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frank.schmidt@cern.ch

Higher Order Dispersion and Momentum Compaction in MAD-X/PTC using NormalForm

F. Schmidt, P. Skowronski [*],
V. Lebedev, A. Valishev [†]

[*]CERN, CH 1211 Geneva 23, Switzerland
[†]Fermilab, P.O. Box500, Batavia 60510, IL

ABSTRACT

Abstract
Dispersion and the momentum compaction factor have been treated since ages in the literature of accelerator physics to first order. Since quite some time dispersion and the momentum compaction factor can be derived via PTC and thereby in MAD-X to arbitrary order (presently hard-wired to fourth order in MAD-X). It was realized that little is known about this high order derivation in the community. This paper attempts to clarify how the higher order momentum compaction factors are derived from first principle allowing for an alternative way to calculate them.

Keywords: MAD-X, General Accelerator Theory, Normal Form, Beam Optics

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Contents

1 Introduction

Dispersion and the momentum compaction factor of higher order have been of interest at many machines, E.G. Ref. [1]. There have also been theoretical attempts for a better understanding (see Ref. [2, 3]).

After establishing codes that can deal with very high order maps representing the accelerator lattice with non-linearities, this should now be settled. In fact, the Polymorphic Tracking Code (PTC) by É. Forest has been part of MAD-X since almost 15 years by now. At CERN it is being used increasingly. The physics of the PTC code but also many of its essential components like the Full Polymorphic Package (FPP), Differential Algebra, Lie Algebra and NormalForm are well described in Forest's books [4, 5]. There is also extensive documentation available for the PTC modules in the MAD-X code [6].

Nevertheless, there remains a gap between the textbooks on accelerator physics and the extended capability of a more modern code like PTC. We therefore believe that the accelerator community would profit as a whole by taking first principal formulae of fundamental accelerator physics concepts and compare them with higher order treatment à la PTC.

The goal of this report is three-fold: first to discuss higher order dispersion and momentum compaction from a theoretical point of view and second to follow it up for the CERN Proton Synchrotron (PS), the Fermilab Booster and Main Injector as examples of existing accelerators with $\beta_0 < 1$. Last, an extensive treatment of the smooth lattice approximation allows for an analytical comparisons with the MAD-X/PTC results.

2 Theoretical Aspects of Dispersion & Momentum Compaction

The fundamental relation between path-length L, the time T and the particle velocity v is:

$$T = L/v. \tag{1}$$

In the literature [7–9] the first order momentum compaction factor α_c is readily derived:

$$\alpha_c = \frac{1}{\gamma_{tr}^2}, \tag{2}$$

where γ_{tr}, an optics property of each ring, refers to that γ_0 where the transition takes place between earlier arrival time for increasing γ_0 ($\gamma_0 < \gamma_{tr}$) and later arrival for ($\gamma_0 > \gamma_{tr}$). This is of course due to the relativistic dynamics and the fact that we consider closed ring accelerators. The relation between α_c and phase-slip factor η is:

$$\alpha_c = \frac{1}{\gamma_0^2} + \eta, \tag{3}$$

while α_c together with γ_{tr} are properties of the ring optics, η is energy dependent and becomes zero at gamma transition γ_{tr}.

Going beyond the first order derivation a treatment with high order maps becomes advantageous. In PTC we can treat a non-linear accelerator with arbitrary high order maps. In particular, with a NormalForm decomposition of such a map M up to order n_o:

$$M = A_1^{-1} \circ A_2^{-1} \circ A_{3-n_o}^{-1} \circ R \circ A_{3-n_o} \circ A_2 \circ A_1, \tag{4}$$

with A_{3-n_o} holding non-linear distortions from order 3 till n_o and R as the amplitude dependent rotation. In lowest order we have matrices:

$$\mathcal{M} = \mathcal{A}_1^{-1} \circ \mathcal{A}_2^{-1} \circ \mathcal{R} \circ \mathcal{A}_2 \circ \mathcal{A}_1, \tag{5}$$

with $\mathcal{R}$ as the linear rotation by the tunes of the accelerator lattice. Please note that the matrix $\mathcal{A}_1$ carries the first order dispersion values, while the map A_1 includes dispersion terms up to arbitrary orders. On the contrary, for the second order terms the matrix and map are the same $A_2 = \mathcal{A}_2$ and hold the lattice dependent Twiss parameters at the longitudinal position s where the map M has been determined.

