$$

for any $X \in U_q(osp(1|2))$ and $f, g \in V$. The index $\epsilon = 0, 1$ defines the class of the representation. The grade adjoint operation $(\star)$ has the following properties

$$(X_1 X_2)^* = (-1)^{deg(X_1)deg(X_2)} X_2^* X_1^*, \quad (X_1^*)^* = (-1)^{deg(X_1)} X_1 . \tag{2.21}$$

Thus, $(\star)$ is an anti-isomorphism of the graded algebra $L(V, V)$ (i.e., it changes the order of the product of operators) and we have

$$(T(X_1)T(X_2)....T(X_n))^* = (-1)^{\sum_{i<j} deg(X_i)deg(X_j)} T(X_n)^* T(X_{n-1})^* ....T(X_1)^*, \tag{2.22}$$

2.2 Finite dimensional irreducible representations.

Let V^l be a finite dimensional rapresentation space with highest weight l (l is a non negative integer). The highest weight vector is denoted by e_l^l and is defined by the following properties

$$T(H)(e_l^l) = \frac{l}{2} e_l^l, \qquad T(v_+)(e_l^l) = 0, \tag{2.23}$$

$$(e_l^l, e_l^l) = (-1)^\psi, \quad \text{with } \psi = 0, 1. \tag{2.24}$$

The last condition is motivated by the fact that, for a tensor product of two irreducible representations of $U_q(osp(1|2))$ with positive definite bilinear Hermitean forms, in the Clebsch-Gordan series appear representation spaces whose Hermitean forms are not positive definite and where the highest weight vector is normalized to -1 [18]. Therefore, in order to study the general case, we consider representation spaces where condition (2.24) holds.
From relations (2.23), it follows that e_l^l belongs either to $V^l(0)$ or to $V^l(1)$, i.e., it has a definite parity. Therefore, we set $e_l^l \equiv e_l^l(\lambda)$ and $V^l \equiv V^l(\lambda)$, where $\lambda = 0, 1$ is the parity of the highest weight vector in the graded representation space.
Thus, any finite dimensional representation of $U_q(osp(1|2))$ is characterised by four parameters : the superspin l (a non negative integer), the parity $\lambda = 0, 1$, the normalization parameter $\psi = 0, 1$ and the class $\epsilon = 0, 1$ of the representation.
One can construct an orthogonal basis in $V^l(\lambda)$ in the usual way, by repeated application of the lowering operator v_-

$$e_m^{lq}(\lambda) = \frac{1}{\sqrt{|N(q, l, m)|}} v_-^{l-m} e_l^l(\lambda). \tag{2.25}$$

Using relation (2.15), the normalization factor $N(q, l, m)$ is determined to be

$$N(q, l, m) = (-1)^{(\epsilon+\lambda+1)(l-m)+\psi} \left(\frac{[l-m]![2l]!}{[l+m]!} \gamma^{l-m} \right), \tag{2.26}$$

and $m = l, l-1, \dots -l+1, -l$, so that the representation space $V(\lambda)$ is $2l+1$ dimensional. The vectors $e_m^{lq}(\lambda)$ are orthogonal and normalized to ± 1; more precisely we have

$$(e_m^{lq}(\lambda), e_{m'}^{lq}(\lambda)) = (-1)^{(\epsilon+\lambda+1)(l-m)+\psi} \delta_{mm'} . \tag{2.27}$$

Now, in order to characterize the representation of $U_q(osp(1|2))$, it is convenient to introduce a new parameter $\varphi = 0, 1$

$$\varphi = \lambda + \epsilon + 1 \quad mod(2) . \tag{2.28}$$

With this parameter, the normalization relation (2.27) can be written in the more compact form

$$(e_m^{lq}(\lambda), e_{m'}^{lq}(\lambda)) = (-1)^{\varphi(l-m)+\psi} \delta_{mm'} . \tag{2.29}$$

In the particular case

$$\psi = 0, \quad \varphi = 0 \quad \Leftrightarrow \quad \lambda = \epsilon + 1 \quad mod(2), \tag{2.30}$$

the basis vectors $e_m^{lq}(\lambda)$ are normalized to $+1$, which means that the Hermitean form (,) is positive definite. This case was considered in Ref.[18]. In the following, we shall consider the general case where φ and ψ are not fixed.
The parity of the basis vectors $e_m^{lq}(\lambda)$ is determined by the values of l, m and λ,

$$deg(e_m^l(\lambda)) = l - m + \lambda \quad mod(2) . \tag{2.31}$$

The operators $T(v_\pm)$ and $T(H)$ act on the basis $e_m^{lq}(\lambda)$ in the following way :

$$T(H)e_m^{lq}(\lambda) = \frac{m}{2} e_m^{lq}(\lambda), \tag{2.32}$$

$$T(v_+)e_m^{lq}(\lambda) = (-1)^{l-m} \sqrt{[l-m][l+m+1]\gamma}\, e_{m+1}^{lq}(\lambda), \tag{2.33}$$

$$T(v_-)e_m^{lq}(\lambda) = \sqrt{[l+m][l-m+1]\gamma}\, e_{m-1}^{lq}(\lambda). \tag{2.34}$$

This action of the operators $T(v_\pm)$ and $T(H)$ does not depend on the parameters λ, ϵ, ψ. The representation T, characterized by the class ϵ and acting in the representation space $V^l(\lambda)$ whose Hermitean form signature is determined by φ and ψ is denoted by $T^{l\epsilon}_{\varphi\psi}$. However, in the following, the indices ϵ, φ, ψ will often be omitted in the notation. Note that from relation (2.28), the parity index λ can be expressed in terms of the other parameters

$$\lambda = \epsilon + \varphi + 1, \quad mod(2). \tag{2.35}$$

In the limit $q \to 1$, we obtain a basis for the superalgebra $osp(1|2)$ representation space

$$\lim_{q \to 1} e_m^{lq}(\lambda) = e_m^l(\lambda), \tag{2.36}$$

however, this basis $e_m^l(\lambda)$ is not the classical basis $f_m^l(\lambda)$ of $osp(1|2)$ built from the reduction chain $sl(2) \subset osp(1|2)$, cf. Ref.[21],[11]. The classical basis can be obtained as the limit $q \to 1$ of another basis $f_m^l(\lambda)$ defined by

$$f_m^{lq}(\lambda) = (-1)^{\frac{(l-m)(l-m-1)}{2}} e_m^{lq}(\lambda), \tag{2.37}$$

i.e., we have

$$f^l_m(\lambda) = \lim_{q\to 1} f^{lq}_m(\lambda). \tag{2.38}$$

The operators $T(v_\pm)$ and $T(H)$ act on the basis $f^{lq}_m(\lambda)$ in the following way :

$$\begin{aligned} T(H)f^{lq}_m(\lambda) &= \frac{m}{2}\, f^{lq}_m(\lambda), & (2.39)\\ T(v_+)f^{lq}_m(\lambda) &= -\sqrt{[l-m][l+m+1]\gamma}\; f^{lq}_{m+1}(\lambda), & (2.40)\\ T(v_-)f^{lq}_m(\lambda) &= (-1)^{(l-m)}\sqrt{[l+m][l-m+1]\gamma}\; f^{lq}_{m-1}(\lambda). & (2.41)\end{aligned}$$

