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Six-Vertex Type Solutions of Yang-Baxter Equation with Color Parameters†

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Abstract: In this paper, we obtain all solutions of Yang-Baxter equation with color parameters for six-vertex model in theory of exactly solved statistical models by five solution transformations, three non-degenerate basic solutions and several degenerate basic solutions. And we show that we can obtain all solutions of Yang-Baxter equation with spectral parameter for six-vertex model from solutions of Yang-Baxter equation with color parameters for six-vertex model.

Keywords: Yang-Baxter equation, solutions with color parameters of six-vertex type, solution transformations and basic solutions.

§1 Introduction

In the research of exact solution of integrable statistical models, C.N. Yang[1] and R.J. Baxter[2] independently proposed a very important non-linear algebraic equation, which is later called Yang-Baxter equation(YBE). (One can also find the history and review of Yang-Baxter equation in [3] and references therein)

Since 1960s, many excellent research works and results on finding solutions of YBE and corresponding theories of exactly solvable statistical models have appeared. Yang gave the first solution of YBE[1], Lieb[4] and Sutherland[5] studied and solved the six-vertex model, Baxter studied eight-vertex model and obtained an elliptic solution of YBE[2], Fan and Wu gave the famous Free-Fermion condition[6], and we can also find discussion of solutions and method to find solutions in [7-15] and references therein.

There are different types of expressions of YBE. Here we introduce three types, the first is the type of YBE with spectral parameter

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u), \tag{1}$$

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where u, v and $u+v$ are spectral parameters,

$$\check{R}_{12}(u) = \check{R}(u)\otimes E, \qquad \check{R}_{23}(u) = E\otimes\check{R}(u),$$

E is the unit matrix of order n, symbol $\otimes$ denotes the tensor product of two matrices. Other types of YBE are

$$\check{R}_{12}(\xi,\eta)\check{R}_{23}(\xi,\lambda)\check{R}_{12}(\eta,\lambda) = \check{R}_{23}(\eta,\lambda)\check{R}_{12}(\xi,\lambda)\check{R}_{23}(\xi,\eta), \tag{2}$$

which is called YBE with color parameters, where ξ, η and λ are color parameters; And

$$\check{R}_{12}(u,\xi,\eta)\check{R}_{23}(u+v,\xi,\lambda)\check{R}_{12}(v,\eta,\lambda) = \check{R}_{23}(v,\eta,\lambda)\check{R}_{12}(u+v,\xi,\lambda)\check{R}_{23}(u,\xi,\eta), \tag{3}$$

which is called the colored YBE with spectral and color parameters, where u, v and $u+v$ are spectral parameters, ξ, η and λ are color parameters. Generally, if a solution of YBE is found, then the corresponding statistical model can be exactly solved.

Equation (1) and (2) are two special cases of equation (3). After simple analysis, one can see the if $\check{R}(u)$ is a solution of (1), then $\check{R}(\xi-\eta)$ must be solution of (2). In [15], we gave all the six-vertex type solutions of (1). In this paper, we will give all the six-vertex type solution of (2) and do some preparation for studying the solutions of (3). To solve the YBE with color parameters is to find a square matrix of order n^2 with entries of two-variable functions, $\check{R}(\xi,\eta) = (a_{ij})_{1\le i,j\le n^2}$, which satisfies equation (2). Six-vertex type solution with color parameters is the solution of (2) with the form

$$\check{R}(\xi,\eta) = \begin{pmatrix} a_1(\xi,\eta) & 0 & 0 & 0 \\ 0 & a_2(\xi,\eta) & a_5(\xi,\eta) & 0 \\ 0 & a_6(\xi,\eta) & a_3(\xi,\eta) & 0 \\ 0 & 0 & 0 & a_4(\xi,\eta) \end{pmatrix}.$$

If $a_i(\xi,\eta)\not\equiv 0$ $(i=1,2,\cdots,6)$, then we call it the non-degenerate six-vertex type solution of YBE with color parameters, otherwise we call it the degenerate six-vertex type solution of YBE with color parameters. For six-vertex type solutions, the Free-Fermion condition can be expressed as

$$a_2(\xi,\eta)a_3(\xi,\eta) - a_1(\xi,\eta)a_4(\xi,\eta) - a_5(\xi,\eta)a_6(\xi,\eta) = 0. \tag{4}$$