Coming back to the path-length we can express $\frac{\Delta L}{L_o}$ as a Taylor series to order n_o:

$$\frac{\Delta L}{L_o}(\delta) = \sum_{n=1}^{n_o} \frac{1}{n!} \frac{\partial^n (\frac{\Delta L}{L_o})}{\partial \delta^n} \cdot \delta^n, \tag{6}$$

where $\delta = \frac{p-p_0}{p_0}$, $\alpha_c = \frac{\partial(\frac{\Delta L}{L_o})}{\partial \delta}$, $\alpha_c' = \frac{\partial^2(\frac{\Delta L}{L_o})}{\partial \delta^2}$, $\alpha_c'' = \frac{\partial^3(\frac{\Delta L}{L_o})}{\partial \delta^3}$ etc.

Let us now present Eq. 1 in a more convenient form:

$$cT(p_t) \cdot \beta(p_t) = L(\delta), \tag{7}$$

where we use the fact that the variables p_t and δ can be expressed as a function of the other variable respectively (see Eq. 9 and Eq. 10).

At this point we need a short digression to discuss the coordinate system in MAD-X/PTC. On the left hand side we find cT and the relativistic β as a function of p_t, which is the canonical momentum coordinate of the longitudinal plane. In MAD-X and in PTC (with the "time" parameter set to "true") the canonical coordinates of the longitudinal phase space are:

$$(cT - cT_0, p_t = \frac{E - E_0}{p_0 \cdot c} = \frac{E - E_0}{E_0 \cdot \beta_0}), \tag{8}$$

In absence of errors the "time" coordinate $cT - cT_0$ becomes zero for the synchronous particle that does not exercise synchrotron oscillations since it arrives in the center of the cavity without accelerating voltage. In the general case, $cT - cT_0$ becomes zero with respect to the 6D closed-orbit.

The necessary canonical choice of p_t has the considerable disadvantage that all dispersive parameters like dispersion are different by factors of the relativistic β_0 causing confusion to many MAD-X users working for small machines where we typically find $\beta_0 < 1$.

The $p_t(\delta)$ and $\delta(p_t)$ variables can be expressed in the following way:

$$p_t(\delta) = \sqrt{(1+\delta)^2 + \frac{1-\beta_0^2}{\beta_0^2}} - \frac{1}{\beta_0} \tag{9}$$

$$\delta(p_t) = \sqrt{1 + 2\frac{p_t}{\beta_0} + p_t^2} - 1 \tag{10}$$

The relativistic β at any energy can be expressed either in p_t or δ:

$$\beta(p_t) = \frac{\beta_0\sqrt{1+2\frac{p_t}{\beta_0}+p_t^2}}{1+p_t\beta_0} \tag{11}$$

$$\beta(\delta) = \frac{\beta_0(1+\delta)}{\sqrt{1+\beta_0^2\delta(2+\delta)}} \tag{12}$$

On the right hand side of Eq. 7 we find the path-length L as a function of δ. PTC is offering an alternative set of canonical variables by setting "time=false":

$$(L-L_o,\delta). \tag{13}$$

In this case L_o is the nominal length of the machine, L the path-length, $L-L_o$ is zero without errors and for the synchronous particle and dispersive parameters like dispersion will have the calculated values. This is also the ideal set of canonical variables for the momentum compaction factor since by definition it is the δ derivation of ΔL divided by L_o.

