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Modelling the interaction of a relativistic beam particle with an electron cloud

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Abstract

We study the transverse and longitudinal forces acting on a relativistic beam particle due to the interaction with an electron cloud present in the beam pipe. It is found to be convenient to compute the electromagnetic field in a boosted reference frame, moving rigidly with the beam. In such a reference frame, charge and current densities are stationary, therefore the electric and magnetic fields are solution of an electrostatic and a magnetostatic problem respectively. It is possible to show that the force acting on the bunch (in the lab frame) is simply proportional to the gradient of the scalar potential and is therefore irrotational. This happens since the non-irrotational part of the electric field force is cancelled exactly by the force due to the magnetic field. For a relativistic beam the scalar potential can be calculated with good approximation as the solution of a 2D Poisson problem. The Hamiltonian of the resulting transformation can be written as a function of the position coordinates, showing that the map is symplectic and can be modelled as a “thin” element in tracking codes.

1 Introduction

We want to compute the transverse and longitudinal forces acting on a relativistic beam particle travelling in an indefinitely long perfectly conducting beam pipe, due to the interaction with an electron cloud [1, 2] present in the chamber.
The particle belongs to a bunch travelling at velocity βc (where c is the speed of light and β the relativistic factor) along the s axis coinciding with the longitudinal direction of the pipe. We want to evaluate the effect of the interaction with a portion of the e-cloud having length L and situated between two sections along the pipe, which are identified by two points at rest, namely $P_1 = (0, 0, -L/2)$ and $P_2 = (0, 0, L/2)$ (see Fig. 1).

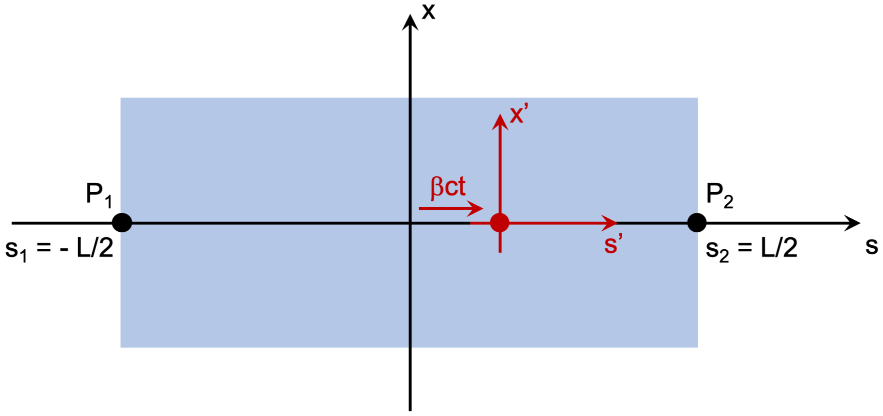

Figure 1: A beam particle (in red) travels within an electron cloud. The effect of the interaction with the electrons in a portion of beam-line of length L needs to be evaluated.

The e-cloud pinch follows the bunch, which means that, if we define $\rho_0 (x, y, t)$ and $\mathbf{J}_0 (x, y, t)$ as the electron charge and current density at the section $s = 0$ as a function of the time t and of the transverse coordinates x and y, we can write the charge and the current density in the entire space as:

$$\rho (x, y, s, t) = \rho_0 \left(x, y, t - \frac{s}{\beta c} \right) \tag{1}$$

$$\mathbf{J} (x, y, s, t) = \mathbf{J}_0 \left(x, y, t - \frac{s}{\beta c} \right) \tag{2}$$

We assume that the electrons do not move along s :

$$J_s = 0 \tag{3}$$

(the more general case is discussed in appendix).
The reference particle moves accordingly to:

$$s (t) = \beta c t \tag{4}$$

A generic particle arriving at the section $s = 0$ with a delay τ with respect to the reference particle will move according to:

$$s (t) = \beta c (t - \tau) \tag{5}$$

We define the distance between the two particles in the lab frame :

$$\zeta = -\beta c \tau \tag{6}$$

which is positive when the particle arrives earlier than the reference particle.
The particles will take a time:

$$T = \frac{L}{\beta c} \tag{7}$$

to cross the e-cloud. Of course here we are assuming that particles have the same momentum and therefore they move at the same speed.