Now we are going to give all solutions with color parameters of six-vertex type. For simplicity, we denote $\check{R}(\xi,\eta) =< a_1(\xi,\eta), a_2(\xi,\eta), a_3(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta) >$. Therefore, equation (2) is equivalent to following 13 equations:

$$a_2(\xi,\eta)a_2(\eta,\lambda)a_3(\xi,\lambda) - a_3(\xi,\eta)a_3(\eta,\lambda)a_2(\xi,\lambda) = 0, \tag{5a}$$

$$\begin{aligned} &a_1(\xi,\eta)a_1(\eta,\lambda)a_2(\xi,\lambda) - a_2(\xi,\eta)a_2(\eta,\lambda)a_1(\xi,\lambda) - a_6(\xi,\eta)a_5(\eta,\lambda)a_2(\xi,\lambda) = 0,\\ &a_1(\xi,\eta)a_2(\eta,\lambda)a_5(\xi,\lambda) - a_5(\xi,\eta)a_2(\eta,\lambda)a_1(\xi,\lambda) - a_3(\xi,\eta)a_5(\eta,\lambda)a_2(\xi,\lambda) = 0,\\ &a_2(\xi,\eta)a_1(\eta,\lambda)a_6(\xi,\lambda) - a_2(\xi,\eta)a_6(\eta,\lambda)a_1(\xi,\lambda) - a_6(\xi,\eta)a_3(\eta,\lambda)a_2(\xi,\lambda) = 0, \end{aligned} \tag{5b}$$

$$\begin{aligned} &a_4(\xi,\eta)a_4(\eta,\lambda)a_2(\xi,\lambda) - a_2(\xi,\eta)a_2(\eta,\lambda)a_4(\xi,\lambda) - a_5(\xi,\eta)a_6(\eta,\lambda)a_2(\xi,\lambda) = 0,\\ &a_4(\xi,\eta)a_2(\eta,\lambda)a_6(\xi,\lambda) - a_6(\xi,\eta)a_2(\eta,\lambda)a_4(\xi,\lambda) - a_3(\xi,\eta)a_6(\eta,\lambda)a_2(\xi,\lambda) = 0,\\ &a_2(\xi,\eta)a_4(\eta,\lambda)a_5(\xi,\lambda) - a_2(\xi,\eta)a_5(\eta,\lambda)a_4(\xi,\lambda) - a_5(\xi,\eta)a_3(\eta,\lambda)a_2(\xi,\lambda) = 0, \end{aligned} \tag{5c}$$

$$\left.\begin{array}{l}
a_1(\xi,\eta)a_1(\eta,\lambda)a_3(\xi,\lambda)-a_3(\xi,\eta)a_3(\eta,\lambda)a_1(\xi,\lambda)-a_6(\xi,\eta)a_5(\eta,\lambda)a_3(\xi,\lambda)=0,\\
a_1(\xi,\eta)a_3(\eta,\lambda)a_5(\xi,\lambda)-a_5(\xi,\eta)a_3(\eta,\lambda)a_1(\xi,\lambda)-a_2(\xi,\eta)a_5(\eta,\lambda)a_3(\xi,\lambda)=0,\\
a_3(\xi,\eta)a_1(\eta,\lambda)a_6(\xi,\lambda)-a_3(\xi,\eta)a_6(\eta,\lambda)a_1(\xi,\lambda)-a_6(\xi,\eta)a_2(\eta,\lambda)a_3(\xi,\lambda)=0,\\
a_4(\xi,\eta)a_4(\eta,\lambda)a_3(\xi,\lambda)-a_3(\xi,\eta)a_3(\eta,\lambda)a_4(\xi,\lambda)-a_5(\xi,\eta)a_6(\eta,\lambda)a_3(\xi,\lambda)=0,\\
a_4(\xi,\eta)a_3(\eta,\lambda)a_6(\xi,\lambda)-a_6(\xi,\eta)a_3(\eta,\lambda)a_4(\xi,\lambda)-a_2(\xi,\eta)a_6(\eta,\lambda)a_3(\xi,\lambda)=0,\\
a_3(\xi,\eta)a_4(\eta,\lambda)a_5(\xi,\lambda)-a_3(\xi,\eta)a_5(\eta,\lambda)a_4(\xi,\lambda)-a_5(\xi,\eta)a_2(\eta,\lambda)a_3(\xi,\lambda)=0.
\end{array}\right\} \quad (5d)$$

One can see that only the sub-indices 2,3 appear in the first equation. If we permute these two sub-indices, (5a) remains the same, but (5b) and (5c) interchange with (5d). Similarly, if we permute sub-indices 1 with 4 and 5 with 6 spontaneously, (5b) interchanges with (5c), but (5a) and (5d) keeps invariant. Thus we have:

(A) If $\check{R}(\xi,\eta) = < a_1(\xi,\eta), a_2(\xi,\eta), a_3(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta) >$ is a solution of (2), then

$$\begin{array}{l}
\check{R}(\xi,\eta) = < a_1(\xi,\eta), a_3(\xi,\eta), a_2(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta) >,\\
\check{R}(\xi,\eta) = < a_4(\xi,\eta), a_2(\xi,\eta), a_3(\xi,\eta), a_1(\xi,\eta), a_6(\xi,\eta), a_5(\xi,\eta) >,\\
\check{R}(\xi,\eta) = < a_4(\xi,\eta), a_3(\xi,\eta), a_2(\xi,\eta), a_1(\xi,\eta), a_6(\xi,\eta), a_5(\xi,\eta) >
\end{array}$$

are also solutions.

