ELLIPTIC GENERA FOR $\mathbb{Z}/k$-MANIFOLDS I

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ABSTRACT. We define and study, for k an odd number, an elliptic genus Φ_k for $\mathbb{Z}/k$-manifolds. We show that this elliptic genus defines a MSO_*-module epimorphism from oriented bordism with coefficients in $\mathbb{Z}/k$ to the ring of mod k-modular forms of Serre and Swinnerton-Dyer.

§0 **Introduction.** $\mathbb{Z}/k$-manifolds are singular manifolds that represent the geometric cycles for bordism with coefficients in $\mathbb{Z}/k\mathbb{Z}$ (see for example [Sul70b], [Bas73], [BRS76]). They can be intuitively described as spaces that look locally like open sets of $\mathbb{R}^n$ or open subsets of $\mathbb{R}^{n-1} \times C(\mathbb{Z}/k)$, where $C(\mathbb{Z}/k)$ is the cone over the space formed by k points. This pattern of singularities is mild enough so that some classical invariants of smooth manifolds that depend on the existence of transversality, such as the signature, admit a natural extension to $\mathbb{Z}/k$-manifolds. The importance of the existence of these extensions is that they can be used to detect torsion information in the theory of characteristic invariants of manifolds. This fact has motivated the use of $\mathbb{Z}/k$-manifolds in geometric topology [MD74](see also [Sul70a] for a general overview of the subject). More recently they were used to detect the torsion part in Witten's anomaly formula [Fre88]. and, based on this work and the results of [APS76], Dan Freed and Richard Melrose developed in [FM92] a version of the index theorem for $\mathbb{Z}/k$-manifolds.

The universal elliptic genus $\Phi : MSO_* \longrightarrow \mathbb{Q}[\delta, \epsilon]$, where MSO_* is the oriented bordism ring and δ and ϵ are two indeterminates of degree 4 and 8 respectively, is the rational genus, in the sense of Hirzebruch, associated to the formal power series

$$\log_\Phi(x) = \sum_{n\geq 0} a_n x^n = \int_0^x \frac{dz}{\sqrt{1 - 2\delta z^2 + \epsilon z^4}}.$$

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This genus is a subtle invariant of manifolds. When X is a closed, compact, oriented, spin manifold representing a class $[X] \in MSO_*$, the universal elliptic genus $\Phi([X])$ evaluated on $[X]$ has a natural [Wit88] geometric interpretation as the formal S^1-equivariant index of the Dirac-Ramond operator on the space of smooth free loops of X. This index, which is a formal power series $\sum_{n\geq 0} a_n q^n$, corresponds to the power series expansion at the cusp $i\infty$ of a modular form for the congruence subgroup $\Gamma_0(2)$ of matrices $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ such that e is even. The ring $\mathrm{Mod}\left(\Gamma_0(2), \mathbb{Z}\left[\frac{1}{2}\right]\right)$ of modular forms for $\Gamma_0(2)$ whose coefficients in the power series expansions at $i\infty$ are in $\mathbb{Z}\left[\frac{1}{2}\right]$ is isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right][\delta,\epsilon]$, where δ and ϵ are two algebraically independent modular forms of weight 2 and 4 respectively[1]. This description of the ring of modular forms provides the link between the algebraic and geometric description of the elliptic genus. In fact, using some results about formal group laws, one can show [LRS] that the image of Φ is equal to $\mathbb{Z}\left[\frac{1}{2}\right][\delta,\epsilon] \subset \mathbb{Q}[\delta,\epsilon]$. The *elliptic cohomology* of a finite CW-complex X is defined by

$$\mathcal{E}\ell\ell^*(X) = MSO^*(X) \bigotimes_{MSO^*} \mathbb{Z}\left[\frac{1}{2}\right]\left[\delta,\epsilon,\Delta^{-1}\right],$$

where $MSO^*(X)$ denotes the oriented cobordism ring of the space X, $\Delta = \epsilon(\delta^2-\epsilon)^2$, and the ring $\mathcal{E}\ell\ell^* = \mathbb{Z}\left[\frac{1}{2}\right][\delta,\epsilon,\Delta^{-1}]$ is considered as a graded module over MSO^* via the canonical ring homomorphism $\Phi_{\mathcal{E}\ell\ell} : MSO^* \longrightarrow \mathcal{E}\ell\ell^*$ induced by the elliptic genus[2].

It this paper we will define, for k an odd number, a natural extension, the $\mathbb{Z}/k$-*elliptic genus* Φ_k, of the universal elliptic genus to $\mathbb{Z}/k$-manifolds. We shall show that Φ_k is a cobordism invariant and that it induces a MSO^*-module homomorphism $\Phi_k : MSO_*(pt,\mathbb{Z}/k\mathbb{Z}) \longrightarrow \mathbb{Z}/k\mathbb{Z}\,[[\,q\,]]$, where $MSO_*(pt,\mathbb{Z}/k\mathbb{Z})$ denotes oriented bordism with coefficients in $\mathbb{Z}/k\mathbb{Z}$ and $\mathbb{Z}/k\mathbb{Z}\,[[\,q\,]]$ is the ring of formal power series in an indeterminate q with coefficients in $\mathbb{Z}/k\mathbb{Z}$. Our main results (theorems 1 and 2) are that Φ_k takes values in the ring $\mathrm{Mod}\,(\Gamma_0(2),\mathbb{Z}/k) \subset \mathbb{Z}/k\mathbb{Z}\,[[\,q\,]]$, where $\mathrm{Mod}\,(\Gamma_0(2),\mathbb{Z}/k)$ denotes the ring of "mod k" modular forms of Serre and Swinnerton-Dyer, and that $\Phi_k : MSO_*(pt,\mathbb{Z}/k\mathbb{Z}) \longrightarrow \mathrm{Mod}\,(\Gamma_0(2),\mathbb{Z}/k)$ is an epimorphism. We shall finally analize the connections between the $\mathbb{Z}/k$-elliptic genera for different k's.

A different approach to "mod k" elliptic genera has been proposed by Peter Braam and Brian Steer. We can, superficially, describe their approach as follows. Instead of using the mod k index theorem of [FM92] they fix a flat bundle L over a smooth spin manifold X and apply the mod k index theorem for flat bundles of Atiyah, Patodi and Singer to the formal power series of elliptic operators $\sum_{n\geq 0} D_n\, q^n$, where D_n

[1]It is important to keep in mind that the grading used by topologists is twice the weight of the modular form.

[2]In this definition one must use the gradings defined by $MSO^q = MSO_{-q}$, $|\delta| = -4$, and $|\epsilon| = -8$.

is the twisted Dirac operator whose index gives the n-th coefficient of the elliptic genus evaluated on X. I would like to express my gratitude to Peter Braam for the communication of their work. There is also a "mod 2" elliptic genus defined by Ochanine in [Och91] using real K-theory.

I wish finally to express my gratitude to Dan Freed for his encouragement, to the Institute des Hautes Études Scientifiques, where this paper was written, for providing a magnificent working environment. And my special gratitude to Alberto Verjovsky who taught me about the importance of geometric methods in topology.