2 Lorentz boost

We call K the lab reference frame in which we have defined all equations above, and we introduce a boosted frame K' moving rigidly with the reference particle. The coordinates in the two systems are related by a Lorentz transformation [3]:

$$ct' = \gamma \left(ct - \beta s \right) \tag{8}$$
$$x' = x \tag{9}$$
$$y' = y \tag{10}$$
$$s' = \gamma \left(s - \beta c t \right) \tag{11}$$

The corresponding inverse transformation is:

$$ct = \gamma \left(ct' + \beta s' \right) \tag{12}$$
$$x = x' \tag{13}$$
$$y = y' \tag{14}$$
$$s = \gamma \left(s' + \beta c t' \right) \tag{15}$$

In the frame K', the kinematic equation of the particle can be obtained by replacing Eqs. 12 and 15 into Eq. 5:

$$s = \gamma \left(s' + \beta c t' \right) = \beta \gamma \left(ct' + \beta s' \right) - \beta c \tau \tag{16}$$

Solving for s' we obtain:

$$s' = -\beta \gamma c \tau = \gamma \zeta \tag{17}$$

Of course for the reference particle we obtain $s' = 0$. We observe that **beam particles are at rest in the reference frame K' and that the distance between them is increased by a factor γ with respect to the lab frame K.**
We now transform the left e-cloud boundary P_1, which in the frame K is at rest:

$$s_1(t) = -\frac{L}{2} \tag{18}$$

Using Eq. 15 this becomes:

$$s_1' \left(t' \right) = -\frac{L}{2\gamma} - \beta c t' \tag{19}$$

Similarly for P_2:

$$s_2'\left(t'\right) = \frac{L}{2\gamma} - \beta c t' \tag{20}$$

We see that the e-cloud is moving along s' with speed $-\beta c$.
The length of the e-cloud in the frame K′ is given by $L' = s_2'\left(t'\right) - s_1'\left(t'\right)$ (difference of the two positions measured at the same time) obtaining:

$$L' = \frac{L}{\gamma} \tag{21}$$

which shows that **in the frame K' the e-cloud is shorter by factor** γ.
In the frame K', the points P_1 and P_2 pass at $s' = 0$ at times t_1' and t_2' which can be obtained from Eqs. 19 and 20:

$$t_1' = -\frac{L}{2\beta c\gamma} \tag{22}$$

$$t_2' = \frac{L}{2\beta c\gamma} \tag{23}$$

The interaction of the e-cloud (moving) with the particle (at rest) therefore lasts $T' = t_2' - t_1'$:

$$T' = \frac{L}{\gamma\beta c} = \frac{T}{\gamma} \tag{24}$$

The interaction lasts γ times less time in the frame K' compared to the frame K.

3 Electrodynamics in the boosted frame

We now want to transform the charge density function $\rho\left(x,y,s,t\right) = \rho_0\left(x,y,t-\frac{s}{\beta c}\right)$.
The quantities $\left(c\rho, J_x, J_y, J_s\right)$ form a Lorentz 4-vector and therefore they are transformed between K and K' by relationships similar to the Eqs. 8-10 [3]:

$$c\rho'\left(\mathbf{r'},t'\right) = \gamma\left[c\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) - \beta J_s\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right)\right] \tag{25}$$

$$J_s'\left(\mathbf{r'},t'\right) = \gamma\left[J_s\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) - \beta c\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right)\right] \tag{26}$$

where the transformations $\mathbf{r}\left(\mathbf{r'},t'\right)$ and $t\left(\mathbf{r'},t'\right)$ are defined by Eqs. 12 and 15 respectively. The transverse components J_x and J_y of the current vector are invariant for our transformation.
Taking into account that we assumed $J_s = 0$ we obtain:

$$\rho'\left(\mathbf{r'},t'\right) = \gamma\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) \tag{27}$$

$$J_s'\left(\mathbf{r'},t'\right) = -\gamma\beta c\rho\left(\mathbf{r}\left(\mathbf{r'},t'\right),t\left(\mathbf{r'},t'\right)\right) = -\beta c\rho'\left(\mathbf{r'},t'\right) \tag{28}$$

Using Eqs. 1 and 12-14, we obtain:

$$\rho'\left(x',y',s',t'\right) = \gamma\rho\left(x',y',s(s',t'),t(s',t')\right) = \gamma\rho_0\left(x',y',t(s',t') - \frac{s(s',t')}{\beta c}\right) \tag{29}$$

