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## Deformed oscillator algebras for two-dimensional quantum superintegrable systems

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### Abstract

Quantum superintegrable systems in two dimensions are obtained from their classical counterparts, the quantum integrals of motion being obtained from the corresponding classical integrals by a symmetrization procedure. For each quantum superintegrable system a deformed oscillator algebra, characterized by a structure function specific for each system, is constructed, the generators of the algebra being functions of the quantum integrals of motion. The energy eigenvalues corresponding to a state with finite dimensional degeneracy can then be obtained in an economical way from solving a system of two equations satisfied by the structure function, the results being in agreement to the ones obtained from the solution of the relevant Schrödinger equation. Applications to the harmonic oscillator in a flat space and in a curved space with constant curvature, the Kepler problem in a flat or curved space, the Fokas-Lagerstrom potential, the Smorodinsky-Winternitz potential, and the Holt potential are given. The method shows how quantum algebraic techniques can simplify the study of quantum superintegrable systems, especially in higher dimensions.

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## 1 Introduction

The idea of studying the properties of physical systems exhibiting degenerate energy levels through use of their symmetries has been exploited since the early days of quantum mechanics. In these cases the symmetry algebra [1] of the system has to be determined, which is a finite-dimensional Lie algebra containing ladder operators which connect all the eigenstates with a given energy, while the Hamiltonian of the system is related to the Casimir operators of the algebra. The set of eigenstates with given energy provides a basis for an irreducible representation (irrep) of the algebra. The energy eigenvalues are then determined by the eigenvalues of the Casimir operators of the algebra in the corresponding irreps. The determination of the symmetry algebra of a given hamiltonian is a quite difficult task, which is not always dictated by an a priori obvious procedure of searching for it. Examples of physical systems having known symmetry algebras are the N-dimensional isotropic harmonic oscillator and the Kepler (Coulomb) system, bearing the symmetries  $SU(N)$  and  $SO(N+1)$  respectively.

The classical counterparts of the quantum isotropic harmonic oscillator and the quantum Kepler problem have another interesting property. They are maximally superintegrable systems in N dimensions. Superintegrable systems in N dimensions have more than N independent classical constants (also called integrals or invariants) of motion, while maximally superintegrable systems in N dimensions have  $2N-1$  independent classical constants of motion, N of which are integrals in involution (the Poisson bracket of each pair of them is zero). This property of the maximally superintegrable systems implies, in classical mechanics, that every closed and bounded trajectory is a periodic trajectory. A detailed review of superintegrable systems in 2 dimensions is given in [2], while examples of superintegrable systems in 3 dimensions can be found in [3, 4].

Higgs [5] and Leemon [6] have shown that in the case of a N dimensional system moving in a space with constant curvature the isotropic harmonic oscillator and the Kepler problem are still maximally superintegrable systems, both in classical and quantum mechanics. Furthermore, they have shown that



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the quantum counterparts of these systems can be described by symmetry algebras isomorphic to  $SU(N)$  and  $SO(N+1)$  respectively. Additional examples of superintegrable classical systems are the Fokas-Lagerstrom potential [7], the Smorodinsky-Winternitz potential [8, 9, 10, 11], the Holt potential [12], the Hartmann potentials [13, 14, 15, 16]. The problem of quantum integrability and its connection to classical integrability is currently under active investigation [17, 18, 19, 20].

In several of the above mentioned cases (see [5, 6], for example) the property of the classical and quantum superintegrability of a physical system coincides with the existence of a symmetry algebra of the system. The energy levels of the system can then be determined by purely algebraic means. However, the identification of a symmetry algebra is not always easy. Furthermore, in some cases (see [16], for example) it seems that the usual Lie algebras do not suffice for this purpose. The recent introduction of quantum algebras [21, 22, 23, 24] (also called quantum groups) opens new possibilities in this direction. Quantum algebras are nonlinear deformations of the usual Lie algebras, to which they reduce when the deformation parameter  $q$  goes to 1. They were initiated as a mathematical tool issued from the study of the quantum inverse problem, the Yang Baxter equation and conformal field theories (see [25] for a collection of original papers). There are already some indications that quantum algebras might be useful as symmetry algebras of certain superintegrable systems. The Higgs algebra (i.e. the symmetry algebra of the Kepler problem in a 2-dimensional space with constant curvature studied in [5]) can be approximated to second order by the quantum algebra  $SU_q(2)$  [26]. The symmetry algebra of the Hartmann potential, for which usual Lie algebras seemed insufficient [16], has been identified as the quadratic Hahn algebra  $QH(3)$  [27]. The quadratic Hahn algebra  $QH(3)$  has also been found to describe the symmetry of the anisotropic singular oscillator (a 3-dimensional harmonic oscillator with an additional term  $\sim 1/(r^2 \sin^2 \theta)$ ) [28]. The quadratic Hahn algebra  $QH(3)$  is a special case of the general quadratic Askey-Wilson algebra  $QAW(3)$ , which is the dynamical symmetry of the potentials having eigenfunctions described by classical polynomials [29]. Létourneau and Vinet [30] have constructed the quadratic algebra de-

scribing the case of a harmonic oscillator potential with a 2 to 1 frequency ratio. Another example of a system described by a nonlinear algebra is the generalized Kepler (Coulomb) system [31]. From these examples it becomes clear that nonlinear algebras can be useful in the description of integrable and superintegrable systems.

In this paper we focus attention on the simplest superintegrable systems, the two dimensional ones, and we propose a method of determining their dynamical symmetries and calculating their spectra by purely algebraic means. It turns out that several quantum superintegrable systems in 2 dimensions can be described in terms of appropriate generalized deformed oscillators, which allow for the direct determination of the energy levels and their degeneracies without any need of solving the Schrödinger equation. A preliminary study of the proposed method can be found in ref. [32], where the method has been used in two cases of potentials, the symmetric harmonic oscillator in a curved space and the asymmetric oscillator with a 2 to 1 frequency ratio. In this paper we study most of the known 2 dimensional superintegrable cases.

