PAR–LPTHE 94–32

INTEGRABLE MAPPINGS FROM MATRIX TRANSFORMATIONS AND THEIR SINGULARITY PROPERTIES

July 12, 1994

A. RAMANI

CPT, Ecole Polytechnique, CNRS, UPR 14, 91128 Palaiseau, France

B. GRAMMATICOS

LPN, Université Paris VII, Tour 24-14, 5ème étage, 75251 Paris, France

J-M. MAILLARD AND G. ROLLET

LPTHE, Université de Paris 6-Paris 7, Tour 16, 1er étage, boîte 126

4 Place Jussieu, F–75252 Paris Cedex 05

Abstract

Various examples of birational transformations having their origin in the theory of exactly solvable vertex-models in lattice statistical mechanics, as well as birational transformations originating from spin edge models, are analyzed using the singularity confinement method. This method provides results concerning the integrable (or not) character of these birational transformations in complete agreement with the results obtained by visualization methods as well as methods based on the analysis of algebraic invariants.

PACS: 02.10, 02.20, 05.20, 05.50

AMS Classification scheme numbers: 82A68, 82A69, 14E05, 14J50, 16A46, 16A24, 11D41

Key-words: Birational transformations, discrete dynamical systems, non-linear recursion relations, bi-quadratic relations, iterations, integrable mappings, elliptic curves, movable singularities.

work supported by CNRS

1 Introduction

The construction of integrable systems is a particularly delicate question. This is all the more true for discrete systems since the latter have only recently attracted the attention of the integrability community. Thus the various techniques that exist for the study of continuous integrability have not yet been fully developed in the discrete case. However, progress is fast, thanks to strong analogies between continuous and discrete systems.

In this paper we intend to study integrable mappings using a combination of methods: derivations based on matrix transformations and integrability assessment with the help of a discrete integrability detector. In a series of papers some of us (JMM, GR) [1, 2, 3] derived and studied mappings associated to tranformations of matrices. The initial motivation of this study stemmed from the fact that these mappings can be interpreted as discrete symmetries of the Yang-Baxter equations [4, 5, 6, 7]. The class of the tranformations considered consists in fact of combinations of involutions. Since we are interested only in rational representations, the resulting mappings are *birational* i.e. rational in both the forward and the backward evolution. Various classes of matrix transformations have indeed been studied and the resulting mappings were identified with respect to their integrability properties [1, 3, 8]. This study of integrability was based on a detailed numerical study of the iteration combined with the explicit construction of invariants for the integrable cases [1, 3, 8].

From a different standpoint, the remaining authors (BG, AR) have developed, over the past few years, a method that makes possible the identification of integrable mappings [9]. Based on the study of singularities of rational mappings, the singularity confinement method requires that, for integrability, any spontaneously appearing singularity should disappear after a few iterations of the mapping. This method has made possible the identification of numerous integrable discrete systems.

In this paper, we set out to apply the singularity confinement approach to systems derived by the matrix transformation method. This is a crucial test for the singularity confinement conjecture and we show indeed that, in every case, it is possible to successfully distinguish between integrable and non-integrable systems. In the case of systems the status of which was not clear at the outset, the singularity confinement was used as a predictor and its prediction was subsequently verified by detailed calculations. Finally, we devote a section of this paper to an analysis of discrete systems proposed by Falqui and Viallet [10], and comment on their results relating singularity to integrability.

2 Some birational transformations

In previous papers, we have analyzed birational representations of discrete groups generated by involutions, having their origin in the theory of exactly solvable vertex (or spin) models in lattice statistical mechanics [4, 5, 6, 7, 11, 12]. These involutions correspond respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. In [1], a particular permutation of the entries was analyzed. For this permutation, it has been shown that the

iteration of the associated birational transformations presents some remarkable factorization properties [1]. These factorization properties explain why the "complexity" of these iterations (degree of the successive iterates) instead of having the exponential growth one would expect, actually has a *polynomial growth* [1]. It has also been shown that the polynomial factors occurring in these factorizations do satisfy remarkable non-linear recursion relations and that the latter were actually integrable, yielding algebraic elliptic curves.

Let us consider the $q \times q$ matrix:

$$R_q = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \cdots \\ m_{21} & m_{22} & m_{23} & \cdots \\ m_{31} & m_{32} & m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.1}$$

We introduce the following transformations: the matrix inverse $\widehat{I}$, the homogeneous matrix inverse I and a transformation t which, in the following, will denote a permutation between two entries of the $q \times q$ matrix, for example t_{12-21} which permutes m_{12} and m_{21}, specifically studied in [1], or t_{12-43} which permutes m_{12} and m_{43} [8]:

$$\widehat{I} : R_q \longrightarrow R_q^{-1} \tag{2.2}$$

$$I : R_q \longrightarrow R_q^{-1} \cdot \det(R_q) \tag{2.3}$$

The *homogeneous* inverse I is a polynomial transformation on each of the entries m_{ij} which associates to each m_{ij} its corresponding cofactor in the framework of the inversion relation. The two transformations t and $\widehat{I}$ are involutions and the homogeneous inverse verifies $I^2 = (\det(R_q))^{q-2} \cdot \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. We also introduce the (generically infinite order) transformations $K = t \cdot I$ and $\widehat{K} = t \cdot \widehat{I}$. Transformation K is a *polynomial* transformation on the entries m_{ij}, while $\widehat{K}$ is clearly a rational transformation on the entries m_{ij}. In fact it is a *birational* transformation since its inverse is $\widehat{I} \cdot t$.

2.1 A first set of birational transformations: permutation t_{12-21}

Considering $q \times q$ matrices, let us first recall the factorization properties for the iteration of the homogeneous transformation K and recursion relations obtained for permutation t_{12-21} which represents one example among a set of permutations denoted class I in [1, 8].

Let us first consider the successive matrices obtained by iteration of the homogeneous transformation K on a generic $q \times q$ matrix R_q (see (2.1)) and the determinants of these various matrices:

$$M_0 = R_q, \qquad M_1 = K(M_0), \qquad f_1 = \det(M_0) \tag{2.4}$$

Remarkably, the determinant of matrix M_1 factorizes enabling us to introduce a homogeneous polynomial f_2:

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}} \tag{2.5}$$

Again, f_1^{q-4} also factorizes in all the entries of the matrix $K(M_1)$, leading to introduce a new "reduced" matrix M_2:

$$M_2 = \frac{K(M_1)}{f_1^{q-4}} \tag{2.6}$$

In fact, similar factorization properties are true at any order. Generally, for $n \geq 1$ and $q \geq 4$, one has[1]:

$$M_{n+3} = \frac{K(M_{n+2})}{f_n^{q-2}\, f_{n+1}^2\, f_{n+2}^{q-4}}, \qquad f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1}\, f_{n+1}^3\, f_{n+2}^{q-3}} \tag{2.7}$$

and the following relation *independent of* q:

$$\frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}} \tag{2.8}$$

From another point of view, transformation K corresponding to this mapping has been shown to yield *algebraic elliptic curves* in $\mathbf{C}P_{q^2-1}$ [1]. In $\mathbf{C}P_{15}$ these algebraic elliptic curves can be seen as intersection of quadrics [3], in a very similar way as for the sixteen vertex model [12].

One important consequence of these factorizations is to introduce the *homogeneous* polynomials f_n. These polynomials *do verify, independently of* q, *a whole hierarchy of non-linear recursion relations* [1] such as:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \tag{2.9}$$

or, for instance, among many other:

$$\frac{f_{n+1} f_{n+4}^2 f_{n+5} - f_{n+2} f_{n+3}^2 f_{n+6}}{f_{n+2}^2 f_{n+3} f_{n+7} - f_n f_{n+4} f_{n+5}^2} = \frac{f_{n+2} f_{n+5}^2 f_{n+6} - f_{n+3} f_{n+4}^2 f_{n+7}}{f_{n+3}^2 f_{n+4} f_{n+8} - f_{n+1} f_{n+5} f_{n+6}^2} \tag{2.10}$$

Let us introduce here variables [1, 8] corresponding to the iteration of the *inhomogeneous* transformation $\widehat{K}$:

$$x_n = \det(\widehat{K}^n(M_0)) \cdot \det(\widehat{K}^{n+1}(M_0)) \tag{2.11}$$

The x_n's also satisfy recursion relations, for instance:

$$R_1 : \qquad \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} = \frac{x_n - 1}{x_{n-1}\, x_n\, x_{n+1} - 1} \cdot x_{n-1}\, x_{n+1} \tag{2.12}$$

Relation R_1 is actually equivalent to:

$$R_2 : \qquad \frac{x_{n+2} - 1}{x_{n+1}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1} \cdot x_n\, x_{n+2}^2 \tag{2.13}$$

These factorizations and recursion relations were shown in [1] *to hold true for arbitrary* q.

