CERN-TH.7385/94

EFFECTIVE THEORIES OF BCS SUPERCONDUCTORS AT T=0

Ian J.R. Aitchison[1,2,†], Ping Ao[2], David J. Thouless[2], and X.-M. Zhu[2]

[1]TH. Division, CERN, CH-1211, Geneva, 23, Switzerland

and [2]Department of Physics, FM-15

University of Washington, Seattle, WA 98195, USA

Abstract

We show that the low frequency, long wavelength dynamics for the phase of the pair field for an BCS-type s-wave superconductor at T=0 is equivalent to that of a time-dependent non-linear Schrödinger Lagrangian (TDNLSL), when terms required by Galilean invariance are included. If the modulus of the pair field is also allowed to vary, the system is equivalent to two coupled TDNLSL's.

PACS#s:74.20.-z, 67.50.Fi

CERN-TH.7385/94

August 1994

† Permanent address: Physics Department, Theoretical Physics

1 Keble Road, Oxford OX1 3NP, UK

The classic Ginzburg-Landau (GL) theory [1] is very successful[2,3] in describing a large class of static superconducting phenomena near the critical temperature T_c, and its form was established by Gorkov [4] shortly after the microscopic BCS theory [5]. Subsequently, a number of attempts [2,3,6,7] were made to obtain a generalized GL theory for time-dependent phenomena, and for temperatures well below T_c, but a consensus has still not been reached on the form of such a theory at T=0. In this letter we shall show that the effective theory at T=0 is equivalent to a time-dependent non-linear Schrödinger Lagrangian (TDNLSL). At first sight, this result might seem almost obvious: after all, the energy density in GL theory looks formally like that of a non-linear Schrödinger theory so that it seems natural to extend it to the corresponding time-dependent theory as, indeed, Feynman assumed [8] in his discussion of the dynamics of superconductors and of the Josephson effects. Yet neither the earlier discussions[2,3,6], nor recent work based on the effective action formalism of quantum field theory [7,9], appears to lead to this conclusion. This is in contrast to the case of a Bose superfluid, such as ^{4}He, which is well described by a TDNLSL near T=0[10]. Indeed, there is considerable current interest in probing the relationship and "crossover" between BCS and Bose superfluidity[11]. Our result implies that both are fundamentally the same, at least near T=0; in particular, the existence of the Magnus force for a vortex line in a superconductor follows naturally. The last point is pertinent to the discussion of vortex dynamics in superconductors within the effective theory formulation [12].

Three of the present authors have, in fact, recently shown [13] that the motion of the condensate is described by a non-linear Schrödinger equation at T=0, using a density matrix approach. But this left open the question how this could be reconciled with the earlier work [2,3,6,7,9], which was generally based on Green's function (or related) techniques, and apparently led to a quite different result. The solution of this problem is contained in the present paper, and it is essentially very simple. We take the Goldstone mode Lagrangian which has recently been derived from BCS theory [9], after being proposed on symmetry grounds [14], and show that it is equivalent to a TDNLSL. This Lagrangian also corresponds precisely to the early results of Kemoklidze and Pitaevskii [15], who started from Gorkov's equations [5]. We also extend this to include variations in the modulus of the pair fields, and show that the dynamics is then that of two coupled TDNLSL's. The thread that unites all these approaches is ultimately Galilean invariance. Since the microscopic starting point is always Galilean invariant, one expects any effective theory to preserve this symmetry, a point emphasized in Ref. 15, and the Schrödinger Lagrangian is the simplest such available.