It is known that the classical (non-deformed) algebras can be constructed using the harmonic oscillator algebra  $\{a, a^+, N\}$  as the basic underlying structure. For quantum algebras, and nonlinear algebras in general, deformed oscillators have to be used. Biedenharn [33], Macfarlane [34], and Sun and Fu [35] constructed the  $q$ -deformed oscillator appropriate for the Schwinger realization of the quantum algebra  $SU_q(2)$ . Many other deformed oscillators can be found in the literature. We mention the  $Q$ -oscillator introduced by Arik and Coon [36] and Kuryshkin [37], the two-parameter deformed oscillator [38, 39], the parafermionic oscillator [40] and its  $q$ -deformation [41, 42], the parabosonic oscillator [40] and its  $q$ -deformation [41, 42], the generalized  $q$ -deformed fermionic algebra [43]. (The  $q$ -deformed fermionic algebra [44] has been proved to be equivalent to the usual fermionic algebra [45].) The common feature of all these deformations is their structural similarity. In all cases an appropriate Fock basis can be constructed, leading to a matrix representation of the algebra.

The structural similarity of the various deformed oscillators implies that

all of them can be described in a unified framework. Among the various alternatives we mention here the generalized deformed oscillator [46], the Odaka-Kishi-Kamefuchi unification scheme [42], the pioneering work of Jannussis *et al.* on the bosonization method [47] and the generalized Q-deformed oscillator [48], the Beckers-Debergh unification scheme [49], and the Fibonacci oscillator scheme [50], while a general treatment of q-algebras is given recently by Fairlie and Nuyts [51].

Among the various equivalent descriptions of the deformed oscillators, in this paper we use the deformed oscillator algebra [46], already used for the study of the energy spectra of one dimensional systems [52, 53, 54].

In section 2 of the present paper we consider the classical superintegrable systems in 2 dimensions, for which we show that the relevant Poisson bracket induced algebra has a structure similar to the deformed oscillator algebra. In section 3 a working hypothesis for the quantum superintegrable systems in 2 dimensions is proposed, which leads to the calculation of the energy eigenvalues and their degeneracies by purely algebraic means. This hypothesis is applied to several quantum superintegrable examples in section 4, while section 5 contains discussion of the present results and plans for further work.

## 2 Classical superintegrable systems in two dimensions

Consider a classical system with two degrees of freedom, described by the Hamiltonian:

$$H = H(x, y, p_x, p_y). \quad (1)$$

If the system is superintegrable there are two independent additional integrals of motion  $I$  and  $C$ , such that:

$$\{H, I\}_{PB} = \{H, C\}_{PB} = 0, \quad \text{and} \quad \{I, C\}_{PB} = F(H, I, C), \quad (2)$$

where  $\{, \}_{PB}$  denotes the Poisson bracket and  $F = F(H, I, C)$  is a constant of motion which depends on the three independent constants of motion  $H, I, C$ . A superintegrable system in two dimensions is necessarily a maximally superintegrable system, which means that all finite and closed classical

trajectories are periodic. Integrable and superintegrable systems in 2 dimensions have been reviewed in [2], while in [3] a systematic study is given of superintegrable systems in 3 dimensions which possess invariants that are quadratic polynomials of the canonical momenta.

Maximally superintegrable systems possess, by definition, the maximum number of independent classical invariants. Therefore any other integral can be expressed as a function of the basic integrals  $H, I, C$ . As a result we can in general choose two new integrals of motion:

$$L = L(H, I, C), \quad \text{and} \quad A = A(H, I, C),$$

such that:

$$\{L, A\}_{PB} = B, \quad \{L, B\}_{PB} = -A. \quad (3)$$

After a calculation we can prove that:

$$B^2 + A^2 = G(H, L),$$

where  $G(H, L)$  is some function depending only on the integrals of motion  $H, L$ , and

$$\{A, B\}_{PB} = \Phi(H, L) = -\frac{1}{2} \frac{\partial G}{\partial L}. \quad (4)$$

The structure of the algebra defined by eqs (3-4) has many similarities to the algebraic structure of the deformed oscillator given in references [46, 52, 53], where  $L$  is some kind of number operator, while  $A, B$  are like the creation and annihilation operators. The deformed oscillator algebra is a non abelian algebra. Therefore it is quite natural to attempt studying the quantum superintegrable systems by applying some similar procedure in order to calculate their quantum properties (eigenvalues and eigenvectors), since the corresponding properties of the deformed oscillators have been already studied.

## 3 Quantum superintegrable systems in two dimensions

The question of integrability in quantum mechanics is under investigation. Many authors [17, 18, 19, 20] have investigated several aspects of the exten-

sion of integrability from classical to quantum mechanics. We should recall at this point that each quantum system with discrete energy spectrum can be considered as a quantum integrable system [17].

In this paper we consider a two dimensional quantum system described by a hamiltonian  $H$  acting on a Hilbert space  $\mathcal{H}$ . The hamiltonian is an autoadjoint operator with range dense in the space  $\mathcal{H}$ . All the operators defined in this section are supposed to be generated by nonlinear combinations of the basic algebra of generators  $x, p_x, y, p_y$  satisfying the usual commutation relations:

$$[x, p_x] = [y, p_y] = i, \quad \text{other commutators} = 0.$$

If the system is integrable, then there is a second autoadjoint operator  $I$  commuting with the hamiltonian and having a range dense in  $\mathcal{H}$

$$[H, I] = 0, \quad (5)$$

the operators  $H$  and  $I$  being linearly independent. The commutativity of these operators implies that there is a family of common eigenvectors for both operators. Let us label the common eigenvectors of these operators using their corresponding eigenvalues, thus obtaining a family of vectors in the Hilbert space  $\mathcal{H}$ . Let us further suppose that this family spans the whole Hilbert space and that there is no simultaneous degeneracy for both labels of the common eigenfunctions (i.e. there could be degeneracy in each label separately, but not in both labels simultaneously). This assumption defines some kind of *completeness* of the system of the two quantum integrals  $H, I$  in involution. The present assumption is also consistent with the fact that the ranges of the operators  $H$  and  $I$  are supposed to be dense in the ambient Hilbert space  $\mathcal{H}$ .

