CERN-ACC-NOTE-2018-0079

11 December 2018

Derivation of the Energy Loss for a Particle passing through an Equipment, from both Beam Impedance Concepts and Electro-Technical Theory.

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Keywords: Beam Impedance, Electro-technical theory, Energy Loss, Accelerator Physics

Summary This note will show that the energy lost when a beam moves through an equipment, when calculated with the wake field approach (which is a beam impedance concept), is the same as when calculated with electro-technical theory.

1 Theory and Introduction

1.1 Introduction to Wake Fields and Wall Currents

A wake field can roughly be described as the electromagnetic field trailing a charged particle. The particle generating the field is called the drive particle and denoted q_d. The wake field is only truly behind the drive particle when the speed of the drive particle equals the speed of light, otherwise the field is also in front of the drive particle. As depicted in Figure 1 there is a drive particle, q_d, moving at the speed of light, which is the one generating the wake field and a point where the wake field is measured. This measuring point is represented by a test charge q_t which is "experiencing" the wake field. The test charge is a fictitious particle,

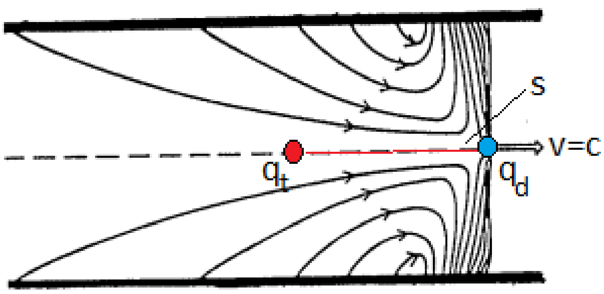

Figure 1: The driving particle, q_d followed by a test particle, q_t, with a constant distance s between them, moving through a resistive equipment. Originating from q_d are seen electric field lines which are terminated at the induced charges in the walls. Adapted from Ref[5]

and it moves so there is always a constant distance ”s” between the drive charge and the test charge. As said, the test charge represent the point were the wake field is measured, but it will also be useful to calculate the energy losses. As can be seen in Figure 1, there are induced charges in the walls. The induced charges in the walls are both positive and negative. Some of the electric field lines in the figure goes between these positive and negative charges. The wake field consists of all the electrical field lines shown (and in addition magnetic field lines that are not shown here). Please notice that some of the electrical field lines goes from positive charges in the wall to negative charges also in the wall. Notice also that because the drive charge is moving, and the induced charges therefore also moves, these moving induced charges represent induced wall currents. Because of the resistance in the walls of the equipment, the induced currents will feel an impedance and so will the beam. This impedance, felt by the beam, is called the beam impedance.

1.2 Characterizing an Equipment via an Impulse Responce or a Lumped Impedance

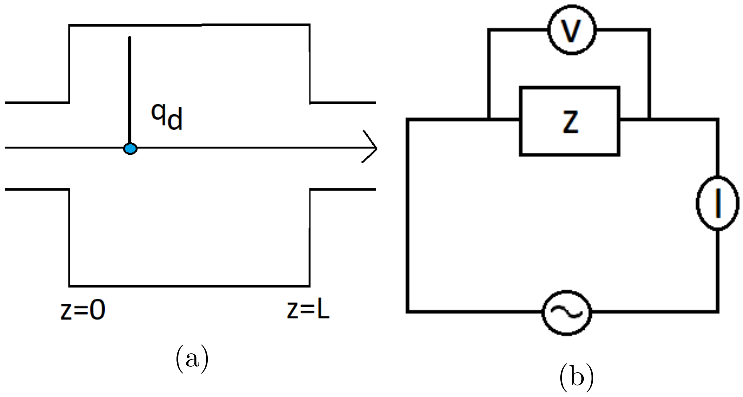

Figure 2: Two identical ways of characterizing the response of an equipment: a) A drive particle, q_d, moves through an equipment with the length L. The drive particle is depicted as a Dirac delta impulse. The response to this Dirac delta function, which is defined as the voltage drop of a test particle that follows the drive particle, fully characterize the equipment. b) A circuit showing an equipment represented by a lumped impedance, Z, and its corresponding current, I, and voltage, V.

An accelerator is build up of many different equipment, e.g. magnets, vacuum chambers, kickers, accelerating cavities, position monitors, gas monitors, electron coolers, collimators, wire scanners, etc. When a particle beam passes through an equipment there will be an electromagnetic interaction between the beam and the equipment. This electromagnetic interaction can be modeled in two ways, either as an impulse response, i.e. as the voltage over the equipment, experienced by a test particle that follows the a Dirac delta pulse (in our case modeled as a single particle, called a drive particle, moving through the equipment), see Figure 2a, or as a lumped impedance, see Figure 2b. Representing the electromagnetic interaction of an equipment as an impulse response is a standard way of characterizing any electromagnetic circuit, see Appendix B. The impulse response of an equipment, to a single particle moving through it, is defined via the concept of a wake function. The wake function will be explained in Section 1.3. Notice that the impulse response of an equipment or an electronic circuit are different in one an important aspect, namely that the wake function considers the length of the equipment, while an electronic circuit is considered as a lumped impedance, which assumes that the length of the impedance is zero. The lumped impedance that models the equipment is what is called the beam impedance.

The purpose of this note is to show that the energy loss can be calculated via both the wake function concept and the beam impedance concept and that both concepts yields identical results. This energy loss will be shown to be given by the formula:

$$U_{loss} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)|\mathcal{I}(\omega)|^2 d\omega \tag{1}$$

where $Z(\omega)$ is the beam impedance and $|\mathcal{I}(\omega)|$ is the absolute value of the Fourier transformation of the current (See appendix A).

1.3 The Wake Function

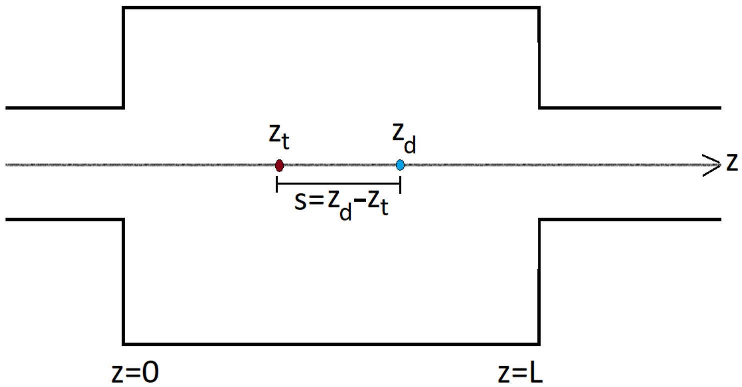

Figure 3: The figure depicts the driving particle, q_d followed by a test particle, q_t, moving through an equipment of length L. The driving particle is positioned in z_d and the test particle is positioned in z_t. The distance between the particles is s.