§1 **The $\mathbb{Z}/k$ elliptic genus.** Let X be a smooth n-dimensional manifold and let ∂X be its boundary. A *$\mathbb{Z}/k$-structure* on X consists of

a: a closed manifold Y of dimension $n-1$,
b: a decomposition $\partial X = \coprod_{i=1}^{k} Y_i$ of the boundary of X into k disjoint manifolds Y_i,
c: a set of diffeomorphisms $\theta_i : Y \longrightarrow Y_i$, $i = 1, \ldots, k$.

An equivalent description of a $\mathbb{Z}/k$-structure on a manifold X is given by a decomposition $\partial X = \coprod_{i=1}^{k} Y_i$ of the boundary of X as in **b** together with a set of diffeomorphisms $\theta_{ij} : Y_j \longrightarrow Y_i$ satisfying the cocycle condition $\theta_{ij}\theta_{jl} = \theta_{il}$, $i, j, l = 1, \ldots, k$. A *$\mathbb{Z}/k$-manifold* is a manifold with a $\mathbb{Z}/k$ structure. To any $\mathbb{Z}/k$-manifold X one can associate a singular space $\overline{X}$ formed by attaching X to Y via the functions θ_i. Spaces like $\overline{X}$ is what Sullivan originally defined as $\mathbb{Z}/k$-manifolds. Two particularly important examples of $\mathbb{Z}/k$-manifolds are $\overline{S^n}$, where X is the sphere S^n with k discs removed and $Y = S^{n-1}$, and Moore's space $M_k = S^1 \cup_k D^2$ (see [MD74] for a description of Moore's space). Removing a point from $\overline{S^n}$ one obtains a non-compact space $\overline{\mathbb{R}^n}$. The "plane" $\overline{\mathbb{R}^2}$ will play for us the intuitive role of a "point" in the sense that, for any $\mathbb{Z}/k$-manifold $\overline{X}$, there is a "collapse map" $\rho^{\overline{X}} : \overline{X} \longrightarrow \overline{\mathbb{R}^2}$ [FM92]. A *$\mathbb{Z}/k$-vector bundle* $\overline{E} \longrightarrow \overline{X}$ over a $\mathbb{Z}/k$-manifold $\overline{X}$ is a vector bundle $E \longrightarrow X$ together with isomorphisms $\tilde{\theta}_{ij} : (E|_{Y_j}) \longrightarrow (E|_{Y_i})$ covering θ_{ij}. The basic example of a $\mathbb{Z}/k$-vector bundle on a $\mathbb{Z}/k$-manifold $\overline{X}$ is the tangent bundle $\overline{TX}$ [FM92]. Recall that for each component Y_i of the boundary of X there is an exact sequence

$$0 \longrightarrow TY_i \longrightarrow T_{Y_i}X \longrightarrow NY_i \longrightarrow 0, \tag{1}$$

where NY_i is the normal bundle of Y_i in X. The differentials $(\theta_{ij})_*$ fit into commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TY_j & \longrightarrow & T_{Y_j}X & \longrightarrow & NY_j & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\theta_{ij})_*} & & \downarrow{\scriptstyle \tilde{\theta}_{ij}} & & \downarrow & & \\ 0 & \longrightarrow & TY_i & \longrightarrow & T_{Y_i}X & \longrightarrow & NY_i & \longrightarrow & 0. \end{array} \tag{2}$$

These commutative diagrams define a structure of $\mathbb{Z}/k$-vector bundle on TX (see [FM92] for the details). The cohomology theory connected to vector bundles is K-theory. From the topological point of view the K-theory, with compact supports, of a $\mathbb{Z}/k$-manifold $\overline{X}$ can be computed using the long exact sequences of the pairs $(X, \partial X)$ and $(\overline{X}, Y)$. For example the reduced K-theory of $\overline{S^n}$ is given by [FM92]

$$(3) \qquad \widetilde{K}^q(\overline{S^n}) = \begin{cases} \mathbb{Z}/k\mathbb{Z}, & if\ q \equiv n \pmod 2 \\ \mathbb{Z}^{k-1}, & if\ q \not\equiv n \pmod 2 \end{cases}$$

and, as we are working with K-theory with compact supports $K^*(\overline{\mathbb{R}^n}) \simeq K^*(\overline{S^n})$. On the other hand the definition of $\mathbb{Z}/k$-vector bundles provides a natural geometric description of the K-theory of $\mathbb{Z}/k$-manifolds that allows us to generalize easily some constructions of K-theory. In particular if $\chi : KO^* \longrightarrow KO^*$ (or $\chi : KO^* \longrightarrow K^*$) is any exponential stable K-theoretical characteristic class, then one can define $\chi(E)$ for any real $\mathbb{Z}/k$-vector bundle $\overline{E}$ in the obvious way. We shall need to work with something slightly more complicated than K-theory. Let us define, for X a topological space, $KO^*_q(X) = \widetilde{KO}^*(X, \mathbb{Z}\left[\frac{1}{2}\right])\,[[\,q\,]]$, where $\widetilde{KO}^*(X, \mathbb{Z}\left[\frac{1}{2}\right])\,[[\,q\,]]$ denotes the ring of formal power series in an indeterminate q with coefficients in $\widetilde{KO}^*(X, \mathbb{Z}\left[\frac{1}{2}\right])$, the KO^* theory with compact supports (with coefficients in $\mathbb{Z}\left[\frac{1}{2}\right]$) of X. We define K^*_q in an analogous way replacing real K-theory by complex K-theory. If $\overline{E}$ is any real $\mathbb{Z}/k$-vector bundle over $\overline{X}$ we define *Witten's characteristic class* $\theta(\overline{E})$ by

$$(4) \qquad \theta(\overline{E}) = \left[\bigotimes_{\substack{n>0 \\ n\ even}} S_{q^n}(\overline{E} \otimes \mathbb{C}) \right] \bigotimes \left[\bigotimes_{\substack{n>0 \\ n\ odd}} \wedge_{-q^t}(\overline{E} \otimes \mathbb{C}) \right],$$

where $S_{q^n}(\overline{E} \otimes \mathbb{C}) = \sum_{k \geq 0} q^{kn} S^n(\overline{E} \otimes \mathbb{C})$, and $\wedge_{q^n}(\overline{E} \otimes \mathbb{C}) = \sum_{k \geq 0} q^{kn} \wedge^n (E \otimes \mathbb{C})$. This class depends only on the isomorphism class $[\overline{E}]$ of $\overline{E}$ and has the properties that $\theta(\overline{E} \oplus \overline{F}) = \theta(\overline{E})\theta(\overline{F})$ and that $\theta([1])$ is a unit in $K_q(\overline{X})$. Therefore θ has a unique extension $\theta : KO^*(\overline{X}) \longrightarrow K_q(\overline{X})$.

