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Neutrino properties from maximally-predictive GUT models and the structure of the heavy Majorana sector.

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Abstract

Starting from a complete set of possible parametrisations of the quark-mass matrices that have the maximum number of texture zeros at the grand unification scale, and the Georgi-Jarlskog mass relations, we classify the neutrino spectra with respect to the unknown structure of the heavy Majorana sector. The results can be casted into a small number of phenomenologically distinct classes of neutrino spectra, characterised by universal mass-hierarchy and oscillation patterns. One finds that the neutrino masses reflect the natural hierarchy among the three generations and obey the quadratic seesaw, for most GUT models that contain a rather unsophisticated Majorana sector. A scenario with ν_τ as the missing hot dark matter component and $\nu_e \leftrightarrow \nu_\mu$ oscillations accounting for the solar neutrino deficit comes naturally out of this type of models and is very close to the experimental limit of confirmation or exclusion. In contrast, in the presence of a strong hierarchy of heavy scales or/and some extra symmetries in the Majorana mass matrix, this natural hierarchy gets distorted or even reversed. This fact can become a link between searches for neutrino oscillations and searches for discrete symmetries close to the Planck scale.

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1 Introduction.

The quest for understanding the Yukawa sector of the Standard Model (SM), which could mean finding, as a first step, simple fermion-mass and quark-mixing relations among the members of the three known families, represents an equally significant challenge to the standard model (SM) as the prospect of unification of the three gauge couplings within the scope of a more fundamental theory. As a matter of fact, the two problems are related to each other, as most grand-unified (GUT) models imply also some partial unification of Yukawa couplings. The by now famous $m_b = m_\tau$ equality between the mass of the bottom quark and the corresponding charged lepton of the third family at M_G, the scale of grand unification, has been one of the early successes of minimal SU(5) [1]. For the first two families, where it is empirically known that $m_d/m_e \simeq 10\, m_s/m_\mu$, rather than $\simeq m_s/m_\mu$, the hope was (and still is) that some other effect may modify these simple mass relations, without significantly altering the two-Yukawa coupling unification scheme. A most promissing attempt in this direction has been the *Ansatz* of Georgi and Jarlskog (GJ)[2], subsequently implemented also into other (so-called predictive) GUT models [3]-[8]. In some models based on the $SO(10)$ group one is even led towards a unification of all three Yukawa couplings of e.g. the third family: [9] $h_t(M_G) = h_b(M_G) = h_\tau(M_G)$.

The structure of the three-generation Yukawa matrices, which parametrise the couplings of the fermions to the Higgs sector, is commonly attributed to the existence of $U(1)$ - axial horizontal symmetries, broken at some intermediate scale between the electroweak and the unification scale, or/and to the existence of discrete symmetries that often appear after compactification of the superstring [10]. In the hope of finding some fundamental symmetry of this type, a different approach has been put forward lately [4, 11, 7]. Instead of trying to accomodate the empirically known mass and mixing parameters in different GUT models, the procedure has been to first find appropriate *Ansätze* for the structure of the Yukawa matrices at M_G which give the correct values at low energies. In order to limit the number of possible choices, and be predictive with respect to an expected improvement of the experimental values of the 13 mass- and mixing-parameters of the SM in the near future, the principle of economicity has again been applied, meaning as few input parameters as possible. One way to meet this requirement is namely to have as many zero entries as possible and/or some extra symmetry, e.g. in flavour space, as this is natural for models of the $SO(10)$ group. *Ansätze* of this type have been made in the past by Fritzsch [12] and Stech [13] for the quark mass matrices at the electroweak scale. More recently, Dimopoulos, Hall and Raby (DHR)[4] and Giudice [11] have proposed new parametrisations of the Yukawa sector at M_G with seven or six parameters only, to describe the six quark masses, the three mixing angles and the CP-violating phase of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. For the running of the Yukawa couplings between M_G and the low energy scale, additional assumptions on the Higgs sector and particle content of the theory are needed. Since it was shown that precision data from LEP are consistent with unification of the three gauge couplings within the minimal supersymmetric standard model (MSSM) [14], it has become the most popular candidate for a description of the physics between the electroweak scale M_Z and the unification scale M_G, which in this context turns out to be of the order of $M_G \simeq 10^{16}$ GeV. However, as most SUSY-GUT models share the same particle content with the MSSM at energies below M_G, while all nonsupersymmetric GUTs suffer from the so-called hierarchy problem, this choice is representative for the evolution of a whole class of models [9].