From Eq. 11 we get:

$$t(s',t') - \frac{s(s',t')}{\beta c} = -\frac{s'}{\gamma\beta c} \tag{30}$$

where the coordinate t' has disappeared.
We can therefore write:

$$\rho'\left(x',y',s',t'\right) = \gamma\rho_0\left(x',y',-\frac{s'}{\gamma\beta c}\right) \tag{31}$$

Similarly from Eq. 26 we can write:

$$\mathbf{J}'\left(x',y',s',t'\right) = \mathbf{J}_0\left(x',y',-\frac{s'}{\gamma\beta c}\right) - \gamma\beta c\rho_0\left(x',y',-\frac{s'}{\gamma\beta c}\right)\hat{\mathbf{i}}_s \tag{32}$$

where $\hat{\mathbf{i}}_s$ is a unit vector identifying the s direction.
We found that, **in the reference frame K′, both the charge density and the current density are not depending on time. As the sources are stationary the fields will also be stationary.**
This means that, in the frame K′, the electric field is solution of the electrostatic problem:

$$\nabla' \times \mathbf{E'} = 0 \tag{33}$$

$$\nabla' \cdot \mathbf{E'} = \frac{\rho'}{\varepsilon_0} \tag{34}$$

and the magnetic field is solution of the magnetostatic problem:

$$\nabla' \times \mathbf{B'} = \mu_0\mathbf{J'} \tag{35}$$

$$\nabla' \cdot \mathbf{B'} = 0 \tag{36}$$

As the magnetic field $\mathbf{B}'$ is solenoidal and the electric field $\mathbf{E'}$ is irrotational, we can introduce a vector potential $\mathbf{A}'$ and a scalar potential ϕ' so that:

$$\mathbf{B'} = \nabla' \times \mathbf{A}' \tag{37}$$

$$\mathbf{E'} = -\nabla'\phi' \tag{38}$$

The potentials can be chosen in order to satisfy the Lorentz gauge condition:

$$\nabla' \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t'} = 0 \tag{39}$$

where the second term of the left hand side vanishes due to the fact that the scalar potential is stationary. The vector potential is therefore solenoidal:

$$\nabla' \cdot \mathbf{A}' = 0 \tag{40}$$

Replacing Eq. 37 into the Eq. 35, we can write:

$$\nabla' \times \left(\nabla' \times \mathbf{A}'\right) = \nabla'\left(\nabla' \cdot \mathbf{A}'\right) - \nabla'^2\mathbf{A}' = \mu_0\mathbf{J'} \tag{41}$$

and using the Eq. 40, we obtain Poisson's equation for the vector potential:

$$\nabla'^2 \mathbf{A}' = -\mu_0 \mathbf{J'} \tag{42}$$

Replacing Eq. 38 into Eq. 34 we obtain Poisson's equation for the scalar potential:

$$\nabla'^2 \phi' = -\frac{\rho'}{\varepsilon_0} \tag{43}$$

To solve Eqs. 42 and 43, appropriate boundary conditions need to be imposed. For a perfectly conducting chamber, this translates into Dirichlet boundary conditions both for ϕ' and $\mathbf{A}'$, as the scalar and vector potentials need to be continuous across boundaries [4].
Projecting the Eq. 42 along s' and using Eq. 28, we obtain:

$$\nabla'^2 A'_s = -\mu_0 J'_s = \mu_0 \beta c \rho' \tag{44}$$

Comparing against Eq. 43 we can write:

$$A'_s = -\frac{\beta}{c}\phi' \tag{45}$$

Using this result and taking into account that the quantities $\left(\frac{\phi}{c}, A_x, A_y, A_s\right)$ form a Lorentz 4-vector, we can show that the s component of the vector potential in the lab frame vanishes:

$$A_s = A'_s + \beta\frac{\phi'}{c} = 0 \tag{46}$$

and that the scalar potential in the lab frame is proportional (with a factor $1/\gamma$) to the scalar potential in the boosted frame:

$$\phi = \gamma\left(\phi' + \beta c A'_s\right) = \gamma(1-\beta^2)\phi' = \frac{\phi'}{\gamma} \tag{47}$$

In the frame K', the beam particle on which we want to evaluate the Lorentz force is at rest, hence we do not need to compute **B'** in order to evaluate the force acting on it. We therefore focus on the calculation of the electric field **E'** from Eqs. 38 and 43.
Equation 43 can be written explicitly as:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\rho'(x', y', s')}{\varepsilon_0} \tag{48}$$