Imagine a particle, called the drive particle, that moves through an equipment. The response of the equipment to this drive particle will now be investigated. The response is defined by a test particle that moves a constant distance, s, behind the drive particle, see Figure 3. The test particle will experience the wake field from the drive particle. *NB in this note only the longitudinal fields will be considered.* The integral, from the start to the end of the equipment, of this electrical field, felt by the test particle, gives a voltage.

$$V_{\|}(s) = -\int_0^L E_{\|}(z_t, s)dz_t \tag{2}$$

where $V_{\|}(s)$ is the longitudinal voltage drop for a given distance s between the drive and test particle, L is the length of the equipment and $E_{\|}(z_t$, s) is the longitudinal electric field evaluated in z_t, which is the position of the test particle. The wake function can now be defined as the voltage drop normalized with the drive charge.

$$w_f(s) = \frac{V_\parallel(s)}{\mathrm{q}_d} = -\frac{1}{\mathrm{q}_d}\int_0^L E_\parallel(z_t, s)dz_t \tag{3}$$

In the definition of the wake function, the reason why the potential is normalized with the drive charge, is that one would like the wake function to represent the impulse response of the equipment. Since an impulse response is the response to a Dirac delta function, which has an integral equal to 1 i.e. $\int_{-\infty}^{\infty}\delta(t)dz_t = 1$, then the response to the drive particle must be scaled down so it represents a charge equal to 1. It is clear from this, that when the potential is normalized by the magnitude of the charge, *the resulting wake function, $w_f(s)$, is the system impulse response.* Another way of describing the wake function is through the wake field which is defined in the following way

$$w_{field}(z_t, s) = \frac{E(z_t, s)}{\mathrm{q}_d} \tag{4}$$

The wake function, which represents the voltage over the equipment as the response to a Dirac delta function, is then the integral of the wake field

$$w_f(s) = -\int_0^L w_{field}(z_t, s)dz_t \tag{5}$$

Just as the wake field is defined by the electrical field and the charge of the drive particle, the electrical field can be defined by the wake field and the charge of the drive particle i.e. $E(z_t, s) = w_{field}(z_t, s)\cdot \mathrm{q}_d$. This is useful in defining the electrical field in a bunch, which is the sum of all the wake fields from all the drive particles in the bunch:

$$E_z(z_t) = \int_{-\infty}^{\infty} w_{field}(z_t, s)d\mathrm{q}_{d,\lambda} \tag{6}$$

1.4 The Beam Impedance and its relation to the wake function

There can be many sources of beam impedance, such as impedance from wall currents, impedances generated by space charge or special cases like electron clouds impedance, however this note will only focus on the impedance from wall currents. Having introduced the wake function for a single particle, the next step would be to introduce the beam impedance and its relation to the wake function.

The beam impedance is defined, as any impedance, in the frequency domain:

$$Z(\omega) = \frac{V(\omega)}{\mathcal{I}(\omega)} \tag{7}$$

where $Z(\omega)$ is the beam impedance, $\mathcal{I}(\omega)$ is the beam current represented by a bunch of particles, and $V(\omega)$ is the voltage drop felt by a test particle that moves through the equipment. All three items are expressed in the frequency domain i.e. they are Fourier transforms of the same objects in the time domain.

The voltage drop, created by a single drive particle, is given by Equation 3:

$$V_{\|}(s) = w_f(s) \cdot \mathrm{q}_d \tag{8}$$

where $V_{\|}(s)$ is the voltage drop of the test particle as it moves through the equipment, $w_f(s)$ is the wake function and q_d is the charge of the driving particle. *NB! Both the voltage drop and the wake function are functions of distance and not of time. The conversion from distance to time will be done later.* For the moment all will be calculated in terms of distance. If, instead of a single drive particle, the voltage drop is created by a bunch of particles, the total voltage drop is the integration of the voltage drops from all the particles in the bunch:

$$V_{\|}(s) = \int_{-\infty}^{\infty} w_f(s + z_b) dq_d(z_b) \tag{9}$$

Where $V_{\|}(s)$ is the voltage drop felt by the test particle at a distance s behind the bunch, i.e. behind the reference position, $z_b = 0$, of the bunch distribution, $w_f(s + z_b)$ is the wake function of a drive particle where the test particle moves at a distance $s + z_b$ behind the drive particle, $dq_d(z_b)$ is the charge of the drive particle and z_b is the position of the drive particle inside the bunch distribution.

The bunch distribution $\lambda(z_b)$ is defined as a function of distance z_b (see figure 4):

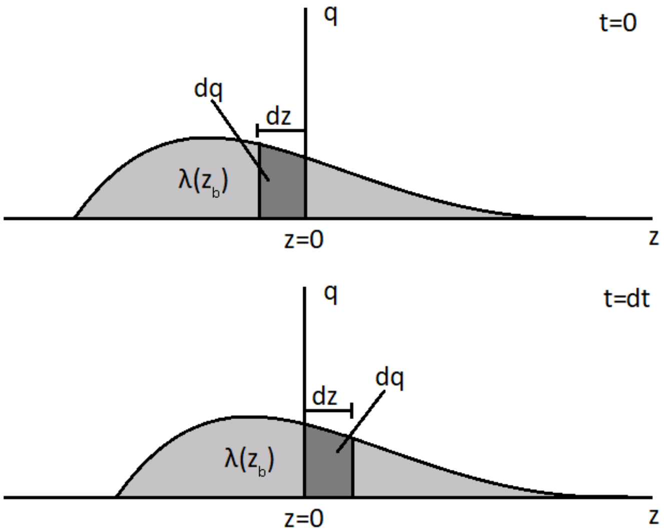

Figure 4: The charge distribution of a bunch is shown at two times, t=0 and t=dt. In the small time interval dt, the bunch moves a distance dz to the right. The speed of the bunch is v, so the distance dz will be dz=v dt. At t=0 the reference point, $z_b = 0$, of the bunch distribution passes the point z=0 on the z-axis. The current in position $z = 0$ at time $t = 0$ is: $i(t = 0) = \frac{dq}{dt} = \frac{\mathrm{q}_{total}\lambda(z_b=0)dz}{dt} = \frac{\mathrm{q}_{total}\lambda(0)vdt}{dt} = \mathrm{q}_{total}\lambda(0)v$

The bunch distribution is normalized, so that the integral of the distribution is 1:

$$1 = \int_{-\infty}^{\infty} \lambda(z_b) dz_b \tag{10}$$

and a drive- or test particle can be represented by the charge:

$$dq(z_b) = q_{total}\lambda(z_b)dz_b \tag{11}$$

where $dq(z_b)$ is the charge as a function of the charge's position in the bunch distribution, q_{total} is the total charge of the bunch, $\lambda(z_b)$ is the distribution of charge inside the bunch and z_b is the position inside the bunch distribution.

The current can now be calculated as:

$$i(t) = \frac{dq(z_b)}{dt} \tag{12}$$

In order to do the Fourier transforms of the voltage drop and the current in equations 9 and 12, all distances must be expressed in terms of time.

The following example will illustrate how to get from distance to time, remembering that there are in fact two types of distances, distances inside the bunch spectrum and distances on the z-axis (since the bunch moves in the z-axis direction). The example consists of a bunch that is made of three drive particles (see figure 5):

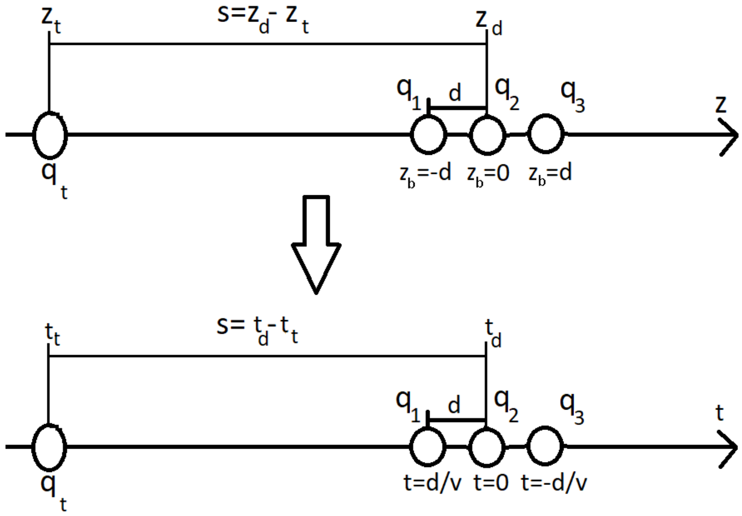

Figure 5: Three charges, q_1, q_2, q_3, represents a bunch distribution that moves in the z direction. The "bunch distribution" has its reference position, $z_b = 0$, at the position of the second particle q_2. At time t=0 this second particle passes a position on the z-axis, which is denominated z_d. The goal is to find out when the two other particles q_1 and q_3 passes this position z_d on the z-axis. The three particles are followed by a test particle q_t, which is at position z_t on the z-axis at time t=0.