In a recent paper Ramond, Roberts and Ross (RRR)[7] have reversed this approach and, starting from what is measured at low energy, have provided a classification of all possible sets of quark-Yukawa matrices, which are hermitian, i.e. symmetric in flavour space, and have five or six texture zeros at M_G. In this way a unified picture in terms of a perturbative generation of the quark-Yukawa sector at the scale M_G has been achieved, which incorporates the Fritzsch, the DHR, and the Giudice *Ansätze*, and sets the level of accuracy needed to discriminate between them by improved measurements of the CKM matrix elements.

By means of the Georgi-Jarlskog Ansatz, which successfully relates the Yukawa couplings of the charged leptons to those of the down quarks at M_G, the DHR [5] and other groups [6] have in addition to the quark sector been able to make predictions concerning also the existence of mixing in the lepton sector, if the three ordinary left-handed neutrinos of the SM were to obtain a small mass through mixing with extra heavy (right-handed) neutrino-like states by the so-called seesaw mechanism. Predictions for this sector are possible only because the very idea of grand unification naturally implies some proportionality relations among the Higgs-Yukawa couplings of the fermions belonging to the same multiplet. Thus in models with an $SO(10)$ symmetry, the neutrino-Yukawa couplings which are of the Dirac type are usually proportional to those for the up-type quarks. However the structure of the Majorana-type Yukawa matrix, responsible for giving large masses to the right-handed neutrino states, thus leading to seesaw-suppressed values for the masses of the ordinary neutrinos, is in general not known [1] and has been chosen, as a matter of convenience, to be diagonal or proportional to the up-quark (or down-quark) mass matrices [6], [17], [18]. Due to its fundamental importance in providing the only known mechanism for suppressing the unacceptably large neutrino masses implied by most GUT models,

[1] For counter examples see refs.([5], [15], [16]).

one would like to know the full phenomenological impact also of this Yukawa sector, whose scale and structure are most likely determined by the physics at the Planck scale. Moreover, it seems to be the case that different models give similar neutrino spectra [5], [6], pointing towards some universality that may follow on one hand from the simple GUT relations and on the other hand from some extra symmetries at the Planck scale. It could well be that a new classification scheme with respect to the structure of the heavy Majorana-mass sector and its predictions for the masses and the mixing of the leptons could shed some more light on the latter. With the prospect of getting close to probing interesting regions of Δm^2, the mass-difference squared of two neutrino species, and $sin^2 2\theta$, the parameter which gives their mutual mixing, in coming neutrino-oscillation experiments (the NOMAD, the CHORUS, the ICARUS and other proposals) [19], the possibility of testing one of the basic ideas of grand unification through a classification of such maximally predictive GUT models could become an interesting project. This paper will be devoted to these questions.

We will start in section (2) with a review of the expectations on neutrino masses and mixing from theory, experiment and observation and discuss the common prejudices. We will then in section (3) extend the RRR approach to the lepton sector such that the structure of the charged-lepton Yukawa matrix is fixed by the GJ relations to the matrix of the down-type quarks and the one of the Dirac-neutrino states to the up-type quarks as implied by the simplest implementation of the GUT idea. The unknown heavy Majorana-neutrino sector is first chosen to be arbitrary and general, to be subsequently classified according to the mass- and mixing-patterns it leads to after diagonalisation of the effective neutrino mass matrix. In section (4) we will examine four cases leading to universal neutrino mass and mixing patterns characterised by a specific hierarchical order. In section (5), we discuss the implications for the coming neutrino oscillation experiments, and in section(6), we give the conclusions.

2 Massive neutrinos between hope and prejudice.

The absence of a gauge or any other principle that could justify a zero mass for the neutrino as it does for the photon may be one of the strongest theoretical prejudices in favour of massive neutrinos. The exasperation with the "overly" successful Standard Model and a general attitude of looking forward to seeing "new physics" has been the prejudice's driving force. Three neutrino-deficit phenomena based on astrophysical observation and cosmological considerations have become the nurishing substance of this hope. The three experimental uper bounds on the masses of the tau- muon- and electron neutrino of 32 MeV, 270 keV and 1 eV (from double-beta decay) respectively have along with other data on neutrino oscillations severly limited the range of beyond-the-SM speculations [20]. The latter have also helped to articulate new questions, as to why for example the electron neutrino is so much lighter than the electron -at least some five orders of magnitude- , adding some extra power to the mass-hierarchy puzzle, to which the seesaw mechanism [21] became a common reply.