For $q=1$ the action of the operators $T(v_\pm)$ and $T(H)$ on the basis $f^l_m(\lambda)$ takes the classical form [11], [21]

$$H f^l_m(\lambda) = \frac{m}{2}\, f^l_m(\lambda), \tag{2.42}$$

$$v_+ f^l_m(\lambda) = \begin{cases} -\frac{1}{2}\sqrt{n}\; f^l_{m+1}(\lambda) & l-m \quad \text{even} \\ -\frac{1}{2}\sqrt{l-n}\; f^l_{m+1}(\lambda) & l-m \quad \text{odd}, \end{cases} \tag{2.43}$$

$$v_- f^l_m(\lambda) = \begin{cases} \frac{1}{2}\sqrt{l-n}\; f^l_{m-1}(\lambda) & l-m \quad \text{even} \\ -\frac{1}{2}\sqrt{n+1}\; f^l_{m-1}(\lambda) & l-m \quad \text{odd}. \end{cases} \tag{2.44}$$

It is quite remarkable that for $q \neq 1$ the action of the generators of the quantum algebra $U_q(osp(1|2))$ can be written in a more compact form than in the classical case $q=1$.
In the following we shall work with the canonical basis $e^{lq}_m(\lambda)$.

2.3 The projection operator P^q for the quantum super-algebra $U_q(osp(1|2))$.

In this Section, we recall the definition and some properties of the projection operator for the quantum superalgebra $U_q(osp(1|2))$. This operator P^q acts linearly in the space V, the direct sum of all representation spaces V^l. It is defined by the following requirements

$$[T(H), P^q] = 0, \quad T(v_+)P^q = 0, \quad (P^q)^2 = P^q, \quad P^q\, e^{lq}_l(\lambda) = e^{lq}_l(\lambda). \tag{2.45}$$

The last condition means that the restriction of P^q to the irreducible representation space V_l, projects on the highest weight vector $e^{lq}_l(\lambda)$. It has been shown in Ref.[18], that the operator P^q can be written in the form of a series

$$P^q = \sum_{r=0}^{\infty} c_r(T(H))(T(v_-))^r(T(v_+))^r, \tag{2.46}$$

where

$$c_r(T(H)) = \frac{[4T(H)+1]!}{[4T(H)+r+1]![r]!\gamma^r}. \tag{2.47}$$

General formulae for the projection operator of quantum orthosymplectic superalgebras have been derived by Koroshkin and Tolstoy [22]. In the limit $q \to 1$, this coefficient and therefore the projection operator are equal to the corresponding $osp(1|2)$ coefficient and projection operator P, cf. [11],

$$\lim_{q\to 1} P^q = P. \tag{2.48}$$

The operator P^q defined by Eqs.(2.46) and (2.47) has the following properties

$$(P^q)^* = P^q, \quad P^q v_- = 0, \tag{2.49}$$

and, by definition, its action on an arbitrary vector E of the representation space V is

$$E = \sum_{l,m} E_{l,m}\, e^{lq}_m(\lambda) \quad \Rightarrow \quad P^q(E) = \sum_l E_{l,l}\, e^{lq}_l(\lambda) \tag{2.50}$$

Note that the bases e^{lq}_m and f^{lq}_m have the same highest weight vector, therefore the operator P^q is the same in both bases.
Let us consider the space W_m of all vectors of weight m, i.e., $W_m = \{f|T(H)f = \frac{m}{2}f\}$. The restriction of P^q to this space is denoted by P^{mq} and it has the form

$$P^{mq} = \sum_{r=0}^{\infty} c_r(m)(T(v_-))^r(T(v_+))^r, \tag{2.51}$$

where the coefficients $c_r(m)$ are now numbers

$$c_r(m) = \frac{[2m+1]!}{[2m+r+1]![r]!\gamma^r}. \tag{2.52}$$

In the sequel, we will use so-called shift operators P^{lq}_{mn}, acting in V_l. For $l \geq m$ and $l \geq n$ they are defined by the expression

$$P^{lq}_{mn} = (-1)^{\frac{(l-n)(l-n+1)}{2}} \sqrt{\frac{[l+m]!}{[2l]![l-m]!}\gamma^{-(l-m)}}\sqrt{\frac{[l+n]!}{[2l]![l-n]!}\gamma^{-(l-n)}}\; (T(v_-))^{l-m}\, P^{lq}\, (T(v_+))^{l-n}, \tag{2.53}$$

and they satisfy the following properties :

$$\begin{aligned} (P^{lq}_{mn})^* &= (-1)^{(l-n)(l-m+1)}(-1)^{\epsilon(m+n)} P^{lq}_{nm}, & (2.54)\\ P^{lq}_{mn}\, P^{l'q}_{kr} &= \delta_{nk}\delta_{ll'} P^{lq}_{mr}, & (2.55)\\ P^{lq}_{mn}\, e^{lq}_n(\lambda) &= e^{lq}_m(\lambda), & (2.56)\end{aligned}$$

with no summation over n in the last relation.

3 The Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$.

3.1 Tensor product of two irreducible representations.

Let $V^{l_1}(\lambda_1)$ and $V^{l_2}(\lambda_2)$ be the representation spaces of two representations $T^{l_1\epsilon}_{\varphi_1\psi_1}$ and $T^{l_2\epsilon}_{\varphi_2\psi_2}$ of the same class ϵ. From Eq.(2.28), this implies that the parities λ_i and signatures φ_i ($i=1,2$) are related by

$$\lambda_1 + \varphi_1 = \lambda_2 + \varphi_2 \quad mod(2). \tag{3.1}$$

The bilinear Hermitean form in the tensor product space $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ is defined by

$$((X_1\otimes X_2),(Y_1\otimes Y_2)) = (-1)^{deg(X_2)deg(Y_1)}(X_1,Y_1)(X_2,Y_2), \tag{3.2}$$

where $X_1, Y_1 \in V^{l_1}(\lambda_1)$, $X_2, Y_2 \in V^{l_2}(\lambda_2)$. It should be stressed that even if both Hermitean forms in the representation spaces $V^{l_1}(\lambda_1)$ and $V^{l_2}(\lambda_2)$ are positive definite, the form (3.2) is not necessarily positive definite.
The vectors

$$e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2), \tag{3.3}$$

for all admissible values of m_1, m_2, form a basis in the representation space $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$, and their parity is

$$deg(e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2)) = l_1+l_2-m_1-m_2+\lambda_1+\lambda_2 \quad mod(2). \tag{3.4}$$

The space $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ is a representation space for the tensor product of representations $T^{l_1\epsilon}_{\varphi_1\psi_1}\otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$ and the action of the generators $v_\pm$ and H in this space is represented by the following operators :

$$v^{\otimes}_{\pm}(1,2) = (T^{l_1}\otimes T^{l_2})\Delta(v_\pm) = T^{l_1}(v_\pm)\otimes q^{T^{l_2}(H)} + q^{-T^{l_1}(H)}\otimes T^{l_2}(v_\pm), \tag{3.5}$$

$$H^{\otimes}(1,2) = (T^{l_1}\otimes T^{l_2})\Delta(H) = T^{l_1}(H)\otimes T^{l_2}(1) + T^{l_1}(1)\otimes T^{l_2}(H), \tag{3.6}$$