There are two lessons to draw from the example distribution in figure 5, and these will later be used to show that the integral in equation 9 represents a convolution. The first lesson is that a drive particle with a **more positive** bunch position has a **more negative** arrival time i.e. an earlier arrival time. F.x. the third particle, q_3, has a position in the bunch distribution that is larger than the the second particle, q_2, but it arrives at a time *before*

the second particle. Specifically it has a position, +d, inside the bunch distribution, but it arrives at position z_d at a time -d/v (where v is the speed of the particles) before particle q_2. Similarly the first particle q_1 has a negative bunch position, -d, but it arrives at z_d at a time +d/v later than the second particle q_2.
The second lesson is that a drive particle with a **more positive** bunch position corresponds to a **more positive** time difference between it and the test particle. Note first that the test particle is at a distance $s_2=z_d-z_t$ behind the q_2 particle and the time difference between it and the test particle is $s_2/v = t_t-t_d$. Then take the example of the q_3 particle that has a larger position in the bunch distribution than the second particle, q_2. The distance between the q_3 particle and the test particle is $s_3=z_d-z_t+d = s_2+d$ — that is larger than s2 — and it also has a larger time difference between it and the test particle, equal to $\Delta t=s_3/v$ or $\Delta t=s_2/v+d/v = t_t-t_d+d/v$.

Using this example, the equations in 9 and 12 can now be expressed in terms of time. First the charge in equation 11 is inserted:

$$\left.\begin{aligned} V_{\|}(s) &= \int_{-\infty}^{\infty} w_f(s+z_b)dq_d(z_b) \\ i(t) &= \frac{dq(z_b)}{dt} \end{aligned}\right\} \Longrightarrow \tag{13}$$

$$\begin{aligned} V_{\|}(s) &= \int_{-\infty}^{\infty} w_f(s+z_b)q_{total}\lambda(z_b)dz_b \\ i(t) &= \frac{q_{total}\lambda(z_b)dz_b}{dt} \end{aligned} \tag{14}$$

Next, the following substitutions are made: $V_{\|,time}(t) = V_{\|}(vt)$, $w_{f,time}(t) = w_f(vt)$, $z_b = vt$ and remembering that a positive position in a bunch corresponds to a negative time (according to the example in figure 5):

$$\left.\begin{aligned} V_{\|,time}(t) &= \int_{-\infty}^{\infty} w_{f,time}(t+t')q_{total}\lambda(vt')vdt' \\ i(t) &= \frac{q_{total}\lambda(-vt)d(vt)}{dt} \end{aligned}\right\} \Longrightarrow \tag{15}$$

$$\begin{aligned} V_{\|,time}(t) &= \int_{-\infty}^{\infty} w_{f,time}(t_s+t')i(-t')dt' \\ i(t) &= q_{total}\lambda(-vt)v \end{aligned} \tag{16}$$

As a final step, the variable t' in the integral is substituted by $-t^*$, which gives $V_{\|,time}(t) = \int_{-\infty}^{\infty} w_{f,time}(t_s - t^*)i(t^*)dt^*$. It can now be seen that the potential is a convolution of the wake field with the current $V_{\|,time}(t) = w_{f,time}(t) \star i(t)$. A Fourier transform is finally done for both voltage drop and the current:

$$\begin{aligned} V_{\|}(\omega) &= \mathcal{F}\left\{w_{f,time}(t)\right\} \cdot \mathcal{F}\left\{i(t)\right\} \\ \mathcal{I}(\omega) &= \mathcal{F}\left\{i(t)\right\} \end{aligned} \tag{17}$$

The impedance is now calculated according to equation 7 as the fraction of the Fourier transforms of the current and voltage

$$\begin{aligned} Z(\omega) &= \frac{\mathcal{F}\left\{w_{f,time}(t)\right\} \cdot \mathcal{F}\left\{i(t)\right\}}{\mathcal{F}\left\{i(t)\right\}} \\ &= \frac{\mathcal{F}\left\{w_{f,time}(t)\right\} \cdot \mathcal{I}(\omega)}{\mathcal{I}(\omega)} \\ &= \mathcal{F}\left\{w_{f,time}(t)\right\} \;\wedge\; \mathcal{I}(\omega) \neq 0 \end{aligned} \tag{18}$$

Here it is shown that the impedance, $Z(\omega)$ is the Fourier transform of the wake function, $w_{f,time}(t)$. Please note that the wake function must be a function in time domain and not a function of distance i.e. not in distance domain. Please note also that if the a bunch passes through the equipment instead of a single particle, then the beam impedance is only measured for the frequencies available in the bunch spectrum i.e. not all frequencies in the equipment impedance $Z(\omega)$. Another problem with a bunch, is that the Fourier coefficients for some frequencies might be so close to zero that the numerical inaccuracies will make the shortening of the fraction $\frac{\mathcal{F}\left\{w_{f,time}(t)\right\} \cdot \mathcal{I}(\omega)}{\mathcal{I}(\omega)}$ imprecise. So, for those bunch frequencies that are close to zero, we cannot calculate the beam impedance for the equipment.

2 Energy loss from the wake function

In this section the energy loss of a bunch of particles that moves through an equipment, will be calculated on the basis of the wake function concept. This type of calculation has been done before, see [6] and [3]. The energy loss is the work done by the bunch, which is defined as the negative integral of the force times distance over the length of the equipment:

$$U_{loss} = -\int_0^L \vec{F} d\vec{r} \tag{19}$$

Where U_{loss} is the energy loss, $\vec{F}$ is the force, $d\vec{r}$ is the distance increment in the direction of the beam and L is the length of the equipment. *Please note that there is a minus sign in front of the integral. The beam impedance works like a friction force, so the integral* $\int_0^L \vec{F} d\vec{r}$ *itself will be negative, so the minus sign is only there to make the energy loss a positive number.* The system is pictured in Figure 6:

The force acting of the test particle is given by the Lorentz force,

$$\vec{F} = \mathrm{q}(\vec{E} + \vec{v} \times \vec{B}) \tag{20}$$

Where $\vec{F}$ is the force, q is the charge that is affected by the force, $\vec{E}$ is the electric field, $\vec{v}$ is the velocity and $\vec{B}$ is the magnetic field. In the present case the velocity is only in the z-direction:

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ v_0 \end{bmatrix} \tag{21}$$

Since the bunch moves in the z-direction, it is only the force in the longitudinal direction that contributes to the energy loss. The vector product of the velocity and the B-field will only be in a direction transverse to the z-direction and therefore can be neglected for the energy loss calculation. In order to calculate the energy loss when the bunch moves through the equipment, the whole bunch is separated into an infinite number of small slices of test charges, and the energy is then the integral of the energy losses of all the slices.