2.1 The seesaw mechanism.

The idea of the seesaw, first implemented in partially or completely unified theories with a left-right symmetry such as $SO(10)$ [21], is based upon the simple fact that for a mass matrix of the type:

$$M = \begin{pmatrix} 0 & a \\ a & b \end{pmatrix}, \tag{1}$$

where $a \ll b$, a simple rotation leads to eigenvalues with a large mass splitting:

$$m_1 \sim \frac{a^2}{b} \qquad m_2 \sim b, \tag{2}$$

and a simple mass and mixing-angle relation:

$$\frac{1}{2} tan 2\theta \simeq \frac{a}{b}, \tag{3}$$

and therefore to:

$$sin\theta \sim \theta \sim \sqrt{\frac{m_1}{m_2}}. \tag{4}$$

So the main lesson of this trivial exercise is the twofold quadratic scaling behaviour of the mass ratio m_1/m_2, that is at the same time proportional to a^2/b^2 and to $sin^2\theta$.

When M represents the mass matrix for one generation of neutrinos, written in the left- and right-handed neutrino basis (ν, N^c), a is a small Dirac mass m_D that is proportional to a quark or lepton mass, and b a large Majorana mass R for the right-handed neutrino that is proportional to some scale $M_X \gg M_Z$:

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D & R \end{pmatrix}. \tag{5}$$

The fact that the entry in the upper left corner of M_ν is usually zero, signifies the absence of an $SU(2)_L$ Higgs triplet that could give a Majorana mass also to the left-handed neutrino. It is interesting to note that the "seesaw" notion, *i.e.* that the mixing of a light state with a heavy state in the mass matrix can render the former even lighter, is employed only in the context of a Majorana-neutrino sector and/or in the presence of heavy singlets, mixing with the quarks [22], but never in the context of the quark-Yukawa matrices which are nevertheless of the same perturbative type (see *e.g* equ.(33)

In the case of a two-generation or three-generation neutrino mixing, where m_D and R are replaced by the corresponding 2×2 or 3×3 matrices M_u and M_R, one finds similar relations among the masses and the mixing of the light neutrinos if, after block diagonalisation of:

$$M_\nu = \begin{pmatrix} 0 & M_u \\ M_u^T & M_R \end{pmatrix} , \tag{6}$$

and assuming that the matrix M_R is not singular, the effective light-neutrino mass matrix:

$$M_\nu^{eff} \simeq M_u \, M_R^{-1} \, M_u^T , \tag{7}$$

is fully or partly of the same perturbative type as M [2]. In this case however, the light-neutrino mass eigenvalues will in general be proportional to a more complicated function of the entries of the matrices M_u and M_R. Considering the case of weak intergenerational mixings in M_R and a perturbative structure (*à la* Fritzsch) for M_u, the ratio of any two light neutrino masses exhibits a simple scaling behaviour:

$$\frac{m_{\nu(x)}}{m_{\nu(y)}} = \frac{(m_{u_i} m_{u_j})_{(x)}}{(m_{u_k} m_{u_l})_{(y)}} \cdot \frac{R_{m(y)}}{R_{n(x)}} \cdot a_{(xy)} , \tag{8}$$

where $i...n$ are generation indices, x and y label the two neutrino species, and $a_{(xy)}$ may be any ratio of additional heavy Majorana scales participating in the seesaw.