In fact one can consider recursions (2.12) and (2.13) independently of the matrix framework previously detailed, that is, independently of (2.11). In this case, it can also be shown that both mappings (2.12) and

[1] Because of factorizations (2.7) one can see that the iteration of the homogeneous transformation K yields a *polynomial growth* of the complexity of the calculations : the degree of the determinant of the matrix M_n, as well as the degree of the polynomials f_n's are *quadratic expressions of* n [1].

(2.13) are *integrable* [1, 8]. From the definition of the x_n's, one can get (see [1]), a very simple expression of the x_n's in terms of the f_n's:

$$x_n = \frac{f_{n-1}^2 \, f_{n+2}}{f_{n+1}^2 \, f_{n-2}} \tag{2.14}$$

Therefore, in analogy to the f_n's, one also has a *whole hierarchy of recursion relations* on the x_n's. The analysis of this hierarchy of compatible nonlinear recursion relations has been sketched in [1] and will be briefly recalled here.

Recursion relations (2.12), (2.13) yield algebraic elliptic curves [1]. This can be shown by relating them to biquadratic relations, introducing the (homogeneous) variables q_n:

$$q_n = \frac{f_n \, f_{n+3}}{f_{n+1} \, f_{n+2}} \tag{2.15}$$

Equation (2.9) or (2.12) or (2.13) can be integrated to the *biquadratic relation*:

$$(\rho - q_n - q_{n+1})(q_n \, q_{n+1} + \lambda) = \mu \tag{2.16}$$

where λ, ρ and μ are integration constants [1][2].

The relation between the algebraic elliptic curves corresponding to the iteration of K in $\mathbf{C}P_{q^2-1}$ and the elliptic curves associated with the recursion on the f_n's or x_n's (see (2.9), (2.12)) or the biquadratic relations (2.16), has been detailed in [1].

The variables x_n's (defined by (2.11)) satisfy a *whole hierarchy of recursion relations* [1], each being valid *for arbitrary values of* q. All the recursion relations on the x_n's can be written in the following general form:

$$\frac{x_{n+1}^{i_1} x_{n+2}^{i_2} \ldots x_{n+t}^{i_t} - 1}{x_n^{j_0} x_{n+1}^{j_1} x_{n+2}^{j_2} \ldots x_{n+t+1}^{j_{t+1}} - 1} \cdot x_n^{k_0} x_{n+1}^{k_1} x_{n+2}^{k_2} \ldots x_{n+t+s}^{k_{t+s}} = \frac{x_{n+1+s}^{i_1} x_{n+2+s}^{i_2} \ldots x_{n+t+s}^{i_t} - 1}{x_{n+s}^{j_0} x_{n+1+s}^{j_1} x_{n+2+s}^{j_2} \ldots x_{n+t+s+1}^{j_{t+1}} - 1} \tag{2.17}$$

with the first and the last exponents i_1, i_t, j_0, j_{t+1}, k_0 and k_{t+s}, being equal to 1. Up to a simple multiplicative factor $x_n^{k_0} x_{n+1}^{k_1} x_{n+2}^{k_2} \ldots$, one has the same rational expression on the left-hand side and the right-hand side of equation (2.17) up to a shift s ($n \longrightarrow n+s$).

Because of this specific form, it is clear that one can get, from one recursion relation, an (infinite) set of other ones combining a recursion relation with itself where the index n has been shifted. For instance, one gets, from (2.12), another one with a shift of 2:

$$R_3 : \quad \frac{x_{n+3} - 1}{x_{n+2} \, x_{n+3} \, x_{n+4} - 1} = \frac{x_{n+1} - 1}{x_n \, x_{n+1} \, x_{n+2} - 1} \cdot x_n \, x_{n+1} \, x_{n+2} \, x_{n+3} \tag{2.18}$$

Recursion R_3, seen independently of any $q \times q$ matrix problem (see (2.11)), is not (generically) integrable [1]. Furthermore, there exists other procedures (symmetries ...) to get, from one recursion relation, a new one (see [1] for more details). For instance, when some conditions are satisfied, other recursion relations, with

[2] Many examples of integrable mappings related to biquadratic elliptic curves have recently been obtained by Quispel and collaborators [13, 14, 15].

the same shift s, can actually be deduced by the following transformation which has been called "procedure Π " [3]:

$$\Pi : \quad x_n \longrightarrow x_n \cdot x_{n+s} \tag{2.19}$$

acting on the left and right-hand side of (2.17), the factor $x_n^{k_0} x_{n+1}^{k_1} x_{n+2}^{k_2} \cdots$ being changed in a different way [1]. For instance, starting from (2.13), procedure Π generates the following new recursion relation:

$$R_4 : \quad \frac{x_{n+2}\, x_{n+3} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3}\, x_{n+4} - 1} = \frac{x_{n+1}\, x_{n+2} - 1}{x_n\, x_{n+1}\, x_{n+2}\, x_{n+3} - 1} \cdot x_n\, x_{n+2}\, x_{n+3} \tag{2.20}$$

This can be shown to be integrable. Let us introduce (homogeneous) variables q_n's as follows:

$$x_n = q_{n+1}/q_n \tag{2.21}$$

Equation (2.20) reads:

$$\frac{q_{n+3} - q_{n+1}}{(q_{n+4} - q_n)\, q_{n+1}\, q_{n+2}\, q_{n+3}} = \frac{q_{n+4} - q_{n+2}}{(q_{n+5} - q_{n+1})\, q_{n+2}\, q_{n+3}\, q_{n+4}} \tag{2.22}$$

which can be "integrated" to get *two* biquadratic relations [1]:

$$\Big(\rho_n\, q_{n+1} - 1\Big) \cdot \Big(\rho_{n+1}\, q_n - 1\Big) = \lambda \cdot q_n\, q_{n+1} \Big(\mu + q_n + q_{n+1}\Big) \tag{2.23}$$

where $\rho_n = \rho_1$, if n is even, and $\rho_n = \rho_2$, if n is odd, as well as λ and μ, are constants of integration.

2.2 Another birational transformation : permutation t_{12-32}

Besides the analysis of permutation t_{12-21} and the corresponding birational transformations[3], similar analysis can be performed on other permutations of two entries [3, 8]. The permutations of two entries with their associated birational transformations have been classified in [3]. Among these (six) classes of transformations, one is of particular interest since it clearly exhibits both *integrability and "weak" chaos.* Let us consider here the birational transformation associated to the particular permutation: t_{12-32}.

The factorizations corresponding to the iterations of this birational transformation K now read:

$$det(M_n) = f_{n+1} \cdot (f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3}^2) \cdot (f_{n-4}^{q-2} \cdot f_{n-5} \cdot f_{n-6}^{q-1} \cdot f_{n-7}^2) \cdots f_1^{\delta_n} \tag{2.24}$$

where δ_n depends on the truncation, and[4]:

$$K(M_n) = M_{n+1} \cdot (f_n^{q-3} \cdot f_{n-2}^{q-2} \cdot f_{n-3}) \cdot (f_{n-4}^{q-3} \cdot f_{n-6}^{q-2} \cdot f_{n-7}) \cdot (f_{n-8}^{q-3} \cdot f_{n-10}^{q-2} \cdot f_{n-11}) \cdots f_1^{\zeta_n} \tag{2.25}$$

where $\zeta_n = q - 3$ for $n = 1 \pmod 4$, $\zeta_n = 0$ for $n = 2 \pmod 4$, $\zeta_n = q - 2$ for $n = 3 \pmod 4$ and $\zeta_n = 1$ for $n = 0 \pmod 4$. One notes that the following factorization, *independent of* q, occurs:

$$\frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{f_1\, f_2 \cdots f_n\, f_{n+1}} \tag{2.26}$$

[3] Called class IV in [3, 8].