We begin by recalling briefly the formalism and results of [9], the latter coinciding with the proposal of [14]. The BCS Lagrangian, for s-wave pairing and in the absence of external fields, is

$$L = \sum_\sigma \psi^*_\sigma(x)\left(i\partial_t + \frac{\nabla^2}{2m} + \mu\right)\psi_\sigma(x) + g\psi^*_\uparrow(x)\psi^*_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x) \tag{1}$$

where ψ_σ describes electrons with spin $\sigma = (\uparrow, \downarrow)$, $\mu = k_F^2/2m$ is the Fermi energy in the normal state, and $x = (\mathbf{x}, t)$. Introducing the auxiliary(pair) fields $\Delta(x)$ and $\Delta^*(x)$, and integrating out the electron fields, one obtains the effective action

$$S[\Delta, \Delta^*] = -iTr\ln G^{-1} - \frac{1}{g}\int d^4x|\Delta(x)|^2 \tag{2}$$

where the Nambu Green function satisfies

$$\begin{pmatrix} O_1 & \Delta(x) \\ \Delta^*(x) & O_2 \end{pmatrix} G(x, x') = \delta(x - x') \tag{3}$$

with $O_1 = i\partial_t + \nabla^2/2m + \mu$, $O_2 = i\partial_t - \nabla^2/2m - \mu$, and Tr includes interval and space-time indices. To obtain from (2) an effective Lagrangian in terms of the degrees of freedom represented by Δ, a possible procedure [7] is to set $\Delta(x) = \Delta_0 + \Delta'(x)$ where Δ_0 is the position of the minimum of S for space-time independent Δ, and where Δ' is assumed to be slowly varying in both space and time. One then expands $Tr\ln G^{-1}$ in powers of derivatives of Δ'. There are, however, two (related) objections to this. First, we are dealing with the spontaneous breaking of a local $U(1)$ phase invariance, which implies that at a temperature far from the transition temperature, the most important degree of freedom is the phase of Δ, which is the relevant Goldstone field. It is this field, rather than the real and/or imaginary parts of Δ, which should carry the low frequency and long wavelength dynamics. Secondly, the ansatz $\Delta(x) = \Delta_0 + \Delta'(x)$ violates the Galilean invariance possessed by (1), which implies [15] that

$$\Delta(r - vt, t)\ \exp(2imv \cdot r - imv^2 t) \tag{4}$$

should satisfy the same equation of motion as $\Delta(r, t)$. We shall return to the question of Galilean invariance below.

We therefore set

$$\Delta(x) = e^{i\theta(x)}|\Delta(x)| \tag{5}$$

and $|\Delta(x)| = |\Delta_0| + \delta|\Delta(x)|$, where we are interested in the low frequency and long wavelength fluctuations of $\theta(x)$ and $\delta|\Delta(x)|$. However, although $\delta|\Delta(x)|/|\Delta_0|$ is expected to be small, and a simple expansion of the sort mentioned above for $Tr \ln G^{-1}$ could easily be set up in terms of derivatives of $\delta|\Delta(x)|$ if $\theta(x)$ were zero, it is crucial to recognize that $\theta(x)$ is not small in general, so that the phase factor in (5) cannot be expanded, but must be treated as a whole. This prevents a straightforward expansion of $Tr \ln G^{-1}$ when (5) is substituted into (2). Fortunately, this difficult can be easily circumvented [9,16]. Defining $U(x) = \exp(i\theta(x)\tau_3/2)$, we can write

$$Tr \ln G^{-1} = Tr \ln G^{-1} U U^{-1} = Tr \ln U \tilde{G}^{-1} U^{-1} = Tr \ln \tilde{G}^{-1} \tag{6}$$

where

$$\tilde{G}^{-1} = G_0^{-1}(1 - G_0 \Sigma)$$

$$G_0^{-1} = \begin{pmatrix} O_1 & |\Delta| \\ |\Delta| & O_2 \end{pmatrix} \tag{7}$$

and

$$\Sigma = i\nabla^2\theta/4m + i\nabla\theta\cdot\nabla/2m + (\dot{\theta}/2 + (\nabla\theta)^2/4m)\tau_3 - \delta|\Delta|\,\tau_1. \tag{8}$$

Minimizing (2) with $\theta = \delta|\Delta| = 0$ yields the usual gap equation for $|\Delta_0|$. The dynamics of θ and $|\Delta|$ is contained in

$$S_{eff}[\theta, \delta|\Delta|] = iTr\sum_{n=1}^{\infty}\frac{1}{n}(G_0\Sigma)^n - \frac{1}{g}\int |\Delta|^2 d^4x\ , \tag{9}$$

where we note that Σ contains just $\delta|\Delta(x)|$ and derivatives of $\theta(x)$, in terms of which (assumed small) quantities a useful expansion can be conducted, following standard techniques [17].