Before we can proceed to calculate α_c and η in higher orders it is mandatory first to transform the one-turn map M onto the δ dependent closed-orbit because the change of the path-length ΔL must be determined with the respect to the closed-orbit. PTC offers to produce a 6D one-turn map with the cavity switched off via "icase=56" (see Annex A for the details). For the sake of the argument we will use an example matrix (of Eq. 14) without coupling, but obviously PTC can do the full case with coupling between all planes. Therefore, the cross matrix parts like R_{13}, R_{14}, R_{23}, R_{24}, are equal to zero. Moreover, the argument holds for both sets of coordinates, except that one gets the true dispersion for $(L-L_o,\delta)$ only.

We start with the traditional matrix $\mathcal{M}$ representation. [1] Note that due to the symplectic nature of the matrix there are also the terms R_{51} and R_{52} besides the dispersive terms R_{16} and R_{26}.

$$\mathcal{M} = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

The first order NormalForm Transformation A_1 is non-linear in the sense that the dispersion terms up to the specified maximum order are included in A_1. Here we need the linear matrix part:

$$\mathcal{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & D_x \\ 0 & 1 & 0 & 0 & 0 & D'_x \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -D'_x & D_x & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1, \end{pmatrix} \tag{15}$$

and its inverse matrix:

$$\mathcal{A}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -D_x \\ 0 & 1 & 0 & 0 & 0 & -D'_x \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ D'_x & -D_x & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{pmatrix} \tag{16}$$

[1] PTC does that in a peculiar way, where the component 5 and 6 are exchanged. As a consequence all components of the line 5 (i.e. relevant for the longitudinal coordinate $cT-cT_0$ or $L-L_o$) change sign, except for $R_{55}=1$.

The significance of the red values seen in matrix 14 and 15 are explained below. When the similarity transformation is carried out the dispersive terms R_{16} and R_{26} but also the R_{51} and R_{52} vanish from the resulting matrix $\hat{\mathcal{M}}$.

$$\hat{\mathcal{M}} = \mathcal{A}_1^{-1} \cdot \mathcal{M} \cdot \mathcal{A}_1 = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \hat{R}_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

You will also notice that the "56" component $\hat{R}_{56}$ has changed, in fact it is the result of the matrix dot product of the line and column with red values in matrix 14 and 15:

$$\hat{R}_{56} = R_{56} + R_{51} \cdot D_x + R_{52} \cdot D'_x \tag{18}$$

At higher orders this transformation will eliminate all purely dispersive terms and via symplecticity also the corresponding terms in the 5^{th} component of the map.

After this transformation on top of the dispersive closed-orbit we may go back to Eq. 7. On the right hand side, i.e. with δ as the longitudinal momentum, the momentum compaction factor α_c can be calculated in the natural way by using the full map M to the desired order. After the similarity transformation with the high order A_1, we may read out the ΔL components to the same order. Last, a division by L_o provides α_c up to the specified order.

For the left hand side, i.e. with p_t as the longitudinal momentum, the situation is a bit more complicated. The single variable map

$$N(p_t) = cT(p_t) \cdot \beta(p_t), \tag{19}$$

where in fact $cT(p_t)$ is the 5^{th} vector of the non-linear map of Eq. 4, after the transformation, will give the η components to the specified order but typically we would rather want the α_c components. We therefore need to make a coordinate transformation from p_t to δ. To this end, we make use of Eq.9 and Eq.12, expand into a Taylor series and simplify.

In first order, we get in Eq. 20 the expression for α_c

$$\alpha_c = 1 - \beta_0^2 - \beta_0^2 \cdot cT^1/L = \frac{1}{\gamma_0^2} + \eta, \tag{20}$$

with cT components expressed order by order as:

$$cT^n = \frac{1}{n!}\frac{\partial^n(cT)}{\partial p_t^n}, \tag{21}$$

and with η given by:

$$\eta = -\beta_0^2 \cdot cT^1/L. \tag{22}$$

Plugging Eq. 17 and Eq. 18 one gets:

$$\eta = -\beta_0^2 \cdot (R_{56} + R_{51} \cdot D_x + R_{52} \cdot D'_x)/L. \tag{23}$$

It is interesting to note that η is calculated in MAD-X in the exact same way (see Annex C).