Additionally, from equations (5a)—(5d), we find following solution transformations. If $\check{R}(\xi,\eta) = < a_1(\xi,\eta), a_2(\xi,\eta), a_3(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta) >$ is a solution of (2), then:

(B) $f(\xi,\eta)\check{R}(\xi,\eta)$ is also a solution of (2), where $f(u)$ is an arbitrary non-zero function;

(C) $\check{R}(\xi,\eta) = < a_1(\xi,\eta), a_2(\xi,\eta), a_3(\xi,\eta), a_4(\xi,\eta), \mu^{-1}a_5(\xi,\eta), \mu a_6(\xi,\eta) >$ is also a solution of (2), where μ is a non-zero constant;

(D) $\check{R}(\xi,\eta) = < a_1(\xi,\eta), \frac{f(\xi)}{f(\eta)}a_2(\xi,\eta), \frac{f(\eta)}{f(\xi)}a_3(\xi,\eta),\ a_4(\xi,\eta),\ a_5(\xi,\eta),\ a_6(\xi,\eta) >$ is also a solution of (2), where $f(u)$ is an arbitrary non-zero function;

(E) $\check{R}(\xi,\eta) = < a_1(\alpha,\beta), a_2(\alpha,\beta), a_3(\alpha,\beta), a_4(\alpha,\beta), a_5(\alpha,\beta), a_6(\alpha,\beta) >$ is also a solution of (2), where $\alpha = f(\xi)$, $\beta = f(\eta)$, f is an arbitrary non-zero function of one variable;

In the following context, we call (A), (B), (C), (D), (E) the solution transformations A, B, C, D, E of six-vertex type YBE with color parameters. And the Free-Fermion condition is invariant with respect to the solution transformations A, B, C, D, E.

§2 Non-degenerate six-vertex type solutions with color parameters

We first consider the non-degenerate solutions. Since $a_2(\xi,\eta)\not\equiv 0$, from (5a) we have

$$\frac{a_3}{a_2}(\xi,\eta) = \frac{a_3}{a_2}(\xi,\lambda)\Big/\frac{a_3}{a_2}(\eta,\lambda). \quad (6)$$

Noticing that the left-hand-side of (6) is independent of λ, up to solution transformation D, we can assume

$$a_2(\xi,\eta) = a_3(\xi,\eta). \quad (7)$$

By considering with solution transformation B, we can assume $a_2(\xi,\eta) = a_3(\xi,\eta) = 1$ in the following discussions of non-degenerate solutions without losing generality. Therefore equations (5a)—(5d) are equivalent to following six equations up to solution transformations B and D:

$$\left.\begin{array}{l}
a_1(\xi,\eta)a_1(\eta,\lambda)-a_1(\xi,\lambda)-a_6(\xi,\eta)a_5(\eta,\lambda)=0\\
a_1(\xi,\eta)a_5(\xi,\lambda)-a_5(\xi,\eta)a_1(\xi,\lambda)-a_5(\eta,\lambda)=0\\
a_1(\eta,\lambda)a_6(\xi,\lambda)-a_6(\eta,\lambda)a_1(\xi,\lambda)-a_6(\xi,\eta)=0
\end{array}\right\} \quad (8a)$$

$$\left.\begin{array}{l}
a_4(\xi,\eta)a_4(\eta,\lambda)-a_4(\xi,\lambda)-a_5(\xi,\eta)a_6(\eta,\lambda)=0\\
a_4(\xi,\eta)a_6(\xi,\lambda)-a_6(\xi,\eta)a_4(\xi,\lambda)-a_6(\eta,\lambda)=0\\
a_4(\eta,\lambda)a_5(\xi,\lambda)-a_5(\eta,\lambda)a_4(\xi,\lambda)-a_5(\xi,\eta)=0
\end{array}\right\} \quad (8b)$$