Weigert [17] has proved that there is another quantum integral  $I'$  possessing the same properties as  $I$ . This is permissible in quantum mechanics but not in classical mechanics. In this sense the choice of complete operators  $H, I'$  instead of  $H, I$  means a different way of labelling of the base spanning the Hilbert space. Here we shall treat the part of the Hilbert space corresponding to a discrete spectrum. In some cases the whole of the Hilbert space can be described by a discrete basis (as in the case of the harmonic oscillator),

while in other cases (as in the case of the Coulomb (Kepler) potential) a part of the Hilbert space is described by a discrete basis (the space of energy eigenvectors with negative energy eigenvalues).

The system is called *superintegrable*, by analogy to the classical definitions, if there is a third operator  $C$ , linearly independent from  $H$  and  $I$ , with range dense in the Hilbert space and commuting with  $H$  but **not** commuting with  $I$

$$[H, C] = 0, \quad [I, C] \neq 0.$$

In this paper we propose the following working hypothesis:

**Hypothesis:** *Let us consider the superintegrable systems for which we can construct an associative algebra:*

$$\begin{aligned} \mathcal{N} &= \mathcal{N}(H, I, C), \\ \mathcal{N}^+ &= \mathcal{N}, \\ \mathcal{A} &= \mathcal{A}(H, I, C), \\ [\mathcal{N}, \mathcal{A}] &= -\mathcal{A}, \\ \mathcal{A}^+ \mathcal{A} &= \Phi(H, \mathcal{N}), \\ [\mathcal{A}^+ \mathcal{A}, \mathcal{A} \mathcal{A}^+] &= 0, \end{aligned} \quad (6)$$

where  $\Phi(E, x)$  is a real positive function definite for  $x \geq 0$  and

$$\Phi(E, 0) = 0. \quad (7)$$

From the above equations we can prove that:

$$\begin{aligned} [\mathcal{N}, \mathcal{A}^+] &= \mathcal{A}^+, \\ \mathcal{A} \mathcal{A}^+ &= \Phi(H, \mathcal{N} + 1). \end{aligned}$$

If this construction is possible we can then define the Fock space for each energy eigenvalue:

$$\begin{aligned} H|E, n\rangle &= E|E, n\rangle, \\ \mathcal{N}|E, n\rangle &= n|E, n\rangle, \quad n = 0, 1, \dots, \\ \mathcal{A}|E, 0\rangle &= 0, \\ |E, n\rangle &= \left(\frac{1}{\sqrt{[n]!}}\right) (\mathcal{A}^+)^n |E, 0\rangle, \end{aligned}$$

where

$$[0]! = 1, \quad [n]! = \Phi(E, n)[n-1]!$$

In the case of the discrete energy eigenvalues, for every energy eigenvalue  $E$  there is some degeneracy of dimension  $N_d + 1$ . Therefore the dimensionality of the Fock space corresponding to that energy eigenfunction should be equal to  $N_d + 1$ . This is equivalent to the condition:

$$\Phi(E, N_d + 1) = 0. \quad (8)$$

As we shall see in the examples given in the following section, *the two conditions (7) and (8), and the positiveness of the structure function  $\Phi(E, x)$*  suffice in order to determine the energy spectrum of the quantum maximally superintegrable systems.

There are only a few quantum 2-dimensional superintegrable systems known. All the examples studied in this paper have a classical counterpart.

## 4 Examples of quantum superintegrable two-dimensional systems

In this section we shall apply the hypothesis of the previous section in order to determine the energy spectrum of some quantum superintegrable two-dimensional systems using purely algebraic methods.

### 4.1 Harmonic oscillator in a flat space

The two-dimensional symmetric harmonic oscillator in euclidean coordinates is described by the hamiltonian:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} \omega^2 (x^2 + y^2). \quad (9)$$

The following Fradkin operators [55] can be defined:

$$B = S_{rx} - S_{yy} = (p_x^2 + \omega^2 x^2) - (p_y^2 + \omega^2 y^2), \quad S_{xy} = p_x p_y + \omega^2 xy. \quad (10)$$

The operator  $B$  in (10) is the quantum mechanical analogue of a third *constant of motion* in the sense of classical hamiltonian mechanics [2], the second one being the angular momentum operator:

$$L = xp_y - yp_x, \quad (11)$$

since:

$$[H, L] = [H, B] = 0, \quad (12)$$

while the operators  $B, L$  do not commute, but they form a closed algebra with the operator  $S_{xy}$ :

$$[L, B] = 4iS_{xy}, \quad [L, S_{xy}] = -iB. \quad (13)$$

The above relations suggest the possibility of expressing the two-dimensional harmonic oscillator algebra by using the deformed oscillator formulation:

$$\begin{aligned} \mathcal{N} &= \frac{L}{2} - u\mathbf{1}, \\ \mathcal{A}^+ &= \frac{B}{2} + iS_{xy}, \\ \mathcal{A} &= \frac{B}{2} - iS_{xy}, \end{aligned} \quad (14)$$