In [FM92] D. Freed and R. Melrose defined for any $\mathbb{Z}/k$ manifold $\overline{X}$ a topological index, $\text{t-index}_K^{\overline{X}} : K(\overline{TX}) \longrightarrow K(\overline{\mathbb{R}^2})$. As we are working with k odd this index admits a factorization

$$(5) \qquad \begin{array}{ccc} K(\overline{TX}) & \xrightarrow{\text{t-index}_K^{\overline{X}}} & K(\overline{\mathbb{R}^2}) \\ {\scriptstyle i}\downarrow & & \| \\ K(\overline{TX}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] & \xrightarrow{\text{t-index}_K^{\overline{X}}} & K(\overline{\mathbb{R}^2}). \end{array}$$

Using this index we can define a topological index t-index$_q^{\overline{X}}$: $K_q(\overline{TX}) \longrightarrow K_q(\overline{\mathbb{R}^2})$ by

$$\text{t-index}_q^{\overline{X}}\left(\sum_{n\geq 0}[E_n]\,q^n\right) = \sum_{n\geq 0}\text{t-index}_K^{\overline{X}}([E])\,q^n.$$

Recall that, as $BSpin \longrightarrow BSO$ is an odd primary homotopy equivalence, any $\mathbb{Z}/k$ manifold $\overline{X}$ that is oriented, in the usual sense of the word, has also an orientation class $\sigma \in K^*(\overline{TX}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$, i.e. there exists $\sigma \in K^*(\overline{TX}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ which restricts to a generator on each fiber of the projection $\overline{TX} \xrightarrow{p} \overline{X}$. This K-theoretical orientation class induces a direct image map $\pi_! : K_q^*(\overline{E}) \longrightarrow K_q^*(\overline{\mathbb{R}^2})$ defined by $\pi_!([E]) = \text{t-index}_q^{\overline{X}}(\sigma.p^*([E]))$. We define the *$\mathbb{Z}/k$-elliptic genus* Φ_k by

$$\Phi_k(\overline{X}) = \pi_!(\theta([\overline{TX}] - [n])) \in \mathbb{Z}/k\mathbb{Z}\;[[\,q\,]]\,, \tag{6}$$

where $[n]$ is the isomorphism class of the trivial real $\mathbb{Z}/k$ vector bundle of dimension equal to the dimension of X.

The other cohomology theory of interest for us is cobordism. The (co)bordism relation for $\mathbb{Z}/k$ manifolds is given by the following definition [3]. In this definition the word manifold will mean manifold with general corners.

Definition 1. A $\mathbb{Z}/k$ manifold $\overline{X}$ obtained from a pair (X, Y) is the boundary of a $\mathbb{Z}/k$ manifold $\overline{M}$ formed from (M, N) if and only if N is a manifold with boundary ∂N diffeomorphic to Y via $\partial N \xrightarrow{f} Y$ and $\partial M = X \bigcup [\cup_{i=1}^n N_i]$ where N_i is a copy of N glued to the ith part ∂X_i of the boundary of X via $\theta_i f$.

This definition of bordism induces a well defined equivalence relation between the isomorphism classes of $\mathbb{Z}/k$-manifolds (see for example [Bas73] lemma 3.1). The bordism groups of $\mathbb{Z}/k$ manifolds are isomorphic to bordism with $\mathbb{Z}/k\mathbb{Z}$ coefficients [MD74], [BRS76]. Let us briefly explain the geometrical origin of this isomorphism. From the point of view of spectra the bordism (cobordism) with coefficients in $\mathbb{Z}/k\mathbb{Z}$ is defined as the bordism (cobordism) of Moore's space M_k. Let $q : S^1 \sqcup D^2 \longrightarrow M_k$ be the projection to the quotient, let $C_k \subset D_2$ be the cone with vertex 0 over the k-th roots of 1 and let $S_k = q(\{1\} \cup C_k) \subset M_k$. Then, if $M \xrightarrow{f} M_k$ is a map from a smooth manifold M to M_k we can, using two stage transversality with respect to S_k, deform f in such a way that $\overline{X} = f^{-1}(S_k)$ has a natural structure of $\mathbb{Z}/k$ manifold. The set S_k is also included in $\overline{\mathbb{R}^2}$ (see [FM92]) and the collapsing map $\rho^{\overline{X}} : \overline{X} \longrightarrow \overline{\mathbb{R}^2}$ is "equal" to $f|_{\overline{X}} : \overline{X} \longrightarrow S_k \subset \overline{\mathbb{R}^2}$. Using the relation between cobordism of $\mathbb{Z}/k$ manifolds and cobordism with coefficients in $\mathbb{Z}/k\mathbb{Z}$ we shall prove in section 4 that the $\mathbb{Z}/k$ elliptic genus is a cobordism invariant of $\mathbb{Z}/k$ manifolds. If we restrict ourselves to certain spin $\mathbb{Z}/k$ manifolds and spin cobordism a direct, analytical proof of the cobordism

[3] See [Bas73] for a more precise definition.

invariance of Φ_k is possible and we shall give it in section 3. In the oriented case what is missing is a "gluing formula" for the elliptic genus.

§2 **Mod k modular forms and the elliptic cohomology of $\overline{S^2}$.** Let $k \in \mathbb{N}$ be any natural number different from 1 and let $\Gamma \subset SL(2,\mathbb{Z})$ be any congruence subgroup of the modular group $SL(2,\mathbb{Z})$. The "mod k" modular forms for Γ were defined in [SD73] and [Ser73] in the following way. Let $\mathcal{O}_k$ be the ring of k-integer rational numbers and let j be an integer. The $\mathcal{O}_k$-module Mod_j is the set of formal power series

$$f(q) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{O}_k \ [[\ q\ ]]$$

for which there exists a modular form g of weight j for the group Γ such that the power series expansion of g at $i\infty$ is $f(q)$. If $f(q) \in \mathrm{Mod}_j$ then the reduction $\tilde{f}(q)$ modulo k belongs to the algebra $\mathbb{Z}/k\mathbb{Z}\ [[q]]$ of formal power series with coefficients in $\mathbb{Z}/k\mathbb{Z}$. In this way, taking the mod k reduction of all the elements in Mod_j, we obtain a $\mathbb{Z}/k\mathbb{Z}$ module $\widetilde{\mathrm{Mod}}_j$ of $\mathbb{Z}/k\mathbb{Z}\ [[\ q\ ]]$. We define[4]

$$\mathrm{Mod}(\Gamma, \mathbb{Z}/k) = \sum_{j \in \mathbb{Z}} \widetilde{\mathrm{Mod}}_j .$$

It is not difficult to show that, if k is odd, then the ring $\mathrm{Mod}\,(\Gamma_0(2), \mathcal{O}_k)$ of modular forms for $\Gamma_0(2)$ whose power series expansions at $i\infty$ have coefficients in $\mathcal{O}_k$ is isomorphic to $\sum_{j\in\mathbb{Z}} \mathrm{Mod}_j \simeq \mathcal{O}_k[\delta, \epsilon]$, where

$$\delta = -\frac{1}{8} - 3\sum_{n\geq 1}\left[\sum_{\substack{d\mid n \\ d \ odd}} d\right] q^n, \qquad \epsilon = \sum_{n\geq 1}\left[\sum_{\substack{d\mid n \\ \frac{d}{n}\ odd}} d^3\right] q^n.$$

and therefore one obtains the same $\mathbb{Z}/k$-algebra $\mathrm{Mod}(\Gamma, \mathbb{Z}/k)$ if one takes only the mod k reductions of the power series expansions of the elements in $\mathrm{Mod}\left(\Gamma_0(2), \mathbb{Z}\left[\frac{1}{2}\right]\right)$ instead of all the elements in $\mathrm{Mod}\,(\Gamma_0(2), \mathcal{O}_k)$.