From Eq. 31 we can write:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\gamma\rho_0(x', y', -\frac{s'}{\gamma\beta c})}{\varepsilon_0} \tag{49}$$

Using the variable ζ defined by Eq. 6, corresponding to the "coordinate along the bunch" in the lab frame, which is related to s' by Eq. 17, we re-define the distribution in the lab frame with respect to ζ:

$$\tilde{\rho}_0\left(x, y, \zeta\right) = \rho_0\left(x, y, -\frac{\zeta}{\beta c}\right) \tag{50}$$

We can rewrite Eq. 49 as:

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} + \frac{\partial^2 \phi'}{\partial s'^2} = -\frac{\gamma \tilde{\rho}_0 \left(x', y', \frac{s'}{\gamma} \right)}{\varepsilon_0} \tag{51}$$

and, using the Eq. 47 we obtain:

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial s'^2} = -\frac{\tilde{\rho}_0 \left(x', y', \frac{s'}{\gamma} \right)}{\varepsilon_0} \tag{52}$$

We observe that ϕ **is the solution of a Poisson problem where the charge distribution is "streched" along the s' direction by a factor γ compared to its definition with respect to the ζ variable in the lab frame. This suggests that, for γ large enough, ϕ is well approximated by the solution of 2D Poisson problem.**
To better visualize this fact we make the substitution:

$$\zeta = \frac{s'}{\gamma} \tag{53}$$

obtained from Eq. 17, which allows to rewrite Eq. 52 as:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{\gamma^2} \frac{\partial^2 \phi}{\partial \zeta^2} = -\frac{\tilde{\rho}_0 (x, y, \zeta)}{\varepsilon_0} \tag{54}$$

Here we have dropped the "$'$" sign from x and y as these coordinates are unaffected by the Lorentz boost.
For large enough values of γ, Eq. 54 can be approximated by:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\tilde{\rho}_0 (x, y, \zeta)}{\varepsilon_0} \tag{55}$$

which is the 2D Poisson equation solved in CERN macroparticle codes like ECLOUD, PyECLOUD, HEADTAIL [5, 6, 7, 8].

4 Transverse kick on the beam particle

We now evaluate the change on the transverse momentum for a beam particle defined in the lab frame by its transverse coordinates x and y and by its delay τ with respect to the reference particle (or equivalently by its ζ coordinate, defined by Eq. 6).
We have seen that in the frame K' the particle is at rest and has longitudinal coordinate $s' = \gamma\zeta$ (see Eq. 17). The x$'$ component of the electric field $\mathbf{E}'$ acting on P is given by (see Eqs. 38 and 47):

$$E'_x = -\frac{\partial \phi'}{\partial x} = -\gamma \frac{\partial \phi}{\partial x} \tag{56}$$

Again, we have dropped the "$'$" sign from x and y as these coordinates are unaffected by the Lorentz boost.

The change in the x component of the momentum, which is an invariant for our Lorentz transformation, is given by :

$$\Delta P_x = \Delta P'_x = qE'_x T' \tag{57}$$

Using Eqs. 56 and 24 we can write:

$$\Delta P_x = -\frac{qL}{\beta c}\frac{\partial \phi}{\partial x}(x, y, \zeta) \tag{58}$$

Normalizing to the momentum of the reference particle:

$$\Delta p_x = \frac{\Delta P_x}{P} = -\frac{qL}{m\gamma\beta^2 c^2}\frac{\partial \phi}{\partial x}(x, y, \zeta) \tag{59}$$

Similarly, for the y-direction we can write:

$$\Delta p_y = \frac{\Delta P_y}{P} = -\frac{qL}{m\gamma\beta^2 c^2}\frac{\partial \phi}{\partial y}(x, y, \zeta) \tag{60}$$

Eqs. 59 and 60 provide the transverse components of the kick from the e-cloud in the form implemented by CERN macroparticle codes HEADTAIL, PyECLOUD-PyHEADTAIL [6, 8].