The grade adjoints of operators $v^{\otimes}_{\pm}(1,2)$ and $H^{\otimes}(1,2)$ are defined by

$$(v^{\otimes}_{\pm}(1,2))^* = (T^{l_1}(v_\pm))^*\otimes q^{(T^{l_2}(H))^*} + q^{-(T^{l_1}(H))^*}\otimes (T^{l_2}(v_\pm))^*, \tag{3.7}$$

$$(H^{\otimes}(1,2))^* = (T^{l_1}(H))^*\otimes T^{l_2}(1)^* + T^{l_1}(1)^*\otimes (T^{l_2}(H))^*, \tag{3.8}$$

i.e., the grade adjoint operation does not change the order in coproduct. With this definition, the tensor product of representations $T^{l_1\epsilon}_{\varphi_1\psi_1}\otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$ is a representation of class ϵ with respect to the Hermitean form (3.2) Since the representations T^l and the coproduct Δ are homomorphisms of the algebra structure we have

$$(v^{\otimes}_{\pm}(1,2))^n = (T^{l_1}\otimes T^{l_2})\Delta(v^n_\pm) \tag{3.9}$$

$$(H^{\otimes}(1,2))^n = (T^{l_1}\otimes T^{l_2})\Delta(H^n) \tag{3.10}$$

This property, together with relations (2.2), imply

$$(v^{\otimes}_{\pm}(1,2))^n = \sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right](q^{\mp 1})\,(T^{l_1}(v_\pm))^{n-k}q^{-kT^{l_1}(H)}\otimes (T^{l_2}(v_\pm))^k q^{(n-k)T^{l_2}(H)}, \tag{3.11}$$

where $\left[\begin{array}{c} n\\ k\end{array}\right]$ is the quantum graded Newton symbol

$$\left[\begin{array}{c} n\\ k\end{array}\right] \equiv \left[\begin{array}{c} n\\ k\end{array}\right](q) = \frac{[n]!}{[k]![n-k]!}. \tag{3.12}$$

Formula (3.11) allows one to calculate the explicit form of the projection operator $P^{lq\otimes}$ for the representation $T^{l_1\epsilon}_{\varphi_1\psi_1}\otimes T^{l_2\epsilon}_{\varphi_2\psi_2}$

$$P^{lq\otimes}(1,2) = \sum_{r=0}^{\infty} c_r(l)(v^{\otimes}_{-}(1,2))^r(v^{\otimes}_{+}(1,2))^r. \tag{3.13}$$

This operator will be used to construct the reduced basis $e^{lq}_m(l_1,l_2,\lambda)$ in $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ and the C-Gc for $U_q(osp(1|2))$. It has been shown in Ref.[18] that the tensor product of representation spaces $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ can be reduced into a direct sum of subspaces $V^l(\lambda)$

$$V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2) = \oplus_l V^l(\lambda), \tag{3.14}$$

where l is an integer satisfying conditions

$$|l_1-l_2|\le l\le l_1+l_2. \tag{3.15}$$

In this reduction, each subspace appears only once, i.e., the tensor product of two representations of the same class is simply reducible.

3.2 $U_q(osp(1|2))$ Clebsch-Gordan coefficients.

By definition, the Clebsch-Gordan coefficients $(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)^\epsilon_q$ relate the pseudo-normalized basis $e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2)$ to the reduced pseudo-normalized basis $e^{lq}_m(l_1,l_2,\lambda)$ in the following way :

$$e^{lq}_m(l_1,l_2,\lambda) = \sum_{m_1,m_2}(l_1m_1\lambda_1,l_2m_2\lambda_2|lm\lambda)^\epsilon_q\, e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2), \tag{3.16}$$

where $m_1+m_2=m$. From the definition (3.2) of the bilinear form in $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ it follows that

$$\begin{aligned}(l_1m_1\lambda_1,l_2m_2\lambda_2|lm\lambda)^\epsilon_q &= (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(\sum_{i=1}^{2}\varphi_i(l_i-m_i)+\psi_i)}\\ &\qquad (e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2), e^{lq}_m(l_1,l_2,\lambda)).\end{aligned} \tag{3.17}$$

In order to calculate the right hand side of this relation, the vector $e^{lq}_m(l_1,l_2,\lambda)$ is represented in the form

$$e^{lq}_m(l_1,l_2\lambda) \cong P^{lq\otimes}_{mn}(e^{l_1 q}_{m_1}(\lambda_1)\otimes e^{l_2 q}_{m_2}(\lambda_2)), \tag{3.18}$$

where $m_1+m_2=n$, and it can be shown, cf, Ref.[18], that without loss of generality, this vector can be chosen as follows

$$e^{lq}_m(l_1,l_2,\lambda) = \frac{1}{\sqrt{|N(q,l,m)|}}P^{lq\otimes}_{ml}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l-l_1}(\lambda_2)). \tag{3.19}$$

Using the properties of the shift operators $v_\pm$, the normalization factor $N(q,l,m)$ is shown to be

$$\begin{aligned}N(q,l,m) &= (P^{lq\otimes}_{ml}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l-l_1}(\lambda_2)), P^{lq\otimes}_{ml}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l-l_1}(\lambda_2))\\ &= (-1)^{\varphi(l-m)+\psi}q^{\frac{(l_1+l_2-l)(l+l_2-l_1+1)}{2}}\frac{[2l+1]![2l_1]!}{[l_1-l_2+l]![l_1+l_2+l+1]!},\end{aligned} \tag{3.20}$$

and therefore, one has

$$\begin{aligned}(e^{lq}_m(l_1,l_2,\lambda)\ &,\ e^{l'q}_{m'}(l_1,l_2,\lambda))\\ &= \frac{1}{|N(q,l,m)|}(P^{lq\otimes}_{ml}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l-l_1}(\lambda_2)\ ,\ P^{l'q\otimes}_{m'l'}(e^{l_1 q}_{l_1}(\lambda_1)\otimes e^{l_2 q}_{l'-l_1}(\lambda_1)),\\ &= \frac{N(q,l,m)}{|N(q,l,m)|}\delta_{ll'}\,\delta_{mm'} = \delta_{ll'}\,\delta_{mm'}(-1)^{\varphi(l-m)+\psi},\end{aligned} \tag{3.21}$$

where

$$\varphi = l_1 + l_2 + l + \lambda_1 + \varphi_2 \quad mod(2), \tag{3.22}$$

$$\psi = (l_1 + l_2 + l + \lambda_2)\lambda_1 + \varphi_2(l_1 + l_2 + l) + \psi_1 + \psi_2 \quad mod(2)\,. \tag{3.23}$$