Last, expanding map (Eq. 19) to higher orders we get the transformations from the η-like terms to α_c in $2^{nd}, 3^{rd}, 4^{th}$ orders, i.e. to the order it is presented implemented in the "ptc_twiss" module in MAD-X. Terms to arbitrary high order are readily available.

$$\alpha_c' = \beta_0^2\{3(-1+\beta_0^2)(1 - cT^1/L) + 2\beta_0 \cdot cT^2/L\} \tag{24}$$

$$\begin{aligned}\alpha_c'' = 3\beta_0^2\{&(-1+6\beta_0^2-5\beta_0^4)(1-cT^1/L)+\\ &+4(\beta_0-\beta_0^3)cT^2/L+2\beta_0^2\cdot cT^3/L))\}\end{aligned} \tag{25}$$

$$\begin{aligned}\alpha_c''' = 3\beta_0^3\{&15\beta_0 - 50\beta_0^3 + 35\beta_0^5+\\ &+(-15\beta_0+50\beta_0^3-35\beta_0^5)\cdot cT^1/L+\\ &+(10-40\beta_0^2+30\beta_0^4)\cdot cT^2/L+\\ &+(20\beta_0-20\beta_0^3)\cdot cT^3/L+8\beta_0^2\cdot cT^4/L\}\end{aligned} \tag{26}$$

This transformation has already been implemented and tested in the MAD-X version of February 2017 [11].

It is always good to try out these tools for a real-world example. To this end we have chosen the PS and collected some results in Annex D, but first we will look at the smooth lattice approximation.

3 SECOND ORDER MOMENTUM COMPACTION AND CHROMATICITY

3.1 Smooth Lattice Approximation

To understand a relationship between the second order momentum compaction and the chromaticity we consider the simplest possible case with the smooth lattice approximation. We assume that the magnetic field depends on the relative radial coordinate, $x = (\rho - \rho_0)/\rho_0$, where ρ_0 is the radius of the reference orbit:

$$B_y(x) = B_0(1 + g \cdot x + s \cdot x^2 + \dots), \tag{27}$$

where g and s are the coefficients which determine the betatron tune and the chromaticity. We assume a constant bending radius of the ring, ρ_0, so that:

$$\frac{1}{\rho_0} = \frac{e \cdot B_0}{p_0 c}. \tag{28}$$

Assuming an absence of betatron motion, we can rewrite Eq. 28 for the orbit with momentum deviation, δ. Using Eq. 27 we obtain:

$$p_0(1+\delta)c = e \cdot \rho_0(1+x)B_0(1+g\cdot x + s\cdot x^2 + \dots). \tag{29}$$

Omitting in the above equation terms above the second order we obtain for the momentum deviation:

$$\delta = (g+1)x + (g+s)x^2. \tag{30}$$

In the smooth lattice approximation, the small amplitude betatron tune around the orbit with the momentum deviation is well-known and is equal to:

$$\begin{aligned}Q_x(x) &= \sqrt{1+\frac{\rho(x)}{B(x)\rho_0}\frac{\partial B}{\partial x}} \approx \sqrt{1+\frac{(1+x)(g+2s\cdot x)}{1+g\cdot x+s\cdot x^2}}\\ &\approx \sqrt{1+g} + \frac{g-g^2+2s}{2\sqrt{1+g}}x.\end{aligned} \tag{31}$$

Consequently, the tune at the reference orbit is equal to:

$$Q_x = \sqrt{1+g}, \tag{32}$$

and the horizontal chromaticity is:

$$Q'_x = \frac{\partial Q_x}{\partial \delta} = \frac{\partial Q_x}{\partial x} \cdot \frac{\partial x}{\partial \delta} = \frac{g - g^2 + 2s}{2Q_x^3} \tag{33}$$

Now we can express the first and second order momentum compaction factors through the betatron tune and chromaticity. Making simple calculations we obtain the first order momentum compaction:

$$\alpha_c \equiv \frac{\partial x}{\partial \delta} = \frac{1}{Q_x^2}. \tag{34}$$

Similarly, the second order momentum compaction is:

$$\alpha'_c \equiv \frac{\partial^2 x}{\partial \delta^2} = \frac{1 - 2Q'_x \cdot Q_x - Q_x^2}{Q_x^4}, \tag{35}$$

where it should be noted that the definition of α'_c is in agreement with the MAD-X definition that strictly considers derivatives and does not include the factorial factors.