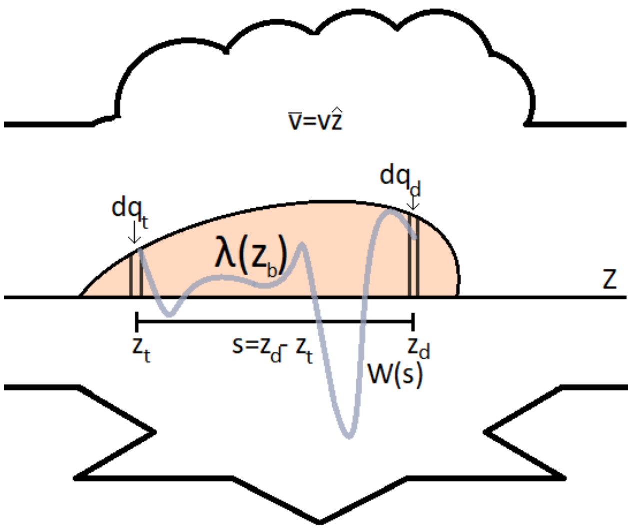

Figure 6: The light orange area depicts a bunch with a linear charge distribution $\lambda(z_b)$. The charges q_d and q_t represents the charges of the drive and test particles, which are positioned at z_d and z_t. The light gray wavy line behind the drive particle, represents the wake field $w_{field}(z_t, s)$ that is generated by the drive particle. s is the distance between the drive and test particles. The bunch moves in the positive z direction. **The thick black lines represents the walls of the equipment.**

The force acting on one small test charge q_t, at position z_t, is:

$$d\vec{F}_z = d\mathrm{q}_t\ E_z(z_t)\hat{z} \tag{22}$$

The energy loss is represented by the work done by the test particle that moves through the equipment i.e. $\mathrm{W} = \int_0^L \mathrm{dw} = -\int_0^L \vec{F}_z dz_t$:

$$dU_{loss} = -\int_0^L d\mathrm{q}_t\ E_z(z_t)dz_t \tag{23}$$

Where L is the length of the equipment, $d\mathrm{q}_t$ is the electric charge of the test particle, dz_t is the small movement of the test particle and $E_z(z_t)$ is the electrical field at the position z_t of the test charge. The energy loss of the whole bunch is then the integral of the energy losses of all the test charges in the bunch.

$$U_{loss} = -\int_{-\infty}^{\infty}\int_0^L d\mathrm{q}_{t,\lambda}\ E_z(z_t)dz_t \tag{24}$$

Here $d\mathrm{q}_{t,\lambda}$ is the electric charge of a test particle in the bunch (the λ index has been added to the charge, in order to be able to refer to different test particles in the bunch and the integral $\int_{-\infty}^{\infty}$ refers to integrating over all the test particles in the bunch), $E_z(z_t)$ is the electrical field at the position of this test particle and dz_t represents the change of the position of this test particle as it moves through the equipment.
The electric field at the position z_t of a test particle, is equal to the sum of all the wake-fields from all the drive particles in the bunch, (see equation 6 in section 1.3):

$$E_z(z_t) = \int_{-\infty}^{\infty} w_{field}(z_t, s)d\mathrm{q}_{d,\lambda} \tag{25}$$

where $d\mathrm{q}_{d,\lambda}$ is the charge of a drive particle in the position z_d (again the λ index refers to the test particle's position in the bunch and the integral $\int_{-\infty}^{\infty}$ refers to integrating over all the particles in the bunch). $w_{field}(z_t, s)$ is the wake field at position z_t, which is due to this drive particle and $s = z_d - z_t$ is the distance from the drive particle to the test particle. This expression for the electrical field can now be substituted in Equation 24:

$$U_{loss} = -\int_{-\infty}^{\infty}\left[\int_0^L d\mathrm{q}_{t,\lambda}\left[\int_{-\infty}^{\infty} w_{field}(z_t, s)d\mathrm{q}_{d,\lambda}\right]\right]dz_t \tag{26}$$

The rigid bunch approximation will now be used to change the order of integration. This step is of great importance and it is not trivial to see the validity of this step. The rigid bunch approximation says that the beam distribution, thereby also the current, is constant as the bunch moves through the device. In other words, the change in particle-particle distance is negligible in terms of the bunch length[1, 2]. Since the bunch charges $d\mathrm{q}_{d,\lambda}$ and $d\mathrm{q}_{t,\lambda}$ are thus independent of z_t, the order of integration can be changed (and the minus sign is moved to the most inner integral):

$$U_{loss} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left[-\int_0^L w_{field,b}(z_t, s)dz_t\right]d\mathrm{q}_{d,\lambda}\right]d\mathrm{q}_{t,\lambda} \tag{27}$$

The most inner integral can now be identified as the wake function for the equipment (see equation 5 in section 1.3):

$$w_f(s) = -\int_0^L w_{field}(z_t, s)dz_t \tag{28}$$

where $w_f(s)$ is the wake function of the equipment. The wake function can now be substituted into equation 27:

$$U_{loss} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} w_f(s) d\mathrm{q}_{d,\lambda} \right] d\mathrm{q}_{t,\lambda} \tag{29}$$

Since the energy loss has to end up being equal to the result from electro-technical theory $U_{loss} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\left|\mathcal{I}(\omega)\right|^2 d\omega$, where both the current $I(w)$ and the impedance $Z(w)$ are functions of frequency, and frequency is related to time domain; then all references to position in the equation must be changed to references to time. The primary components of the equation is the magnitude of the drive and test charges and the distance between them. A natural way of transforming the charges into time is via the current i.e. $\frac{dq}{dt} = i(t)$ and a just as natural is it to transform a distance into a time interval i.e. $\Delta z = v\Delta t$

The change from position to time is done in the following way:

$$\begin{aligned} d\mathrm{q}_{d,\lambda} &= i(t_d)dt \\ d\mathrm{q}_{t,\lambda} &= i(t_t)dt \\ w_{f,time}(t_t - t_d) &= w_f(s) \end{aligned} \tag{30}$$

Please note that a new wakefunction $w_{f,time}(\Delta t)$, which is a function of a time interval, is now defined. Please note also, that the time interval equals $t_t - t_d$ is a positive time interval if the corresponding distance $s = z_d - z_t$ is also positive. The reason for the change of the sequence of indexes i.e. $z_d - z_t$ is changed into $t_t - t_d$ is that a charge in the front of the bunch arrives at an earlier time, i.e. a smaller time value, than a charge in the back of the bunch. (see figure 5 in section 1.4).

The equations in equation 30 are now inserted into the energy loss equation:

$$U_{loss} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} w_{f,time}(t_t - t_d) i(t_d) dt_d \right] \cdot i(t_t)dt_t \tag{31}$$

here t_d and t_t are the times of respectively the drive- and the test particles (NB! both are seen in the lab frame, so there is no need for relativistic calculations) and $i(t_d)$ and $i(t_t)$ are the currents evaluated at these times.