Let us now study the elliptic cohomology of the $\mathbb{Z}/k$ manifold $\overline{S^2}$. We shall use, in general, elliptic cohomology with compact supports, i.e. for a $\mathbb{Z}/k$ manifold $\overline{X}$ we define $\mathcal{E}\ell\ell^*(\overline{X}) = \widetilde{\mathcal{E}\ell\ell}^*(\overline{X}^+)$, where $\overline{X}^+$ is the one point compactification of $\overline{X}$ and $\widetilde{\mathcal{E}\ell\ell}^*$ denotes reduced elliptic cohomology.

[4] It is important to keep in mind that the sum is not a direct sum, power series expansions of modular forms of different weights can have the same reduction mod k.

Proposition 1. *The elliptic cohomology $\mathcal{E}\ell\ell^*(\overline{S^2})$ of $\overline{S^2} = (X^2, S^1)$, where $X^2 = S^1\setminus\left(\sqcup_{i=1}^k D_i\right)$ and D_i is a disk, is given by*

$$\widetilde{\mathcal{E}\ell\ell}^q(\overline{S^2}) = \begin{cases} \mathcal{E}\ell\ell^{q-2}(pt)\otimes_{\mathbb{Z}}\mathbb{Z}/k\mathbb{Z} & \text{if } q \text{ is even} \\ \{\mathcal{E}\ell\ell^{q-1}(pt)\}^{\oplus\,(k-1)} & \text{if } q \text{ is odd.}\end{cases}$$

Proof. The idea of the proof is similar to the idea of the proof of proposition 1.7 of [FM92]. Let us remove from S^2 a set U which is the union of k disjoint open disks such that each one of them contains in its interior one of the excised disks D_i, and let V denote the interior of X^2. Then $V\cap U$ is homotopically equivalent to the disjoint union of k copies of S^1 and hence

$$\widetilde{\mathcal{E}\ell\ell}^q(U\cap V)\simeq\widetilde{\mathcal{E}\ell\ell}^q(\partial X^2) = \begin{cases}\{\mathcal{E}\ell\ell^q\}^{\oplus k-1} & \text{if } q \text{ is even}\\ \{\mathcal{E}\ell\ell^{q-1}(pt)\}^{\oplus\,k} & \text{if } q \text{ is odd.}\end{cases}$$

The Mayer Vietoris exact sequence in elliptic cohomology associated to $S^2 = U\cup V$ is, for q odd,

$$(7)\quad \ldots\longrightarrow\mathcal{E}\ell\ell^q(S^2)\longrightarrow\mathcal{E}\ell\ell^q(U)\oplus\mathcal{E}\ell\ell^q(V)\longrightarrow\mathcal{E}\ell\ell^q(U\cap V)\longrightarrow$$
$$\longrightarrow\widetilde{\mathcal{E}\ell\ell}^{q+1}(S^2)\longrightarrow\widetilde{\mathcal{E}\ell\ell}^{q+1}(U)\oplus\widetilde{\mathcal{E}\ell\ell}^{q+1}(V)\longrightarrow\ldots.$$

From the suspension isomorphism and the long exact sequence for the pair (S^2, pt) it follows that $\mathcal{E}\ell\ell^{\text{odd}}(S^2)\simeq\mathcal{E}\ell\ell^{\text{odd}} = 0$. Therefore, taking q odd, we obtain from (7) an exact sequence

$$(8)\quad 0\longrightarrow\mathcal{E}\ell\ell^q(V)\longrightarrow\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k}\xrightarrow{s}\mathcal{E}\ell\ell^{q-1}\longrightarrow$$
$$\longrightarrow\{\mathcal{E}\ell\ell^{q+1}\}^{\oplus k-1}\oplus\widetilde{\mathcal{E}\ell\ell}^{q+1}(V)\longrightarrow\{\mathcal{E}\ell\ell^{q+1}\}^{\oplus k-1}\longrightarrow 0,$$

where the homomorphism s assigns to an element $(a_1,\ldots,a_k)\in\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k}$ its sum $\sum a_i\in\mathcal{E}\ell\ell^{q-1}$. As s is an epimorphism the exact sequence (8) splits into two short exact sequences and we can conclude that, for q odd, $\mathcal{E}\ell\ell^q(V) = \{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k-1}$ and $\widetilde{\mathcal{E}\ell\ell}^{q+1}(V) = 0$. Therefore $\mathcal{E}\ell\ell^q(X^2) = \{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k-1}$ and $\widetilde{\mathcal{E}\ell\ell}^{q-1}(X^2) = 0$. Then the exact sequence, for q odd, for the pair $(X^2,\partial X^2)$ is equal to

$$(9)\quad 0\longrightarrow\widetilde{\mathcal{E}\ell\ell}^{q-1}(\partial X^2)\longrightarrow\mathcal{E}\ell\ell^q(X^2,\partial X^2)\longrightarrow\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k-1}\xrightarrow{i}$$
$$\xrightarrow{i}\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k}\longrightarrow\widetilde{\mathcal{E}\ell\ell}^{q+1}(X^2,\partial X^2)\longrightarrow 0,$$

where the homomorphism i can be seen as the isomorphism of $\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k-1}$ into the subgroup of $\{\mathcal{E}\ell\ell^{q-1}\}^{\oplus k}$ of the elements $(a_1,\ldots,a_k)$ such that $\sum a_i = 0$. From this exact sequence it follows that $\mathcal{E}\ell\ell^q(X^2,\partial X^2)\simeq\widetilde{\mathcal{E}\ell\ell}^{q-1}(\partial X^2) = \{\mathcal{E}\ell\ell^{q-1}\}^{\oplus(k-1)}$

and $\widetilde{\mathcal{E}\ell\ell}^{q+1}(X^2, \partial X^2) \simeq \mathcal{E}\ell\ell^{q-1}$. The long exact sequence for the pair $(\overline{S^2}, S^1)$ is, as $\partial X^2 \to S^1 \subset \overline{S^2}$ is the trivial k covering, isomorphic to

$$0 \to \{\mathcal{E}\ell\ell^{q-1}\}^{\oplus(k-1)} \to \mathcal{E}\ell\ell^{q}(\overline{S^2}) \to \mathcal{E}\ell\ell^{q-1} \xrightarrow{m_k} \mathcal{E}\ell\ell^{q-1} \to \widetilde{\mathcal{E}\ell\ell}^{q+1}(\overline{S^2}) \to 0,$$

where m_k is multiplication by k. The proposition follows immediately from this sequence. □

Recall that there exists a natural transformation, the elliptic character of H. Miller [Mil89], of cohomology theories $\lambda_0 : \mathcal{E}\ell\ell^*(X) \to KO^*(X, \mathbb{Z}\left[\frac{1}{2}\right])[[q]]$. When X is a point the elliptic character of $\theta \in \mathcal{E}\ell\ell^*$ is the power series expansion of θ at $i\infty$. The elliptic character composed with the transformation $c : KO^* \to K^*$ induced by the complexification of vector bundles induces a natural transformation $\lambda = c\lambda_0 : \mathcal{E}\ell\ell^* \to K_q^*$. This "character" can be used to give a modular interpretation of $\mathcal{E}\ell\ell^*(\overline{\mathbb{R}^2})$.

Proposition 2. *The image* $\lambda(\mathcal{E}\ell\ell^{even}(\overline{S^2}))$ *is equal to the ring* $\mathrm{Mod}(\Gamma_0(2), \mathbb{Z}/k)$.