3.2 Simplified FODO Lattice

To understand how the second order momentum compaction depends on the chromaticity for the cases when the beta-function and dispersion are varied along the ring we considered two simple models.

In the first model, FOBO, one lattice period consists of a zero length focusing lens, rectangular dipole and two equal length drifts separating the lens and dipole. In the second model, FBDB, one period includes two sector dipoles, and one focusing, F, and one defocusing, D, zero-length lenses with equal focusing strengths of opposite signs.

3.3 Effect of Sextupoles on the Second Order Momentum Compaction

First, we consider an effect of single sextupole on the Second Order Momentum Compaction. Let a zero length sextupole with the integrated strength:

$$(k_2 l) = \frac{e}{pc} \frac{\partial^2 B_y}{\partial x^2} = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2}, \tag{36}$$

where l is the effective length of the sextupole being placed at a location with the horizontal dispersion D_x and beta-function β_x. In the first order the orbit offset due to relative momentum change results in particle trajectory bending by the sextupole with an angle

$$\Delta\theta = -(k_2 l) \cdot (D_x \delta)^2. \tag{37}$$

That results in the particle location and angle immediately after the sextupole being related by the following equation:

$$\begin{pmatrix} x \\ \theta - \Delta\theta \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}. \tag{38}$$

The solution of the equation is:

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \frac{1}{2 - R_{11} - R_{22}} \cdot \begin{pmatrix} 1 - R_{22} & R_{12} \\ R_{21} & 1 - R_{11} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}. \tag{39}$$

The corresponding orbit lengthening is

$$\Delta s = R_{51} x + R_{52} \theta. \tag{40}$$

The transfer matrix elements for a ring can be parameterized by the ring Twiss parameters:

$$R_{11} = c_v + \alpha_x s_v, R_{12} = \beta_x s_v,$$

$$
\begin{aligned}
R_{21} &= -(1+\alpha_x^2)\frac{s_v}{\beta_x}, R_{22} = c_v - \alpha_x s_v, \\
R_{51} &= D'_x(1 - c_v - \alpha_x s_v) - \frac{1+\alpha_x^2}{\beta_x} D_x s_v, \\
R_{52} &= D_x(c_v - \alpha_x s_v - 1) - D'_x \beta_x s_v, \\
c_v &= cos(2\pi Q_x), s_v = sin(2\pi Q_x), \\
D'_x &= \frac{\partial D_x}{\partial s}, \alpha_x = -\frac{1}{2}\frac{\partial \beta_x}{\partial s},
\end{aligned} \tag{41}
$$

where R_{11}, R_{12}, R_{21}, R_{22}, R_{51} and R_{52} correspond to the matrix terms of Eq. 14. Here, Q_x is the horizontal betatron tune of the ring. Substituting Equations (37) to (39) and Eq. 41 into Eq. 40 and simplifying we obtain:

$$
\Delta s = -\frac{(k_2 l) D_x^3 \delta^2}{2}. \tag{42}
$$

Then the change in the second order momentum compaction is

$$
\Delta \alpha'_c \equiv \frac{1}{L_o}\frac{\partial^2 \Delta s}{\partial \delta^2} = -\frac{(k_2 l) D_x^3}{L_o}, \tag{43}
$$

where L_o is the ring circumference. Contributions of different sextupoles are added linearly. That yields for the total change of $\Delta\alpha'_c$ as a sum over all sextupoles:

$$
\Delta \alpha'_c = -\frac{1}{L_o}\sum_n (k_2 l)_n D_{x_n}^3. \tag{44}
$$

The contribution of sextupoles to the horizontal chromaticity change is well-know and is equal to:

$$
\Delta Q'_x = \frac{1}{4\pi}\sum_n (k_2 l)_n D_{x_n} \beta_{x_n}, \tag{45}
$$

where D_{x_n} and β_{x_n} are the horizontal dispersion and beta-functions at the location of corresponding sextupoles. In the case when all sextupoles are located at the same beta- and dispersion functions Eqs. 44 and 45 yield:

$$
\frac{\partial \alpha'_c}{\partial Q'_x} = -\frac{4\pi D_x^2}{L_o \beta_x}. \tag{46}
$$

Substituting dispersion $D_x \approx \frac{\rho_o}{Q_x^2}$ and beta-function $\beta_x \approx \frac{\rho_o}{Q_x}$ for the smooth lattice approximation one obtains $\frac{\partial \alpha'_c}{\partial Q'_x} = -\frac{2}{Q_x^3}$. The same result follows directly from Eq. 35.

3.4 MAD-X Comparison with the Smooth Lattice Approximation

In Appendix B one finds a minimalistic MAD-X example for a smooth approximation lattice. In Tab. 1 one finds the α_c and α'_c of Eqs. 34 and 35 of the smooth approximation and the MAD-X results respectively.

Table 1: First and second order momentum compaction in comparison between the smooth lattice approximation and MAD-X.

	α_c		α'_c	
Q'	Smooth Appr.	MAD-X	Smooth Appr.	MAD-X
–	3.16373318	3.16373318	–	–
0.0	–	–	6.8454779	6.845341731
10.0	–	–	-105.70047	-105.6998606

4 Acknowledgments

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5 Conclusions

This paper shows that MAD-X/PTC can now derive the dispersion and the momentum compaction factor both for “time=false” and “time=true” as generally expected with respect to δ, for the time being implemented up to fourth order (changeable on user request). To convince the accelerator community we have made a comparison with the smooth lattice approximation for which α' can be derived analytically. Equally relevant, we have shown for a realistic low-β CERN PS lattice that the δ dependent closed orbit components and the path-length ΔL can be calculated with the Taylor series expanded up the order $n_o = 5$.

Appendix

A PTC Template

PTC template to create 5^{th} order maps for both sets of canonical coordinates. The δ attribute can be set to produce a 5D closed-orbit or path-length difference ΔL.

```
!*********************************************
ptc_create_universe;
 ptc_create_layout,model=2,method=6,nst=20,
   exact=true, closed_layout=true,time=%%time%%;
 select,flag=twiss,clear;
 select,FLAG=twiss,column=name ,s,l,beta11,
    beta22,beta33,disp1,disp2,disp3,disp4,
    beta12,beta13;

 ptc_twiss,table=twiss,icase=56,no=5,
    closed_orbit, writetmap,deltap=%%deltap%%;
ptc_end;
!*********************************************
```

There are four settings required for the cases studied in this note.

1 Produce η components to 5^{th} order by setting:
   a) %%time%% =true
   b) %%deltap%% =0.

2 Produce α_c components to 5^{th} order by setting:
   a) %%time%% =false
   b) %%deltap%% =0.

3 Produce 5D closed-orbit for $\delta = 0.0001$:
   a) %%time%% =true
   b) %%deltap%% =0.0001

4 Produce ΔL for $\delta = 0.0001$:
   a) %%time%% =false
   b) %%deltap%% =0.0001

B Simple MAD-X Example for a Smooth Approximation Lattice

```
!*********************************************
title, 'Smooth Model';
option, -echo, warn, info;

beam,particle=electron,energy:=5.3933035235,
exn:=2.5E-6,eyn:=2.5E-6;

k2c:=0;
bm : SBEND, L=300, ANGLE=twopi, K1=-0.0003,
K2:=k2c;
ring : LINE=(bm);
use,period=ring;
match,sequence=ring;
  vary, NAME=k2c, step = 0.00001;
```

```
  global, DQ1=0;
  lmdif, calls = 10000, tolerance = 1.0E-21;
endmatch;
twiss;
ptc_create_universe;
  ptc_create_layout,model=2,method=6,nst=20,
  exact=true,time=false,closed_layout=true;
  ptc_twiss,closed_orbit,no=5,icase=56,
  file=ptc_twiss_thick,writetmap;
ptc_end;

stop;
!**********************************************
```