Now in the case where M_R has only one scale, equ.(8) reduces either to the quadratic seesaw:

$$\frac{m_{\nu(x)}}{m_{\nu(y)}} = \frac{m_{u_i}^2}{m_{u_j}^2} , \tag{9}$$

or to the linear seesaw:

$$\frac{m_{\nu(x)}}{m_{\nu(y)}} = \frac{m_{u_i}}{m_{u_j}} , \tag{10}$$

or to the mixed case:

$$\frac{m_{\nu(x)}}{m_{\nu(y)}} = \frac{m_{u_i} m_{u_j}}{m_{u_k}^2} . \tag{11}$$

Futhermore it is easy to check that when the matrix M_R can be diagonalised simultaneously with the matrix M_u one obtains the quadratic seesaw with a neutrino mass hierarchy corresponding to the quark or charged-lepton mass hierarchy, *i.e.* $x = i$ and

[2]When M_u is chosen to be a real matrix, as this is a common choice for the mass matrix of the up-quark sector, M_u^T is its transpose. In general though, M_ν might not even be a hermitian matrix, in which case one will have to diagonalise the matrix $M_\nu M_\nu^\dagger$ which is hermitian for any complex matrix M_ν.

$y = j$, and a simple proportionality relation between the quark- and lepton-mixing matrices:

$$V_l \simeq V_{CKM} \,. \tag{12}$$

This is indeed the simplest version of seesaw that one can have. In contrast, in the presence of a strong hierarchy and/or strong intergeneration mixings in M_R, one would rather expect a distortion of this rather "natural" hierarchy pattern, up to the point that it actually gets reversed. This is an interesting possibility to explore that would have important implications for future neutrino-oscillation experiments [23].

2.2 The neutrino-deficit problem.

Let us now come to the observed neutrino deficits. The most prominent one, since it has been confirmed by four different experiments involving three different targets (Cl^{37}, e^- in H_20, and Ga^{71}) [24], is the solar neutrino deficit: The flux of the neutrinos coming from the sun and measured sofar is half to one third of the expected number of SNUs calculated by two groups using the standard solar model (SSM) [25]. Apart from a ever decreasing probability of resolving this discrepancy by altering the parameters of the SSM [26] and while waiting for the calibration of the two Ga^{71} experiments, the most promissing solution seems to be the Pontecorvo or the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism of vacuum or matter-enhanced oscillations of the electron neutrinos from the sun into some other neutrino species in their way to the earth [27]. Analysing the latest data of the four experiments Krastev and Petcov have found three disconnected areas in the Δm^2 and $sin^2 2\theta$ plot where such two- or even three-neutrino-flavour oscillations are allowed [29]. The vacuum-oscillation solution (VOS) is characterized by very small mass differencies and a large mixing angle:

$$\begin{aligned} \Delta m^2 &\simeq (0.5 - 1.) \times 10^{-10} eV^2 \\ sin^2 2\theta_{\nu_e - \nu_x} &\simeq 0.75 - 1. \,, \end{aligned} \tag{13}$$

while in the case of matter-enhanced oscillations there is a small-angle non-adiabatic MSW solution:

$$\begin{aligned} \Delta m^2 &\simeq (0.3 - 1.2) \times 10^{-5} eV^2 \\ sin^2 2\theta_{\nu_e - \nu_x} &\simeq (0.5 - 1.6) \times 10^{-2} \,, \end{aligned} \tag{14}$$

and a large angle solution:

$$\begin{aligned} \Delta m^2 &\simeq (0.3 - 3.) \times 10^{-5} eV^2 \\ sin^2 2\theta_{\nu_e - \nu_x} &\simeq 0.6 - 0.7 \,, \end{aligned} \tag{15}$$

where the upper range of $sin^2 2\theta$ in the last equation has been reduced from 0.9 to $\simeq 0.7$ due to the non-observed effect of double conversion of the electron neutrinos from the supernova SN87A [30]. The quoted numbers strictly hold for transitions of ν_e's into ν_μ's or ν_τ's. For a transition into a sterile neutrino state (with respect to the electroweak interaction) ν_s, the allowed range for $sin^2 2\theta$ shrinks to smaller values in the case of a small-angle solution and to larger values in the case of a large-angle MSW solution [29].