We now concentrate on $\theta(x)$, and set $\delta|\Delta| = 0$ for the time being. The results of Ref.9 (see also[14]) then give

$$L_{eff}(\theta) = \rho_0(\dot{\theta} + (\nabla\theta))^2/4m) - \frac{1}{2}N(0)(\dot{\theta} + (\nabla\theta)^2/4m)^2 \tag{10}$$

where $\rho_0 = k_F^3/3\pi^2$ is the electron density at T=0, $N(0)$ is the density of states (for one spin projection) at the Fermi surface and we have adopted a convenient normalization; note that $N(0) = \rho_0/2mv_a^2$ where $v_a = v_F/\sqrt{3}$ is the velocity of propagation of the Bogoliubov-Anderson mode. We proceed to our main result, which is the demonstration that (10) is equivalent to a TDNLSL. The equation of motion which follows from (10) is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot j = 0 \tag{11}$$

where

$$\rho - \rho_0 = -N(0)(\dot{\theta} + (\nabla\theta)^2/4m) \equiv \delta\rho \tag{12}$$

and

$$j = \rho\nabla\theta/2m. \tag{13}$$

Equations (11)-(13) are, in fact, precisely those obtained (to this order in derivatives, and changing an overall sign) by putting $\delta|\Delta| = 0$ in Eqns.(21)-(23) of Ref.15. Consider now the non-linear Schrödinger Lagrangian

$$L_\psi = i\psi^*\dot{\psi} - \frac{1}{4m}\nabla\psi^* \cdot \nabla\psi - V \tag{14}$$

where the mass has been chosen to be $2m$, and V will be assumed to be a function of $|\psi|$ only. Our observation is that if we set

$$\psi = \sqrt{\rho}\ \exp(i\theta) \tag{15}$$

where ρ and θ are as defined above, then the equations of motion that follow from L_ψ are (up to the given order in derivatives) the same as (11)-(13). This is easy to verify: putting (15) into (14) and discarding a total derivative we obtain

$$L_\psi = -\rho\dot{\theta} - \rho(\nabla\theta)^2/4m - (\nabla\rho)^2/16m\rho - V(\rho) \tag{16}$$

leading to the equation of motion (11) with j given by (13), and

$$\frac{dV}{d\rho} = -(\dot{\theta} + (\nabla\theta)^2/4m) - (\nabla\rho)^2/16m\rho^2 + \nabla^2\rho/8m\rho^2. \tag{17}$$

We now choose

$$V = (\rho - \rho_0)^2/2N(0) \tag{18}$$

and solve (17) by expanding in derivatives. The lowest order solution is exactly (12), so that all of (11)-(13) have been recovered. We have shown that the dynamics of the Goldstone field $\theta(x)$ is given (to the relevant order in derivatives) by the TDNLSL (14), where ψ is given by (15), ρ by (12) and V by (18).

Before discussing the inclusion of the field $\delta|\Delta(x)|$ we comment further on (10)-(13). We first note that the Galilean invariance requirement (see Eqn.(4)) implies that

$$\theta'(r', t') = \theta(r, t) + mv^2 t - 2mv \cdot r \tag{19}$$

where $r' = r - vt$, $t' = t$. From (19) we easily find

$$\delta\rho'(r,t') = \delta\rho(r,t),\ j'(r',t') = j(r,t) - v\rho(r,t) \tag{20}$$

so that L_{eff} is a Galilean scalar, as is the original L of (1), while $\delta\rho$ and j transform covariantly. Indeed, a simple alternative route to (10) is to start from a Lagrangian which describes just the Bogoliubov-Anderson mode, viz

$$L_a = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}v_a^2(\nabla\theta)^2. \tag{21}$$