Thus, the basis $e_m^{lq}(l_1, l_2, \lambda)$ is orthogonal but not positive definite. Its signature is the same as in the classical $osp(1|2)$ case [14]. If we set $\varphi_i = \psi_i = 0$, $i = 1,2$, we obtain the same formula as in Ref.[18].
Using the recursion relations (2.2, 2.3) one can check that the operators $H^{\otimes}(12)$ and $v_{\pm}^{\otimes}(12)$ act on the basis $e_m^{lq}(l_1, l_2, \lambda)$ in the standard way given by formulae (2.32, 2.34).
From relation (3.19), it follows immediately that the parity λ of the reduced basis $e_m^{lq}(l_1, l_2, \lambda)$ can be expressed as

$$\lambda = l_1 + l_2 + l + \lambda_1 + \lambda_2 \quad mod(2)\,. \tag{3.24}$$

By substitution of expression (3.19) into relation (3.17), one gets

$$\begin{aligned}(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^e \;=\; & (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(\sum_{i=1}^2 \varphi_i(l_i-m_i)+\psi_i)}\frac{1}{\sqrt{|N(q,l,m)|}} \times \\ & (e_{m_1}^{l_1q}(\lambda_1) \otimes e_{m_2}^{l_2q}(\lambda_2), \Delta(P_{ml}^{lq})e_{l_1}^{l_1q}(\lambda_1) \otimes e_{l-l_1}^{l_2q}(\lambda_2))\,,\end{aligned} \tag{3.25}$$

i.e., the Clebsch-Gordan coefficients are proportional to the matrix elements of the operator $\Delta(P_{ml}^{lq})$ in the basis $e_{m_1}^{l_1q}(\lambda_1) \otimes e_{m_2}^{l_2q}(\lambda_2)$ of the space $V_{l_1} \otimes V_{l_2}$.
Using the properties of the projection operators, after a laborious calculation, we get the following analytical expression for the Clebsch-Gordan coefficients of the quantum superalgebra $U_q(osp(1|2)$

$$\begin{aligned}&(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^e = \\ &\quad (-1)^{\lambda_1(l+l_1-m_2)}(-1)^{\frac{(l_1+l_2-l)(l_1+l_2-l+1)}{2}} q^{-\frac{(l_1+l_2-l)(l+l_2+l_1+1)}{4}+\frac{l_1m_2-l_2m_1}{2}} \times \\ &\quad \left([2l+1]\frac{[l_1-l_2+l]![l+m]![l+l_2+l_1+1]![l_2-m_2]![l_1+l_2-l]!}{[l-m]![l_1+m_1]![l_2+m_2]![l_2-l_1+l]![l_1-m_1]!}\right)^{\frac{1}{2}} \times \\ &\quad \sum_z (-1)^{\frac{z(z+1)}{2}} q^{\frac{z(l_1+m_1)}{2}} \frac{[l_2+l_1-m-z]![2l_2-z]!}{[z]![l_1+l_2-l-z]![l_1+l_2+l+1-z]![l_2-m_2-z]!}\,,\end{aligned} \tag{3.26}$$

where the summation index z runs over all possible values such that the arguments of the symbol $[n]$ are non negative. The Clebsch-Gordan coefficients do not depend on the parameters φ_i, ψ_i, $(i = 1,2)$ and ϵ, i.e., C-Gc do not depend on the signature of the representation spaces. Exactly the same formula has been obtained in Ref.[18] where the particular case $\varphi_i = \psi_i = 0$ $(i = 1,2)$ was considered.
It is quite noticeable that this formula differs from the corresponding formula for $U_q(sl(2))$ Clebsch-Gordan coefficients only by the phase factor and by the definition of the symbol $[n]$. More precisely, setting $\lambda_1 = 0$ in the analytical formula (3.26) and making the following substitutions, for the symbol $[n]$

$$[n] \;\to\; [n]_- = \frac{q^{-\frac{n}{2}} - q^{\frac{n}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\,, \tag{3.27}$$

and for the phases

$$(-1)^{\frac{a(a-1)}{2}} \;\to\; (-1)^a\,, \tag{3.28}$$

then, Eq.(3.26) becomes identical to the analytical formula of Clebsch-Gordan coefficients for the quantum algebra $U_q(su(2))$ [7].
Using methods similar to those described in Ref.[7], one can show that for the particular case $m = l$ the Clebsch-Gordan coefficients take the form

$$\begin{aligned}&(l_1m_1\lambda_1, l_2m_2\lambda_2|ll\lambda)_q^e = \\ &\quad (-1)^{\lambda_1(l-l_1-m_2)}(-1)^{(l_1-m_1)(l_2-m_2)+\frac{(l_1-m_1)(l_1-m_1+1)}{2}} q^{\frac{(l_1+l_2-l)(l+l_2-l_1+1)}{4}} q^{-\frac{(l_1-m_1)(l+1)}{2}} \times \\ &\quad \left(\frac{[2l+1]![l_2+m_2]![l_1+m_1]![l_1+l_2-l]!}{[l_1-m_1]![l_2-m_2]![l_2-l_1+l]![l_1-l_2+l]![l_1+l_2+l+1]!}\right)^{\frac{1}{2}}\,.\end{aligned} \tag{3.29}$$

This formula has also been derived using a recursion relation in Ref.[17].
The analytical values of the simplest Clebsch-Gordan coefficients $(l_1\, m_1, 1\, m_2|l\; m)_q^e$ are given in table 1. Note that

$$(l_1\, l_1\, \lambda_1,\; l_2\, l_2\, \lambda_2|\; l_1 + l_2 \;\; l_1 + l_2\; \lambda_1 + \lambda_2)_q^e \;=\; 1 \tag{3.30}$$

and that the Clebsch-Gordan coefficients

$$(l_1\, l_1\, \lambda_1,\; l_2\, m_2\, \lambda_2|l\; l\; \lambda)_q^e \tag{3.31}$$

are always positive. Therefore these Clebsch-Gordan coefficients satisfy the classical Condon-Shortley convention.
Relation (3.25) and the limits (2.36), (2.48) give, as limit of the $U_q(osp(1|2))$ C-Gc the $osp(1|2)$ C-Gc in the basis $e_m^l(\lambda)$,

$$\lim_{q\to 1}\,(l_1m_1\lambda_1,\; l_2m_2\lambda_2|lm\lambda)_q^e = (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)^e\,. \tag{3.32}$$

Substituting relation (2.37) in equation (3.25) the Clebsch-Gordan coefficients in the bases $f_m^{lq}(\lambda)$ and $e_m^{lq}(\lambda)$ are related by

$$\begin{aligned}(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^f \;=\; & (-1)^{(l_1-m_1)(l_2-m_2)}(-1)^{(l-m)(l_1+l_2+l)} \times \\ & (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^e\,,\end{aligned} \tag{3.33}$$

and it is obvious that the same relation holds, for $q = 1$, between $osp(1|2)$ Clebsch-Gordan coefficients in both basis $f_m^l(\lambda)$ and $e_m^l(\lambda)$.