Proof. The K_q^*-theory of $\overline{S^2}$ is given by

$$K_q^n(\overline{\mathbb{R}^2}) = \begin{cases} \mathbb{Z}/k\mathbb{Z}\,[[\,q\,]] & \text{if } n \text{ is even} \\ \{\mathbb{Z}\left[\frac{1}{2}\right]\,[[\,q\,]]\}^{\oplus(k-1)} & \text{if } n \text{ is odd.} \end{cases}$$

This result can be computed using the Mayer Vietoris sequence in K_q-theory for the decomposition $S^2 = V \cup U$, where V and U are the same open sets that we used for the computation of the elliptic cohomology of $\overline{S^2}$. Using the same argument that we used in the proof of proposition 1 we obtain an exact sequence

$$(10) \quad 0 \to \mathbb{Z}\,[[\,q\,]]^{k-1} \to K_q^{odd}(\overline{S^2}) \to \mathbb{Z}[[\,q\,]] \xrightarrow{m_k} \mathbb{Z}\,[[\,q\,]] \to \widetilde{K}_q^{even}(\overline{S^2}) \to 0,$$

where m_k denoted multiplication by k. The elliptic character λ induces an homomorphism of exact sequences

$$\begin{array}{ccccccc} \longrightarrow & \mathcal{E}\ell\ell^{even} & \xrightarrow{m_k} & \mathcal{E}\ell\ell^{even} & \longrightarrow & \widetilde{\mathcal{E}\ell\ell}^{even}(\overline{S^n}) & \longrightarrow 0 \\ & \lambda\downarrow & & \lambda\downarrow & & \lambda\downarrow & \\ \longrightarrow & \mathbb{Z}\,[[\,q\,]] & \xrightarrow{m_k} & \mathbb{Z}\,[[\,q\,]] & \longrightarrow & \mathbb{Z}/k\mathbb{Z}\,[[\,q\,]] & \longrightarrow 0 \end{array}$$

The proposition follows immediately from the fact that the first two terms in the morphism of exact sequences correspond to taking the power series expansion of a modular form at the cusp $i\infty$. □

§3 **The $\mathbb{Z}/k$-elliptic genus and Dirac operators.** Let us suppose now that X is an even dimensional spin $\mathbb{Z}/k$-manifold with a metric g that is a product near the boundary and let us write

$$\theta(TX) = \sum_{n\geq 0} R_n(TX \otimes \mathbb{C})\; q^n.$$

The metric g induces an hermitian structure and a compatible connection in each one of the complex vector bundles $R_n(TX \otimes \mathbb{C})$. We can then consider the formal power series of Dirac operators

$$\mathrm{D}_{ell}(X) = \sum_{n\geq 0} \mathrm{D}_n(X)\; q^n = \sum_{n\geq 0} \mathrm{D}_X(R_n(TX \otimes \mathbb{C}))\; q^n, \tag{11}$$

where $\mathrm{D}_n(X) = \mathrm{D}_X(R_n(TX\otimes\mathbb{C}))$ is the Dirac operator on X twisted by $R_n(TX\otimes\mathbb{C})$. This formal power series can be heuristically interpreted as the "localization" of the Dirac-Ramond operator on the loop space $\mathcal{L}X$ [Wit88], [Wit87] at the set of points fixed under the natural S^1 action on $\mathcal{L}X$. A more rigorous geometric interpretation can be given using the operator defined by Taubes in [Tau89].

Due to the assumption on the metric, the formal power series of twisted Dirac operators $\mathrm{D}_{ell}(X)$ induces a formal power series [FM92]

$$\mathrm{D}_{ell}(Y) = \sum_{n\geq 0} \mathrm{D}_n(Y)\; q^n = \sum_{n\geq 0} \mathrm{D}_Y(R_n(TX \otimes \mathbb{C}))\; q^n, \tag{12}$$

of self-adjoint operators on Y with numerable spectra. Let $Sp(\mathrm{D}_n(Y)) \subset \mathbb{R}$ be the spectrum of $\mathrm{D}_n(Y)$ and let $Sp(\mathrm{D}_{ell}(Y)) = \cup_{n\geq 0} Sp(\mathrm{D}_n(Y))$. Then $Sp(\mathrm{D}_{ell}(Y))$ is a numerable set and fixing $r \in \mathbb{R}$ in the complement of this set we can define a coherent family of global boundary conditions[5] such that all the operators $\mathrm{D}_n(X)$ have a well defined index mod k. We can now define

$$\eta_{ell}(Y) = \sum_{n\geq 0} \eta_n(Y)\; q^n, \tag{13}$$

where η_n is the η invariant [APS75] of the operator $\mathrm{D}_Y(R_n(TX\otimes\mathbb{C}))$. Corollary (5.4) of [FM92] applied to this formal series gives an analytical formula for the mod k elliptic genus

$$\Phi_k(\overline{X}) = \int_X \hat{A}(X)\mathrm{ch}\left(\Omega^L/2\right)\mathrm{ch}\left(\Omega^{\theta(TX)}\right) - k\; \eta_{ell}(Y) \;(\mathrm{mod}\ 1), \tag{14}$$

where L is the line bundle corresponding to the spin^c-structure induced on X, Ω^L is its curvature, and $\Omega^{\theta(TX)} = \sum_{n\geq 0} \Omega(R_n(TX \otimes \mathbb{C}))\; q^n$ is the formal power series whose coefficients are the curvature forms $\Omega(R_n(TX \otimes \mathbb{C}))$ of the vector bundles $R_n(TX \otimes \mathbb{C})$. It is not difficult to see (see for example [Bry88],[Zag88]) that the expression $\int_X \hat{A}(X)\mathrm{ch}\left(\Omega^L/2\right)\mathrm{ch}\left(\Omega^{\theta(TX)}\right)$ is the power series expansion of a modular

[5] See for example [FM92] section 3 or [APS75] .

form, hence its reduction mod k is a mod k modular form; in the next section we shall show that also $\Phi_k(\overline{X})$ is a mod k modular form, we can therefore conclude that

Proposition 3. *The formal power series $k\eta_{ell}(\overline{X}) \in \mathbb{Z}/k\mathbb{Z}\,[[\,q\,]]$ is a mod k modular form.*

Suppose now that the spin $\mathbb{Z}/k$ manifold $\overline{X}$ is the boundary[6] of a spin $\mathbb{Z}/k$ manifold $\overline{M}$ with ∂M smooth and let us fix an extension of the metric g and the spin structure to M. This choice induces a metric and spin structure on each of the copies N_i of N included in the boundary of M and we can suppose that the metric on N_i is a product near the boundary. Then the disjoint union $\overline{N} = \sqcup_{i=1}^{k} N_i$ has a structure of $\mathbb{Z}/k$ spin manifold such that the orientation in the boundary is the opposite to the orientation of the boundary of X and therefore $\eta_{ell}(\partial\overline{N}) = -k\eta_{ell}(Y)$. It follows that if we denote by $L_i \longrightarrow N_i$ the line bundle corresponding to the spin^c structure induced on N_i, then

$$\begin{aligned}\Phi_k(\overline{N}) &= \sum_i \int_N \hat{A}(N_i)\mathrm{ch}(\Omega^{L_i}/2)\mathrm{ch}(\Omega^{\theta(TN_i)}) + k\eta_{ell}(N) \mod 1\\ &= k(\mathrm{index}\, \mathrm{D}_{\mathcal{E}ll}(N)) = 0 \mod 1.\end{aligned}$$