[4] From these equations it can be shown that one has an *exponential growth* of the calculations: the entries of the successive matrices M_n's, as well as the polynomials f_n's grow like λ^n with $\lambda \propto 1.465571226\cdots$

The f_n's do not satisfy simple recursions (like (2.9)) *but "pseudo-recursions" where products from* f_n *all the way down to* f_1 *occur* [8, 3]. One of these "pseudo-recursions" can be written as follows :

$$\frac{(f_{n+2} - f_{n-1} f_{n+1})}{(f_n - f_{n-3} f_{n-1})} \cdot \frac{f_{n-6}\, f_{n-10}\, f_{n-14} \cdots}{f_{n-4}\, f_{n-8}\, f_{n-12} \cdots}$$

$$= \frac{f_n\,(f_{n-1}\, f_{n-5}\, f_{n-9} \cdots) - (f_{n+1}\, f_{n-3}\, f_{n-7} \cdots)}{f_{n-2}\,(f_{n-3}\, f_{n-7}\, f_{n-11} \cdots) - (f_{n-1}\, f_{n-5}\, f_{n-9} \cdots)} \tag{2.27}$$

Though one does not have recursions on the f_n's but pseudo-recursions such as (2.27), the variables x_n's previously defined (see (2.11)) *remarkably satisfy* [8] *simple recursions of the form* (2.17) *independent of* q:

$$\frac{x_{n+3} - 1}{x_{n+2}\, x_{n+4} - 1} = x_n\, x_{n+3} \cdot \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1} \tag{2.28}$$

One can actually show [8] that the x_n's, corresponding to permutation t_{12-32} through (2.11), do satisfy a *whole hierarchy of recursion relations* in the same way it has been proved for permutation t_{12-21} in [1]. It has also been shown [8] that a recursion of the form (2.17) (*but with some of the exponents* k_{t+s}*'s negative*) is actually satisfied for the birational transformations associated with permutation t_{12-32}.

We will study another recursion relation, related to (2.28) (which is a a consequence of it):

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1} \cdot \frac{x_n\, x_{n+2}}{x_{n+1}} \tag{2.29}$$

This recursion relation is actually equivalent to another one, namely:

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+3} - x_{n+2}} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - x_{n+1}} \cdot x_n\, x_{n+2} \tag{2.30}$$

These new recursions are not necessarily satisfied by the x_n's, corresponding to permutation t_{12-32} through (2.11) : they are only satisfied when the initial matrix in the iteration satisfies a particular condition (see [8]). We will come back to this point in the following section.

Following [1], one can consider these recursion relations for themselves, *without referring to our birational transformations acting on* $q \times q$ *matrices anymore.* Again one can see that some of these recursions *are integrable* (for instance (2.30)) while some *are* (generically) *not integrable* (for instance (2.28)).

Recursion relation (2.29), or equivalently (2.30), is an integrable one yielding elliptic curves [8], however one notes that the recursion deduced by a shift of two of (2.29), namely (2.28), *is not integrable* (it has however a very regular behaviour corresponding to (very) weak chaos, a situation which has been called "almost integrable" [3]).

In fact, equation (2.28), though generically not integrable, can be partially integrated [8]. Instead of the variables x_n's, let us come back to homogeneous variables q_n's defined as follows:

$$x_n = \frac{q_{n+2}}{q_n} \tag{2.31}$$

From (2.28) bearing on the x_n's, one recovers the "almost integrable"relations studied in section (8) of [3]:

$$\frac{q_{n+3} - q_{n+1}}{q_{n+4} - q_n} \cdot \frac{1}{q_{n+3}\, q_{n+1}} = \frac{q_{n+5} - q_{n+3}}{q_{n+6} - q_{n+2}} \cdot \frac{1}{q_{n+5}\, q_{n+3}} \tag{2.32}$$

3 Singularity confinement analysis of selected examples of mappings

In this section, we will analyze mappings obtained from matrix transformation, as explained in section 2, using the integrability detector developped in [16]. The method is based on the conjecture that the movable (i.e. initial condition dependent) singularities of an integrable discrete system are *confined*, i.e. they do not extend *ad infinitum* under the iteration of the mapping.

One of the most interesting examples is the simpler mapping of [1], namely (2.32). A numerical study of this mappings is shown in the (q_n, q_{n+1})-plane (figure 1), and from the observed regularity one would be tempted to surmise that (2.32) is indeed an integrable recursion relation. One first integration of (2.32) is staightforward. Noting that the left-hand-side is twice upshifted with respect to the right-hand-side, we can integrate (2.32) to:

$$(\frac{1}{q_{n+1}} - \frac{1}{q_{n+3}}) \cdot \frac{1}{q_{n+4} - q_n} = \frac{1}{\lambda_n} \tag{3.1}$$

where λ_n is a parity dependent parameter: $\lambda_{2n} = \lambda_e$ and $\lambda_{2n+1} = \lambda_o$. We rewrite (3.1) as

$$q_{n+4} + \lambda_n \frac{1}{q_{n+3}} = q_n + \lambda_n \frac{1}{q_{n+1}} \tag{3.2}$$

Adding q_{2n+2} to both sides and using the fact that $\lambda_{n+2} = \lambda_n$ we can integrate once more to:

$$q_{n+2} + q_n + \frac{\lambda_n}{q_{n+1}} = \rho_n \tag{3.3}$$

where ρ_n is also parity dependent: $\rho_{2n} = \rho_e$ and $\rho_{2n+1} = \rho_o$.

So the study of the integrability of (3.1) is reduced to that of (3.3). Clearly the only singularity of (3.4) occurs when the denominator q_{n+1} vanishes. So let us assume that, for some n, we have q_{n-1} finite and $q_n = \epsilon$, where ϵ is a small quantity, (and without loss of generality one can assume that n is odd). The singularity sequence is then the following (to dominant order):

$$q_n = \epsilon$$

$$q_{n+1} \approx \frac{\lambda_e}{\epsilon}$$

$$q_{n+2} \approx \rho_o$$

$$q_{n+3} \approx -\frac{\lambda_e}{\epsilon}$$

$$q_{n+4} \approx \frac{\lambda_e - 2\lambda_o}{\lambda_e}\epsilon$$

$$q_{n+5} \approx \frac{2(\lambda_e - \lambda_o)}{\epsilon}\frac{\lambda_e}{\lambda_e - 2\lambda_o}$$

$$q_{n+6} \approx \rho_o$$

We remark that the sequence q_{n+4}, q_{n+5}, q_{n+6} reproduce the initial sequence q_n, q_{n+1}, q_{n+2} with ϵ replaced by $\tilde{\epsilon} = (\lambda_e - 2\lambda_o)/\lambda_e \cdot \epsilon$ and λ_e/ϵ replaced by $2(\lambda_e - \lambda_o)/\tilde{\epsilon}$. Thus the singularity will propagate

indefinitely *unless* its sequence is broken at the q_{n+5} level by assuming $\lambda_e = \lambda_o$. Indeed, with this assumption we find:

$$q_{n+4} \approx -\epsilon$$

$$q_{n+5} \approx q_{n-1}$$

$$q_{n+6} \approx \rho_o + \frac{\lambda_o}{q_{n-1}}$$

and thus the singularity is confined: no singular terms appear beyond q_{n+3} and, moreover, the memory of the initial condition is recovered in q_{n+6} through q_{n-1}. So our prediction is that (2.32), or equivalently (3.3), is not integrable, *unless* for the latter, $\lambda_e = \lambda_o$, in which case it can be integrated in terms of elliptic functions.