The inner bracket in equation 31 is a convolution (see appendix C). It can be written as $i(t_t) * w_{f,time}(t_t) = \int_{-\infty}^{\infty} w_{f,time}(t_t - t_d)i(t_d)dt_d$. Substituting this gives:

$$U_{loss} = \int_{-\infty}^{\infty} [i(t_t) \star w_{f,time}(t_t)] \cdot i(t_t)dt_t \tag{32}$$

Equation 32 can now be expressed in the frequency domain with the help of the Plancherel theorem $\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)\overline{\mathcal{G}(\omega)}d\omega$ (see appendix D). Since the second function in the Plancehel theorem, i.e. $\overline{g(t)}$ in time domain and $\mathcal{G}(\omega)$ in frequency domain, is a complex conjugated function, the second function $i(t_t)$, i.e. the current, in equation 32 must also be a complex conjugated function. Since the current is a real valued function, it is equal to the complex conjugate of itself i.e. $i(t_t) = \overline{i(t_t)}$, and the Plancherel theorem can be applied:

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\left\{ i(t_t) \star w_{f,time}(t_t) \right\} \cdot \overline{\mathcal{F}\left\{ i(t_t) \right\}} d\omega \tag{33}$$

The Fourier transform of the convolution of the current with the wake function is equal to the Fourier transform of the two functions multiplied with each other, according to the convolution theorem (see appendix C):

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\left\{ i(t_t) \right\} \cdot \mathcal{F}\left\{ w_{f,time}(t_t) \right\} \cdot \overline{\mathcal{F}\left\{ i(t_t) \right\}} d\omega \tag{34}$$

The product of the a number with it's conplex conjugate is equal to the square of the amplitude of the number i.e. the amplitude of the Fourier transform of the current $|\mathcal{F}\left\{ i(t_t) \right\}|$ will be squared:

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\left\{ w_{f,time}(t_t) \right\} \cdot \left| \mathcal{F}\left\{ i(t_t) \right\} \right|^2 d\omega \tag{35}$$

As shown in Section 1.4 the Fourier transform of the wakefunction is the impedance. Through this result Equation 35 can be transformed (the Fourier transform of the current will here be written as $\mathcal{I}(\omega)$):

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) \left| \mathcal{I}(\omega) \right|^2 d\omega \tag{36}$$

From here it is obvious that this is identical to the classical circuit theory approach which states (see equation 43 in section 3). The above result in equation 36 is also identical to the result derived in [6] and [3]):

$$\Delta E = -\frac{q_{bunch}^2}{2\pi} \int_{-\infty}^{\infty} Z(\omega) \left| \Lambda(\omega) \right|^2 d\omega \tag{37}$$

Where ΔE is the change of energy when the bunch moces through the equipment, q_{bunch} the total charge of the bunch, $Z(\omega)$ the beam impedance and $\Lambda(\omega)$ is the bunch spectrum.

The bunch spectrum is defined as the Fourier transformation of the line density times the speed the particles $\Lambda(\omega) = \mathcal{F}(\lambda(vt) \cdot v)$ (see equation 11 in section 1.4 for a definition of the line density $\lambda(z)$). The bunch spectrum times the bunch charge therefore turns out to be equal to the complex conjugate of the Fourier transform of the bunch current:

$$q_{bunch} \cdot \Lambda(\omega) = q_{bunch} \cdot \mathcal{F}(v \cdot \lambda(vt)) = \mathcal{F}\left(q_{bunch} \cdot v \cdot \lambda(vt) \right) = \mathcal{F}\left(i(-t) \right) = \overline{\mathcal{I}(\omega)} \tag{38}$$

The equality $\mathcal{F}\left(i(-t) \right) = \overline{\mathcal{I}(\omega)}$ is true because i(t) is a real function.

Thus, because $q_{bunch}^2 \cdot |\Lambda(\omega)|^2 = |\overline{\mathcal{I}(\omega)}|^2 = |\mathcal{I}(\omega)|^2$, it has been shown that equations 36 and 37 are identical.

3 The energy loss through the lumped impedance model

Power loss, as a function of time, is defined as:

$$P(t) = V(t)I(t) \tag{39}$$

where P(t) is the power, I(t) is the amplitude of the current and V(t) is the amplitude of the voltage. All three terms are functions of time and $P(t)$ therefore gives the power of the system at a specific time. To find the energy loss, the power must be integrated over time from $-\infty$ to ∞:

$$U_{loss} = \int_{-\infty}^{\infty} V(t)I(t)\ dt \tag{40}$$

Through Plancherel's theorem, which is derived in Appendix D, this can be described in the frequency domain. Plancherel's theorem requires one of the factors to be a complex conjugate. Since both $I(t)$ and $V(t)$ are real value functions, any of them can be complex conjugated without changing the expression e.g. $\overline{I(t)} = I(t)$. The Plancherel transformation will then yield:

$$U_{loss} = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\overline{\mathcal{I}(\omega)}d\omega \tag{41}$$

Ohms law (in frequency domain) will now be used to substitute the voltage:

$$V(\omega) = \mathcal{I}(\omega)Z(\omega) \tag{42}$$

The energy loss now takes the form:

$$\begin{aligned} U_{loss} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{I}(\omega)Z(\omega)\overline{\mathcal{I}(\omega)}\ d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)|\mathcal{I}(\omega)|^2\ d\omega \end{aligned} \tag{43}$$

This result for the energy loss is equal to the one found through the wake function method in section 3. In equation 43, it does not matter if the impedance $Z(\omega)$ is replaced with the real part of the impedance $Re\{Z(\omega)\}$, because the complex part of the impedance has opposite sign for positive and negative frequencies (as the impedance is the Fourier transform of a real valued function) and the complex parts of the integral will eliminate each other.

Please note that normally one sees the above equation (43) in the following form (without the factor $\frac{1}{2\pi}$):

$$U_{loss} = \int_{-\infty}^{\infty} Z(\omega)|\mathcal{I}(\omega)|^2\ d\omega \tag{44}$$

The reason is that the Fourier transform normally used to calculate $I(w)$ is $\mathcal{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i(t)e^{-i\omega t}dt$ instead of the one used in this report $\mathcal{I}(\omega) = \int_{-\infty}^{\infty} i(t)e^{-i\omega t}dt$ (see appendix A).

3.1 Energy Loss Through the Lumped Impedance model - Example

The energy loss for a system with lumped components is given as:

$$U_{loss} = \int_{-\infty}^{\infty} I(t)V(t)dt \tag{45}$$

where U is the energy loss, I(t) is the current as a function of time, t, and V(t) is the potential drop as a function of time. Assuming that the lumped impedance is constant, then $V(t) = R\ I(t)$ from Ohms law, and the energy loss can now also be calculated in frequency domain from equation 43:

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R\ |I(\omega)|^2\ d\omega \tag{46}$$

An example of the calculations in respectively time and frequency domain, will be done in the following:

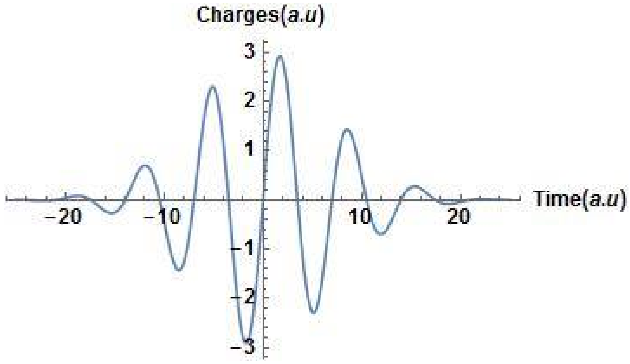

Figure 7: An example of a bunch, represented by the current $I(t) = 3sin(\omega_0 t)e^{-0.01t^2}$, plotted in Wolfram Mathematica 11.2. The value of the current is extremely close to zero for $|i(t)| > 25$. Both the time and current is in arbitrary units (a.u)