If $\equiv$ denotes congruence mod k, then

$$\begin{aligned}\mathrm{ind}_k(\mathrm{D}_{ell}(X)) &\equiv \int_X \hat{A}(X)\mathrm{ch}\left(\Omega^L/2\right)\mathrm{ch}\left(\Omega^{\theta(TX)}\right) - k\eta_{ell}(Y)\\ &\equiv \int_X \hat{A}(X)\mathrm{ch}\left(\Omega^L/2\right)\mathrm{ch}\left(\Omega^{\theta(TX)}\right) - k\eta_{ell}(Y)\\ &\qquad + \Phi_k(\overline{N})\\ &\equiv \int_X \hat{A}(X)\mathrm{ch}\left(\Omega^L/2\right)\mathrm{ch}\left(\Omega^{\theta(TX)}\right) - k\eta_{ell}(Y)\\ &\quad + \sum_i \int_{N_i} \hat{A}(N_i)\mathrm{ch}\left(\Omega^{L_i}/2\right)\mathrm{ch}\left(\Omega^{\theta_G(TN_i)}\right) + k\eta_{ell}(Y)\\ &\equiv \mathrm{ind}\,(\mathrm{D}_{ell}(\partial M)) = \Phi(\partial M) = 0.\end{aligned}$$

This shows that the elliptic genus for $\mathbb{Z}/k$-spin manifolds is a spin cobordism invariant.

Remark 1. It should be possible to give a more conceptual approach to the proof using the gluing formula for the index of the Dirac operator of [Fre92].

[6] The conventions will be the same ones as in section 1 definition 1.

§4 **Modular properties of the $\mathbb{Z}/k$-elliptic genus.** We shall now define, for any $\mathbb{Z}/k$ compact manifold $\overline{X}$, a topological index,

$$\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell} : \mathcal{E}\ell\ell^*(\overline{TX}) \longrightarrow \text{Mod}(\Gamma_0(2), k).$$

The definition of $\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell}$ follows closely the definition of the topological index in [FM92] so we shall only outline the construction. The precise details can be found in [FM92] or [AS68]. Let $\overline{X} = (X, Y)$ be a compact $\mathbb{Z}/k$-manifold and let $i : \overline{X} \longrightarrow \overline{X}'$ be an inclusion of $\overline{X}$ into a $\mathbb{Z}/k$-manifold $\overline{X}' = (X', Y')$. Let N be an open tubular neigborhood of X in X' and let N' be an open tubular neigborhood of Y in Y' such that $\overline{N} = (N, N')$ is an open tubular neigborhood of $\overline{X}$ in $\overline{X}'$. By the tubular neigborhood theorem $\overline{N}$ can be seen as the normal bundle over $\overline{X}$. The bundle $\overline{TN} \longrightarrow \overline{TX}$ has a natural structure of $\mathbb{Z}/k$ complex vector bundle (see [FM92] pg 287). The morphism

$$Ti_!^{\mathcal{E}\ell\ell} : \mathcal{E}\ell\ell^*(\overline{TX}) \longrightarrow \mathcal{E}\ell\ell^*(\overline{TX}') \tag{15}$$

is defined as the composition of the Thom isomorphism $\mathcal{E}\ell\ell^*(\overline{TX}) \longrightarrow \mathcal{E}\ell\ell^*(\overline{TN})$ with extension by zero $\mathcal{E}\ell\ell^*(\overline{TN}) \longrightarrow \mathcal{E}\ell\ell^*(\overline{TX}')$. Let V be a real vector space such that there exist an embedding $i : X \longrightarrow V$. Then the map i induces an embedding $Ti \times \rho^{\overline{TX}} : \overline{TX} \longrightarrow TV \times \overline{\mathbb{R}^2}$. The bundle $TV \times \overline{\mathbb{R}^2} \longrightarrow \overline{\mathbb{R}^2}$ admits a complex structure and hence there is a Thom isomorphism $\phi : \mathcal{E}\ell\ell^*(\overline{T\mathbb{R}}^2) \longrightarrow \mathcal{E}\ell\ell^*(TV \times \overline{T\mathbb{R}}^2)$. We define

$$\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell} = \phi^{-1} \circ (Ti \times \rho^{\overline{TX}})_!^{\mathcal{E}\ell\ell}.$$

Remark 2. If E^* is any complex oriented generalized cohomology theory and $\overline{X}$ is a $\mathbb{Z}/k$ manifold we can define a topological index

$$\text{t-index}^{\overline{X}}_{E^*} : E^*(\overline{TX}) \longrightarrow E^*(\overline{\mathbb{R}^2})$$

in exactly the same way as we did for elliptic cohomology. In particular we can define a topological index in oriented cobordism which we shall denote by $\text{t-index}^{\overline{X}}_{MSO}$. Recall that there exists a natural transformation $\Phi_{\mathcal{E}\ell\ell} : MSO^*(-) \longrightarrow \mathcal{E}\ell\ell^*(-)$ that sends Thom's classes into Thom's classes and therefore one can fix a choice of Thom's classes in cobordism and elliptic cohomology such that $\Phi_{\mathcal{E}\ell\ell} \circ \text{t-index}^{\overline{X}}_{MSO} = \text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell} \circ \Phi_{\mathcal{E}\ell\ell}$.

Theorem 1. *Let X be any oriented $\mathbb{Z}/k$-manifold. Then the $\mathbb{Z}/k$elliptic genus Φ_k evaluated on X is a $\mathbb{Z}/k$-modular form for the group $\Gamma_0(2)$.*

Proof. The proof will be based on a "Riemann-Roch" formula relating the topological indices in elliptic cohomology and K_q-theory. Let us fix a choice of Thom's classes $t_{\mathcal{E}\ell\ell}$ in elliptic cohomology and t_K in K_q^*-theory. We shall do this by giving their values on the universal line bundle $L \longrightarrow \mathbb{CP}^\infty$. Once this is done it is easy to show (see for example [Dye69]) that there exists a characteristic class R_q in K_q^* theory with the property that for any complex bundle $E \xrightarrow{p} X$ we have

$$p^*(R_q(E)) \cup (t_K(E)) = \lambda(t_{\mathcal{E}\ell\ell}(E)). \tag{16}$$

The class R_q also has the property that if $x \xrightarrow{j} X$ is the inclusion of any point, then $j^*(R_q(E)) = R_q(j^*(E)) = 1$. Let us note that due to the difference of conventions our class $R_q(E)$ is the multiplicative inverse of the class considered in [Mil89], therefore, if V is a real vector bundle, then $(R_q(E \otimes_{\mathbb{R}} \mathbb{C}))^{-1} = \theta(E)$. As the natural transformation $\lambda : \mathcal{E}\ell\ell^* \longrightarrow K_q^*$ commutes with extension by zero we can deduce from (16) that if $\overline{X} \xrightarrow{i} \overline{X}'$ is an inclusion and $a \in \mathcal{E}\ell\ell^*(T\overline{X})$, then