C Calculation of η in MAD-X

In MAD-X the longitudinal momentum coordinate is p_t, i.e. we will get η-like terms. The MAD-X code snippet is given below:

```
**********************************************
     sd = rt(5,6)
     do i = 1, 4
        sd = sd + rt(5,i) * disp(i)
     enddo
     eta = - sd * betas**2 / circ
     alfa = one / gammas**2 + eta
**********************************************
```

Compared to Eq. 23:

$$\eta = -\beta_0^2 \cdot (R_{56} + R_{51} \cdot D_x + R_{52} \cdot D'_x)/L,$$

and with different but obvious notation conventions one finds agreement, except that in our case the vertical dispersion $disp(3)$ and $disp(4)$ are zero.

The momentum compaction α_c is equal to Eq. 3.

D The CERN PS as an Example

The PS is studied at a kinetic energy of 2 GeV (see Tab. D.1). The first test is how well the closed-orbit

Table D.1: PS Beam Parameters.

Parameter	Value
Length [m]	628.32
Kinetic energy [GeV]	2
Relativistic β	0.9486
Relativistic γ	3.13

$(cox, copx)$ can be reproduced by the following Taylor expansion to order n_o:

$$cox(p_t) = \sum_{n=1}^{n_o} \frac{1}{n!} \frac{\partial^n (cox)}{\partial p_t^n} \cdot p_t^n, \tag{D.1}$$

$$copx(p_t) = \sum_{n=1}^{n_o} \frac{1}{n!} \frac{\partial^n (copx)}{\partial p_t^n} \cdot p_t^n, \tag{D.2}$$

with $D_x = \frac{\partial(cox)}{\partial p_t}$, $D_x^2 = \frac{\partial^2(cox)}{\partial p_t^2}$, $D_x^3 = \frac{\partial^3(cox)}{\partial p_t^3} \cdots$ and $D'_x = \frac{\partial(coxp)}{\partial p_t}$, $D'^2_x = \frac{\partial^2(coxp)}{\partial p_t^2}$, $D'^3_x = \frac{\partial^3(coxp)}{\partial p_t^3} \cdots$.

The reader is reminded that here the dispersion is defined for "time=true" and will be different by factors of β_0.

The second and maybe more interesting test is how well the change of path-length ΔL can calculated from the α_c components to order n_o:

$$\Delta L = \sum_{n=1}^{n_o} \frac{1}{n!} \frac{\partial^n(\frac{\Delta L}{L_o})}{\partial \delta^n} \cdot \delta^n \cdot L_o, \tag{D.3}$$

with the derivatives replaced by the α_c components as shown in Eq. 6, $\alpha_c = \frac{\partial(\frac{\Delta L}{L_o})}{\partial \delta}$, $\alpha'_c = \frac{\partial^2(\frac{\Delta L}{L_o})}{\partial \delta^2}$, $\alpha''_c = \frac{\partial^3(\frac{\Delta L}{L_o})}{\partial \delta^3}$ etc.

We have chosen a momentum deviation of $\delta = 0.0001$ with the values in Tab. D.2:

Table D.2: Closed-orbit at $\delta = 0.0001$.

Parameter	Value
cox	3.303E-04
copx	2.283E-06
p_t	9.476E-05
ΔL	0.1745E-02

For $\delta = 0.0001$ Fig. D.1 shows the evaluated Taylor series Equations (D.1) to (D.3) for *cox*, *coxp* and ΔL respectively. The tailoring off of the curves at higher orders of n_o should be addressed to expected numerical problems at those small absolute errors.