Now (21) is clearly not a scalar under (19), but it can be made so by adding terms of the form $\alpha\dot{\theta}(\nabla\theta)^2$ and $\beta((\nabla\theta)^2)^2$. The requirement that the resulting L be a scalar under (19) (up to constants and total derivatives) determines α and β uniquely to be $1/4m$ and $1/32m^2$ respectively, and the Lagrangian is then proportional to $L_{eff}(\theta)$. Further, simple linear response theory (assuming, as always, a derivative expansion) gives[18]

$$\delta\rho \approx -N(0)\dot{\theta},\ j \approx \rho_0\nabla\theta/2m. \tag{22}$$

The first relation can be converted into (12) by requiring that $\delta\rho$ is a Galilean scalar, while the second has to be replaced by (13) to ensure that j transforms covariantly. The requirement that the Galilean symmetry possessed by the original theory (1) should be respected by the effective theory is a powerful constraint.

In view of its relative unfamiliarity, it maybe worth noting that $L_{eff}(\theta)$ (or equivalently L_ψ) embodies the usual phenomenology of superfluid dynamics at T=0 (see, for example [19,20]). We identify $\nabla\theta/2m$ with the superfluid velocity v_s, and multiply ρ and j of (12), (13) by m to convert them to mass density and flux, ρ_m and j_m. Eq. (11) is then the law of mass conservation, following from the fact that L_{eff} does not depend explicitly on θ. Eq.(12) is equivalent [15] to Bernoulli's equation, if we make use of $\delta\rho \approx 2N(0)\delta\mu$ and $\delta p \approx \rho_0\delta\mu$.

Since L_{eff} does not depend explicitly on t, we have the energy conservation relation

$$\frac{\partial E}{\partial t} + \nabla \cdot Q = 0 \tag{23}$$

where using the canonical definitions (with suitable normalization), one finds

$$E \approx \frac{1}{2}\rho_m v_s^2, \; Q = j_m \left(\frac{1}{2}v_s^2 + \delta\mu\right), \tag{24}$$

and we have dropped a quantity of order $\delta\rho \; \delta\mu$ in E. Finally, since L_{eff} is translation invariant we have the momentum conservation relation

$$\frac{\partial j_m}{\partial t} + \nabla \cdot \Pi = 0 \tag{25}$$

where the momentum flux density tensor is

$$\Pi_{ij} = \rho_m v_{si} v_{sj} + \delta p \; \delta_{ij}. \tag{26}$$

Eq.(25) is equivalent to Euler's equation. In Ref.14, the proportionality between the momentum density and the number current j (defined by $\partial L/\partial(\nabla\theta)$), which is included in (25), was taken as a constraint on possible Lagrangians L. If L is a function solely of the Galilean scalar $g = \dot{\theta} + (\nabla\theta)^2/4m$, then

$$\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial g}, \; \frac{\partial L}{\partial(\nabla\theta)} = \frac{\partial L}{\partial g}\frac{\nabla\theta}{2m}. \tag{27}$$

Since θ is a phase variable, we can interpret $\partial L/\partial\dot{\theta}$ and $\partial L/\partial(\nabla\theta)$ as being proportional to a conserved number density ρ and number current density j respectively, so that (27) becomes just (13). The momentum density is then automatically proportional to j. Once again, Galilean invariance is the essential principle.