$$(Ti)_!^q(\lambda(\alpha) \cup R_q(\overline{TN})) = \lambda\left((Ti)_!^{\mathcal{E}\ell\ell}(\alpha)\right), \tag{17}$$

where $(Ti)_!^q$ is the analogue in K_q^* theory of the morphism $(Ti)_!^{\mathcal{E}\ell\ell}$ of (15). Let $\overline{\mathbb{R}^2} \xrightarrow{j} pt$ be the projection of $\overline{\mathbb{R}^2}$ onto a point let $TV \longrightarrow x$ be TV considered as a vector bundle over a point. Then $TV \times \overline{\mathbb{R}^2} = j^*(TV)$ and by naturality $R_q(TV \times \overline{\mathbb{R}^2}) = j^*(R_q(TV)) = 1$. Therefore if $\varphi : K_q^*(\overline{\mathbb{R}^2}) \longrightarrow K_q^*(TV \times \overline{\mathbb{R}^2})$ denotes the Thom isomorphism in K_q^* theory, then $\varphi(\lambda(a)) = \lambda(\phi(a))$, $\forall a \in \mathcal{E}\ell\ell^*(\overline{\mathbb{R}^2})$. From this equality and (17) follows immediately that

$$\lambda(\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell}(a)) = \text{t-index}^{\overline{X}}_q(\lambda(a) \cup R_q(\overline{TN})). \tag{18}$$

Considering V as a trivial bundle over X and using that

$$1 = R_q(V \otimes_{\mathbb{R}} \mathbb{C}) = R_q((N \oplus \overline{TX}) \otimes_{\mathbb{R}} \mathbb{C}) = R_q(N \otimes_{\mathbb{R}} \mathbb{C}) R_q(\overline{TX} \otimes_{\mathbb{R}} \mathbb{C})$$

and that $R_q(\overline{TX} \otimes_{\mathbb{R}} \mathbb{C})^{-1} = \theta(\overline{TX})$ we find that

$$\lambda(\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell}(a)) = \text{t-index}^{\overline{X}}_{KO}(\lambda(a) \cup \theta(\overline{TX})), \tag{19}$$

and therefore, if $\sigma \in \mathcal{E}\ell\ell^*(\overline{TX})$ is an orientation class, then

$$\Phi_k(\overline{X}) = \lambda\left(\text{t-index}^{\overline{X}}_{\mathcal{E}\ell\ell}(\sigma)\right) \in \lambda(\mathcal{E}\ell\ell^{\text{even}}).$$

The theorem follows immediately from this formula and proposition 3. □

Theorem 2. *Let $\overline{X}$ be an oriented $\mathbb{Z}/k$-manifold which is the boundary of a oriented $\mathbb{Z}/k$-manifold $\overline{M}$. Then $\Phi_k(\overline{X}) = 0$.*

Proof. The isomorphisms

$$MSO^n(pt,\mathbb{Z}/k\mathbb{Z}) \simeq MSO_{-n}(pt,\mathbb{Z}/k\mathbb{Z}),$$
$$MSO_n(pt,\mathbb{Z}/k\mathbb{Z}) \simeq MSO_n(M_k),$$
$$MSO^n(pt,\mathbb{Z}/k\mathbb{Z}) \simeq MSO^n(M_k)$$

induce an isomorphism $MSO^n(M_k) \simeq MSO_{-n}(M_k)$. We can therefore represent any class $[X] \in MSO^{-n}(M_k)$ by a pair (X,t) formed by an n-dimensional smooth manifold with an orientation in the stable normal bundle together with a map $t : X \longrightarrow M_k$. It is easy to see that the class $[X]$ is zero if and only if there exists a smooth manifold M' and a map $t' : M' \longrightarrow M_k \times \mathbb{R}$ such that $t'^{-1}(M_k \times \{0\}) = X$, $t'|_X = t$ and $t'^{-1}(M_k \times \{1\}) = \emptyset$. Let $X \xrightarrow{t} M_k$ the class in $MSO^{-n}(M_k)$ that corresponds to the $\mathbb{Z}/k$ manifold $\overline{X}$ by the geometric construction defined in section 1 and let $M \xrightarrow{t'} M_k \times \mathbb{R}$ be the manifold that represents the cobordism from X to 0. Let $M \xrightarrow{i} V$ be an embedding of M into a real vector space V and define $\tilde{p} : M \longrightarrow V \times M_k \times \mathbb{R}$ by $\tilde{p} = i \times t'$. For each $s \in \mathbb{R}$ we can, *by an arbitrary small perturbation of* $\tilde{p}$, *make it* transversal to $V \times S_k \times \{s\}$. *In this way we obtain an oriented* $\mathbb{Z}/k$*-manifold* $\overline{X}_s$ *and* an embedding $i_s : \overline{X}_s \longrightarrow V$ with normal bundle $N_s \longrightarrow \overline{X}_s$. This family can be defined in such a way so that $\overline{X}_0 = \overline{X}$ and $\overline{X}_1 = \emptyset$. Let σ_s be the Thom class in cobordism of the complex bundle $TN_s \longrightarrow \overline{TX}_s$. Then, by the homotopy invariance of cobordism, the one parameter family of classes $\rho_s = (Ti_s \times \rho^{TX_s})_!(\sigma_s) \in MSO_*(S^V \times \overline{\mathbb{R}^2})$ is constant. Therefore $\rho_0 = \rho_1 = 0$. The theorem follows immediately using that

$$\Phi_k(\overline{X}) = \lambda\left(\phi^{-1}\Phi_{\mathcal{E}\ell\ell}(\rho_0)\right).$$

□

It is trivial to see that if $\overline{X}$ and $\overline{X}'$ are two $\mathbb{Z}/k$-manifolds then $\Phi_k(\overline{X} \sqcup \overline{X}') = \Phi_k(\overline{X}) + \Phi_k(\overline{X}')$ and that if M is a smooth manifold considered as a $\mathbb{Z}/k$-manifold with empty Bockstein then $\Phi_k(M)$ is equal to the mod k reduction of $\Phi_{\mathcal{E}\ell\ell}(M)$. From the multiplicativity properties of the topological index in K-theory (axiom $\overline{B3}$ of [FM92]) and the fact that R_q is an exponential characteristic class it follows that $\Phi_k(M \times \overline{X}) = \Phi_k(M)\Phi_k(\overline{X})$. Therefore the mod k elliptic genus is a MSO_*-module homomorphism.

Theorem 3. *The* MSO_**-module homomorphism*

$$\Phi_k : MSO_*(pt,\mathbb{Z}/k) \longrightarrow \mathrm{Mod}(\Gamma_0(2),\mathbb{Z}/k)$$

is an epimorphism.