Compared to the Cabibbo quark-mixing angle $sin\theta_c \equiv s_{12} \simeq 0.22$, the value of the neutrino-mixing suggested by the small-angle MSW solution is smaller and could indeed correspond to a lepton-mixing angle: [3] $s^l_{12} \simeq 0.04 - 0.07 \simeq \sqrt{\frac{m_e}{m_\mu}}$, unless it represents the mixing between the first and third generation, in which case it is larger than the corresponding one in the quark sector. In contrast, the values suggested by the large-angle solution always exceed the values of the CKM matrix elements and, given the very small mass differences they correspond to, they would rather represent an anomaly in any attempt to find universal relations for all fermion sectors among the fermion masses and the mixing angles. A second general remark concerns the range of the masses that one would expect for neutrinos participating in such oscillations. Unless there is a symmetry to guarrantee small mass differences between masses whose scale lies much higher, and in order to avoid fine tuning, one would expect at least one of the neutrinos to have a mass in the range of $m_{\nu_i} \simeq (\Delta m^2)^{1/2}$, which in the case of matter-enhanced oscillations implies a mass scale $\Lambda_1 \simeq (2-3) \times 10^{-3}$ eV.

A second neutrino deficit - a more controversial topic due to the large experimental error bars and counter evidence from two experiments - has been reported by several experimental groups [31] with respect to the expected ratio of muon- to electron-neutrino flux produced by hadronic collisions in the upper atmosphere and measured deep underground. Again a straight forward explanation of this can be given if one assumes that the muon neutrinos oscillate into other light-neutrino species with the values of:

$$\begin{aligned} \Delta m^2 &\simeq (0.5 - 0.005) eV^2 \\ sin^2 2\theta_{\nu_\mu - \nu_x} &\simeq 0.5 \end{aligned} \qquad (16)$$

Since the option of a $\nu_\mu - \nu_e$ oscillation in the range of $\Delta m^2 < 0.007 eV^2$ for $sin^2 2\theta \simeq 1$ and of large Δm^2 for $sin^2 2\theta \leq 4 \times 10^{-3}$ has been excluded from reactor and accelerator experiments [33], there remain only two possibilities, *i.e.* a transition into a tau neutrino or a sterile neutrino-like state. However the possibility of ν_μ or any other active neutrino component oscillating into a light sterile neutrino state according to the range of parameters suggested by equ.(16) seems to be excluded from the data on nucleosynthesis, which imply that the effective neutrino degrees of freedom consistent

[3] We use the following short-hand notations: $sin\theta_{ij} \equiv s_{ij}$ and $cos\theta_{ij} \equiv c_{ij}$.

with the measured H_e abundance is less than 3.3, and lead to a constraint of [34]:

$$\Delta m^2 \, sin^2 2\theta \leq 3.6 \times 10^{-6} eV^2 \, .$$

It should also be noted that the latest results of the BAKSAN and IMB background measurements of upcoming muons seem to narrow down the allowed range for such oscillations to a vanishingly small area [20].

Now if both neutrino deficits were to be confirmed and interpreted as neutrino oscillations between massive SM neutrinos this would naturally imply the following mass hierarchy in the neutrino sector: $m_{\nu_e} \ll m_{\nu_\mu} \sim 10^{-3}$ eV $< m_{\nu_\tau} \sim 10^{-1}$ eV. This scenario is however at odds with the possibility of resolving as well the third observed deficit, known as the hot dark matter (HDM) problem, in a scenario with three light neutrinos only. The COBE results on the anisotropy of the cosmic microwave background radiation and the data on the angular correlations of galaxies and galactic clusters can be best fitted by a "coctail" of 70% cold dark matter and 30% hot dark matter [32], the latter consisting of neutrinos with mass of a few electronvolts:

$$\Sigma_i m_{\nu i} \simeq 7eV \, , \tag{17}$$

the sum being over all light neutrino components. Assuming a "natural-hierarchy" scenario for the neutrino masses [23], to be discussed in more detail later on, this would point towards a 7 eV tau neutrino which can therefore not fulfill its role as a participant in a $\nu_\mu - \nu_\tau$ oscillation in the earth's atmosphere [35]. If all three observed deficits were to be of the same origin, *i.e.* related to non-standard neutrino properties and in particular to oscillations between different mass eigenstates, another light (sterile) neutrino state would be needed in order to avoid a fine-tuned mass matrix and one would be left with two scenarios [35]: Either the three known neutrinos participate in oscillations which allow for a resolution of the solar and atmospheric neutrino deficits so that a massive sterile neutrino would be needed to account for the HDM, or the muon and tau neutrinos have masses of a few electronvolts so as to be the main components of the HDM and participate in oscillations such as required for the atmospheric neutrino puzzle to be resolved, in which case there should exist $\nu_e - \nu_s$ oscillations, responsible for the solar neutrino deficit. It seems that in the first scenario it is very hard to reconcile the nucleosynthesis constraint with the HDM requirement. On the other hand the second scenario requires a slight fine tuning of the muon and tau neutrino masses. So there is no scenario which would naturally satisfy all three requirements. Of course once the requirement of resolving all three puzzles via neutrino masses and oscillations, which is by no means compelling, is dropped, more scenarios are available to speculation.