Motivated by these results we have systematically considered the iteration of transformation K, for $\lambda_1 \neq \lambda_2$ in the (q_n, q_{n+1})-plane which, for aesthetical reasons, will be given in the (x_n, x_{n+1})-plane . For a quite large set of initial conditions (satisfying $\lambda_1 \neq \lambda_2$) one gets curves. For initial conditions such that $\lambda_1 \sim \lambda_2$ these curves look similar to the biquadratic equations mentioned in section (2): see figure (1). However a systematic examination of these orbits shows that, though in some domain of initial conditions such that $\lambda_1 \sim \lambda_2$, one gets most of the time regular curves, one does find (although rarely) very stretched "bubbles" which correspond to islands of regularity. However with a proper choice of initial values, we can obtain an orbit which looks like a curve with 16 self intersections. This gives a rough idea of the frontier between the dominant regular curves and sixteen islands of regularity: see figure (2a) and (2b). Actually if one iterates K^{16} instead of K the orbits are regular curves inside a single one of the 16 islands. One does not jump from one island to another one, but rather one can restrict oneself to one of these island, and then the whole situation reproduces "self-similarly", K being replaced by K^{16}. A very carefull and drastic magnification of the space between the islands shows that one has a situation similar to the one of the *hyperbolic-versus-elliptic* points encountered in the Hénon-Heiles mapping [17, 18, 19, 20, 21][5]. Figure (3) is an illustration of this elliptic-versus-hyperbolic situation. Regularity largely dominates for initial conditions such that $\lambda_1 \sim \lambda_2$ (in the sense of a measure on the initial conditions), and the chaos corresponding to the hyperbolic points needs an extremely careful numerical study (we have called such mappings "almost integrable [3]). However one can actually find initial conditions where chaos clearly occurs. Figure (2b) provides an example of well established chaos.

A large family of mappings has been studied in [3], where the recursion relations obtained have been classified with respect to their integrable character. One of the simplest integrable cases obtained is the mapping R_1. Solving for x_{n+2} we obtain:

$$x_{n+2} = \frac{x_{n-1}\, x_{n+1} \cdot (x_n\, x_{n+1} - 1) + 1 - x_{n+1}}{x_{n-1}\, x_n\, x_{n+1}^2 \cdot (x_n - 1)} \tag{3.4}$$

[5]At this point it is important to make the following comment: since they are generated by involutions, all our birational transformations are such that $\widehat{K}$ and $\widehat{K}^{-1}$ are conjugated ($\widehat{K} = t \cdot \widehat{I} = t \cdot \widehat{K}^{-1} \cdot t$). When transformation $\widehat{K}$ (or more precisely $\widehat{K}^2$) can be reduced to a mapping on only two variables this means that one has some *area preserving properties* and one can recover the features of two-dimensional dynamics (elliptic versus hyperbolic points, Arnold's diffusion ... [18, 19, 20, 21]). This explains, to some extend, the regularities one encounters here with permutation t_{12-32}, even when the mapping is not integrable.

A singularity of x_{n+2} appears when one of the x's in the denominator vanishes or when x_n=1. Let us start with the first case and assume that x_{n-1} and x_n are regular while x_{n+1} vanishes, i.e. $x_{n+1} \sim \epsilon$. We obtain the following singularity pattern:

$$x_{n+1} = \epsilon$$
$$x_{n+2} \approx \frac{a}{\epsilon^2}$$
$$x_{n+3} = -1 + \mathcal{O}(\epsilon)$$
$$x_{n+4} \approx \frac{\epsilon^2}{a}$$
$$x_{n+5} \approx -\frac{1}{\epsilon}$$

where a depends on x_n, x_{n+1}. Moreover we find that (as $\epsilon \to 0$) $x_{n+2}\,x_{n+4} = 1$, $x_{n+1}\,x_{n+5} = -1$, and indeed, $x_{n+6} = 1/x_n$, $x_{n+7} = -1/x_{n-1}$. Thus the singularity is confined and, as we see, its effect on the iteration is indeed particularly simple. The second type of singularity may appear when $(x_n - 1)$ vanishes. So let us assume that x_{n-2} and x_{n-1} are finite and that $x_n = 1$. First we compute x_{n+1} with this assumption (using the downshifted form of (3.4)) and we find $x_{n+1} = 1/x_{n-1}$. Using these values we obtain $x_{n+2} = 1/x_{n-2}$ and thus no singularity develops despite the vanishing of the $(x_n - 1)$ factor in the denominator.

Thus, both the most obvious singularity patterns of (3.4) lead to confined singularities. Before concluding on the integrability of this mapping, we must investigate the possibility of existence of more intricate singularity patterns. One such pattern could have been described by the set of values $(f, 0, 0)$ for (x_{n-1}, x_n, x_{n+1}), where f stands for a finite value. However, such a pattern is impossible since, after a zero value (preceded by finite values), we can only have ∞. The same holds for a pattern $(f, 0, 1)$. The pattern $(f, 1, 0)$ is equally impossible since after a value 1 preceded by a finite value f the only possible value is $1/f$. The only possibility that cannot be rejected offhand is pattern $(f, 1, 1)$ but the same argument tells us that one can only have $f = 1$. That means that, in fact, we are blocked on the constant solution $x_n = 1$ for *all* n. This singularity is not confined but it is not movable either, so this is not incompatible with integrability . Now we can indeed state that (3.4) has only confined *movable* singularities. From our conjecture, it must be integrable, and this is in fact the case.

Another integrable mapping is R_4 which can be solved to:

$$x_{n+4} = \frac{1 + x_{n+2}\,x_{n+3} \cdot (x_n\,x_{n+1}\,x_{n+2} \cdot (1 + x_{n+3}) - 1 - x_n - x_n\,x_{n+1})}{x_n\,x_{n+1}x_{n+2}^2x_{n+3}^2 \cdot (x_{n+1}\,x_{n+2} - 1)} \tag{3.5}$$

As in the previous case, a singularity may develop when any of the x_n in the denominator vanishes or when $x_{n+1}x_{n+2} = 1$. The latter situation, in analogy to $x_n = 1$ for (3.4) above, is a case where the singularity in fact never develops. Indeed, computing x_{n+3} based on the assumption that x_{n-1}, x_n, x_{n+1}, x_{n+2} are regular with $x_{n+1}x_{n+2} = 1$, we find $x_{n+3} = 1/x_n$, and iterating $x_{n+4} = 1/x_{n-1}$ and so on.

Let us now turn to the study of a singularity where x_n, x_{n+1}, x_{n+2} are regular and x_{n+3} vanishes. We find the following sequence:

$$x_{n+3} = \epsilon$$

$$x_{n+4} \approx \frac{a}{\epsilon^2}$$

$$x_{n+5} = -1 + b\epsilon + \mathcal{O}(\epsilon^2)$$

$$x_{n+6} \approx c\,\epsilon^2$$

$$x_{n+7} \approx -\frac{1}{a^2c^2\epsilon}$$

where a, b and c have complicated expressions in terms of x_n, x_{n+1} and x_{n+2}. Iterating further, we obtain finite values for x_{n+8}, x_{n+9}, x_{n+10}, x_{n+11} and we also find that $x_{n+k}\,x_{n+k+1} \neq 1$ (in this range) and thus the singularity is confined. Special singularity sequences may appear, in particular whenever the initial conditions are such that $a^2\,c\cdot(ac+b)=1$, but also these ones are confined. Thus (3.5) has only confined singularities in agreement with its integrable character.

Our last example will be chosen among the non-integrable mappings. We have considered the recursion relation R_3:

$$x_{n+4} = \frac{x_n\,x_{n+1}\,x_{n+2}\cdot(x_{n+1}\,x_{n+3}-1)+1-x_{n+3}}{x_n\,x_{n+1}x_{n+2}^2x_{n+3}^2\cdot(x_{n+1}-1)} \tag{3.6}$$

As previously, the vanishing of $(x_{n+1}-1)$ does not lead to a singularity. So let us assume that x_n, x_{n+1} and x_{n+2} are finite while $x_{n+3} \sim \epsilon$. Then we find the following sequence:

$$x_{n+4} \approx \frac{a^2c^2}{\epsilon^2}$$

$$x_{n+5} \approx -\frac{1}{ac}$$

$$x_{n+6} \approx \epsilon^2$$

$$x_{n+7} \approx \frac{a}{\epsilon^2}$$

$$x_{n+8} \approx b\epsilon$$

$$x_{n+9} \approx \frac{1+c}{ab^2}$$

where a, b and c depend on the initial conditions. Conversely, we may introduce a, b and c as free parameters and ask for the conditions for the backward iteration to lead to finite x_{n+3} (we find $c \neq 0$ or $x_{n+7}x_{n+8}^2x_{n+9} \neq 1$) and finite x_{n+2}, x_{n+1}, x_n (we find $a^2c\cdot(c+1)=1$ or $x_{n+6}^2\,x_{n+7}^3\,x_{n+8}^2\,x_{n+9}\cdot(x_{n+7}\,x_{n+8}^2\,x_{n+9}-1)\neq 1$). It is then easier to postulate the singularity sequence $x_{n+6},\ldots x_{n+10}$ (the latter turns out to be $x_{n+10}\sim 1/\epsilon$) with a, b and c given and iterate forward. We find:

$$x_{n+11} \sim \epsilon^2$$

$$x_{n+12} \sim \frac{1}{\epsilon^2}$$

$$x_{n+13} \sim \epsilon$$

$$x_{n+14} \quad \text{finite}$$

$$x_{n+15} \sim \frac{1}{\epsilon}$$

i.e. the same sequence as for $x_{n+6}, \ldots x_{n+10}$ (up to the precise coefficients). Thus, the basic singular pattern propagates itself and the singularity is not confined, as was expected, given the non-integrable character of (3.7).