Next, from equations (8) we will use $a_i(\xi,\lambda)$ and $a_i(\eta,\lambda)$ $(i=1,2,\cdots,6)$ to represent $a_i(\xi,\eta)$ $(i=1,2,\cdots,6)$. From (8a) we have

$$\begin{array}{l}
a_1(\xi,\eta) = \frac{a_1(\xi,\lambda)}{a_1(\eta,\lambda)}(1-a_5(\eta,\lambda)a_6(\eta,\lambda)) + a_5(\eta,\lambda)a_6(\xi,\lambda),\\
a_5(\xi,\eta) = \frac{a_5(\xi,\lambda)}{a_1(\eta,\lambda)}(1-a_5(\eta,\lambda)a_6(\eta,\lambda)) - \frac{a_5(\eta,\lambda)}{a_1(\xi,\lambda)}(1-a_5(\xi,\lambda)a_6(\xi,\lambda)),\\
a_6(\xi,\eta) = a_1(\eta,\lambda)a_6(\xi,\lambda) - a_1(\xi,\lambda)a_6(\eta,\lambda),
\end{array} \quad (9)$$

and from (8b) we have

$$\begin{array}{l}
a_4(\xi,\eta) = \frac{a_4(\xi,\lambda)}{a_4(\eta,\lambda)}(1-a_5(\eta,\lambda)a_6(\eta,\lambda)) + a_5(\xi,\lambda)a_6(\eta,\lambda),\\
a_5(\xi,\eta) = a_4(\eta,\lambda)a_5(\xi,\lambda) - a_4(\xi,\lambda)a_5(\eta,\lambda),\\
a_6(\xi,\eta) = \frac{a_6(\xi,\lambda)}{a_4(\eta,\lambda)}(1-a_5(\eta,\lambda)a_6(\eta,\lambda)) - \frac{a_6(\eta,\lambda)}{a_4(\xi,\lambda)}(1-a_5(\xi,\lambda)a_6(\xi,\lambda)).
\end{array} \quad (10)$$

By comparing the expressions of $a_5(\xi,\eta)$ and $a_6(\xi,\eta)$ in (9) and (10), we have

$$\begin{array}{rl}
& a_1(\xi,\lambda)a_5(\xi,\lambda)(1-a_1(\eta,\lambda)a_4(\eta,\lambda)-a_5(\eta,\lambda)a_6(\eta,\lambda))\\
= & a_1(\eta,\lambda)a_5(\eta,\lambda)(1-a_1(\xi,\lambda)a_4(\xi,\lambda)-a_5(\xi,\lambda)a_6(\xi,\lambda)),\\
& a_4(\xi,\lambda)a_6(\xi,\lambda)(1-a_1(\eta,\lambda)a_4(\eta,\lambda)-a_5(\eta,\lambda)a_6(\eta,\lambda))\\
= & a_4(\eta,\lambda)a_6(\eta,\lambda)(1-a_1(\xi,\lambda)a_4(\xi,\lambda)-a_5(\xi,\lambda)a_6(\xi,\lambda)),
\end{array}$$

i.e.

$$\begin{array}{l}
\frac{(1-a_1(\xi,\lambda)a_4(\xi,\lambda)-a_5(\xi,\lambda)a_6(\xi,\lambda))}{a_1(\xi,\lambda)a_5(\xi,\lambda)} = C_1(\lambda),\\
\frac{(1-a_1(\xi,\lambda)a_4(\xi,\lambda)-a_5(\xi,\lambda)a_6(\xi,\lambda))}{a_4(\xi,\lambda)a_6(\xi,\lambda)} = C_2(\lambda),
\end{array} \quad (11)$$

where C_1, C_2 are functions of λ. Now we fix λ to a value and denote $f_i(\xi) = a_i(\xi,\lambda)$ $(i=1,4,5,6)$, then $f_i(\xi)$ $(i=1,4,5,6)$ satisfy

$$\begin{array}{l}
1-f_1(\xi)f_4(\xi)-f_5(\xi)f_6(\xi) = C_1f_1(\xi)f_5(\xi),\\
1-f_1(\xi)f_4(\xi)-f_5(\xi)f_6(\xi) = C_2f_4(\xi)f_6(\xi),
\end{array} \quad (12)$$

where C_1, C_2 are constants. Therefore, from the combination of (9), (10) and (12), the expressions of $a_i(\xi,\eta)$ $(i=1,4,5,6)$ by $f_i(\xi), f_i(\eta)$ $(i=1,4,5,6)$ are as follows

$$\begin{aligned} a_1(\xi,\eta) &= f_1(\xi)(f_4(\eta)+C_1 f_5(\eta)) + f_5(\eta)f_6(\xi), \\ a_4(\xi,\eta) &= f_4(\xi)(f_1(\eta)+C_2 f_6(\eta)) + f_5(\xi)f_6(\eta), \\ a_5(\xi,\eta) &= f_4(\eta)f_5(\xi) - f_4(\xi)f_5(\eta), \\ a_6(\xi,\eta) &= f_1(\eta)f_6(\xi) - f_1(\xi)f_6(\eta). \end{aligned} \tag{13}$$

Since we consider non-degenerate solutions, there are only two cases for C_1 and C_2, one is $C_1 = C_2 = 0$, the other is $C_1 \neq 0$ and $C_2 \neq 0$. And by solution transformation C, we can assume $C_1 = C_2 = -2\cos(C)$.