In this example the current, in time domain, will be chosen as:

$$I(t) = 3sin(\omega_0 t)e^{-0.01t^2} \tag{47}$$

Where ω_0=0.6. For this configuration, the parameters of the equipment are chosen as

$$R = 100\Omega \quad \wedge \quad L = 1mH \tag{48}$$

This should be evaluated over an infinite range but from Figure 7, which is a plot of the current, reasonable boundaries can be chosen. From the graph it can be seen that the bunch distribution is tending to zero around ±25. Therefore the boundaries for Equation 46, are chosen to be ±25. Equation 2 then becomes

$$U_{loss} = \int_{-25}^{25} \left(3sin(\omega_0 t)e^{-0.01t^2}\right)^2 100\Omega dt = \underline{\underline{5639.91J}} \tag{49}$$

It will now be shown that this is equal to the standard electro-technical method from equation 46 i.e. the energy loss will now be calculated in frequency domain:

$$U_{loss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R\ |I(\omega)|^2 d\omega \tag{50}$$

First of all a Fourier transform must be performed on the current

$$I(\omega) = \mathcal{F}\left\{I(t)\right\} = 6 \cdot 10^{-3} i e^{-25\omega^2} sinh(30\omega) \tag{51}$$

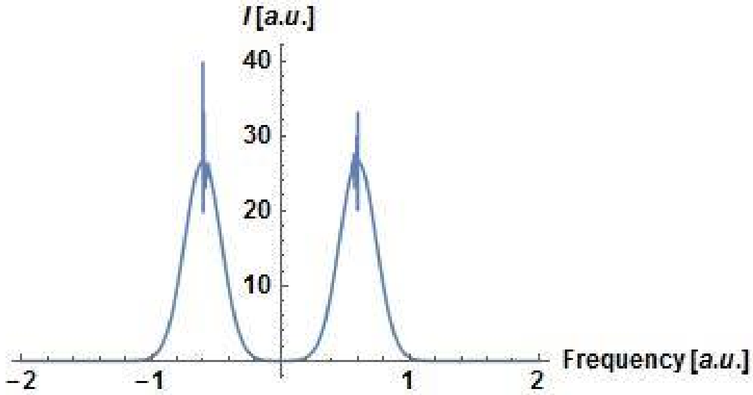

Figure 8: The absolute squared Fourier transform of the current $|I(\omega)|^2$ from figure 7. The plot is done in Wolfram Mathematica 11.2. $|I(\omega)|^2$ is in praxis zero for $|\omega| > 1$.

Next, to make the computation doable, boundaries needs to be chosen from Figure 8. The frequency ω boundaries are chosen to be ± 1 and the calculation is done in Wolfram Mathematica 11.2:

$$\begin{aligned} U &= \frac{1}{2\pi}\int_{-1}^{1} |6 \cdot 10^{-3} i e^{-25\omega^2} sinh(30\omega)|^2 d\omega \\ &= \underline{\underline{5639.73 J}} \end{aligned} \tag{52}$$

As the energies are practically identical for the two types of calculation in equations 49 and 52, it has now been illustrated that to calculate the energy loss in time- or frequency domain will always give the same outcome.

Part I
Appendix

A Notes on notation

In this note different ambiguous terms are used, and to avoid confusion a these terms will be identified, and defined for the course of this note.

The Fourier transform will be defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{53}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{54}$$

Where the important change is to note the scaling constant.
The notation of charge propagating through a chamber will be separated in 2 different cases.

The wake field associated with the trajectory for a single particle, will be the electric field, E, normalized with the charge of the drive particle, q_d, seen by the test particle located in z_t

$$w_{field}(z_t, s) = \frac{E(z_t, s)}{q_d} \tag{55}$$

The wake function, $w_f(s)$, is the negative integral over the length of the chamber, of the wake field of a single particle:

$$w_f(s) = -\int_0^L w_{field}(z_t, s)dz \tag{56}$$

where L is the length of the chamber and z is the direction of propagation.

Similarly to the wake function for a single particle, also the wake potential of a bunch $w_p(s)$ of particles is defined as the negative integral of the wake field of the bunch over the length of the chamber:

$$w_p(s) = -\int_0^L w_{field,bunch}(z_t, s)dz_t \tag{57}$$

where L is the length of the chamber, $w_{field,bunch}(z_t, s)$ is the wake field of the bunch i.e. the electrical field $E_z(z_t, s)$ divided by the charge of the whole bunch $q_{d,bunch}$ i.e. $w_{field,bunch}(z_t, s) = \frac{E(z_t,s)}{q_{d,bunch}}$ and z is the direction of propagation. The electrical field from a whole bunch was defined in equation 6 in section 1.3

$$E_z(z_t, s) = \int_{-\infty}^{\infty} w_{field}(z_t, s)d\mathrm{q}_{d,\lambda} \tag{58}$$

B Impulse response

A signal, x(t) through a system, h(t), will give an output y(t), see figure 9. The output y(t) can be calculated by a convolution of the input function x(t) and the system response function h(t):

$$y(t) = h(t) \star x(t)$$

The convolution is calculated as following:

$$y(t) = \int_{-\infty}^{\infty} h(t^{'})x(t - t^{'})dt^{'}$$

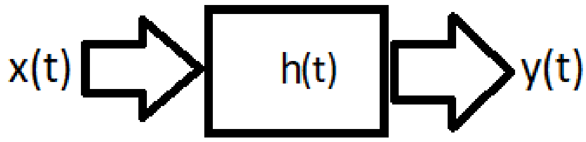

Figure 9: A signal x(t) is exciting a system with the response function h(t). The output signal is y(t).

Notice, if the input function is a Dirac delta function i.e. $x(t) = \delta(t)$, then the output, $y(t)$, will be the response function, $h(t)$, for the system itself

$$y(t) = \int_{-\infty}^{\infty} h(t^{'})\delta(t - t^{'})dt^{'} = h(t)$$

It is now clear that any system is described its impulse response. This is indeed the standard way of describing a system response function [4]:

$$\boxed{y(t) = h(t) \qquad \text{if} \quad x(t) = \delta(t)}$$

That the system response function is defined as its impulse response can also be seen in the Fourier domain

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

where $Y(\omega)$ is the output function in the frequency domain, $H(\omega)$ is the system response function in the frequency domain and $X(\omega)$ is the input function in the frequency domain. H(ω) is multiplied with $X(\omega)$ to get the output, $Y(\omega)$.
If the input function is a Dirac delta function, then its Fourier transform will be equal 1, see Figure 10 i.e. $X(\omega) = 1$ if $x(t) = \delta(t)$ and the system response is thus equal to the impulse response:

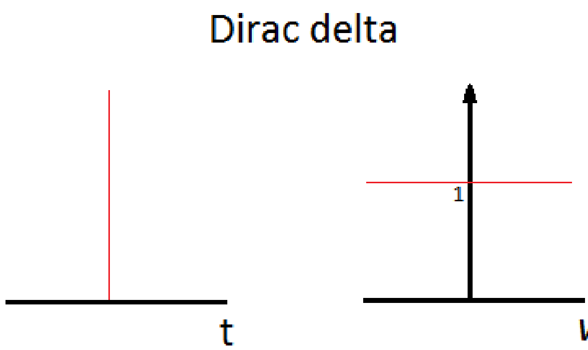

Figure 10: The Fourier spectrum of a Dirac delta pulse is flat i.e. it contains all frequencies and each frequency have the same Fourier amplitude (=1).

$$\boxed{Y(\omega) = H(\omega) \qquad \text{if} \quad x(t) = \delta(t)}$$

From here it can be seen that the Dirac delta function excites all possible frequencies of the system and that a system is completely defined by it's impulse response i.e. it's response to the Dirac delta function.