Proof. Let $\overline{\vartheta}$ be a mod k modular form and let $\vartheta \in \mathcal{E}\ell\ell^*$ be a modular form whose mod k reduction is $\overline{\vartheta}$. Then there exist an oriented manifold M such that $\Phi_{\mathcal{E}\ell\ell}(M) = \vartheta$ and therefore $\Phi_k(M) = \overline{\vartheta}$. □

Remark 3. The universal coefficient theorem implies that there is an isomorphism $\mathcal{E}\ell\ell^*(pt, \mathbb{Z}/k) = \mathcal{E}\ell\ell^*(pt) \otimes \mathbb{Z}/k \not\simeq \mathrm{Mod}(\Gamma_0(2), \mathbb{Z}/k)$ so there is a difference between the mod k elliptic genus and the usual one. The difference is due to the fact that $\lambda : \mathcal{E}\ell\ell^*(pt, \mathbb{Z}/k) \longrightarrow \mathrm{Mod}(\Gamma_0(2), \mathbb{Z}/k)$ is not an isomorphism. In order to explain the difference, and give a modular interpretation to elliptic cohomology with coefficients in $\mathbb{Z}/k$, let us recall an algebro-geometric definition of modular forms (see [Kat73]). *A modular form* f of weight k for $\Gamma_0(2)$ is a rule that assigns to each triple $(E|spec(R),\ \phi,\ \overline{M})$ formed by an elliptic curve $E \xrightarrow{p} spec(R)$ over the spectrum of a ring R such that $\frac{1}{2} \in R$ together with a cyclic subgroup $G[2] \in E[2]$, where $E[2] = \ker[2] : E \longrightarrow E$ is the kernel of multiplication by two in the group-scheme structure of E, and a basis ω of $\omega_{E|spec(R)} = p_*(\Omega_{E|specR})$ an element of R in such a way that the following conditions are fulfilled

1: the element $f(E|spec(R),\ \phi,\ \omega)$ depends only on the R-isomorphism class of the triple $(E|spec(R),\ \varphi,\ \omega)$,
2: f is homogeneous of degree $-k$ in the third variable,
3: the formation of f commutes with arbitrary extensions of scalars.

Let us call $H^0(M|_{specR}, \omega^k)$ this group. Then the ring of modular forms (in this language) is $mod(R) = \oplus_{k\geq 0} H^0(M|_{specR}, \omega^k)$. There is, for each k, a group homomorphism (we shall call it the expansion) $H^0(M|_{specR}, \omega^k) \longrightarrow R\ [[\ q\ ]]$ which is obtained by evaluating a modular form on Tate's curve with its canonical subgroup of order two. The difference with the usual case is that modular forms of different weights can have the same expansions (see for example Deligne's congruence $E_{p-1} = 1 \bmod p$). It is not difficult to show that if $\frac{1}{2} \in R$ then $mod(R) = R[\delta, \epsilon]$. In particular taking $R = \mathbb{Z}/k$ we obtain a modular interpretation of the elliptic cohomology of a point with coefficients in $\mathbb{Z}/k$.

§5 **Final Remarks.** Let us make some final remarks about the relation of the elliptic genera for $\mathbb{Z}/k$ manifolds and the theory developed in [Sul70b] chapter 6. This remarks are only the "top of the iceberg" of a relation between elliptic cohomology and Sullivan's theory that we shall explain elsewhere. Let n be an odd number. Then there are two natural homomorphisms of groups $r : \mathbb{Z}/kn\mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z}$ and $i : \mathbb{Z}/k \longrightarrow \mathbb{Z}/kn\mathbb{Z}$ induced by the homomorphisms $\mathbb{Z} \xrightarrow{id} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. These homomorphisms can be extended, for each j, in the obvious way to group homomorphisms $\mathrm{Mod}_j(\Gamma_0(2), \mathbb{Z}/kn) \xrightarrow{rm} \mathrm{Mod}_j(\Gamma_0(2), \mathbb{Z}/k)$ and $\mathrm{Mod}_j(\Gamma_0(2), \mathbb{Z}/k) \xrightarrow{im}$

$\mathrm{Mod}_j(\Gamma_0(2), \mathbb{Z}/kn)$. Sullivan showed in [Sul70b] that r_G and i_G can also be geometrically defined on $\mathbb{Z}/kn$-manifolds and $\mathbb{Z}/k$-manifolds and that they induce "coefficient homomorphisms" $MSO_j(\mathbb{Z}/kn\mathbb{Z}) \xrightarrow{r_G} MSO_j(\mathbb{Z}/k)$ and $MSO_j(\mathbb{Z}/k) \xrightarrow{i_G} MSO_j(\mathbb{Z}/kn\mathbb{Z})$ and an exact ladder

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & MSO_*(-) & \xrightarrow{.n} & MSO_*(-) & \longrightarrow & MSO_*(-,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{.k} & & \downarrow & & \\ \cdots & \longrightarrow & MSO_*(-) & \xrightarrow{.kn} & MSO_*(-) & \longrightarrow & MSO_*(-,\mathbb{Z}/kn\mathbb{Z}) & \longrightarrow & \cdots. \end{array}$$

It is easy to see that there exists a commutative diagram

$$\begin{array}{ccc} MSO_*(\mathbb{Z}/k) & \xrightarrow{i_G} & MSO_*(\mathbb{Z}/kn\mathbb{Z}) \\ \Phi_k \downarrow & & \Phi_{kn} \downarrow \\ \mathrm{Mod}(\Gamma_0(2), \mathbb{Z}/k) & \xrightarrow{i_m} & \mathrm{Mod}(\Gamma_0(2), \mathbb{Z}/kn\mathbb{Z}). \end{array} \tag{20}$$

Recall now that

$$\mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right] = \bigoplus_{p \text{ odd prime}} \mathbb{Z}_{p^\infty} = \bigoplus_{p \text{ odd prime}} \varinjlim_{n} \mathbb{Z}_{p^n}.$$

and that one can [Sul70b] define, using the exact ladder and the coefficient homomorphisms, oriented bordism with coefficients in $\mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]$ by the equality

$$MSO_*\left(-, \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]\right) = \varinjlim_{n \text{ odd}} MSO_*(-, \mathbb{Z}/n\mathbb{Z}).$$

If we define

$$\mathrm{Mod}_j\left(\Gamma_0(2), \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]\right) = \varinjlim_{n \text{ odd}} \mathrm{Mod}(\Gamma_0(2), \mathbb{Z}_n),$$

then by diagram (20) we obtain a MSO^* module homomorphism

$$\Phi_{tor} : MSO_j(pt, \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]) \to \mathrm{Mod}(\Gamma_0(2), \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]).$$

Taking the limit in the exact sequence given by the exact ladder we obtain an exact sequence

$$\cdots \to MSO_*(-, \mathbb{Z}\left[\frac{1}{2}\right]) \to MSO_*(-, \mathbb{Q}) \xrightarrow{t} MSO_*(-, \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]) \to \cdots.$$

From the definition of mod k modular forms it follows that there exists a sequence

$$\mathrm{Mod}_j(\Gamma_0(2), \mathbb{Z}\left[\frac{1}{2}\right]) \to \mathrm{Mod}_j(\Gamma_0(2), \mathbb{Q}) \xrightarrow{t'} \mathrm{Mod}_j(\Gamma_0(2), \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]).$$

From these exact sequences and the compatibility of all the constructions involved we see that the elliptic genera considered fit into a commutative diagram

$$\begin{array}{ccc} MSO_j(pt,\mathbb{Q}) & \xrightarrow{\Phi} & \mathrm{Mod}(\Gamma_0(2),\mathbb{Q}) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ MSO_j(pt,\mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]) & \xrightarrow{\Phi_{tor}} & \mathrm{Mod}(\Gamma_0(2),\mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right]). \end{array}$$

The existence of this square is an "integrality" condition for the polynomials in the Pontrjagin classes that defines the elliptic genus.

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