For the case $C_1 = C_2 = 0$,

$$1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) = 0.$$

Equivalently, $f_i(\xi)$ $(i=1,4,5,6)$ can be parameterized as

$$\begin{aligned} f_1(\xi) &= (f(\xi)+1)g(\xi), \\ f_4(\xi) &= (-f(\xi)+1)/g(\xi), \\ f_5(\xi) &= f(\xi)h(\xi), \\ f_6(\xi) &= f(\xi)/h(\xi), \end{aligned} \tag{14}$$

where f, g, h are arbitrary functions of one variable. And corresponding expressions of $a_i(\xi,\eta)$ $(i=1,2,\cdots,6)$ are

$$\begin{aligned} a_1(\xi,\eta) &= (f(\xi)+1)(-f(\eta)+1)\frac{g(\xi)}{g(\eta)} + f(\xi)f(\eta)\frac{h(\eta)}{h(\xi)}, \\ a_2(\xi,\eta) &= a_3(\xi,\eta) = 1, \\ a_4(\xi,\eta) &= (f(\eta)+1)(-f(\xi)+1)\frac{g(\eta)}{g(\xi)} + f(\xi)f(\eta)\frac{h(\xi)}{h(\eta)}, \\ a_5(\xi,\eta) &= f(\xi)(-f(\eta)+1)\frac{h(\xi)}{g(\eta)} - f(\eta)(-f(\xi)+1)\frac{h(\eta)}{g(\xi)}, \\ a_6(\xi,\eta) &= f(\xi)(f(\eta)+1)\frac{g(\eta)}{h(\xi)} - f(\eta)(f(\xi)+1)\frac{g(\xi)}{h(\eta)}. \end{aligned} \tag{15}$$

This solution satisfies the Free-Fermion condition (4).

For the case C_1, C_2 are non-zero, by solution transformation C, we can assume $C_1 = C_2 = -2\cos(C) \neq 0$,

$$1 - f_1(\xi)f_4(\xi) - f_5(\xi)f_6(\xi) = -2\cos(C)f_1(\xi)f_5(\xi) = -2\cos(C)f_4(\xi)f_6(\xi). \tag{16}$$

Since $-2\cos(C) \neq 0$,

$$f_1(\xi)f_5(\xi) = f_4(\xi)f_6(\xi). \tag{17}$$

By (17), $f_i(\xi)$ $(i=1,4,5,6)$ can expressed as follows

$$\begin{aligned} f_1(\xi) &= h(\xi)/g(\xi), \quad & f_4(\xi) &= h(\xi)g(\xi), \\ f_5(\xi) &= f(\xi)g(\xi), \quad & f_6(\xi) &= f(\xi)/g(\xi). \end{aligned} \tag{18}$$

Then we can rewrite (16) as

$$h^2(\xi) - 2\cos(C)h(\xi)f(\xi) + f^2(\xi) - 1 = 0. \tag{19}$$

Considering (19) as a second order equation of $h(\xi)$, we have the solutions as

$$h(\xi) = \cos(C)f(\xi) \pm \sqrt{1 - \sin^2(C)f^2(\xi)}. \tag{20}$$

1) If $\cos(C) = \pm 1$, then $\sin(C) = 0$,

$$h(\xi) = \pm f(\xi) \pm 1.$$

After considering $f_i(\xi)$ $(i=1,4,5,6)$ and $h(\xi), f(\xi)$ with the influence of solution transformations B,C,E and other equivalent conditions, we find that we need only assume

$$\cos(C) = 1, \qquad h(\xi) = f(\xi) + 1.$$

Thus

$$\begin{aligned} f_1(\xi) &= (f(\xi)+1)/g(\xi), \quad & f_5(\xi) &= f(\xi)g(\xi), \\ f_4(\xi) &= (f(\xi)+1)g(\xi), \quad & f_6(\xi) &= f(\xi)/g(\xi), \end{aligned} \tag{21}$$

The solution of (2) is

$$\begin{aligned} a_1(\xi,\eta) &= (f(\xi) - f(\eta) + 1)\frac{g(\eta)}{g(\xi)}, \\ a_2(\xi,\eta) &= a_3(\xi,\eta) = 1, \\ a_4(\xi,\eta) &= (f(\xi) - f(\eta) + 1)\frac{g(\xi)}{g(\eta)}, \\ a_5(\xi,\eta) &= (f(\xi) - f(\eta))g(\xi)g(\eta), \\ a_6(\xi,\eta) &= (f(\xi) - f(\eta))\frac{1}{g(\xi)g(\eta)}. \end{aligned} \tag{22}$$