From all these examples we see that there is a perfect agreement between the singularity confinement conjecture predictions and the integrable (or not) character, analytically and/or numerically established, of the systems under consideration.

4 Comparison with the approach of Falqui and Viallet

In a recent publication [10], Falqui and Viallet have addressed the problem of the relation between the singularities of birational mappings in the projective 2-plane and their integrable (or not) character. The birational transformations they consider are, in close analogy to ours, realizations of Coxeter groups, generated by involutions. We will concentrate here in the part of their work in direct relation to the present one, namely the case where the group is generated by just two involutions I, J. Here J is the Hadamard inverse, $[x] \to [1/x]$, and I is related to J through some collineation matrix C: $I = C^{-1}JC$. The method of Falqui and Viallet is based on the examination of the set of points that "blow up" under the action of *some* operator of the group (i.e. $(IJ)^n$, $(IJ)^nI$, $J(IJ)^n$ or $(JI)^n$), called the singular locus of the transformation. Given a point in the projective 2-plane we will say that it "blows up" if its iteration under some operator of the group leads to an indeterminate $(0,0,0)$. If this happens for an infinity of distinct points (infinite singular locus) then Falqui and Viallet consider this as an indication of non-integrability and in fact with some technical precautions, they cast their result in the form of a theorem. The precise setting is crucial to the proof:

- projective two-plane

- to have properly singular birational transformations [6].

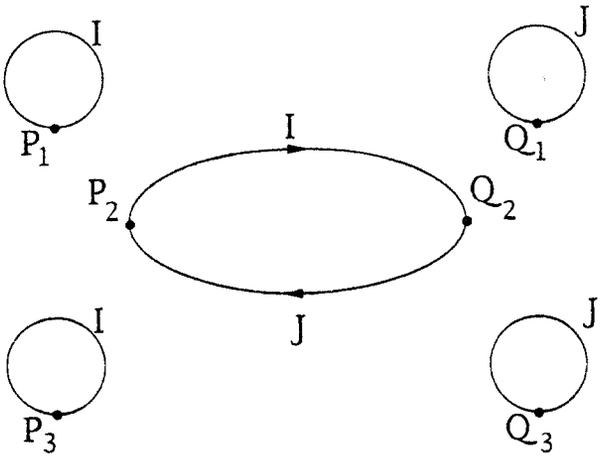
Does this mean that when the singular locus is finite the mapping should be integrable? Not necessarily so, as can be seen from the examples they offer. The example in their paragraph 6.3 is based on the collineation matrix:

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

The singular points of the transformation are the singular points P_i of the Hadamard inversion J ([1,0,0], [0,1,0], [0,0,1]) and the singular points Q_i of the involution I ([1,1,1], [1,0,-1], [1,-1,1]).

[6] This properly singular requirement is automatically fulfilled in their examples, since their birational transformations are generated by two involutions, one involution being precisely the Hadamard inverse J and the other one being intertwined to the Hadamard inverse by a collineation.

The singular locus is given by the diagram below:



This mapping is integrable and possesses the invariant:

$$\Delta = \frac{y^2 - xz}{y(x-z)}$$

Putting $x = uy$ and $z = vy$ we can write it simply as:

$$u' = \frac{v-u}{uv+u^2-2}, \quad v' = \frac{u-v}{uv+v^2-2} \tag{4.1}$$

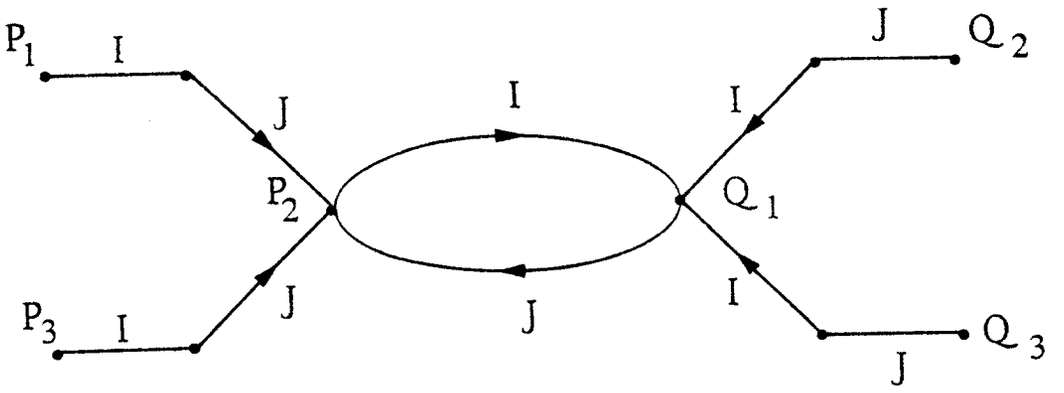
The example of their paragraph 6.6 is based on the collineation matrix:

$$C = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

Putting $x = uy$ and $z = vy$ we can write it as:

$$u' = \frac{8(u+v)}{2uv-5u^2+3v^2+1}, \quad v' = \frac{8(u+v)}{2uv+3u^2-5v^2+1} \tag{4.2}$$

The singular locus is:

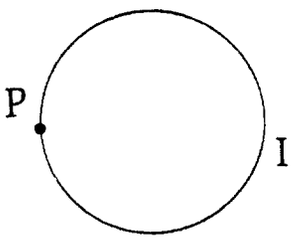


In this case, the mapping does not possess any invariant.

How can one understand those results in terms of our approach of singularity confinement? First let us apply our method of analysis to the mapping (4.2) above. A singularity appears whenever the Jacobian of (u', v') with respect to u and v vanishes. This happens for $u+v=0$, $u-v-1=0$ and $u-v+1=0$. If $u+v=0$ then $u'=v'=0$ and this remains true for all the subsequent steps. This singularity is not confined. To check that it is indeed a movable singularity we ask how one can have $u'+v'=0$ without having first $u+v=0$. This is precisely the case when $u-v=\pm 1$, which gives $u'=-v'=\pm 2$ and indeed $u''=v''=0$. Moreover the lines $u-v=\pm 1$ have preimages which are regular all the way: this is a typical case of a non confined movable singularity. Thus our prediction is that mapping (4.2) is not integrable, in agreement with the absence of invariant and the exponential growth of the calculations noted by Falqui and Viallet [10] (even though the singularity locus is finite).

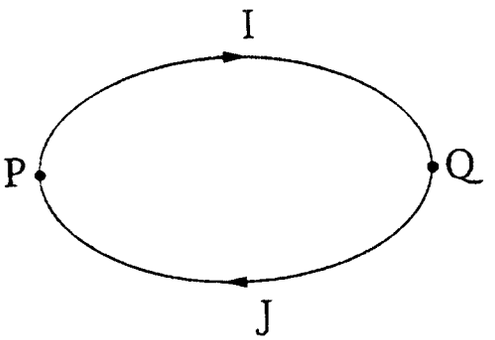
Let us now turn to (4.1). The analysis looks superficially similar to the one of (4.2). The Jacobian vanishes for $u-v=0$ and $u+v=\pm 2$. The singularity for $u-v=0$ gives $u'=v'=0$ and this remains true for subsequent iterations. Such a singularity would not be confined but the only way to have $u'=v'$ for *finite* u, v that do not already satisfy $u=v$ at the previous step is to have $u+v=\pm 2$. This does indeed lead to $u'=v'$ but the common value is ± 1 leading to an indeterminate form $0/0$ for u'', v'' rather than the value 0. In fact, u'' and v'' have finite values and depend on the initial conditions. This is a typical *confined* movable singularity: one degree of freedom was apparently lost over one single iteration step at the point $(\pm 1, \pm 1)$. Still, before concluding on the integrability of this mapping, one must consider all possible singularities, including those where a denominator vanishes. This can happen whenever $u(u+v)=2$ (or $v(u+v)=2$). In this case u' (resp v') diverges while v' (resp u') remains finite. At the next step we find $u''=0$ (resp $v''=0$) and everything remains finite thereafter. This is again a confined singularity. (The case where both denominators vanish corresponds to $u=v=\pm 1$ already studied). Finally let us consider the singularity in the "backwards" iteration when $u'=v'=a$ with $a \neq (0, \pm 1)$. We find that both u, v diverge with a *finite* sum and all the previous preimages also diverge with a *vanishing* sum. So the singularity pattern is the following: we start from $(\infty, -\infty)$ go through (a, a) and then get blocked on $(0,0)$. This is again a nonmovable singularity (or if the emergence of a degree of freedom (a, a) that was previously absent and subsequently vanishes is considered as a singularity, the latter is movable and confined). So this mapping passes the singularity confinement test, as expected, given its integrable character.