where  $u$  is a constant to be determined and

$$\begin{aligned} [\mathcal{N}, \mathcal{A}^+] &= \mathcal{A}^+, \\ [\mathcal{N}, \mathcal{A}] &= -\mathcal{A}, \\ \mathcal{A}^+ \mathcal{A} &= H^2 - \omega^2 (L - 1)^2 \\ &= H^2 - \omega^2 (2\mathcal{N} + 2u - 1)^2 \\ &= \Phi(H, \mathcal{N}), \end{aligned} \quad (15)$$

where the function  $\Phi(E, x)$  is given by:

$$\Phi(E, x) = E^2 - \omega^2 (2x + 2u - 1)^2, \quad (16)$$

and we can see that

$$\mathcal{A} \mathcal{A}^+ = \Phi(H, \mathcal{N} + 1).$$

The existence of a finite dimensional representation of the oscillator algebra is equivalent to the existence of a maximum number  $N + 1$  which is a root of the structure function, with  $N$  being the dimensionality of the algebra representation, coinciding with the dimensionality of the appropriate Fock space. This restriction, combined with the annihilation of the structure function for  $x = 0$ , is written as:

$$\begin{aligned} \Phi(E, 0) &= 0, \\ \Phi(E, N + 1) &= 0, \\ \Phi(E, x) &> 0 \quad \text{for } x = 1, 2, \dots, N. \end{aligned} \quad (17)$$

Solving this system of two equations with two unknowns,  $E$  and  $u$ , one obtains the eigenvalues of the harmonic oscillator in a flat space:

$$u = -\frac{N}{2}, \quad E = E_N = \omega(N+1). \quad (18)$$

The angular momentum values allowed for each energy level can then be determined by inserting the value of the constant  $u$  just obtained into the first of eq. (14), the result being:

$$L = -N, -N+2, \dots, N-2, N. \quad (19)$$

The structure function of the deformed oscillator is calculated to be:

$$\Phi(E_N, x) = 4\omega^2 x(N+1-x).$$

Clearly the above energy (18) and angular momentum spectra (19) are the same as the ones obtained by classical means, i.e. by solving the appropriate Schrödinger equation, or by using the SU(2) symmetry. The existence of a finite dimensional algebra representation should be attributed to the existence of stable periodical trajectories in the corresponding classical case.

The connection to the usual SU(2) symmetry becomes clear by observing that the operators

$$J_+ = \frac{\mathcal{A}^+}{2\omega}, \quad J_- = \frac{\mathcal{A}}{2\omega}, \quad J_0 = \mathcal{N} + u\mathbf{1} = \frac{L}{2},$$

satisfy the usual SU(2) commutation relations

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_0,$$

the Casimir operator being

$$J^2 = J_+ J_- + J_0(J_0 - 1) = \frac{\mathcal{A}^+ \mathcal{A}}{4\omega^2} + \frac{L}{2} \left( \frac{L}{2} - 1 \right).$$

## 4.2 Harmonic oscillator in a space with constant curvature

Higgs [5] has studied the symmetries of a harmonic oscillator in a non-flat space, a space with constant curvature in particular. A typical example of such a space is the surface of the sphere in a three dimensional space.

The curved space is geometrically described by the metric:

$$ds^2 = \frac{dx^2 + dy^2 + \lambda(xdy - ydx)^2}{(1 + \lambda(x^2 + y^2))^2},$$

the flat space corresponding to  $\lambda = 0$ . The harmonic oscillator in this space is defined in ref. [5] by the Hamiltonian:

$$H = \frac{1}{2}(\pi_x^2 + \pi_y^2 + \lambda L^2) + \frac{\omega^2}{2}(x^2 + y^2), \quad (20)$$

where the angular momentum operator  $L$  is given by eq. (11) and

$$\begin{aligned} \pi_x &= p_x + \frac{\lambda}{2}(x(xp_x + yp_y) + (xp_x + yp_y)x), \\ \pi_y &= p_y + \frac{\lambda}{2}(y(xp_x + yp_y) + (xp_x + yp_y)y). \end{aligned} \quad (21)$$

By analogy to the harmonic oscillator in a flat space, Higgs[5] has defined the Fradkin-like operators:

$$B = S_{xx} - S_{yy} = (\pi_x^2 + \omega^2 x^2) - (\pi_y^2 + \omega^2 y^2), \quad (22)$$

$$S_{xy} = \frac{1}{2}\{\pi_x, \pi_y\} + \omega^2 xy, \quad (23)$$

which are symmetrized versions of eq. (10).

The commutators of  $H$  with  $L$  and  $B$  are given in eq. (12), while the operators  $B, L$  do not commute and their commutator is the same as in equation (13), where the coordinates of the momentum  $p$  should be replaced by the coordinates of the extended momentum  $\pi$ , defined by eq. (21) and already used in the symmetrized Fradkin operators given above.

The operators  $H, L, B, S_{xy}$  define again a closed non-linear algebra as in the flat harmonic oscillator case. Therefore we have another example of a superintegrable system in a non-flat space.

In the present case we can also define the corresponding deformed oscillator as in eq. (14). The deformed algebra is then completely defined by the structure function:

$$\begin{aligned} \Phi(E, x) &= \\ E^2 - \left( \omega^2 + \frac{\lambda}{4} + \lambda E \right) (2x + 2u - 1)^2 + \frac{\lambda^2}{4} (2x + 2u - 1)^4. \end{aligned} \quad (24)$$

Following the same methodology as in the case of the harmonic oscillator in a flat space, we can generate the corresponding Fock space for the curved harmonic oscillator. By assuming then the existence of a finite dimensional deformed algebra representation the restrictions corresponding to eq. (17) are valid. These equations determine the energy eigenvalues:

$$E = E_N = \sqrt{\omega^2 + \frac{\lambda^2}{4}}(N+1) + \frac{\lambda}{2}(N+1)^2, \quad (25)$$

while the constant  $u$  turns out to have the same values as in eq. (18). The angular momentum eigenvalues are again given by eq. (19).