3.3 Properties of the Clebsch-Gordan coefficients.

From equations (3.2) and (3.21), it follows that the Clebsch-Gordan coefficients satisfy the following pseudo-orthogonality relations :

$$\begin{aligned}&\sum_{m_1m_2}(-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(\sum_{i=1}^2 \varphi_i(l_i-m_i)+\psi_i)} \times \\ &(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^e(l_1m_1\lambda_1, l_2m_2\lambda 2|l'm'\lambda)_q^e \;=\; (-1)^{(\varphi(l-m)+\psi)}\delta_{ll'}\delta_{mm'},\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\sum_{lm}(-1)^{(\varphi(l-m)+\psi)}(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q^e(l_1m_1'\lambda_1, l_2m_2'\lambda_2|lm\lambda)_q^e = \\ &\qquad (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(\sum_{i=1}^2 \varphi_i(l_i-m_i)+\psi_i)}\delta_{m_1m_1'}\delta_{m_2m_2'}\,.\end{aligned} \tag{3.35}$$

l	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$l_1 + 1$	$q^{\frac{l_1-m+1}{2}} \times \sqrt{\frac{[l_1+m+1][l_1+m]}{[2l_1+1][2l_1+2]}}$	$(-1)^{\lambda_1} q^{-\frac{m}{2}} \times \sqrt{\frac{[2][l_1+m+1][l_1-m+1]}{[2l_1+1][2l_1+2]}}$	$q^{\frac{-l_1-m-1}{2}} \times \sqrt{\frac{[l_1-m+1][l_1-m]}{[2l_1+1][2l_1+2]}}$
l_1	$(-1)^{\lambda_1+1} q^{-\frac{m}{2}} \times \sqrt{\frac{[2][l_1-m+1][l_1+m]}{[2l_1][2l_1+2]}}$	$q^{-\frac{m}{2}} \times \frac{(-1)^{l_1-m+1} q^{\frac{l_1}{2}}[l_1+m+1] + q^{\frac{-l_1}{2}}[l_1-m+1]}{\sqrt{[2l_1][2l_1+2]}}$	$(-1)^{\lambda_1} q^{-\frac{m}{2}} \times \sqrt{\frac{[2][l_1+m+1][l_1-m]}{[2l_1][2l_1+2]}}$
$l_1 - 1$	$-q^{\frac{-l_1-m}{2}} \times \sqrt{\frac{[l_1-m][l_1-m+1]}{[2l_1][2l_1+1]}}$	$(-1)^{\lambda_1} q^{-\frac{m}{2}} \times \sqrt{\frac{[2][l_1-m][l_1+m]}{[2l_1][2l_1+1]}}$	$q^{\frac{l_1-m}{2}} \times \sqrt{\frac{[l_1+m][l_1+m+1]}{[2l_1][2l_1+1]}}$

Table 1: Clebsch-Gordan Coefficients $(l_1\, m_1 \lambda_1, 1\, m_2 \lambda_2 | l\ m\ \lambda)^e_q$

where φ, ψ are given by formulae (3.22, 3.23). These pseudo-orthogonality properties reflect the fact that the bases (3.3) and (3.19) are not positive definite. Using relations (3.22, 3.23, 3.24) one can simplify the pseudo-orthogonality relations and get

$$\sum_{m_1 m_2} (-1)^{(l_1-m_1)(l_2-m_2)} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)^e_q (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda)^e_q = (-1)^{(l-m)L} \delta_{ll'} \delta_{mm'}, \tag{3.36}$$

$$\sum_{lm} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda) e_q (l_1 m_1' \lambda_1, l_2 m_2' \lambda_2 | l m \lambda)^e_q = (-1)^{(l_1-m_1)(l_2-m_2)} \delta_{m_1 m_1'} \delta_{m_2 m_2'}, \tag{3.37}$$

where $L = l_1 + l_2 + l$. Thus actually, the pseudo-orthogonality relations do not depend on the parameters φ_i, ψ_i, λ_i and ϵ.
From relation (3.33), it follows also that the Clebsch-Gordan coefficients $(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m_\lambda)^f_q$ in basis $f^{lq}_m(\lambda)$ have the same pseudo-orthogonality relations as the C-Gc $(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)^e_q$ in basis $e^{lq}_m(\lambda)$. Note that the classical C-Gc for $osp(1|2)$ superalgebra satisfy also the same pseudo-orthogonality relations [14].
Considering the action of operators $v^q_\pm(1,2)$ on the defining relations for Clebsch-Gordan coefficients, one can derive the following recursion relations

$$\begin{aligned} &\sqrt{[l+m][l-m+1]}\gamma(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m-1\ \lambda)^e_q = \\ &\quad q^{\frac{m_2}{2}} \sqrt{[l_1-m_1][l_1+m_1+1]}\gamma(l_1 m_1+1\ \lambda_1, l_2 m_2 \lambda_2 | l m\ \lambda)^e_q + \\ &\quad (-1)^{l_1-m_1+\lambda_1} q^{\frac{-m_1}{2}} \sqrt{[l_2-m_2][l_2+m_2+1]}\gamma(l_1 m_1\ \lambda_1, l_2 m_2+1\ \lambda_2 | l m\ \lambda)^e_q, \end{aligned} \tag{3.38}$$

$$\begin{aligned} &(-1)^{l_1+l_2+l+\lambda_1} \sqrt{[l-m][l+m+1]}\gamma(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m+1\ \lambda)^e_q = \\ &\quad (-1)^{l_2-m_2+\lambda_1} q^{\frac{m_2}{2}} \sqrt{[l_1+m_1][l_1-m_1+1]}\gamma(l_1 m_1-1\ \lambda_1, l_2 m_2 \lambda_2 | l m\ \lambda)^e_q + \\ &\quad q^{\frac{-m_1}{2}} \sqrt{[l_2+m_2][l_2-m_2+1]}\gamma(l_1 m_1\ \lambda_1, l_2 m_2-1\ \lambda_2 | l m\ \lambda)^e_q. \end{aligned} \tag{3.39}$$

Such recursion relations were used in Ref.[17] to derive an analytical formula for Clebsch-Gordan coefficients equivalent to Eq.(3.26).
In the theory of the classical Racah-Wigner calculus, a very important role is played by the Clebsch-Gordan coefficient $(jm, jn|00)$ which defines an invariant metric in the representation space. In the case of the quantum superalgebra $U_q(osp(1|2)$, the corresponding coefficient has the form

$$C^{lq}_{mn}(\lambda) = (lm\lambda, ln\lambda|000)^e_q = (-1)^{(\lambda)(l-m)}(-1)^{\frac{(l-m)(l-m-1)}{2}} q^{\frac{m}{2}} \frac{1}{\sqrt{[2l+1]}} \delta_{m,-n}. \tag{3.40}$$

It defines also an invariant metric and it satifies the properties :

$$C^{lq}_{mn}(\lambda) = (-1)^m C^{lq^{-1}}_{nm}(\lambda), \qquad C^{lq}_{mn}(\lambda) C^{lq^{-1}}_{pn}(\lambda) = \frac{\delta_{mp}}{[2l+1]}. \tag{3.41}$$

This invariant metric will be used to construct the symmetric $3-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. One easily checks that in the limit $q = 1$, the invariant metric (3.40) becomes the invariant metric for the classical superalgebra $osp(1|2)$ [14].