2) If $\cos(C) \neq \pm 1$, then $\sin(C) \neq 0$. Up to solution transformation E, We assume

$$f(\xi) = \sin(\xi)/\sin(C).$$

Thus

$$h(\xi) = \cos(C)\sin(\xi)/\sin(C) \pm \cos(\xi) = \sin(\xi \pm C)/\sin(C).$$

After considering the influence of solution transformations B,C,E and other equivalent conditions, we find that we need only assume

$$f(\xi) = \sin(\xi)/\sin(C) \qquad h(\xi) = \sin(\xi + C)/\sin(C).$$

Therefore

$$f_1(\xi)=\frac{\sin(\xi+C)}{\sin(C)g(\xi)},\qquad f_5(\xi)=\frac{\sin(\xi)g(\xi)}{\sin(C)},$$
$$f_4(\xi)=\frac{\sin(\xi+C)g(\xi)}{\sin(C)},\qquad f_6(\xi)=\frac{\sin(\xi)}{\sin(C)g(\xi)},\tag{23}$$

The solution of (2) for this case is

$$\begin{aligned}
a_1(\xi,\eta)&=\frac{\sin(\xi-\eta+C)}{\sin(C)}\frac{g(\eta)}{g(\xi)},\\
a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\
a_4(\xi,\eta)&=\frac{\sin(\xi-\eta+C)}{\sin(C)}\frac{g(\xi)}{g(\eta)},\\
a_5(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)}g(\xi)g(\eta),\\
a_6(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)}\frac{1}{g(\xi)g(\eta)},
\end{aligned}\tag{24}$$

where C is arbitrary constant.

From above discussions, we finally have the following theorem.

Theorem Up to solution transformations A,B,C,D,E, any non-degenerate six-vertex type solution of YBE (2) with color parameters is equivalent to one of the three sets of basic solutions: (15), (22) and (24).

Remark 1: If we take $f(\xi)=\xi$, $g(\xi)=h(\xi)=1$ in (15), solution (15) becomes

$$\begin{aligned}
a_1(\xi,\eta)&=\xi-\eta+1,\\
a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\
a_4(\xi,\eta)&=\eta-\xi+1,\\
a_5(\xi,\eta)&=\xi-\eta,\\
a_6(\xi,\eta)&=\xi-\eta.
\end{aligned}\tag{25}$$

If we take $f(\xi)=\sin(\xi)/\sin(C)$, $g(\xi)=\sin(\xi+C)/(\sin(xi)+\sin(C))$ and $h(\xi)=1$ in (15), solution (15) becomes

$$\begin{aligned}
a_1(\xi,\eta)&=\frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\
a_4(\xi,\eta)&=\frac{\sin(\eta-\xi+C)}{\sin(C)},\\
a_5(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)},\\
a_6(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)}.
\end{aligned}\tag{26}$$

If we take $f(\xi)=\xi$, $g(\xi)=1$ in (22), solution (22) becomes

$$\begin{aligned}
a_1(\xi,\eta)&=\xi-\eta+1,\\
a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\
a_4(\xi,\eta)&=\xi-\eta+1,\\
a_5(\xi,\eta)&=\xi-\eta,\\
a_6(\xi,\eta)&=\xi-\eta.
\end{aligned}\tag{27}$$

If we take $g(\xi)=1$ in (24), solution (24) becomes

$$\begin{aligned}
a_1(\xi,\eta)&=\frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_2(\xi,\eta)&=a_3(\xi,\eta)=1,\\
a_4(\xi,\eta)&=\frac{\sin(\xi-\eta+C)}{\sin(C)},\\
a_5(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)},\\
a_6(\xi,\eta)&=\frac{\sin(\xi-\eta)}{\sin(C)}.
\end{aligned}\tag{28}$$

In (25), (26), (27) and (28), $a_i(\xi,\eta)$ $(i=1,2,\cdots,6)$ are in forms of functions of two variables, but they can also be regarded as one-variable functions of $\xi-\eta$. Therefore from three basic solutions (15), (22) and (24) of (2), we obtain the four basic solutions of (1).(please see [15])

§3 Degenerate six-vertex type solutions with color parameters

Now that we have discussed the non-degenerate.solution, we come to the degenerate solutions of equations (5a)—(5d).

1. If $a_2(\xi,\eta)=a_3(\xi,\eta)\equiv 0$, then $a_1(\xi,\eta), a_4(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta)$ can be arbitrary functions.