5 Longitudinal kick on the beam particle

In the frame K', the longitudinal component of the electric field is given by:

$$E'_s = -\frac{\partial \phi'}{\partial s'} = -\gamma\frac{\partial \phi}{\partial \zeta}\frac{\partial \zeta}{\partial s'} = -\frac{\partial \phi}{\partial \zeta} \tag{61}$$

where we have used the fact that $\frac{\partial \zeta}{\partial s'} = \frac{1}{\gamma}$ (see Eq. 53).
As the particle was not moving along s' before the interaction with the e-cloud, its longitudinal momentum after the interaction is given by:

$$P'_s = qE'_s T' = -\frac{qL}{\gamma\beta c}\frac{\partial \phi}{\partial \zeta} \tag{62}$$

The total energy of the particle in the frame K' is given by:

$$\mathcal{E}' = \sqrt{m^2c^4 + c^2\left(P'^2_s + P'^2_x + P'^2_y\right)} \tag{63}$$

Assuming that in K' after the kick the particle remains not relativistic, i.e.:

$$\left(P'^2_s + P'^2_x + P'^2_y\right) \ll m^2c^2 \tag{64}$$

we can approximate Eq. 63 as follows:

$$\mathcal{E}' \simeq mc^2\left(1 + \frac{P'^2_s + P'^2_x + P'^2_y}{2m^2c^2}\right) \tag{65}$$

The quantities $\left(\mathcal{E}/c, P_x, P_y, P_s\right)$ form a Lorentz 4-vector [3] and therefore can be transformed as follows:

$$\frac{\mathcal{E}}{c} = \gamma\left(\frac{\mathcal{E}'}{c} + \beta P_s'\right) \tag{66}$$

$$P_s = \gamma\left(P_s' + \beta\frac{\mathcal{E}'}{c}\right) \tag{67}$$

Replacing Eq. 65 into Eq. 66 we obtain:

$$\mathcal{E} = c\gamma\left(mc\left(1 + \frac{P_s'^2 + P_x'^2 + P_y'^2}{2m^2c^2}\right) + \beta P_s'\right) \tag{68}$$

Neglecting second order terms this can be rewritten as:

$$\mathcal{E} = mc^2\gamma\left(1 + \beta\frac{P_s'}{mc}\right) \tag{69}$$

The energy change due to the interaction with the e-cloud is given by:

$$\Delta\mathcal{E} = \mathcal{E} - mc^2\gamma = \beta\gamma c P_s' \tag{70}$$

Replacing Eq. 62 into Eq. 70 and normalizing to the reference energy $\mathcal{E}_0 = m\gamma c^2$ we obtain:

$$\frac{\Delta\mathcal{E}}{\mathcal{E}_0} = -\frac{qL}{m\gamma c^2}\frac{\partial\phi}{\partial\zeta} \tag{71}$$

From this we can easily compute the change in total momentum. Taking into account that $\frac{dP}{d\beta} = mc\gamma^3$, $\frac{d\mathcal{E}}{d\beta} = mc^2\beta\gamma^3$ and $\frac{P}{\mathcal{E}} = \frac{\beta}{c}$, we can write:

$$\frac{\Delta P}{P_0} = \frac{1}{\beta^2}\frac{\Delta\mathcal{E}}{\mathcal{E}_0} \tag{72}$$

Combining Eqs. 71 and 72 we obtain:

$$\frac{\Delta P}{P_0} = -\frac{qL}{m\gamma\beta^2 c^2}\frac{\partial\phi}{\partial\zeta} \tag{73}$$

6 Practical steps to evaluate the kick

We now recollect the main results found above that are useful for a numerical implementation:

1. We have a particle having phase space coordinates [9]:

$$x \tag{74}$$

$$p_x = P_x/P_0 \tag{75}$$

$$y \tag{76}$$

$$p_y = P_y/P_0 \tag{77}$$

$$\zeta = (\beta/\beta_0)s - \beta ct \tag{78}$$

$$\delta = (P - P_0)/P_0 \tag{79}$$

2. We want to compute the interaction of the particle with an e-cloud pinch described by the evolution of the charge density at one section ($s = 0$):

$$\rho_0(x, y, t) \tag{80}$$

or equivalently as a function of $\zeta = -\beta ct$:

$$\tilde{\rho}_0\left(x, y, \zeta\right) = \rho_0\left(x, y, -\frac{\zeta}{\beta c}\right) \tag{81}$$

The e-cloud pinch follows the bunch over a length L along the accelerator. The effect of the the speed difference between the particle and the bunch ($\beta \neq \beta_0$ is neglected.