3.4 Symmetries of Clebsch-Gordan coefficients

Using techniques similar to those described in Ref.[7], or using the recursion relations (3.38, 3.39), one can prove that the Clebsch-Gordan coefficients possess the following symmetry properties :

$$\begin{aligned} (l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2 | l_3\ m_3\ \lambda_3)^e_q \ &= \ (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)} \times \\ &(-1)^{(\lambda_1+\lambda_2)(l_1+l_2+l_3)+\lambda_1\lambda_2} (-1)^{\frac{(l_1+l_2-l_3)(l_1+l_2-l_3+1)}{2}} (l_2\ m_2\ \lambda_2,\ l_1\ m_1 \lambda_1 | l_3\ m_3\ \lambda_3)^e_{q^{-1}}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} (l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2 | l_3\ m_3\ \lambda_3)^e_q \ &= \ (-1)^{\frac{(l_2+m_2)(l_2+m_2-1)}{2}} (-1)^{\lambda_1(l_1+l_3-m_2)} \times \\ &(-1)^{(\lambda_2+L)(l_1+l_2-m_3)} q^{-\frac{m_2}{2}} \left(\frac{[2l_3+1]}{[2l_1+1]}\right)^{\frac{1}{2}} (l_2\ -m_2\ \lambda_2,\ l_3\ m_3 \lambda_3 | l_1\ m_1\ \lambda_1)^e_{q^{-1}}, \end{aligned} \tag{3.43}$$

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^e_q = (-1)^{\frac{(l_1-m_1)(l_1-m_1-1)}{2}}(-1)^{\lambda_3(l_2+l_3-m_1)} \times$$
$$(-1)^{(\lambda_1+L)(l_1+l_3-m_2)} q^{\frac{m_1}{2}} \left(\frac{[2l_3+1]}{[2l_2+1]}\right)^{\frac{1}{2}} (l_3\ m_3\ \lambda_3,\ l_1\ -m_1\lambda_1|l_2\ m_2\ \lambda_2)^e_{q^{-1}}. \tag{3.44}$$

The Clebsch-Gordan coefficients satisfy also the "mirror" symmetry

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^e_q =$$
$$(-1)^{\sum_{i=1}^{3}\frac{(l_i-m_i)(l_i-m_i-1)}{2}}(l_1\ -m_1\ \lambda_1,\ l_2\ -m_2\ \lambda_2|l_3\ -m_3\ \lambda_3)^e_{q^{-1}}, \tag{3.45}$$

where $L = l_1 + l_2 + l_3$. All these symmetries have the same structure as the symmetries of Clebsch-Gordan coefficients for quantum algebra $U_q(su(2))$ [7], excepted that the phases are non linear in l_i, m_i and that they depend on the parities λ_i, $(i = 1,2,3)$.
Another similarity between both cases is the existence of Regge symmetry. However, in the case of quantum superalgebra $U_q(osp(1|2))$, Regge symmetry is realized only under some condition. Assume that in the analytical formula (3.26) for Clebsch-Gordan coefficients the condition

$$l_1 + l_2 + m_1 + m_2 = 0, \quad mod(2) \tag{3.46}$$

is satisfied, and consider the following linear transformation on the arguments :

$$l_1' = \frac{1}{2}(l_1 + l_2 + m_1 + m_2), \quad m_1' = \frac{1}{2}(l_1 - l_2 + m_1 - m_2) \tag{3.47}$$

$$l_2' = \frac{1}{2}(l_1 + l_2 - m_1 - m_2), \quad m_2' = \frac{1}{2}(l_1 - l_2 - m_1 + m_2) \tag{3.48}$$

$$l_3' = l_3, \qquad m_3' = l_1 - l_2. \tag{3.49}$$

The superspins l_i', $i = 1,2,3$ satisfy relation (3.15), the projections m_i' satisfy $m_1' + m_2' = m_3'$ and condition (3.46) guarantees that l_i are integers. Thus, the transformation (3.47-3.49) is an admissible transformation on superspins l_i and projections m_i in the tensor product $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$. If we substitute now in the analytical formula (3.26) the values of l_i' and m_i', then the phase is invariant and all expressions in symbols $[n]$ either remain invariant or are exchanged pairwise so that the expression (3.26) remains unchanged. Therefore, the $U_q(osp(1|2))$ C-Gc satisfy Regge symmetry

$$(l_1'm_1'\lambda_1, l_2'm_2'\lambda_2|l_3'm_3'\lambda_3)^e_q = (l_1m_1\lambda_1, l_2m_2\lambda_2|l_3m_3\lambda_3)^e_q \tag{3.50}$$

Let us observe that the numerical value of the right hand side of expression (3.26) is invariant under transformation (3.47-3.49) even if condition (3.46) is not satisfied. But in this case, the transformation introduces superspins which are half-odd-integers which do not correspond to irreducible representations of $U_q(osp(1|2))$. Thus, the analytical formula (3.26), obtained by application of the projection operator method, exhibits in a natural way the Regge symmetry of the coefficients. Regge symmetry of Clebsch-Gordan coefficients for a tensor product of representations with $\varphi = \psi = 0$ has been discussed in Ref.[23].
Using relation (2.27) one can easily check, that Clebsch-Gordan coefficients in basis $f^{lq}_m(\lambda)$ have similar symmetry properties

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^f_q = (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)} \times$$
$$(-1)^{(\lambda_1+\lambda_2)(l_1+l_2+l_3)+\lambda_1\lambda_2}(-1)^{\frac{(l_1+l_2-l_3)(l_1+l_2-l_3+1)}{2}}(l_2\ m_2\ \lambda_2,\ l_1\ m_1\lambda_1|l_3\ m_3\ \lambda_3)^f_{q^{-1}}, \tag{3.51}$$

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^f_q = (-1)^{\frac{(l_2+m_2)(l_2+m_2-1)}{2}}(-1)^{(\lambda_1+1)(l_1+l_3-m_2)} \times$$
$$(-1)^{(\lambda_2+L)(l_1+l_2-m_3)} q^{-\frac{m_2}{2}} \left(\frac{[2l_3+1]}{[2l_1+1]}\right)^{\frac{1}{2}} (l_2\ -m_2\ \lambda_2,\ l_3\ m_3\lambda_3|l_1\ m_1\ \lambda_1)^f_{q^{-1}}, \tag{3.52}$$

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^f_q = (-1)^{\frac{(l_1-m_1)(l_1-m_1-1)}{2}}(-1)^{(\lambda_3+1)(l_2+l_3-m_1)} \times$$
$$(-1)^{(\lambda_1+L)(l_1+l_3-m_2)} q^{\frac{m_1}{2}} \left(\frac{[2l_3+1]}{[2l_2+1]}\right)^{\frac{1}{2}} (l_3\ m_3\ \lambda_3,\ l_1\ -m_1\lambda_1|l_2\ m_2\ \lambda_2)^f_{q^{-1}}, \tag{3.53}$$

$$(l_1\ m_1\ \lambda_1,\ l_2\ m_2\ \lambda_2|l_3\ m_3\ \lambda_3)^f_q =$$
$$(-1)^{\sum_{i=1}^{3}\frac{(l_i-m_i)(l_i-m_i-1)}{2}}(l_1\ -m_1\ \lambda_1,\ l_2\ -m_2\ \lambda_2|l_3\ -m_3\ \lambda_3)^f_{q^{-1}}. \tag{3.54}$$

The first symmetry and "miror" symmetry are the same in both bases. In the limit $q \to 1$, these symmetry properties become symmetry properties of Clebsch-Gordan coefficients for the classical superalgebra $osp(1|2)$ derived in Ref.[13].