2. If $a_2(\xi,\eta)\equiv 0$ or $a_3(\xi,\eta)\equiv 0$, then up to solution transformation A, we assume $a_2(\xi,\eta)\not\equiv 0$, $a_3(\xi,\eta)\equiv 0$ without losing generality, equation (2) is equivalent to following six equations:

$$\begin{aligned}
&a_1(\xi,\eta)a_1(\eta,\lambda)-a_1(\xi,\lambda)-a_6(\xi,\eta)a_5(\eta,\lambda)=0,\\
&a_1(\xi,\eta)a_5(\xi,\lambda)-a_5(\xi,\eta)a_1(\xi,\lambda)=0,\\
&a_1(\eta,\lambda)a_6(\xi,\lambda)-a_6(\eta,\lambda)a_1(\xi,\lambda)=0,\\
&a_4(\xi,\eta)a_4(\eta,\lambda)-a_4(\xi,\lambda)-a_5(\xi,\eta)a_6(\eta,\lambda)=0,\\
&a_4(\xi,\eta)a_6(\xi,\lambda)-a_6(\xi,\eta)a_4(\xi,\lambda)=0,\\
&a_4(\eta,\lambda)a_5(\xi,\lambda)-a_5(\eta,\lambda)a_4(\xi,\lambda)=0.
\end{aligned}\tag{29}$$

a. If we have $a_5(\xi,\eta)=a_6(\xi,\eta)\equiv 0$ additionally, then up to solution transformations, the basic solutions for this case are

$$\begin{cases}
a_1(\xi,\eta)=\epsilon_1 f_1(\xi)/f_1(\eta),\\
a_2(\xi,\eta)=1,\\
a_4(\xi,\eta)=\epsilon_4 f_4(\xi)/f_4(\eta),\\
a_3(\xi,\eta)=a_5(\xi,\eta)=a_6(\xi,\eta)=0,
\end{cases}\tag{30}$$

where f_1, f_4 are arbitrary functions, ϵ_1, ϵ_4 are 1 or 0.

b. If we have $a_5(\xi,\eta) \equiv 0$ or $a_6(\xi,\eta) \equiv 0$ additionally, then up to solution transformation A, we assume $a_5(\xi,\eta) \not\equiv 0$, $a_6(\xi,\eta) \equiv 0$ without losing generality, therefore for this case there are four sets of basic solutions up to the solution transformations:

i) $a_1(\xi,\eta) = a_3(\xi,\eta) = a_4(\xi,\eta) = a_6(\xi,\eta) \equiv 0$, $a_2(\xi,\eta), a_5(\xi,\eta)$ are arbitrary functions,

ii) $a_1(\xi,\eta) = a_3(\xi,\eta) = a_6(\xi,\eta) \equiv 0$, $a_4(\xi,\eta) = f_4(\xi)/f_4(\eta)$, $a_2(\xi,\eta) = 1$, $a_5(\xi,\eta) = f_4(\xi)f_5(\eta)$,

iii) $a_3(\xi,\eta) = a_4(\xi,\eta) = a_6(\xi,\eta) \equiv 0$, $a_1(\xi,\eta) = f_1(\xi)/f_1(\eta)$, $a_2(\xi,\eta) = 1$, $a_5(\xi,\eta) = f_5(\xi)/f_1(\eta)$,

iv) $a_3(\xi,\eta) = a_6(\xi,\eta) \equiv 0$, $a_1(\xi,\eta) = f_1(\xi)/f_1(\eta)$, $a_4(\xi,\eta) = f_4(\xi)/f_4(\eta)$, $a_2(\xi,\eta) = 1$, $a_5(\xi,\eta) = f_4(\xi)/f_1(\eta)$,

where f_1, f_4, f_5 are arbitrary functions.

c. If we have $a_5(\xi,\eta) \not\equiv 0$ and $a_6(\xi,\eta) \not\equiv 0$ additionally, up to solution transformations, the basic solutions for this case are

$$\begin{cases} a_1(\xi,\eta) = \dfrac{f_6(\xi)}{f_6(\eta)}(g(\xi,\eta)+1/2), \\ a_2(\xi,\eta) = 1, \\ a_3(\xi,\eta) = 0, \\ a_4(\xi,\eta) = \dfrac{f_5(\xi)}{f_5(\eta)}(-g(\xi,\eta)+1/2), \\ a_5(\xi,\eta) = \dfrac{f_5(\xi)}{Cf_6(\eta)}(g(\xi,\eta)+1/2), \\ a_6(\xi,\eta) = -\dfrac{Cf_6(\xi)}{f_5(\eta)}(-g(\xi,\eta)+1/2), \end{cases} \tag{31}$$

and

$$\begin{cases} a_1(\xi,\eta) = \dfrac{C_1C_4}{1+C_1C_4}\dfrac{f_6(\xi)}{f_6(\eta)}, \\ a_2(\xi,\eta) = 1, \\ a_3(\xi,\eta) = 0, \\ a_4(\xi,\eta) = \dfrac{C_1C_4}{1+C_1C_4}\dfrac{f_5(\xi)}{f_5(\eta)}, \\ a_5(\xi,\eta) = \dfrac{C_4}{1+C_1C_4}\dfrac{f_5(\xi)}{f_6(\eta)}, \\ a_6(\xi,\eta) = \dfrac{C_1}{1+C_1C_4}\dfrac{f_6(\xi)}{f_5(\eta)}, \end{cases} \tag{32}$$

where f_1, f_4, f_5 are arbitrary one-variable functions, g is arbitrary two-variable function, C_1, C_4 are non-zero constants. And solution (31) satisfies Free-Fermion condition (4).