C The convolution theorem

To properly being able to evaluate Fourier transforms, the convolution theorem is important to keep in mind. It states that the Fourier transform of a convolution is the product of the individual functions transformed

$$\mathcal{F}\left\{(f \star g)(t)\right\} = \mathcal{F}\left\{f(t)\right\} \cdot \mathcal{F}\left\{g(t)\right\} = F(\omega) \cdot G(\omega) \tag{59}$$

To prove the convolution theorem, first the definition of a convolution is used:

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(p) g(t-p) dp \tag{60}$$

$g(t-p)$ is now substituted by its inverse Fourier transform $g(t-p) = \int_{-\infty}^{\infty} \mathcal{G}(\omega) e^{i\omega(t-p)} d\omega$:

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(p) \left[\int_{-\infty}^{\infty} \mathcal{G}(\omega) e^{i\omega(t-p)} d\omega\right] dp \tag{61}$$

The order of integration is now interchanged:

$$(f \star g)(t) = \int_{-\infty}^{\infty} \mathcal{G}(\omega) \left[\int_{-\infty}^{\infty} f(p) e^{i\omega(t-p)} dp\right] d\omega \tag{62}$$

$e^{i\omega(t-p)}$ is now separated into the product $e^{i\omega t} \cdot e^{-i\omega p}$:

$$(f \star g)(t) = \int_{-\infty}^{\infty} \mathcal{G}(\omega) \left[\int_{-\infty}^{\infty} f(p) e^{-i\omega p} dp\right] e^{i\omega t} d\omega \tag{63}$$

The inner bracket is identified as the Fourier transform of $f(t)$, $\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(p) e^{-i\omega p} dp$:

$$(f \star g)(t) = \int_{-\infty}^{\infty} \mathcal{G}(\omega) \cdot \mathcal{F}(\omega)\ e^{i\omega t} d\omega \tag{64}$$

Finally, the right hand side is identified as the inverse Fourier transform of the product of the functions $\mathcal{G}(\omega) \cdot \mathcal{F}(\omega)$

$$(f \star g)(t) = \mathcal{F}^{-1}\left[\mathcal{G}(\omega) \cdot \mathcal{F}(\omega)\right] \tag{65}$$

Taking the Fourier transform on both sides, gives the convolution theorem:

$$\mathcal{F}\left[(f \star g)(t)\right] = \mathcal{F}(\omega) \cdot \mathcal{G}(\omega) \tag{66}$$

D The Plancherel theorem

The Plancherel theorem states that the integrated inner product of two functions is equal to the integrated inner product of their Fourier transforms:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)\overline{\mathcal{G}(\omega)}d\omega \tag{67}$$

where $\mathcal{F}(\omega)$ and $\mathcal{G}(\omega)$ are the Fourier transforms of the functions $f(t)$ and $g(t)$ and $\overline{g(t)}$ and $\overline{\mathcal{G}(\omega)}$ denotes the complex conjugate of the functions $g(t)$ and $\mathcal{G}(t)$.

The first step in the proof of the Plancherel theorem is to make the following substitutions:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega t}d\omega \\ g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{G}(\omega)e^{i\omega t}d\omega \end{aligned} \tag{68}$$

The Plancherel equation 67 now becomes:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega t}d\omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathcal{G}(\omega')}e^{-i\omega' t}d\omega'\right]dt \tag{69}$$

in the above equation is used that $\overline{e^{i\omega t}} = e^{-i\omega t}$.

Because of the "Constant factor rule in integration" i.e. $\int a dx = a\int dx$, the above equation can be re-written in the following way:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \mathcal{F}(\omega)\overline{\mathcal{G}(\omega')}\left[\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt\right]d\omega\right]d\omega' \tag{70}$$

A special equation, derived in the footnote[1] will be applied to the most inner bracket:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \mathcal{F}(\omega)\overline{\mathcal{G}(\omega')}\delta(\omega-\omega')d\omega\right]d\omega' \tag{71}$$

The integration over $d\omega'$ only yield a result for $\omega' = \omega$ and Plancherels theorem is derived:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\overline{\mathcal{G}(\omega)}d\omega \tag{72}$$

[1] The special equation: $\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt$ can be derived from the inverse Fourier transform of the Dirac delta function: $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\ e^{i\omega t}d\omega$ The variable names ω and t will now be interchanged. (*This is allowed because the Fourier transform is just a mathematical transformation from one domain into another - the time and frequency domains have no special significance*): $\delta(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\ e^{i\omega t}dt$. Finally ω is substituted with $(\omega-\omega')$ and the special equation is obtained.

E The frequency-domain definitions of current and voltage in electro-technical theory and beam-impedance theory are different

The definition of current (in frequency domain) in electro-technical theory is the amplitude of the time domain current; while in beam impedance theory it is the Fourier transform of the time domain current:

$$\mathcal{I}_{electro-technical}(\omega_0) = I_0 \qquad , \text{ where } i(t) = I_0 Cos(\omega_0 t)$$

$$\mathcal{I}_{beam-impedance}(\omega_0) = \int_{-\infty}^{\infty} i(t) e^{-i\omega t} dt \text{ , where } i(t) = I_0 Cos(\omega_0 t)$$
$$= \pi I_0 \delta(w - w_0) + \pi I_0 \delta(w + w_0))$$

The two definitions are used for different purposes. The electro-technical definition is only used for single-frequency signals i.e. for currents or voltages that are sine waves, while the beam-impedance definition is mostly used for currents or voltages that are shaped like a pulse:

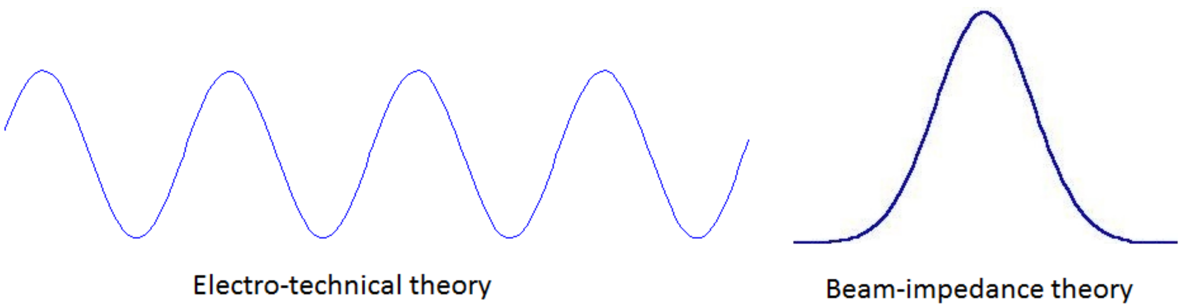

Figure 11: Current and voltage shapes in electro-technical theory are always sine waves; while in beam-impedance theory, current or voltage are mostly pulse shaped

Both theories use the same impedance definition:

$$R(\omega) = \frac{V(\omega)}{I(\omega)}$$

However, for beam-impedance theory, the sign of the imaginary part of the impedance is dependent of the Fourier transform that is used.