Another interesting point arises from the comparison between the structure function (16) in a flat space and eq. (24) in a curved space: The geometry of the space affects the algebra characterizing the harmonic oscillator. For  $\lambda = 0$  eq. (24) reduces to eq. (16), as it should.

Finally, the structure function can be written as

$$\begin{aligned} \Phi(E_N, x) &= \\ &= 4x(N+1-x) \left( \lambda(N+1-x) + \sqrt{\omega^2 + \lambda^2/4} \right) \left( \lambda x + \sqrt{\omega^2 + \lambda^2/4} \right). \end{aligned}$$

The symmetries of the harmonic oscillator in a curved space have been studied in ref. [56] using the notion of the quadratic Racah algebras QR(3).

### 4.3 The Kepler problem in a curved space

The case of the Kepler problem in a space with constant curvature has been studied by Higgs [5]. The hamiltonian is given by:

$$H = \frac{1}{2} (\pi_x^2 + \pi_y^2 + \lambda L^2) - \frac{\mu}{r}, \quad r = \sqrt{x^2 + y^2}, \quad (26)$$

where the angular momentum operator  $L$  is given by eq. (11) and the  $\pi_x, \pi_y$  are defined in eq. (21).

The Runge-Lenz vectors in the curved space can be defined by:

$$R_x = -\frac{1}{2} \{L, \pi_y\} + \mu \frac{x}{r}, \quad R_y = \frac{1}{2} \{L, \pi_x\} + \mu \frac{y}{r}. \quad (27)$$

This system is a quantum superintegrable system in a curved space because:

$$[H, L] = 0, \quad [H, R_x] = 0,$$

and the operators  $L, R_x, R_y$  form a closed algebra:

$$[L, R_x] = iR_y, \quad [L, R_y] = -iR_x.$$

Using the same hypothesis as previously we can define the deformed oscillator algebra:

$$\begin{aligned} \mathcal{N} &= L - u, \\ \mathcal{A}^+ &= R_x + iR_y, \\ \mathcal{A} &= R_x - iR_y, \\ \mathcal{A}^+ \mathcal{A} &= \mu^2 + 2H(L - 1/2)^2 - \lambda(L - 1/2)^2 ((L - 1/2)^2 - 1/4) \\ &= \mu^2 + 2H(\mathcal{N} + u - 1/2)^2 - \lambda(\mathcal{N} + u - 1/2)^2 ((\mathcal{N} + u - 1/2)^2 - 1/4) \\ &= \Phi(H, \mathcal{N}). \end{aligned} \quad (28)$$

The structure function in this case is defined by:

$$\begin{aligned} \Phi(E, x) &= \mu^2 + 2E(x + u - 1/2)^2 \\ &\quad - \lambda(x + u - 1/2)^2 ((x + u - 1/2)^2 - 1/4). \end{aligned}$$

The solution of eqs (17) is given by:

$$u = -\frac{N}{2}, \quad E_N = -\frac{2\mu^2}{(N+1)^2} + \lambda \frac{N(N+2)}{8}. \quad (29)$$

The permitted eigenvalues of the angular momentum operator  $L$  are given by:

$$L = -\frac{N}{2}, -\frac{N}{2} + 1, \dots, \frac{N}{2} - 1, \frac{N}{2}.$$

This means that the symmetries of the Kepler problem are compatible with the existence of angular momenta equal to  $0, 1/2, 1, 3/2, \dots$ . In physical situations, however, only integer angular momenta appear, which means that  $N = 2n$ . In this case the spectrum given by eq. (29) is the same to that obtained by Higgs [5]. In the case of zero curvature, i.e.  $\lambda = 0$ , we obtain the usual Coulomb energy spectrum.

The structure function corresponding to the Kepler problem is given by:

$$\Phi(E_N, x) = x(N+1-x) \left( \frac{4\mu^2}{(N+1)^2} + \lambda \frac{(N+1-2x)^2}{4} \right).$$

Zhedanov [26] has proven that the nonlinear algebra of eq. (28) can be approximated to second order by the  $SU_q(2)$  algebra [33, 34].

The symmetries of the Kepler potential in a curved space have been studied in ref. [57] using the notion of the quadratic Racah algebras QR(3).

#### 4.4 Fokas-Lagerstrom potential

In classical mechanics the superintegrable system described by the Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{x^2}{2} + \frac{y^2}{18} \quad (30)$$

has been studied by Fokas and Lagerstrom [7]. This system has two additional classical invariants of motion,

$$J = p_x^2 + x^2, \quad \text{and} \quad C = (xp_y - yp_x)p_y^2 + \frac{y^3 p_x}{27} - \frac{xy^2 p_y}{3}, \quad (31)$$

the second of which ( $C$ ) is a cubic function of the coordinates. The quantum version of the hamiltonian (30) corresponds to a quantum superintegrable system with two additional integrals:

$$J = p_x^2 + x^2, \quad \text{and} \quad B = \frac{1}{2} \{ xp_y - yp_x, p_y^2 \} + \frac{y^3 p_x}{27} - \frac{\{ xy^2, p_y \}}{6},$$

where  $\{ , \}$  is the usual anticommutator. It is clear that the quantum integral  $B$  is the symmetrized version of the classical integral  $C$ . From the above definitions we can verify that:

$$[H, J] = 0, \quad [H, B] = 0,$$

$$[J, B] = R, \quad [J, R] = 4B,$$

and

$$[R, B] = 8J^3 - 36J^2H + 48JH^2 - 16H^3 + \frac{56}{9}J - \frac{92}{9}H,$$

$$R^2 - 4B^2 = 4J^4 - 24J^3H + 48J^2H^2 - 32JH^3 + \frac{200}{9}J^2 - \frac{616}{9}JH + 48H^2 + \frac{20}{9}.$$