3. We compute the scalar potential $\phi(x, y, \zeta)$ by solving Eq. 54:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial \zeta^2} = -\frac{\tilde{\rho}_0\left(x, y, \zeta\right)}{\varepsilon_0} \tag{82}$$

For large values of γ, this can be approximated by the 2D equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{\tilde{\rho}_0\left(x, y, \zeta\right)}{\varepsilon_0} \tag{83}$$

4. The interaction of the beam particle with the e-cloud is modelled by the following map (defined by Eqs. 59, 60, and 73):

$$x \mapsto x \tag{84}$$

$$p_x \mapsto p_x - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial x}\left(x, y, \zeta\right) \tag{85}$$

$$y \mapsto y \tag{86}$$

$$p_y \mapsto p_y - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial y}\left(x, y, \zeta\right) \tag{87}$$

$$\zeta \mapsto \zeta \tag{88}$$

$$\delta \mapsto \delta - \frac{qL}{P_0\beta c}\frac{\partial\phi}{\partial \zeta}\left(x, y, \zeta\right) \tag{89}$$

The CERN codes PyECLOUD-PyHEADTAIL and HEADTAIL implement only the transverse part of this map [6, 7].

7 Hamiltonian of the e-cloud interaction

The map defined by Eqs. 84-89 is generated by the Hamiltonian:

$$H = \frac{qL}{P_0\beta c}\phi\left(x, y, \zeta\right)\delta(s) \tag{90}$$

This can be easily verified using Hamilton's equations:

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x} \tag{91}$$

$$\frac{dp_y}{ds} = -\frac{\partial H}{\partial y} \tag{92}$$

$$\frac{d\delta}{ds} = -\frac{\partial H}{\partial \zeta} \tag{93}$$

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} \tag{94}$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial p_y} \tag{95}$$

$$\frac{d\zeta}{ds} = \frac{\partial H}{\partial \delta} \tag{96}$$

$$\tag{97}$$

which, for the Hamiltonial defined in Eq. 90, coincide with Eqs. 84-89.
The fact that it is generated by an Hamiltonian proves that **the map is symplectic**.

8 Where did the magnetic field go?

It is tempting to interpret ϕ as the electrostatic potential in the lab frame and therefore interpret the kicks defined by Eqs. 84-89 as the exclusive effect of the electric field. This would be puzzling, since the time-changing sources defined by Eqs. 1 and 2 should in general generate also magnetic fields, which should be visible on the force acting on the moving particle. Still, observing Eqs. 84-89, we cannot recognize anything that looks like the vector product from the Lorentz force expression.
To understand this apparent contradiction it is worth stating explicitly that in general in the lab frame:

$$\mathbf{E} \neq -\nabla\phi \tag{98}$$

Instead, as the sources are not stationary, the electric field depends on both electromagnetic potentials $\mathbf{A}$ and ϕ:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{99}$$

and the magnetic field can be written as:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{100}$$

where the Lorentz gauge has been assumed.
The Lorenz force on a particle travelling along the s axis with speed βc can be written as:

$$\mathbf{F} = q\left(\mathbf{E} + \beta c\, \hat{\mathbf{i}}_s \times \mathbf{B}\right) \tag{101}$$

Combining Eqs. 99, 100 and 101 we obtain:

$$\mathbf{F} = q\left(\underbrace{-\nabla\phi}_{\substack{\text{Irrorational}\\\text{part of }\mathbf{E}}} \underbrace{-\frac{\partial \mathbf{A}}{\partial t}}_{\substack{\text{Non-irrotational}\\\text{part of }\mathbf{E}}} + \beta c\, \hat{\mathbf{i}}_s \times \underbrace{(\nabla \times \mathbf{A})}_{\mathbf{B}\text{ field}}\right) \tag{102}$$

Taking into account that $A_s = 0$ (see Eq. 46), by expressing the curl in Cartesian coordinates, we can write:

$$\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = -\frac{\partial A_x}{\partial s}\hat{\mathbf{i}}_x - \frac{\partial A_y}{\partial s}\hat{\mathbf{i}}_y = -\frac{\partial \mathbf{A}}{\partial s} \tag{103}$$

As the e-cloud is following the bunch, the potentials will have the same form as the sources (this can be shown explicitly using the Lorentz transformations):

$$\mathbf{A}(x, y, s, t) = \mathbf{A}_0\left(x, y, t - \frac{s}{\beta c}\right) \tag{104}$$