In the light of our results, we can now present our interpretation of the approach of Falqui and Viallet [10]. A singular diagram of the type



corresponds to confined singularities. Indeed, since the point P blows up under J, this means that we have a one dimensional manifold (for simplicity we shall call it a "line") that reduces to P under the action of J. Thus one degree of freedom is lost: a singularity appears. Next the action of I brings us back to P, which, at the following step blows up under J and thus we recover the lost degree of freedom: the singularity is confined. The presence of longer "arms" leading to a one-step singular loop as in the case of their paragraph 6.2 of Falqui and Viallet [10] just means that it can take more steps to confine the singularity since one may "enter" at the beginning of the arm and "exit" only after some wandering about, but still the singularity is always confined.

On the other hand a singular diagram of the type:



indicates a priori a non-confined singularity. Indeed, a "line" shrinks down, under J, to the point P and the action of I transforms P into Q. Then we have to apply J. Since Q blows up under I, not J, the action of J does not lead to a recovery of the lost degree of freedom but rather sends us back to the point P and so on. Thus, at this point, we are stuck on this singularity, which is *not* confined. We can now understand why mapping (4.2) is *not* integrable. The fact that the singular loop is connected to other nonsingular points does not change anything to this reasoning: this just means that there are several "entry points" to one singular loop.

The case of the singular loop of mapping (4.1) is more subtle: there is indeed a non-confined singularity but it is not a *movable* one. The only way to reach the line L that "blows down" to P was to be on L at the previous step, because $I(L) = L$. (This was not the case in the preceding example: the singularity there was indeed movable). But then at the preceding step, where we have to use J, we again blow down to P and thereafter we alternate between P and Q. So we did not lose any degree of freedom: a "spurious" degree of freedom appears for two steps (in fact two half-steps) then disappears again. If one considers that the singularity is precisely the appearance of this spurious degree of freedom (rather than the loss of one, as usual) then we are indeed in the presence of a *movable* and *confined* singularity. This singularity is precisely the one found by our method: $\ldots \to (\infty, -\infty) \to (a, a) \to (0, 0) \to \ldots$.

The main difference between our approach and that of Falqui and Viallet [10] lies in the fact that in applying the singularity confinement criterion we distinguish between movable and non-movable singularities. Only the non-confined *movable* singularities are incompatible with integrability. Since Falqui and Viallet only examine the iteration where a "line" blows down to a point, but never look "backwards" to see where

this line comes from, they cannot distinguish between movable and non-movable singularities. A movable singularity is one where the "line" that blows down to a point, leads, when iterated backwards, to an infinite set of other "lines". The non-movable case corresponds to a line that, upon backwards iteration, leads to only a *finite* number of lines and then blows down to a point. It is thus crucial to know what are the preimages of every singular point.

Although in a general setting the application of the singularity confinement criterion must be carried along the lines we explained above, it may turn out that, in the restricted setting of Falqui and Viallet, the examination of the singular locus may suffice to give a necessary criterion for integrability (though not a sufficient one, as their example 6.6 shows). In principle, even an infinite singular locus may be compatible with integrability from the point of view of the singularity confinement. In the case of their example 6.3 above, the singularity was the apppearance of a spurious degree of freedom for some iterations preceded and followed by two fixed points between which the mapping alternates. In analogy to this case, we can imagine a situation where an *infinity* of (distinct) points exists before the appearance of some additional degree of freedom and that the latter disappears after some iterations leading again to an infinity of distinct points. From the point of view of Falqui and Viallet this is an infinite singular locus (and they would predict nonintegrability), while from our viewpoint the singularity is a confined movable one (compatible with integrability). However we have not been able to construct a mapping with this behaviour within the setting of Falqui and Viallet. This is an indication that, *in this restricted setting*, this situation may never occur and thus the finiteness of a singular locus would indeed be a necessary condition for integrability.

5 Conclusion

Various examples of birational transformations, originating from vertex or spin edge models, have been analyzed using the singularity confinement method. The singularity confinement method confirms the integrable character of transformations K corresponding to t_{12-21}. A particular care has been devoted to the analysis of the iteration of K corresponding to t_{12-32} where both regularity and weak chaos occur. Again this method provides results concerning the integrable (or not) character of these birational transformations in complete agreement with the results obtained by systematic search of algebraic invariants for the action of the group. The encoding of the integrable (or not) character of a birational transformation by a graph of the singularity locus is a tempting idea at first sight: still the last section of this paper shows that such a graph, though giving precious indication on the very nature of the transformation, is not sufficient for such an encoding.

Acknowledgement: We thank C.M. Viallet for many animated discussions.

References

[1] S. Boukraa, J-M. Maillard, and G. Rollet, *Determinantal identities on integrable mappings.* Int. J.

Mod. Phys. **B** (1994). to be published.

[2] S. Boukraa, J-M. Maillard, and G. Rollet. *A new approach for lattice modes in statistical mechanics.* LPTHE preprint 94-27, (1994).

[3] S. Boukraa, J-M. Maillard, and G. Rollet, *Almost integrable mappings.* Int. J. Mod. Phys. **B8** (1994), pp. 137–174.

[4] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Integrable Coxeter Groups.* Physics Letters **A 159** (1991), pp. 221–232.

[5] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Higher dimensional mappings.* Physics Letters **A 159** (1991), pp. 233–244.

[6] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Infinite Discrete Symmetry Group for the Yang-Baxter Equations: Spin models.* Physics Letters **A 157** (1991), pp. 343–353.

[7] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Infinite Discrete Symmetry Group for the Yang-Baxter Equations: Vertex Models.* Phys. Lett. B **260** (1991), pp. 87–100.

[8] S. Boukraa, J-M. Maillard, and G. Rollet, *Integrable mappings and polynomial growth.* Physica A **208** (1994). to be published.

[9] B. Grammaticos, A. Ramani, and V. Papageorgiou, *Do integrable mappings have the Painlevé property?* Phys. Rev. Lett. **67** (1991), pp. 1825–1827.

[10] C.M. Viallet G. Falqui, *Singularity, Complexity, and Quasi-Integrability of Rational Mappings.* Comm. Math. Phys. **154** (1993), pp. 111–125.

[11] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Rational Mappings, Arborescent Iterations, and the Symmetries of Integrability.* Physical Review Letters **67** (1991), pp. 1373–1376.

[12] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Quasi integrability of the sixteen-vertex model.* Phys. Lett. B **281** (1992), pp. 315–319.

[13] G.R.W. Quispel and F. W. Nijhoff. Phys. Lett. A **161** (1992), p. 419.

[14] G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson, *Integrable Mappings and Soliton Equations.* Phys. Lett. A **126** (1988), p. 419.

[15] G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson, *Integrable Mappings and Soliton Equations II.* Physica **D34** (1989), pp. 183–192.

[16] B. Grammaticos and A.Ramani. Investigating the integrability of discrete systems. In J-M.Maillard, editor, *Yang-Baxter equations in Paris*, pages 79–93. World Scientific, (1993).

[17] M.R. Herman. *Recent results and some open questions on Siegel's linearisation theorem of germs of complex analytic diffeomorphisms of* $\mathbf{C}^n$ *near a fixed point.* World Scientific Publishers, Singapor, (1987). Proceedings VIIIth Int. Conf. Math. Phys.