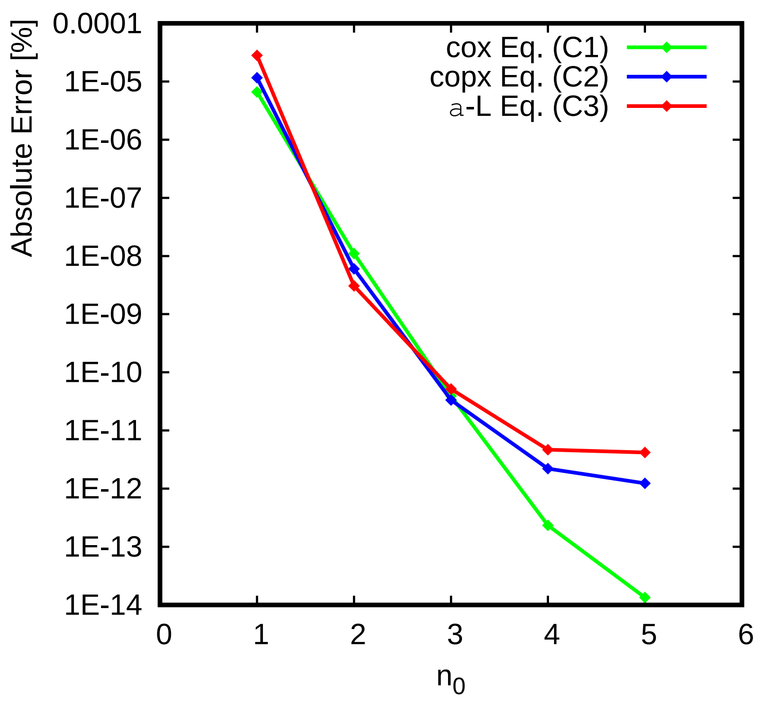

Figure D.1: The closed orbit components *cox, coxp* calculated via Equations (D.1) and (D.2) are depicted in green and blue color respectively. In red is shown the path-length ΔL calculated via Eq. D.3.

Bibliography

[1] P. Zisopoulos, F. Antoniou, Y. Papaphilippou, "Parametric Study of the CLIC Damping Rings Delay Ring for reaching Isochronicity Conditions", Proceedings of IPAC2012, New Orleans, Louisiana, USA.

[2] J. Shan, S.G. Peggs and S.A. Bogacz, "Analytical Evaluation of the Second Order Momentum Compaction Factor and Comparison with MAD Results", Particle Accelerators, 1994, Vol. 45, pp. 1-16.

[3] Hitoshi Tanaka, Masaru Takao, Kouichi Soutome, Hiroyuki Hama, Masahito Hosaka, "A perturbative formulation of nonlinear dispersion for particle motion in storage rings", Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment,Volume 431, Issue 3, 21 July 1999, Pages 396-408.

[4] É. Forest, "Beam Dynamics: A New Attitude and Framework", Hardwood Academic / CRC Press, 1998, ISBN: 90-5702-558-2.

[5] É. Forest, "From Tracking Code to Analysis: Generalised Courant-Snyder Theory for Any Accelerator Model", Springer, Publishing Company, Incorporated ©2016 ISBN:4431558020 9784431558026.

[6] MAD-X web page: http://mad.web.cern.ch/mad/

[7] H. Wiedemann, "Particle Accelerator Physics", 3^{rd} ed. Springer Berlin Heidelberg New York, ISBN-13 978-3-54043-2.

[8] S.Y.Lee, "Accelerator Physics", 3^{rd} edition, World Scientific, ISBN-13: 978-9814374941.

[9] F.Zimmermann, G. Minty, "Measurement and control of charged particle beams", Springer Berlin Heidelberg, ISBN: 978-3-642-07914-6.

[10] K.-Y. Ng, "Higher order momentum compaction for a simplified FODO lattice and comparison with SYNCH", Fermilab-FN-578 (1991).

[11] P. Showronski, private communication.