For a function in this form we can write:

$$\frac{\partial \mathbf{A}}{\partial s} = -\frac{1}{\beta c}\frac{\partial \mathbf{A}}{\partial t} \tag{105}$$

Replacing Eq. 105 into Eq. 103, we obtain:

$$\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = \frac{1}{\beta c}\frac{\partial \mathbf{A}}{\partial t} \tag{106}$$

which shows that in Eq. 102 **the magnetic component of the Lorentz force cancels exactly the non-irrotational part of the electric component** (as shown in [10] for the special case of a circular symmetric geometry). Hence the force is proportional to the gradient of the scalar potential:

$$\mathbf{F} = -q\nabla\phi \tag{107}$$

from which the map defined by Eqs. 84-89 can be easily derived.

9 Conclusions

The transverse and longitudinal forces acting on a beam particle due to the effect of an electron cloud, can be conveniently calculated in a reference frame moving rigidly with the particle. In such a reference frame, charge and current densities are stationary, therefore the electric and magnetic fields are solution of an electrostatic and a magnetostatic problem respectively.
It is possible to show that the force acting on the bunch (in the lab frame) is simply proportional to the gradient of the scalar potential and is therefore irrotational. This happens since the force due to the non-irrotational component of the electric field is cancelled exactly by the force due to the magnetic field.
For a very relativistic beam the scalar potential can be calculated with good approximation as the solution of a 2D Poisson problem. The Hamiltonian of the resulting transformation can be written as a function of the position coordinates, showing that the map is symplectic and can be modelled as a "thin" element in tracking codes.

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Appendix: Extension to the case $J_s \neq 0$

We now briefly discuss how the results described above need to be changed in case the hypothesis $J_s = 0$ is not verified and therefore A_s can in general be non-zero.
In this case Eq. 103 can be generalized as follows:

$$\begin{aligned}
\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y \\
&= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y + \underbrace{\left(\frac{\partial A_s}{\partial s} - \frac{\partial A_s}{\partial s}\right)}_{=0}\hat{\mathbf{i}}_s \\
&= \nabla A_s - \frac{\partial \mathbf{A}}{\partial s}
\end{aligned} \tag{108}$$

Using Eq. 105 this can be rewritten as:

$$\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) = \nabla A_s + \frac{1}{\beta c}\frac{\partial \mathbf{A}}{\partial t} \tag{109}$$

Eq. 109 can be replaced in the expression of the Lorentz force, obtaining:

$$\begin{aligned}
\mathbf{F} &= q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \beta c\, \hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right) \\
&= q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \beta c \nabla A_s + \frac{\partial \mathbf{A}}{\partial t}\right) \\
&= -q\nabla\left(\phi - \beta c A_s\right)
\end{aligned} \tag{110}$$

This shows that also in this case the force can be written as the gradient of a scalar potential, but it is not anymore simply proportional to ϕ.
The potential A_s and ϕ are related to ϕ' by a Lorentz transformation:

$$\phi' = \gamma\left(\phi - \beta c A_s\right) \tag{111}$$

Hence Eq. 110 can be rewritten as:

$$\mathbf{F} = -\frac{q}{\gamma}\nabla\phi' \tag{112}$$

where ϕ' is the electrostatic potential calculated in the boosted frame and remapped to the lab frame.
Following the same reasoning exposed in Sec. 3, the potential ϕ' can be calculated as the solution of Poisson's equation (Eq. 43):

$$\nabla'^2\phi' = -\frac{\rho'}{\varepsilon_0} \tag{113}$$

where ρ' is related to ρ and J_s by the Lorenz transformation in Eq. 25. This results in an additional term in Eq. 82, which becomes:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial \zeta^2} = -\frac{1}{\varepsilon_0}\left[\tilde{\rho}_0\left(x,y,\zeta\right) - \frac{\beta}{c}\tilde{J}_{0s}\left(x,y,\zeta\right)\right] \tag{114}$$

where, $\tilde{J}_{0s}$ is defined similarly to $\tilde{\rho}_0$ (see Eq. 81):

$$\tilde{J}_{0s}\left(x,y,\zeta\right) = J_{0s}\left(x,y,-\frac{\zeta}{\beta c}\right) \tag{115}$$

Also in this case, for large values of γ, Eq. 114 can be approximated by a 2D Poisson equation.

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