We now turn to the inclusion of the field $\delta|\Delta(x)|$. $L_{eff}(\theta, \delta|\Delta|)$ can be extracted from (9), up to a given order in derivatives, but calculations rapidly become laborious. For our present purpose, we will simply use the result of Ref.15 which, using the normalization of (10), gives

$$L_{eff}(\theta,\epsilon)=\rho_0\left[\dot{\theta}+\frac{(\nabla\theta)^2}{4m}+\frac{(\nabla\epsilon)^2}{4m}\right]-\frac{N(0)}{2}\left[\dot{\theta}+\frac{(\nabla\theta)^2}{4m}+\frac{(\nabla\epsilon)^2}{4m}\right]^2$$

$$-\frac{N(0)}{2}\left[\dot{\epsilon}+\frac{\nabla\epsilon\cdot\nabla\theta}{2m}\right]^2+6N(0)|\Delta_0|^2\epsilon^2 \tag{28}$$

where $\epsilon(x)\equiv\delta|\Delta(x)|/(\sqrt{3}|\Delta_0|)$ and we have retained corresponding terms in ϵ and θ. The quadratic terms in ϵ yield the amplitude collective mode found in Ref.18 (and are also in agreement with the result of Ref.7); we have omitted higher powers of ϵ. The term in $\dot{\epsilon}$ is made Galilean invariant by the addition of $\nabla\epsilon\cdot\nabla\theta/2m$, since $\epsilon(x)$ is a scalar.

There are now clearly two independent degrees of freedom involved, and correspondingly we find that (28) is equivalent to two TDNLSL's. That is, the equations of motion for θ and ϵ which follow from (28) are identical to those arising from

$$L_{\psi_1,\psi_2}=i\psi_1^*\dot{\psi}_1-\frac{1}{4m}\nabla\psi_1^*\cdot\nabla\psi_1-(|\psi_1|^2-\rho_0/2)/N(0)$$

$$+i\psi_2^*\dot{\psi}_2-\frac{1}{4m}\nabla\psi_2^*\cdot\nabla\psi_2-(|\psi_2|^2-\rho_0/2)/N(0)+\frac{3}{2}N(0)|\Delta_0|^2[Im\ln(\psi_1/\psi_2)]^2 \tag{29}$$

where

$$\psi_1=\sqrt{(\rho_\theta+\rho_\epsilon)/2}\ \exp(i(\theta+\epsilon)),\ \psi_2=\sqrt{(\rho_\theta-\rho_\epsilon)/2}\ \exp(i(\theta-\epsilon)). \tag{30}$$

For example, corresponding to (12), we have

$$\rho_\theta=\rho_0-N(0)\left(\dot{\theta}+\frac{(\nabla\theta)^2}{4m}+\frac{(\nabla\epsilon)^2}{4m}\right)=\frac{\partial L_{eff}(\theta,\epsilon)}{\partial\dot{\theta}} \tag{31}$$

and

$$\rho_\epsilon=-N(0)\left(\dot{\epsilon}+\frac{\nabla\epsilon\cdot\nabla\theta}{2m}\right)=\frac{\partial L_{eff}(\theta,\epsilon)}{\partial\dot{\epsilon}}. \tag{32}$$

Eqn.(29) represents a system of two TDNLSL's coupled via the "mass" term in (28). Expressions for all the conserved quantities can be found as before, and will include quantum corrections to the semiclassical results of (23)-(26).

The inclusion of electromagnetism in the above formalism is straightforward. Consider the formulation in terms of $L_{eff}(\theta, \epsilon)$. Since θ is the phase of a field with charge $-2e$, gauge invariance implies that $\dot{\theta}$ and $\nabla\theta$ must appear in the combinations $\dot{\theta} - 2eA_0$ and $\nabla\theta + 2eA$ $(e > 0)$, where A_0 and A are the electromagnetic potentials. The field ϵ, on the other hand, is electromagnetically neutral. The leading order electromagnetic charge and current densities are obtained by multiplying $\delta\rho$ and j in (22) by $-e$ and making the above replacements for $\dot{\theta}$ and $\nabla\theta$. One then obtains the usual results [7]. In terms of the Schrödinger formulation, one simply makes the expected minimal coupling substitutions: $i\partial_t \to i\partial_t + 2eA_0$ and $-i\nabla \to -i\nabla + 2eA$ in (14) or (29) (note from (30) that both ψ_1 and ψ_2 have charge $-2e$).