3. If $a_2(\xi,\eta) \not\equiv 0$ and $a_3(\xi,\eta) \not\equiv 0$, we also have following cases,

a. If we have $a_5(\xi,\eta) = a_6(\xi,\eta) \equiv 0$ additionally, then up to solution transformations, the basic solution for this case is

$$\begin{cases} a_1(\xi,\eta) = f_1(\xi)/f_1(\eta), \\ a_2(\xi,\eta) = a_3(\xi,\eta) = 1, \\ a_4(\xi,\eta) = f_4(\xi)/f_4(\eta), \\ a_5(\xi,\eta) = a_6(\xi,\eta) = 0, \end{cases} \tag{33}$$

where f_1, f_4 are arbitrary functions.

b. If we have $a_5(\xi,\eta) \equiv 0$ or $a_6(\xi,\eta) \equiv 0$ additionally, then up to solution transformation A, we assume $a_5(\xi,\eta) \not\equiv 0$, $a_6(\xi,\eta) \equiv 0$ without losing generality, equation (2) is equivalent to following six equations:

$$\begin{aligned} &a_1(\xi,\eta)a_1(\eta,\lambda) - a_1(\xi,\lambda) = 0, \\ &a_1(\xi,\eta)a_5(\xi,\lambda) - a_5(\xi,\eta)a_1(\xi,\lambda) - a_5(\eta,\lambda) = 0, \\ &a_4(\xi,\eta)a_4(\eta,\lambda) - a_4(\xi,\lambda) = 0, \\ &a_4(\eta,\lambda)a_5(\xi,\lambda) - a_5(\eta,\lambda)a_4(\xi,\lambda) - a_5(\xi,\eta) = 0. \end{aligned}$$

Then up to solution transformations, the basic solutions for this case are

$$\begin{cases} a_1(\xi,\eta) = \dfrac{f_1(\xi)}{f_1(\eta)}, \\ a_2(\xi,\eta) = a_3(\xi,\eta) = 1, \\ a_4(\xi,\eta) = \dfrac{f_1(\eta)}{f_1(\xi)}, \\ a_5(\xi,\eta) = \dfrac{f_5(\xi)}{f_1(\eta)} - \dfrac{f_5(\eta)}{f_1(\xi)}, \\ a_6(\xi,\eta) = 0, \end{cases} \tag{34}$$

and

$$\begin{cases} a_1(\xi,\eta) = \dfrac{f_1(\xi)}{f_1(\eta)}, \\ a_2(\xi,\eta) = a_3(\xi,\eta) = 1, \\ a_4(\xi,\eta) = \dfrac{f_4(\xi)}{f_4(\eta)}, \\ a_5(\xi,\eta) = \dfrac{f_4(\eta)(1-f_1(\xi)f_4(\xi))}{Cf_1(\xi)} - \dfrac{f_4(\xi)(1-f_1(\eta)f_4(\eta))}{Cf_1(\eta)}, \\ a_6(\xi,\eta) = 0, \end{cases} \tag{35}$$

where f_1, f_4, f_5 are arbitrary functions, C is non-zero constant. And solution (34) satisfied Free-Fermion condition (4).

c. If we have $a_5(\xi,\eta) \not\equiv 0$ and $a_6(\xi,\eta) \not\equiv 0$ additionally, there is no degenerate solution for this case. Otherwise, let $a_4(\xi,\eta) \equiv 0$ for example, then from equations (5c) one can see at least one of $a_2(\xi,\eta), a_3(\xi,\eta), a_5(\xi,\eta), a_6(\xi,\eta)$ must be zero, that gives the contradiction, i.e. there are only non-degenerate solutions for this case.

In fact, from above discussion, we give all the six-vertex type degenerate solution with color parameters up to solution transformations A,B,C,D,E.

Remark 2: In both references [12,13], a solution of equations (2) is mentioned and can be expressed in the notations of this paper as:

$$\check{R} = \begin{pmatrix} p(\xi) & 0 & 0 & 0 \\ 0 & \frac{p^2(\xi)-1}{p(\eta)} & 1 & 0 \\ 0 & \frac{p(\xi)}{p(\eta)} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{p(\eta)} \end{pmatrix}, \tag{36}$$

If we let

$$g(\xi,\eta) = \frac{p^2(\eta)+1}{2(p^2(\eta)-1)}, \quad f_5(\xi) = \frac{1}{p^2(\xi)-1}, \quad f_6(\xi) = \frac{p(\xi)}{C(p^2(\xi)-1)},$$

in the degenerate basic solution (31) discussed in this paper, the above solution (36) is obtained.

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