From the above closed algebra we can define:

$$\mathcal{N} = J/2 - u, \quad \mathcal{A}^+ = B + R/2, \quad \mathcal{A} = B - R/2,$$

where  $u$  is a constant to be determined. These operators correspond to a deformed oscillator algebra:

$$[\mathcal{N}, \mathcal{A}^+] = \mathcal{A}^+, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A},$$

$$\begin{aligned} \mathcal{A}^+ \mathcal{A} &= \frac{1}{9}(J-1)(2H-J+1)(6H-3J+1)(6H-3J+5) \\ &= \frac{1}{9}(2\mathcal{N}-1+2u)(2H-2u+1-2\mathcal{N}) \\ &\quad (6H-6u+1-6\mathcal{N})(6H-6u+5-6\mathcal{N}) \\ &= \Phi(H, \mathcal{N}), \\ \mathcal{A} \mathcal{A}^+ &= \Phi(H, \mathcal{N}+1). \end{aligned} \quad (32)$$

The corresponding structure function is defined by:

$$\Phi(E, x) = \frac{1}{9}(2x-1+2u)(2E-2u+1-2x) \\ (6E-6u+1-6x)(6E-6u+5-6x).$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (17) and it is a positive function. Therefore we can find the possible energy eigenvalues having degeneracy equal to  $N+1$ :

Case i)

$$u = 1/2, \quad \text{and} \quad E_N = N+1,$$

corresponding to the structure function:

$$\Phi(E_N, x) = 16x(N+1-x) \left( N + \frac{2}{3} - x \right) \left( N + \frac{4}{3} - x \right).$$

Case ii)

$$u = 1/2, \quad \text{and} \quad E_N = N+2/3,$$

corresponding to the structure function:

$$\Phi(E_N, x) = 16x(N+1-x) \left( N + \frac{2}{3} - x \right) \left( N + \frac{1}{3} - x \right).$$

Case iii)

$$u = 1/2, \quad \text{and} \quad E_N = N+4/3,$$



corresponding to the structure function:

$$\Phi(E_N, x) = 16x(N+1-x) \left(N + \frac{5}{3} - x\right) \left(N + \frac{1}{3} - x\right).$$

In all cases the degeneracy is determined by  $J = 2(\mathcal{N} + u)$ . Since  $\mathcal{N}$  obtains the  $N+1$  values  $0, 1, \dots, N$ , as a result  $J$  also obtains  $N+1$  values.

The Hamiltonian (30) corresponds to the linear combination of two harmonic oscillators which have the above described energy spectrum. This case has a special significance, since it is an example of a superintegrable potential which is not a separable one in two different coordinate systems. The proposed method does not depend on the separability of the variables in two systems.

#### 4.5 Smorodinsky–Winternitz potential

The classical superintegrable Smorodinsky–Winternitz system [8, 9, 10, 11] corresponds to the Hamiltonian:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + k(x^2 + y^2) + \frac{c}{x^2}. \quad (33)$$

This system has two additional classical invariants of motion,

$$T = p_y^2 + 2ky^2, \quad \text{and} \quad C = x^2 p_y^2 + y^2 p_x^2 - 2xy p_x p_y + 2c \frac{y^2}{x^2}, \quad (34)$$

the second of which ( $C$ ) is a quartic function of the coordinates. Evans [11] has proved that the Winternitz–Smorodinsky potential in  $N$  dimensions is an example of a superintegrable system. The quantum version of the hamiltonian (33) corresponds to a quantum superintegrable system with two additional integrals:

$$T = p_y^2 + 2ky^2, \quad \text{and} \quad B = x^2 p_y^2 + y^2 p_x^2 - \{xy, p_x p_y\} + 2c \frac{y^2}{x^2}.$$

It is clear that the quantum integral  $B$  is the symmetrized version of the classical integral  $C$ . From the above definitions we can verify that:

$$[H, T] = 0, \quad [H, B] = 0,$$

and

$$[T, B] = R, \quad [T, R] = 32kB + 8T^2 - 16HT - 16k,$$

$$[R, B] = 16BT - 16BH + 32(c-1)T + 8R + 32H,$$

$$R^2 = 32kB^2 + 224kB + 32(c-1)T^2 + 64HT + 16RT - 16RH - 48H^2 + 16BT^2 - 32BT^2H + 192k(c-1).$$

From the above closed non-linear algebra we can define:

$$\begin{aligned} \mathcal{N} &= \frac{1}{\sqrt{32k}} T + u, \\ \mathcal{A}^+ &= 4kB + \sqrt{\frac{k}{2}} R + T^2 - 2HT - 2k, \\ \mathcal{A} &= 4kB - \sqrt{\frac{k}{2}} R + T^2 - 2HT - 2k, \end{aligned}$$

where  $u$  is a constant to be determined. These operators correspond to a deformed oscillator algebra:

$$\begin{aligned} [\mathcal{N}, \mathcal{A}^+] &= \mathcal{A}^+, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \\ \mathcal{A}^+ \mathcal{A} &= 24H^2 k + 3 \cdot 2^{\frac{3}{2}} H k^{\frac{3}{2}} + 36k^2 \\ &\quad - 96ck^2 - 2^{\frac{3}{2}} H^2 \sqrt{k} T - 88HkT \\ &\quad - 3 \cdot 2^{\frac{3}{2}} k^{\frac{3}{2}} T + 2^{\frac{3}{2}} ck^{\frac{3}{2}} T + 4H^2 T^2 \\ &\quad + 3 \cdot 2^{\frac{3}{2}} H \sqrt{k} T^2 + 44kT^2 - 16ckT^2 \\ &\quad - 4HT^3 - 2^{\frac{3}{2}} \sqrt{k} T^3 + T^4 \\ &= \Phi(H, \mathcal{N}), \\ \mathcal{A} \mathcal{A}^+ &= \Phi(H, \mathcal{N} + 1). \end{aligned} \quad (35)$$