When the above analysis is extended to higher order derivative terms, it is clear on dimensional grounds that some characteristic scale must enter. In fact, such higher terms enter in the form $\partial_t/|\Delta_0|$ and $v_F\nabla/|\Delta_0| \sim \xi\nabla$ (see for example Eqn.(35) of [9]), where ξ is the coherence length. The basis of the expansion is therefore the usual assumption [15] that the characteristic frequency ω, and wavenumber k, of variations of $\Delta(x)$ satisfy $\omega \ll |\Delta_0|$, $k \ll \xi^{-1}$. Indeed, (28) already yields a static solution for ϵ which decays exponentially over a characteristic distance $\xi/6$. Such a solution is of the type expected far from a vortex core. Inclusion of appropriate higher derivative terms should make possible some predictions about the vortex core structure.

The TDNLSL formulation provides, we believe, a simple and unifying framework for the discussion of dynamical effects in BCS superconductors at T=0. Results which have been known for many years [2,3,6,15], as well as those obtained by quite different methods only recently [9,14], are all seen to be in agreement with each other, and with the TDNLSL formulation.

Acknowledgements: This work was supported in part by the US National Science

Foundation under grant No. DMR-9220733. The work of I.J.R.A, while at the University of Washington, was also supported by a grant from the US Department of Energy to the nuclear theory group at UW, and by the physics Physics Department, UW. I.J.R.A. is grateful to the members of the UW Physics Department for their hospitality and support.

References

1. V. I. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).

2. See, for example, N. R. Werthamer in *Superconductivity*, Vol.1, edited by R.D. Parks, M. Dekker, New York, 1969.

3. M. Cyrot, Rep. Prog. Phys. **36**, 103(1973) gives a critical review.

4. L. P. Gorkov, Zh. Eksp. Teor. **36**, 1918 (1959).

5. J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

6. M. Dreschler and W. Zwerger, Ann. Phys. **1**, 15(1992).

7. H. T. C. Stoof, Phys. Rev. **47**, 7979 (1993).

8. R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics*, vol. III, Addison-Wesley, 1965. See also R. P. Feynman, *Statistical Mechanics*, Reading, Massachusetts, 1972.

9. A. M. J. Schakel, Mod. Phys. Lett, **B** 14, 927 (1990); see also Ch. G. van Weert, in *Thermal Field Theories*, Proc. 2nd Workshop on Thermal Field Theories and Their Applications, Tsukuba, Japan, July 1990, edited by H. Ezawa, T. Arimitsu and Y. Hashimoto (North Holland, 1991).

10. E. P. Gross, Nuovo Cimento **20**, 454 (1961); L. P. Pitaevskii, Sov. Phys. JETP **12**, 155 (1961).

11. See, for example, C. A. R. Sá de Melo, Mohit Randeira and Jan R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).

12. A. T. Dorsey, Phys. Rev. **B46**, 8376 (1992).

13. P. Ao, D. J. Thouless, and X.-M. Zhu, preprint.

14. M. Greiter, F. Wilczek and E. Witten, Mod. Phys. Lett. **B3**, 903 (1989).

15. M. P. Kemoklidze and L. P. Pitaevskii, Zh. Eksp. Teor. **50**, 160(1966).

16. U. Eckern, G. Schön and V. Ambegaokar, Phys. Rev. **30**, 6419 (1984).

17. C. M. Fraser, Z. Phys. **C 28**, 101 (1985); I. J. R. Aitchison and C. M. Fraser, Phys. Rev. **D 31**, 2605 (1985).

18. See, for example, E. Abrahams and T. Tsuneto, Phys. Rev. **152**, 416(1966).

19. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press Inc, New York, 1959, Section 130.

20. M. J. Stephen, Phys. Rev. **139A**, 197 (1965); O. Betbeder-Matibet and P. Nozieres, Ann. Phys. **51**, 392 (1969); A. G. Aronov, Y. M. Gal'perin, V. L. Gurevish and V I. Kozub, Adv. Phys. **30**, 539 (1981).