[18] A.M. Ozorio de Almeida. *Hamiltonian systems Chaos and Quantization.* Cambridge University Press, Cambridge, New-York, (1988).

[19] V.I. Arnold. *Les Méthodes Mathématiques de la Mécanique Classique.* Mir, Moscou, (1976).

[20] V.I. Arnold. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* Mir, Moscou, (1980).

[21] T. Mullin. *The Nature of Chaos.* Clarendon Press Oxford, (1993).

6 Figure captions

Figure 1: Orbit of an iteration of K in the (x_n, x_{n+1})-plane close to the $\lambda_1 = \lambda_2$ integrability condition.

Figure 2a: A set of ten orbits corresponding to the iteration K in the (x_n, x_{n+1})-plane: one sees between regular concentric orbits nine "regular" islands and, at the frontier, a separatrix with 16 bubbles.

Figure 2b: A set of orbits corresponding to the iteration K in the (x_n, x_{n+1})-plane showing the transition from regular concentric orbits to a chaotic orbit at the frontier.

Figure 3: An illustration of the elliptic-versus-hyperbolic points situations on a set of orbits in a region between islands of regularity.

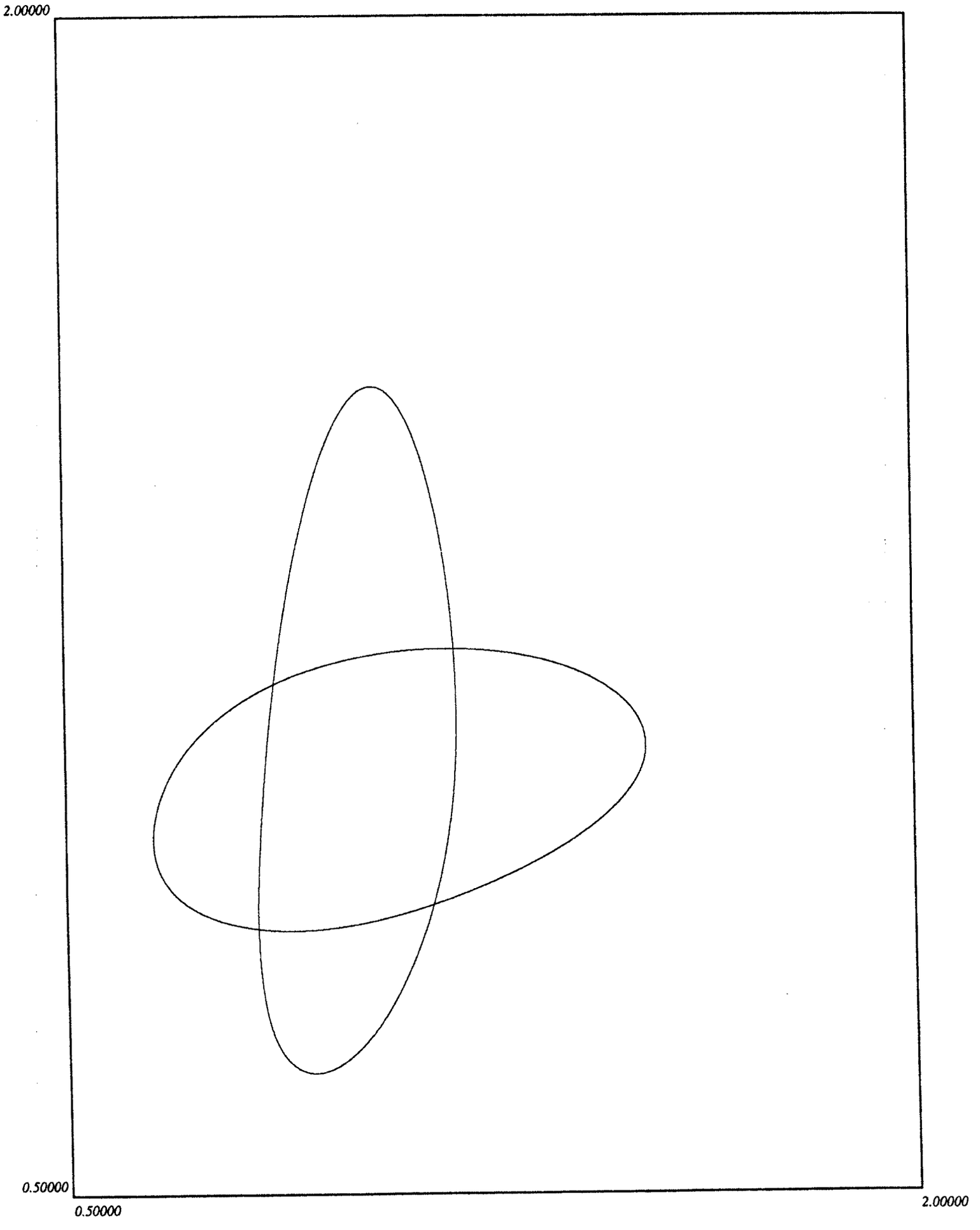


Figure 1

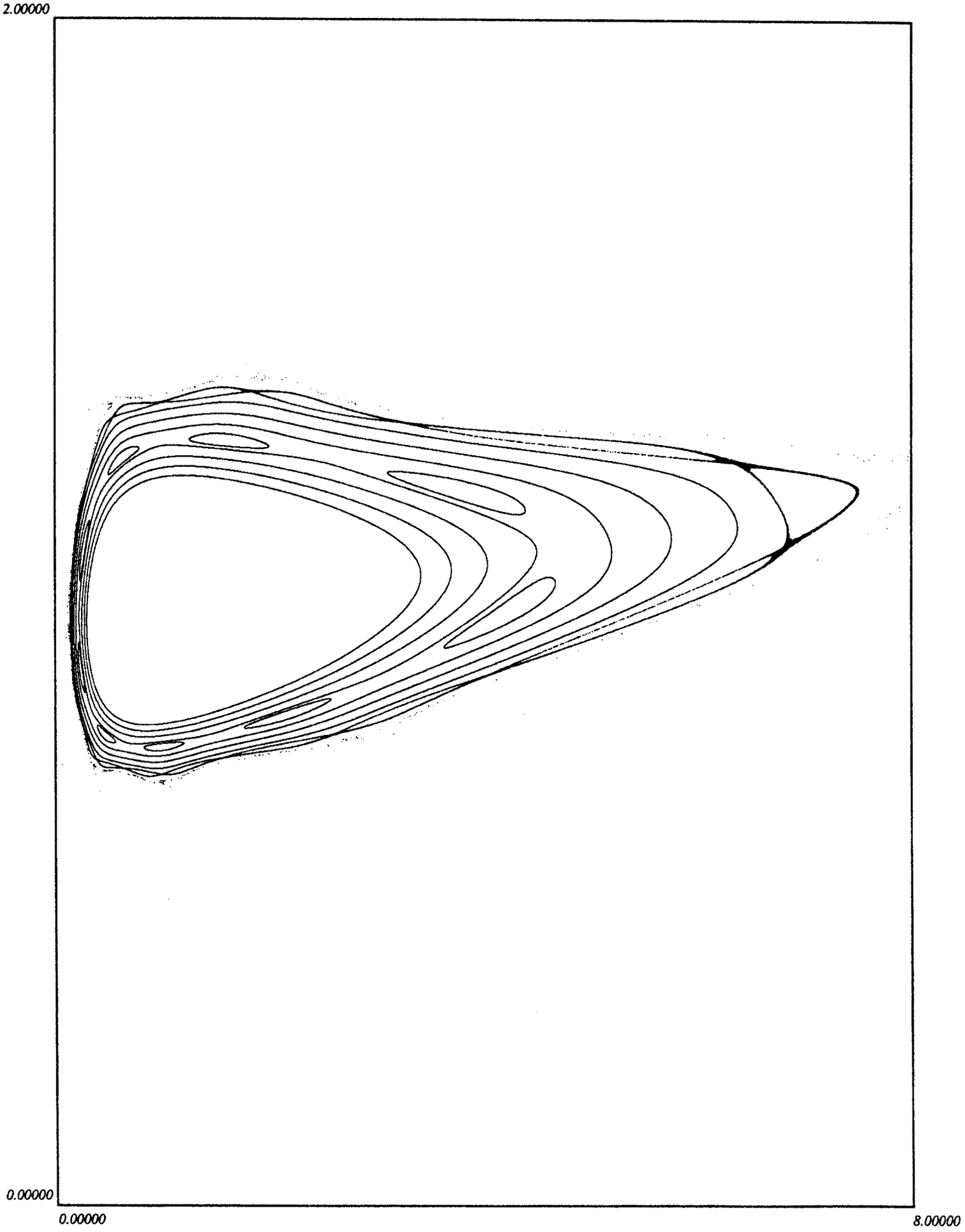


Figure 2a

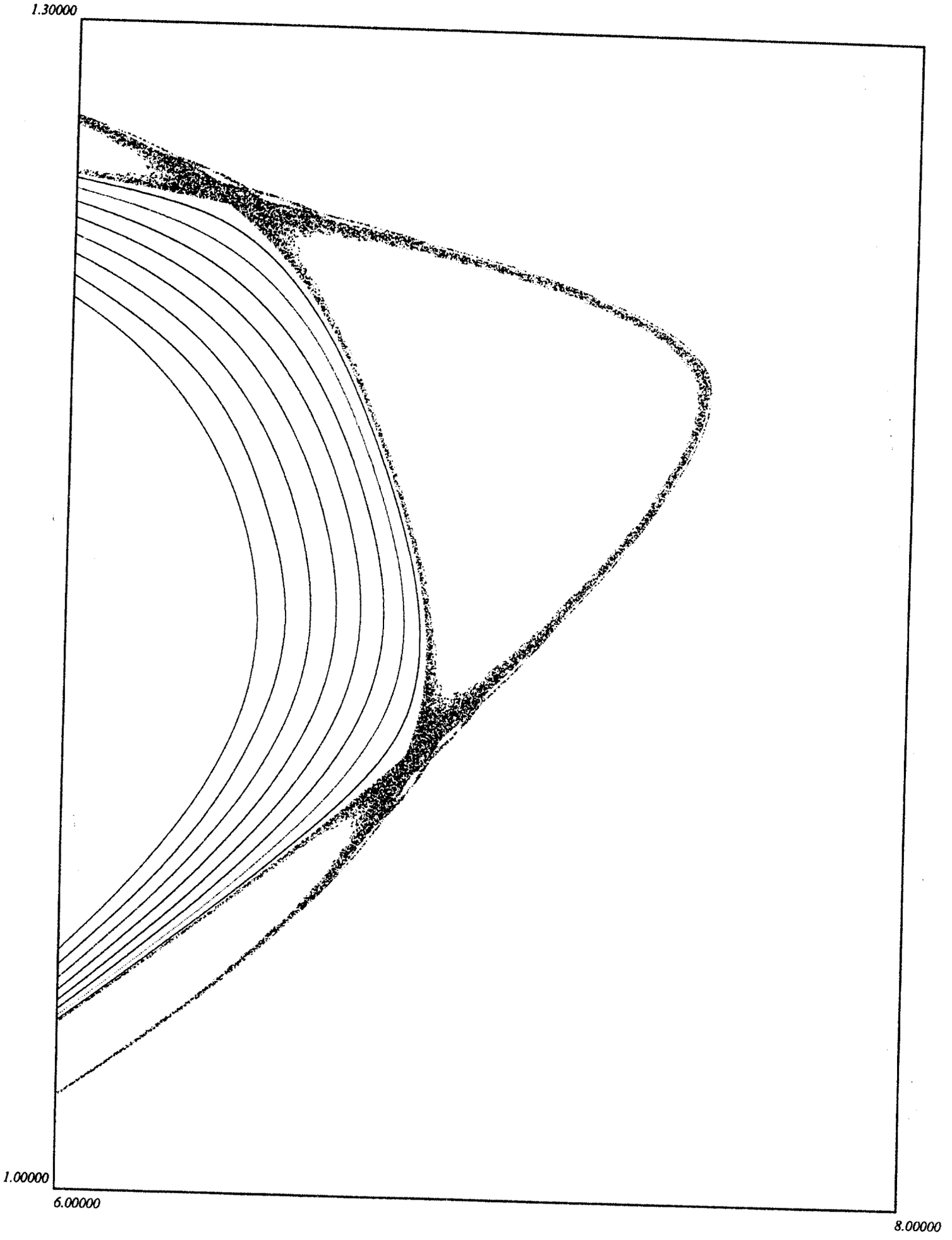


Figure 2b

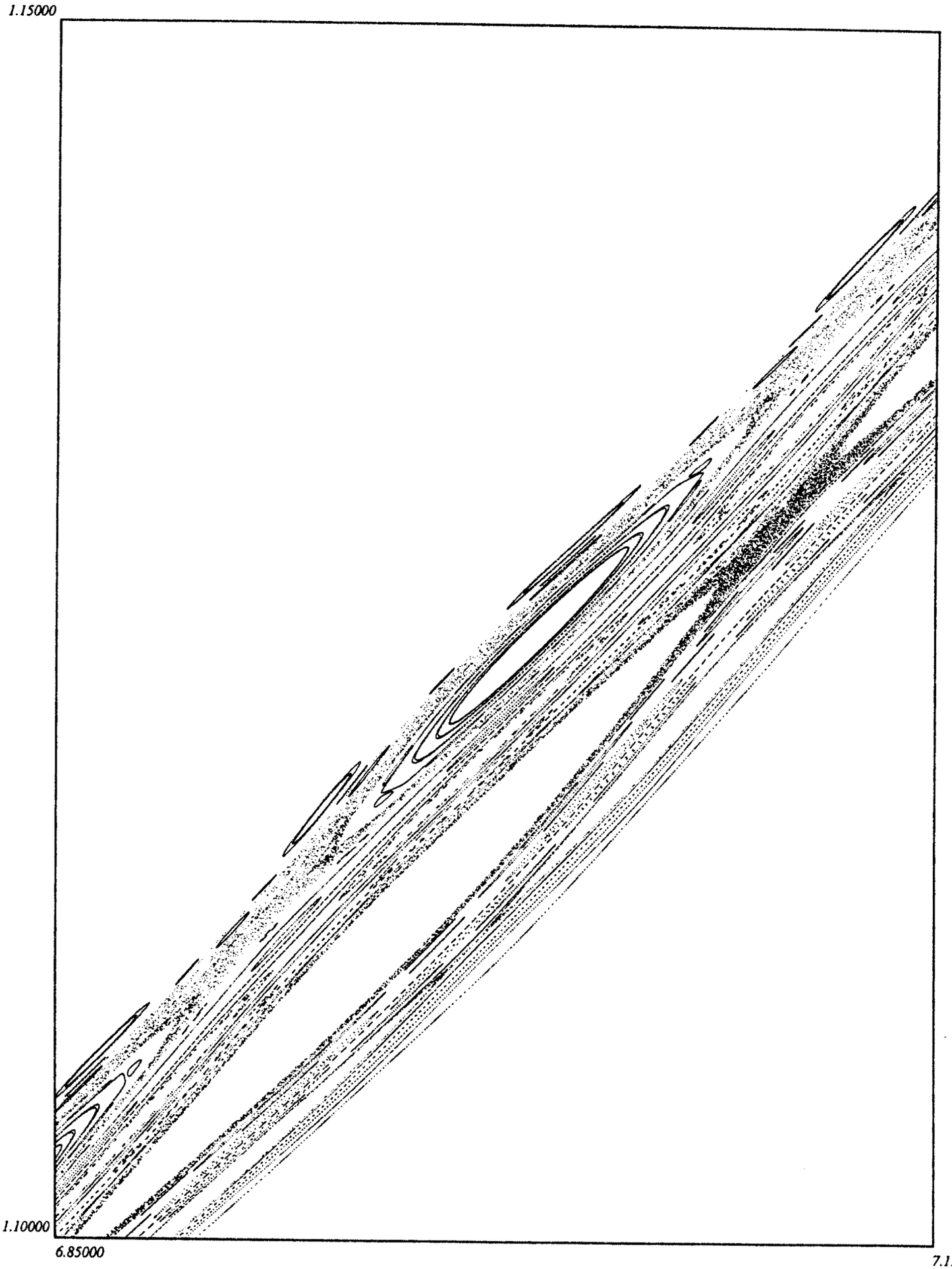


Figure 3