Let's illustrate this by a small example with an R,L circuit on the next page.

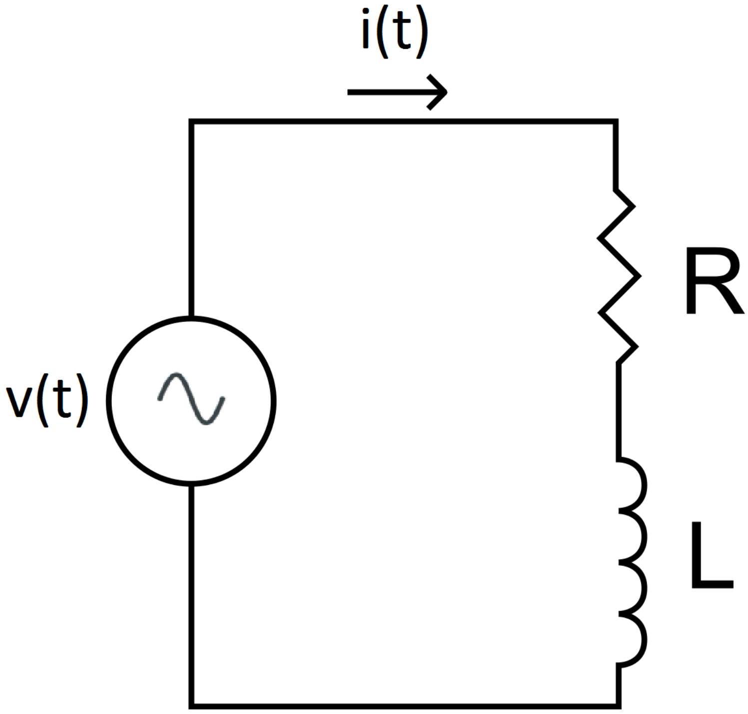

Figure 12: R,L circuit that illustrates the difference in impedance between electro-technical and beam-impedance theory.

From electro-technical theory, the impedance is:

$$Z(\omega) = R + \boldsymbol{j}\omega L$$

While, for beam-impedance theory, one must take the Fourier transform of the network equation in time domain and from that calculate the impedance $Z(\omega)$. Please note that the Fourier transform $I(\omega) = \int_{-\infty}^{\infty} i(t)e^{-\boldsymbol{i}\omega t}dt$ has a minus sign in the exponential i.e. $e^{-\boldsymbol{i}\omega t}$:

$$\begin{aligned} v(t) &= R \cdot i(t) + L \cdot \frac{di(t)}{dt} && \Longleftrightarrow \\ V(\omega) &= R \cdot I(\omega) - \boldsymbol{i}\omega L \cdot I(\omega) && \Longleftrightarrow \end{aligned}$$

$$\begin{aligned} Z(\omega) &= \frac{V(\omega)}{I(\omega)} = \frac{R \cdot I(\omega) - \boldsymbol{i}\omega L \cdot I(\omega)}{I(\omega)} \\ &= R - \boldsymbol{i}\omega L \end{aligned}$$

Had one used the American way of making a Fourier transform $I(\omega) = \int_{-\infty}^{\infty} i(t)e^{+\boldsymbol{j}\omega t}dt$, which has a plus sign in the exponential i.e. $e^{+\boldsymbol{j}\omega t}$ (and also use a $\boldsymbol{j}$ instead of an $\boldsymbol{i}$ for the imaginary number), then the impedance from beam-impedance theory would have been identical to the electo-technical definition:

$$Z(\omega) = R + \boldsymbol{j}\omega L$$

The calculation of energy loss in beam-impedance theory is compatible with the electro-technical definition:

$$U_{loss,beam-impedance} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\{Z(w)\}\ |I(w)|^2 d\omega$$

$$\begin{aligned} U_{loss,electro-technical} &= \int_{-\infty}^{\infty} R(\omega)\ i(t)^2 dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)\ |I(\omega)|^2 d\omega \end{aligned}$$

where this second equation for the energy loss is derived with the help of the Plancherel theorem. Please note that $I(\omega)$ is the Fourier transform of $i(t)$

One should note that the equation $U_{loss,electro-technical} = \int_{-\infty}^{\infty} R(\omega)\ i(t)^2 dt$ is questionable, because it mixes together quantities that are defined in one another's exclusive domains. $R(\omega)$ is defined in frequency domain while $i(t)$ is defined in time domain. The equation only works because $R(\omega)$ is a constant for all frequencies.

The electro-technical definition of current is unique because it can be used to easily calculate the effect (for one time period T):

$$\begin{aligned} P_T &= V_{effective} \cdot I_{effective} \\ &= \frac{1}{2}\ V_0 \cdot I_0 \end{aligned}$$

, where $V_{effective} = \frac{1}{\sqrt{2}}V_0$, $I_{effective} = \frac{1}{\sqrt{2}}I_0$ and the time period is $T = \frac{2\pi}{\omega}$

One can of course do the same calculation based on beam-impedance theory, but this is more complicated. To calculate the effect, one needs to cut out one time period See figure 13, calculate the energy loss for this time period and then divide this by the length of the time period. The result from electro-technical theory and beam-impedance theory of course yields the same results.

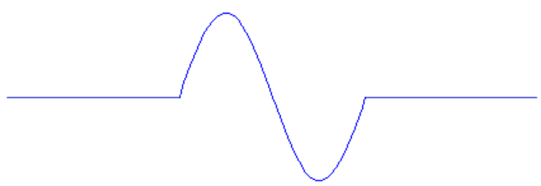

Figure 13: When calculating the effect for one time period, with the help of beam-impedance theory, the current shape consists of a zero current followed by one time period and finished by zero current.

In the following example, the effect (for one period) is calculated with the beam-impedance theory. To make matters simple, it is assumed that the impedance $Z(\omega)$ is a constant R:

$$
\begin{aligned}
P_T &= \frac{U_{loss,beam-impedance}}{T} \\
&= \frac{1}{T} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} Re\{Z(w)\}\ |I(w)|^2 d\omega \\
&= \frac{R}{2\pi T} \int_{-\infty}^{\infty} \ |I(w)|^2 d\omega
\end{aligned}
$$

This calculation of the integral has been done with *Mathematica*®:

```
Clear[f]; $Assumptions = T > 0 && I0 > 0;
f[t_] := Piecewise[ {{0, t < 0}, {I0 * Sin[2 π * t / T], 0 ≤ t ≤ T}, {0, t > T}}]
```

```
√(2 π) FourierTransform[f[t], t, w]
```

$$-\frac{2\left(-1+e^{i T w}\right) I0\, \pi\, T}{4\pi^2 - T^2 w^2}$$

$$\int_{-\infty}^{\infty} \mathrm{Abs}\left[-\frac{2\left(-1+e^{i T w}\right) I0\, \pi\, T}{4\pi^2 - T^2 w^2}\right]\text{^}2\, dw$$

$$I0^2\, \pi\, T$$

And the result of calculating the effect with beam-impedance theory then, as expected, gives exactly the same result as the electro-technical theory:

$$
\begin{aligned}
P_T &= \frac{R}{2\pi T} * (I_0^2 \pi T) \\
&= \frac{1}{2} R I_0^2 \\
&= \frac{1}{2}\, V_0 \cdot I_0
\end{aligned}
$$

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