The corresponding structure function can be factorized as:

$$\Phi(E, x) = 1024k^2 \left(x - \left(u + \frac{3}{4}\right)\right) \left(x - \left(u + \frac{1}{4}\right)\right) \left(x - \left(u + \frac{1}{2} + \frac{E}{\sqrt{8k}} + \frac{\sqrt{1+8c}}{4}\right)\right) \left(x - \left(u + \frac{1}{2} + \frac{E}{\sqrt{8k}} - \frac{\sqrt{1+8c}}{4}\right)\right).$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (17). The positiveness of the structure function for every  $0 < x \leq N$  implies:

$$u = -\frac{3}{4},$$

while the energy eigenvalues are given by:

$$E_N = \sqrt{8k} \left(N + \frac{5}{4} + \frac{\sqrt{1+8c}}{4}\right), \quad N = 1, 2, \dots,$$

with  $-\frac{1}{8} \leq c$ . The structure function is given by:

$$\Phi(E_N, x) = 1024k^2 x \left(x + \frac{1}{2}\right) (N+1-x) \left(N+1 + \frac{\sqrt{1+8c}}{2} - x\right).$$

If the following restriction is valid:

$$-\frac{1}{8} \leq c \leq \frac{3}{8}, \quad (36)$$

the following energy eigenvalues are also permitted:

$$E_N = \sqrt{8k} \left(N + \frac{5}{4} - \frac{\sqrt{1+8c}}{4}\right), \quad N = 1, 2, \dots$$

corresponding to the structure function:

$$\Phi(E_N, x) = 1024k^2 x \left(x + \frac{1}{2}\right) (N+1-x) \left(N+1 - \frac{\sqrt{1+8c}}{2} - x\right).$$

It is worth noticing that in the case of solving the problem using the Schrödinger equation, the restriction (36) is introduced by the assumption that the eigenfunctions should be square integrable functions on the plane  $(x, y)$ . In the Schrödinger equation solution the additional restriction of finiteness of the potential energy restricts the choice of  $c$  to positive values only.

## 4.6 The Holt potential

The classical superintegrable Holt [12] system corresponds to the Hamiltonian:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + (x^2 + 4y^2) + \frac{\delta}{x^2}. \quad (37)$$

This potential is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. This system has two additional classical invariants of motion,

$$T = p_y^2 + 8y^2, \quad \text{and} \quad C = p_x^2 p_y + 8xy p_x - 2x^2 p_y + \frac{2\delta}{x^2} p_y, \quad (38)$$

the second of them ( $C$ ) being a cubic function of the momenta. The quantum version of the hamiltonian (37) corresponds to a quantum superintegrable system with two additional integrals:

$$T = p_y^2 + 8y^2, \quad \text{and} \quad B = p_x^2 p_y + 4\{xy, p_x\} - 2x^2 p_y + \frac{2\delta}{x^2} p_y.$$

It is clear that the quantum integral  $B$  is the symmetrized version of the classical integral  $C$ . From the above definitions we can verify that:

$$[H, T] = 0, \quad [H, B] = 0,$$

and

$$[T, B] = R, \quad [T, R] = 32B,$$

$$[R, B] = -96 + 256\delta - 64H^2 + 128HT - 48T^2,$$

$$R^2 - 32B^2 = 1024H - 704T + 512\delta T - 128TH^2 + 128T^2H - 32T^3.$$

From the above closed non-linear algebra we can define:

$$\mathcal{N} = \frac{T}{\sqrt{32}} - u, \quad \mathcal{A}^+ = 8B + \sqrt{2}R, \quad \mathcal{A} = 8B - \sqrt{2}R,$$

where  $u$  is a constant to be determined. These operators correspond to a deformed oscillator algebra:

$$\begin{aligned} [\mathcal{N}, \mathcal{A}^+] &= \mathcal{A}^+, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \\ \mathcal{A}^+ \mathcal{A} &= 2^6 (T - 2\sqrt{2}) \\ &= \left(H - \frac{T}{2} + \sqrt{2} + \sqrt{\frac{1+8\delta}{2}}\right) \left(H - \frac{T}{2} + \sqrt{2} - \sqrt{\frac{1+8\delta}{2}}\right) \quad (39) \\ &= \Phi(H, \mathcal{N}), \\ \mathcal{A} \mathcal{A}^+ &= \Phi(H, \mathcal{N} + 1). \end{aligned}$$

The corresponding structure function is defined by:

$$\begin{aligned} \Phi(E, x) &= 2^{\frac{3}{2}} \left((x+u) - \frac{1}{2}\right) \\ &\quad \left(\frac{E}{\sqrt{8}} - (x+u) + \frac{1}{2} + \frac{\sqrt{1+8\delta}}{4}\right) \\ &\quad \left(\frac{E}{\sqrt{8}} - (x+u) + \frac{1}{2} - \frac{\sqrt{1+8\delta}}{4}\right). \end{aligned}$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (17). Therefore we can find the possible energy eigenvalues having degeneracy equal to  $N+1$ :

$$u = \frac{1}{2}, \quad \text{and} \quad E_N = \sqrt{8} \left(N + 1 + \frac{\sqrt{1+8\delta}}{4}\right),$$

where  $(1 + 8\delta) \geq 0$ . The corresponding structure function is:

$$\Phi(E_N, x) = 2^{\frac{23}{2}} x(N+1-x) \left( N+1-x + \frac{\sqrt{1+8\delta}}{2} \right). \quad (40)$$

In the special case where  $-\frac{1}{8} \leq \delta \leq \frac{3}{8}$  there are energy eigenvalues given by:

$$u = \frac{1}{2}, \quad \text{and} \quad E_N = \sqrt{8} \left( N+1 - \frac{\sqrt{1+8\delta}}{4} \right),$$

and the structure function is:

$$\Phi(E_N, x) = 2^{\frac{23}{2}} x(N+1-x) \left( N+1-x - \frac{\sqrt{1+8\delta}}{2} \right), \quad (41)$$

which is positive for  $0 < x \leq N$  if  $-\frac{1}{8} \leq \delta \leq \frac{3}{8}$ .