4 Symmetric $3-j$ symbols for quantum superalgebra $U_q(osp(1|2))$

4.1 The parity dependent $sq3-j\lambda$ symbols.

In analogy with the classical case of $su(2)$ algebra, one can define for quantum superalgebra $U_q(osp(1|2))$, $sq3-j\lambda$ symbols that possess good symmetry properties

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^e_q =$$
$$(-1)^{L\lambda_3}(-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}}(-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}}(-1)^{\frac{(l_3+m_3)(l_3+m_3-1)}{2}} \times$$
$$q^{\frac{1}{2}m_3-\frac{1}{6}(m_1-m_2)} C^{l_3q}_{m_3'm_3}(\lambda_3)(l_1m_1\lambda_1, l_2m_2\lambda_2|l_3m_3'\lambda_3)^e_q, \tag{4.1}$$

where $C^{l_3q}_{m_3n_3}(\lambda_3)$ is the invariant metric defined by relation (3.40). Using the explicit form of the invariant metric, the definition of the symbol $sq3-j\lambda$ may be written

$$\begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^e_q = (-1)^{(l_1+l_2-m_3)\lambda_3}(-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}}(-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \times$$
$$\frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}}(l_1m_1\lambda_1, l_2m_2\lambda_2|l_3-m_3\lambda_3)^e_q \tag{4.2}$$

The symbols $sq3-j\lambda$ satisfy the same constraints as Clebsch-Gordan coefficients, namely

$$|l_1 - l_2| < +l_3 < l_1 + l_2\,, \tag{4.3}$$

$$m_1 + m_2 + m_3 = 0\,, \tag{4.4}$$

$$l_1 + l_2 + l_3 = \lambda_3 + \lambda_1 + \lambda_2 \quad mod(2)\,. \tag{4.5}$$

Taking into account the symmetry properties of Clebsch-Gordan coefficients, one can show that $sq3-j\lambda$ symbols are invariant under even permutations of columns

$$\left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q = \left(\begin{array}{ccc} l_3\lambda_3 & l_1\lambda_1 & l_2\lambda_2 \\ m_3 & m_1 & m_2 \end{array}\right)^e_q = \left(\begin{array}{ccc} l_2\lambda_2 & l_3\lambda_3 & l_1\lambda_1 \\ m_2 & m_3 & m_1 \end{array}\right)^e_q . \tag{4.6}$$

Under an odd permutation of columns and simultaneous change $q \to q^{-1}$, they are multiplied by a phase factor β

$$\left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q = \beta \left(\begin{array}{ccc} l_2\lambda_2 & l_1\lambda_1 & l_3\lambda_3 \\ m_2 & m_1 & m_3 \end{array}\right)^e_{q^{-1}} =$$
$$\beta \left(\begin{array}{ccc} l_1\lambda_1 & l_3\lambda_3 & l_2\lambda_2 \\ m_1 & m_3 & m_2 \end{array}\right)^e_{q^{-1}} = \beta \left(\begin{array}{ccc} l_3\lambda_3 & l_2\lambda_2 & l_1\lambda_1 \\ m_3 & m_2 & m_1 \end{array}\right)^e_{q^{-1}} , \tag{4.7}$$

where the phase is

$$\beta \;=\; (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i-\lambda_i)(l_i-m_i-\lambda_i-1)}{2}} . \tag{4.8}$$

Under "mirror" symmetry, the $sq3-j\lambda$ symbols transform in a slightly different way

$$\left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ -m_1 & -m_2 & -m_3 \end{array}\right)^e_q \;=\; (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i)(l_i-m_i-1)}{2}} \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_{q^{-1}} . \tag{4.9}$$

Finally, if condition

$$l_1 + l_2 + m_1 + m_2 = 0, \quad mod(2)\,, \tag{4.10}$$

is satisfied, then the symbols satisfy Regge symmetry

$$\left(\begin{array}{ccc} l'_1\lambda_1 & l'_2\lambda_2 & l'_3\lambda_3 \\ m'_1 & m'_2 & m'_3 \end{array}\right)^e_q \;=\; \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q , \tag{4.11}$$

where $l'_i, m'_i, i = 1,2,3$ are of the form

$$l'_1 = \frac{1}{2}(l_1 + l_2 + m_1 + m_2), \quad m'_1 = \frac{1}{2}(l_1 - l_2 + m_1 - m_2)\,, \tag{4.12}$$

$$l'_2 = \frac{1}{2}(l_1 + l_2 - m_1 - m_2, \quad m'_2 = \frac{1}{2}(l_1 - l_2 - m_1 + m_2)\,, \tag{4.13}$$

$$l'_3 = l_3, \qquad m'_3 = l_2 - l_1 . \tag{4.14}$$

From the definition of $sq3-j\lambda$ symbols, and from pseudo-orthogonality relations for Clebsch-Gordan coefficients, it follows that these symbols satisfy modified pseudo-orthogonality relations, as follows

$$\sum_{m_1 m_2} (-1)^{(l_1-m_1)(l_2-m_2)} q^{\frac{1}{3}(m_1-m_2)} \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l'_3\lambda_3 \\ m_1 & m_2 & m'_3 \end{array}\right)^e_q =$$
$$(-1)^{(l_3-m_3)L} \frac{\delta_{l_3 l'_3}\delta_{m_3 m'_3}}{[2l_3+1]}, \tag{4.15}$$

$$\sum_{l_3 m_3} (-1)^{(l_3-m_3)L} [2l_3+1] \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m'_1 & m'_2 & m_3 \end{array}\right)^e_q =$$
$$(-1)^{(l_1-m_1)(l_2-m_2)} q^{-\frac{1}{3}(m_1-m_2)} \delta_{m_1 m'_1}\delta_{m_2 m'_2} . \tag{4.16}$$

We have also the following relation between $sq3-j\lambda$ symbols and Clebsch-Gordan coefficients

$$C^{l_3 q^{-1}}_{m_3 m'_3}(\lambda_3) \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m'_3 \end{array}\right)^e_q \;=\; (-1)^{L\lambda_3} (-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}} (-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \times$$
$$\frac{q^{-\frac{1}{6}(m_1-m_2)}}{[2l_3+1]} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l_3 m_3 \lambda_3)^e_q . \tag{4.17}$$

Let us now consider $sq3-j\lambda$ symbols in the basis $f^{lq}_m(\lambda)$. They are defined by

$$\left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^f_q \;=\; (-1)^{(l_1+l_2-m_3)(\lambda_3+1)} (-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}} (-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \times$$
$$\frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l_3 - m_3 \lambda_3)^f_q \tag{4.18}$$

The first phase is different from the corresponding one in definition (4.2) of $sq3-j$ symbols in basis $e^{lq}_m(\lambda)$, and therefore $sq3-j\lambda$ symbols in bases $e^{lq}_m(\lambda)$ and $f^{lq}_m(\lambda)$ differ by a phase factor

$$\left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^f_q = (-1)^{(l_3-m_3)(L+1)+L} (-1)^{(l_1-m_1)(l_2-m_2)} \left(\begin{array}{ccc} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{array}\right)^e_q . \tag{4.19}$$

Both types of $sq3-j\lambda$ symbols (in basis $f^{lq}_m(\lambda)$ and in basis $e^{lq}_m(\lambda)$) possess the same symmetry properties, i.e., they transform with the same phase β under an odd permutation of colums and with the same phase under "mirror" symmetry. They satisfy also the same pseudo-orthogonality relations.
In the limit $q \to 1$, $sq3-j\lambda$ symbols in basis $f^{lq}_m(\lambda)$ are identical to the $s3-j\lambda$ symbols for superalgebra $osp(1|2)$ defined in Ref.[14].
The $sq3-j\lambda$ symbols have better symmetry properties than Clebsch-Gordan coefficients, but they still depend on parities λ_i, so these symbols are not real analogues of $q3-j$ symbols for quantum algebra $U_q(su(2))$. It has been shown in Ref.[14] that in case of superalgebra $osp(1|2)$, the dependence on λ_i can be factored out, so that it was possible to define $s3-j$ symbols that do not depend on parities λ_i. In the next Section, we will show that such a factorization is also possible in case of quantum superalgebra $U_q(osp(1|2))$, which gives the possibility to define parity independent $sq3-j$ symbols.