In the case of  $\delta = 0$  the hamiltonian of eq. (37) corresponds to the hamiltonian of an anisotropic harmonic oscillator with ratio of frequencies 1/2. This system, studied in ref. [58], is separable in parabolic coordinates. The system is described by two deformed oscillators, with structure functions obtained by putting  $\delta = 0$  in equations (40) and (41). The corresponding energy eigenvalues with degeneracy  $N+1$  are given by:

$$E_N = \sqrt{8} \left( N + \frac{5}{4} \right), \quad E_N = \sqrt{8} \left( N + \frac{3}{4} \right).$$

This case should be compared with the case 1/3 (Fokas-Lagerstrom potential), where three energy eigenvalues correspond to degeneracy equal to  $N+1$ . The general problem of finding the symmetry algebra of the harmonic oscillator with rational ratio of frequencies equal to  $m/n$  is a difficult problem, implicitly posed by Jauch and Hill [59] already in 1940, in a paper where the general form of the integrals in both the classical and quantum mechanical cases has been given.

The quantum Holt potential has also been studied recently by using quadratic algebras by Létourneau and Vinet [30].

## 5 Discussion

In this paper, starting from classical superintegrable systems, we have shown that the corresponding quantum systems are superintegrable ones, the quan-

tum integrals (quantum constants of motion) being obtained from the classical ones using a symmetrization procedure. Furthermore, the quantum superintegrable systems can be described in terms of a deformed oscillator algebra. The operators of the deformed oscillator algebra are constructed from the quantum integrals. The deformed oscillator algebra is characterized by a structure function  $\Phi(E, N)$ , which takes a specific form for each superintegrable system. The eigenvalues of the energy and their degeneracies are determined in an economical way directly from equations satisfied by the structure function, the results being in agreement with these coming from the independent solution of the relevant Schrödinger equation.

A few comments and some open problems are now in place:

i) In all of the examples considered in this paper, quantum superintegrability is induced by classical superintegrability, the quantum integrals of motion being symmetrized versions of the corresponding classical integrals. The extend to which classical superintegrability implies in general quantum superintegrability as well, has to be tested.

ii) In all of the examples considered in this paper, quantum superintegrability manifests itself in the degeneracy of the energy levels, a fact already noticed [4, 59].

iii) The hypothesis that to each superintegrable system corresponds a deformed oscillator algebra, i.e.

$$\text{superintegrability} \rightarrow \text{deformed oscillator algebra}$$

is an exact proposition in the classical case, as shown in section 2. In the quantum case, however, a general formal proof is still lacking. In section 3 a working hypothesis was made, which was proved successful in the examples considered in section 4.

iv) The list of two-dimensional quantum superintegrable systems given in this paper is not exhaustive. There are classical superintegrable systems for which the quantum superintegrability has to be proven, as, for example, the Calogero system ([2], eq. (3.5.9)), which possesses a sixth order invariant. Furthermore, there are two-dimensional systems for which the quantum superintegrability has been shown, but the determination of the corresponding

deformed oscillator algebra requires heavy computation, as, for example, the Winternitz-Sinorodinsky potential of ref. [9], given also in [2], eq. (3.2.36).

v) The extension of the present method to three-dimensional quantum superintegrable systems is under investigation. It should be mentioned that classical superintegrable systems in 3 dimensions having invariants which are quadratic polynomials in the canonical momenta have been recently studied in [11].

vi) Another interesting point is the semiclassical study of two-dimensional superintegrable systems. The algebra characterizing the two-dimensional classical superintegrable systems, studied in section 2, can be quantized by using the correspondence { Poisson bracket  $\rightarrow$  commutator }. Through this procedure, from the classical algebra a quantum deformed algebra is obtained, which is the semiclassical counterpart of the exact quantum oscillator algebra considered in this paper, these two algebras being slightly different.

vii) The example of the Fokas Lagerstrom potential, which is the oscillator with ratio of frequencies 1:3, shows that quantum superintegrability implies a dynamical symmetry. This example was examined using algebraic methods for the first time. All the other examples have been already studied by other authors, as it has been indicated in the text. The difference of the proposed treatment is that we do not use the separability in two coordinate systems in order to calculate the dynamical symmetries. The set of two-dimensional systems separable in two different coordinate systems is a subset of the class of the superintegrable systems in two dimensions. The study of other superintegrable systems non-separable in more than one coordinate system seems to be very interesting. A class of such systems already known consists of the oscillators with rational ratio of frequencies. The general problem of finding the symmetry algebra describing the anisotropic oscillator with rational ratio of frequencies is an important problem implicitly posed by Jauch and Hill [59] already in 1940, in a paper where the integrals in both the classical and quantum cases are given. The method of the present paper is based on the assumption that the symmetry algebra is constructed by applying a symmetrization procedure to the classical integrals. In the general case described by Jauch and Hill [59], both the classical and quantum

integrals are known, but they lead to complicated algebraic structures. The construction of the symmetry algebra in the general case is an open problem already receiving attention.

viii) Many of these examples [30, 56, 57] have been studied by using cases of quadratic Askey-Wilson algebras QAW(3) [29]. An open problem is if the general quadratic Askey-Wilson algebra can be expressed by a deformed oscillator. One can also notice that the algebra of the Fokas-Lagerstrom problem is a cubic algebra, while all the other examples in this paper correspond to quadratic algebras.

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