4.2 Parity independent $sq3-j$ symbols.

Parity independent $sq3-j$ symbols are defined in the following way

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e = (-1)^{\sum_{circ}(l_i-m_i)(l_{i+1}+\lambda_{i+1})} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e , \tag{4.20}$$

with the short notation

$$\sum_{circ} x_i y_{i+1} = x_1y_2 + x_2y_3 + x_3y_1 . \tag{4.21}$$

Using relation (4.2), the $sq3-j$ symbols may be expressed in terms of Clebsch-Gordan coefficients

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e = (-1)^{\lambda_1(l_1+l_3-m_2)}(-1)^{l_1(l_2+l_3-m_1)}(-1)^{l_2(l_1+l_2-m_3)}(-1)^{L(l_1+l_2-m_3)} \times$$

$$(-1)^{\frac{(l_1+m_1)(l_1+m_1-1)}{2}}(-1)^{\frac{(l_2-m_2)(l_2-m_2-1)}{2}} \frac{q^{-\frac{1}{6}(m_1-m_2)}}{\sqrt{[2l_3+1]}} (l_1\, m_1\ \lambda_1, l_2\, m_2\ \lambda_2 | l_3\ -m_3\ \lambda_3)_q^e . \tag{4.22}$$

The symbols satisfy constraints (4.3) and (4.4). Using the analytical formula for Clebsch-Gordan coefficients, one can easily check that $sq3-j$ symbols do not depend on parities λ_i. The symmetry properties satisfied by the parity independent $sq3-j$ symbols are similar to the symmetry properties of $sq3-j\lambda$ symbols :

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix}_q^e = \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix}_q^e , \tag{4.23}$$

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e = \alpha \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix}_{q^{-1}}^e = \alpha \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix}_{q^{-1}}^e$$

$$= \alpha \begin{pmatrix} l_3 & l_2 & l_1 \\ m_3 & m_2 & m_1 \end{pmatrix}_{q^{-1}}^e , \tag{4.24}$$

where the phase α is

$$\alpha = (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i)(l_i-m_i-1)}{2}}(-1)^{\sum_{i=1}^{3} l_i m_i} . \tag{4.25}$$

Under "mirror" symmetry, $sq3-j$ symbols transform in the same way as $sq3-j\lambda$ symbols

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e = (-1)^{\sum_{i=1}^{3} \frac{(l_i-m_i)(l_i-m_i-1)}{2}} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}_{q^{-1}}^e . \tag{4.26}$$

Besides, there exists a conditional Regge symmetry, but in this case the symbol is not invariant, it is changed by a phase factor

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}_q^e = (-1)^{(l_1+l_3+m_2)(l_2+(1/2)(l_1+l_2+m_3))} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e , \tag{4.27}$$

where $l'_i, m'_i, i = 1,2,3$ are given by formulas (4.12-4.14) and l_i and m_i satisfy the condition

$$l_1 + l_2 + m_1 + m_2 = 0, \quad mod(2) , \tag{4.28}$$

which guarantees that the phase is real.

The $sq3-j$ symbols satisfy pseudo-orthogonality relations identical to those of $sq3-j\lambda$ symbols

$$\sum_{m_1m_2} (-1)^{(l_1-m_1)(l_2-m_2)} q^{\frac{1}{3}(m_1-m_2)} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e \begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix}_q^e =$$

$$(-1)^{(l_3-m_3)L} \frac{\delta_{l_3 l'_3}\delta_{m_3 m'_3}}{[2l_3+1]} , \tag{4.29}$$

$$\sum_{lm} (-1)^{(l_3-m_3)L}[2l_3+1] \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e \begin{pmatrix} l_1 & l_2 & l_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix}_q^e =$$

$$(-1)^{(l_1-m_1)(l_2-m_2)} q^{-\frac{1}{3}(m_1-m_2)} \delta_{m_1m'_1}\delta_{m_2m'_2} . \tag{4.30}$$

From the parity independent $sq3-j$ symbol, one can define an invaraint metric that is independent of λ

$$C^{lq}_{mn} = q^{\frac{1}{3}m} \begin{pmatrix} l & l & 0 \\ m & n & 0 \end{pmatrix}_q^e = (-1)^{l(l-m)}(-1)^{\frac{(l-m)(l-m-1)}{2}} \frac{q^{\frac{m}{2}}}{\sqrt{[2l+1]}} \delta_{m,-n} . \tag{4.31}$$

It is related to the invariant metric $C^{lq}_{mn}(\lambda)$ defined by relation (3.40) in the following way

$$C^{lq}_{mn} = (-1)^{(l-m)(\lambda+l)} C^{lq}_{mn}(\lambda) . \tag{4.32}$$

One can also define $sq3-j$ symbols in the basis $f^{lq}_m(\lambda)$ in the same way as in basis $e^{lq}_m(\lambda)$, i.e.,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^f = (-1)^{\sum_{circ}(l_i-m_i)(l_{i+1}+\lambda_{i+1})} \begin{pmatrix} l_1\lambda_1 & l_2\lambda_2 & l_3\lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e , \tag{4.33}$$

and they are related by

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^f = (-1)^{(l_3-m_3)(L+1)+L}(-1)^{(l_1-m_1)(l_2-m_2)} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q^e . \tag{4.34}$$

These $sq3-j$ symbols in basis $f^{lq}_m(\lambda)$ have exactly the same symmetry properties and pseudo-orthogonality relations as symbols $sq3-j$ in basis $e^{lq}_m(\lambda)$.
In the limit $q \to 1$, $sq3-j$ symbols become $s3-j$ symbols for the superalgebra $osp(1|2)$. In particular, for $q=1$, the last symbols ($sq3-j$ symbols in basis $f^{lq}_m(\lambda)$) are identical to the $s3-j$ symbols defined in Ref.[14].

5 Conclusion.

The quantum superalgebra $U_q(osp(1|2))$ can be considered either as the quantum analogue of $osp(1|2)$ superalgebra or as the super-analogue of $U_q(su(2))$ quantum algebra.

In this article, it has been shown that the irreducible representations of the quantum superalgebra $U_q(osp(1|2))$ have the same structure as those of the not deformed superalgebra $osp(1|2)$. In particular, Clebsch-Gordan coefficients have been defined such that they satisfy the same symmetry properties and pseudo-orthogonality relations as in the non deformed case. Moreover, after factorization of the parity dependence, we have defined symmetric $sq3-j$ symbols which are, at the same time, quantum deformations of $s3-j$ symbols for superalgebra $osp(1|2)$ and supersymmetric analogues of $q3-j$ symbols for quantum algebra $U_q(su(2))$.
In a forthcoming publication, this analysis of the Racah-Wigner calculus for $U_q(osp(1|2))$ quantum superalgebra will be continued with the definition and analysis of $sq6-j$ symbols.

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