Since the existence of an atmospheric neutrino deficit is experimentally disputable we would like in what follows to focus on scenarios which could do away only with the solar neutrino puzzle and provide the relativistic component needed in order to resolve the dark matter problem, assuming the existence of three light neutrinos only. Let us denote by m_{ν_1}, m_{ν_2} and m_{ν_3} the corresponding mass eigenstates in increasing mass order. We first notice that in order to satisfy both requirements there should be at least two mass scales, *e.g.*, [4] $\Lambda_1 \simeq (2-3) \times 10^{-3}$ eV and $\Lambda_2 \simeq 7$ eV and thus a hierarchy of:

$$\Lambda \equiv \frac{\Lambda_1}{\Lambda_2} \sim (3-4) \times 10^{-4} . \tag{18}$$

This hierarchy could then be realised in two different ways: One possibility would be that the light-neutrino masses follow a similar hierarchy pattern as the three up- or down-type quarks,

$$m_{\nu_e} \leq m_{\nu_\mu} \sim \Lambda_1 \ll m_{\nu_\tau} \sim \Lambda_2 , \tag{19}$$

where $m_{\nu_{e,\mu}}$ is the dominant component of $m_{\nu_{1,2}}$ and $m_{\nu_\tau} \simeq m_{\nu_3}$. We shall refer to this as the "natural hierarchy"pattern. The alternative then would be to have a reversal of the natural hierarchy pattern, *i.e* to have:

$$m_{\nu_e} \leq m_{\nu_\tau} \sim \Lambda_1 \ll m_{\nu_\mu} \sim \Lambda_2 , \tag{20}$$

or [5]

$$m_{\nu_\tau} \leq m_{\nu_e} \sim \Lambda_1 \ll m_{\nu_\mu} \sim \Lambda_2 . \tag{21}$$

Such an "inverse hierarchy" would naturally imply a strong $\nu_e - \nu_\tau$ mixing.

We would like now to consider the possibility that there is indeed a natural hierarchy among the neutrino masses and see whether it could more likely originate from a quadratic or a linear seesaw mechanism. Assuming that the heavy Majorana sector has only one scale R, the quadratic seesaw would yield a neutrino-mass ratio in terms of the masses of the up-type quarks,

$$\frac{m_{\nu_2}}{m_{\nu_3}} \sim \frac{m_c^2}{m_t^2} \sim (0.5-3) \times 10^{-4} \simeq \mathcal{O}(\Lambda) , \tag{22}$$

that is compatible with the value of Λ as required by the two MSW solutions to the solar neutrino problem, while the linear seesaw in terms of masses of any of the three fermion sectors leads to values that are considerably larger ($\sim 10^{-2}$), and the quadratic seesaw in terms of down-quark or charged-lepton masses leads to smaller

[4]For the moment we consider only the MSW solutions to the solar neutrino problem.

[5]Notice that the upper bound on the electron-neutrino mass is such that it cannot at the same time be the heaviest neutrino and an HDM candidate.

values of order 10^{-3}. Despite the uncertainty coming from the charm- and top-quark masses - evaluated at the electroweak scale M_Z -, clearly the hierarchy suggested by equ.(19) seems to follow from the quadratic rather than the linear seesaw and involves the masses of the up quarks. On the other hand, if one would try to understand the small-angle MSW mixing in equ.(14) in terms of some power of the mass ratio of the corresponding up-type quarks one would find that:

$$s^l_{12} \simeq \sqrt{\frac{m_{\nu_1}}{m_{\nu_2}}} \simeq \sqrt{\frac{m_u}{m_c}}, \tag{23}$$

a relation which is typical for a linear seesaw. Apparently the way neutrino masses are generated, seems to differ when going from the third to the second generation or from the second to the first. The simplest mass pattern giving rise to such a scenario could *e.g.* be:

$$m_{\nu_1} \sim \frac{m_u m_c}{R} \tag{24}$$

$$m_{\nu_2} \sim \frac{m_c^2}{R} \tag{25}$$

$$m_{\nu_3} \sim \frac{m_t^2}{R}. \tag{26}$$

Using again the naive mass and mixing-angle relations of equ.(4), one can estimate the remaining two neutrino-mixing angles and compare them to the corresponding quark-mixing angles [36]:

$$s^l_{23} \sim \frac{m_c}{m_t} \quad < \quad s_{23} \sim 0.03 - 0.05 \tag{27}$$

$$s^l_{13} \sim \sqrt{\frac{m_u m_c}{m_t^2}} \quad \ll \quad s_{13} \sim (2. - 7.) \times 10^{-3}. \tag{28}$$

The scale of the Majorana sector can be fixed by requiring that the heaviest neutrino be the HDM candidate:

$$m_{\nu_3} \simeq m_{\nu_\tau} \simeq 7eV. \tag{29}$$

This gives:

$$R \sim \mathcal{O}(10^{12})\, GeV, \tag{30}$$

and masses to the other neutrinos:

$$m_{\nu_1} \sim 10^{-5} eV \tag{31}$$

$$m_{\nu_2} \sim 10^{-3} eV. \tag{32}$$

We turn next to alternative scenarios, that could also provide a solution to the solar neutrino problem and the HDM, but do not follow directly from the simple seesaw relations, eqs.(9,10,11). Under this category fall the two large-angle solutions to the solar neutrino problem, eqs.(13,15), simply because they indicate that the two neutrinos participating in such oscillations are linear combinations of mass-degenerate eigenstates: $\nu_{1/2} \simeq \frac{1}{2}(\nu_e \pm \nu_x)$ with $m_{\nu_1} \simeq m_{\nu_2}$. This is only possible in the context of a more elaborate heavy Majorana sector, where for example M_R or some of its subdeterminants could be singular, and/or the presence of a large-scale hierarchy can compensate the hierarchy in M_u. Obviously the same is true also of any inverse-hierarchy scenario. Therefore these options require a full treatment of the neutrino mass matrix within the context of a particular model or at least of an *Ansatz.*

On the other hand, the semi-quantitative treatment of the natural-hierarchy scenario should be also viewed with a great deal of caution, for two main reasons: First because even in the quark sector the simple relations we have employed are only valid in the approximation of two-generation mixing. As an example we take the original Fritzsch model [12], where the mass matrices for the up- and down-type quarks were parametrised according to:

$$M_F = \begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & C \end{pmatrix}, \tag{33}$$

with A, B, C written in terms of the up-quark (down-quark) masses:

$$A \simeq \sqrt{m_{u(d)} \cdot m_{c(s)}} \ll B \simeq \sqrt{m_{c(s)} \cdot m_{t(b)}} \ll C \simeq m_{t(b)} . \tag{34}$$

For the large Cabibbo-mixing angle one does indeed recover the simple Gatto-Sartori-Tonin-Oakes relation [37]:

$$s_{12} \simeq \sqrt{\frac{m_d}{m_s}}, \tag{35}$$

but for the small mixing angles s_{23} and s_{13} a fine-tuning of the quark phases ϕ_i is needed

$$s_{23} \simeq -\sqrt{\frac{m_s}{m_b}} + e^{i\phi_1}\sqrt{\frac{m_c}{m_t}} \qquad s_{13} \simeq -\sqrt{\frac{m_u}{m_c}} e^{i\phi_2} \cdot s_{23} , \tag{36}$$

for obtaining the values that have been measured.

The second and more substantial criticism concerns the very existence of any such relations that could be scale independent. There is namely no apriori reason why any relation among the various observables of the fermion-mass and mixing matrices should remain invariant under the renormalization group (RG) equations which relate them to the structure of the Higgs-Yukawa interaction at some more fundamental scale. In fact, Olechowski and Pokorski [38] have shown that such approximate low-energy relations can be preserved in the presence of a strong top-Yukawa coupling, or more generally, when the Yukawa couplings of one fermion family become predominant, a fact that holds true for the third generation. This was shown by expressing the masses and mixing angles in terms of invariants of the Yukawa matrices and then study their evolution, assuming that the latter is governed by the radiative corrections to the gauge and the Higgs-Yukawa couplings coming from the MSSM, or other models containing two doublets of Higgs bosons, or simply the SM. Interestingly, they obtained some universal results. First, the evolution of the Cabibbo angle is for all models negligible. Writing the CKM matrix in the Wolfenstein parametrisation [39]:

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho+i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho+i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{37}$$

where the small expansion parameter $\lambda \simeq 0.22$ is approximately the Cabibbo angle, one finds that only the parameter $A \simeq 0.9$ (and the CP phase) evolves when going from a low-energy scale M_0 to a high-energy scale M_X:

$$\frac{dA}{dx} \simeq -\frac{3c_0}{2}(h_t^2 + h_b^2)A\,, \tag{38}$$

while:

$$\frac{d\lambda}{dx} \sim \mathcal{O}(\lambda^4)\,, \tag{39}$$

where $x = 1/16\pi^2\, ln(M_X/M_0)$, h_t and h_b are the top and bottom Yukawas, and the constant c_0 is determined by the gauge couplings of the model, *e.g.* $c_0 = 2/3$ for the MSSM. An interesting consequence of this evolution behaviour is that in the MSSM the small mixing elements $|V_{13}|$ and $|V_{23}|$ become smaller with increasing energy. This may suggest a mass-generation mechanism where due to some yet unknown symmetry principle at ultra-high energies, *i.e* the Planck scale, only the fermions of the third generation are having a mass, but as their mixing with the fermions of the first and second generation gets stronger at lower energies they too develop a (smaller) mass. As for the evolution of the quark-mass ratios the following approximate equations hold [38]:

$$\frac{d(m_{u,c}/m_t)}{dx} \simeq -\frac{3}{2}(c_1 h_t^2 + c_0 h_b^2)\,(m_{u,c}/m_t)\,, \tag{40}$$

and

$$\frac{d(m_{d,s}/m_b)}{dx} \simeq -\frac{3}{2}(c_0 h_t^2 + c_1 h_b^2)\,(m_{d,s}/m_b)\,, \tag{41}$$

where $m_{u/c}$ stands for m_u and m_c, and $m_{d/s}$ for m_d and m_s, and where for the MSSM $c_1 = 2$. So in the approximation where only the Yukawa couplings of the third generation are considered, the different mixing elements run in the same way as the corresponding fermion-mass ratios and talking about the existence of such approximate relations makes indeed sense.

3 Implementing the neutral-lepton sector into the Yukawa quilt.

While the search for Ansätze for the up- and down-quark mass matrices and for the charged leptons beyond the electroweak scale needs some extra motivation, for the neutral-lepton mass matrices, this approach represents a necessity, as they are not part of the SM. The particular choice of the grand-unification scale as the scale where such Ansätze are formulated follows of course from the same arguments that motivated the RRR work [7], namely the unification of the gauge couplings within the MSSM and some of the Yukawa couplings in grand-unified theories. The major difficulty that one is faced with is the lack of uniqueness in the choice of the three 3×3 Yukawa matrices Y_u, Y_d and Y_e whose diagonalisation should lead to the observable quark and charged-lepton masses:

$$\begin{aligned} M_u^{diag} &= 2^{-1/2} v_u U_u^L \, Y_u \, U_u^{R\dagger} \\ &= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c'/\lambda^4 & 0 \\ 0 & 0 & m_t'/\lambda^8 \end{pmatrix}, \end{aligned} \tag{42}$$

$$m_c' \sim m_c \lambda^4 \qquad m_t' \sim m_t \lambda^8 \,,$$

$$\begin{aligned} M_d^{diag} &= 2^{-1/2} v_d U_d^L \, Y_d \, U_d^{R\dagger} \\ &= \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s'/\lambda^2 & 0 \\ 0 & 0 & m_b'/\lambda^4 \end{pmatrix}, \end{aligned} \tag{43}$$

$$m_s' \sim m_s \lambda^2 \qquad m_b' \sim m_b \lambda^4 \,,$$

and:

$$\begin{aligned} M_e^{diag} &= 2^{-1/2} v_d U_e^L \, Y_e \, U_e^{R\dagger} \\ &= \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu'/\lambda^4 & 0 \\ 0 & 0 & m_\tau'/\lambda^6 \end{pmatrix}, \end{aligned} \tag{44}$$

$$m'_\mu \sim m_\mu \lambda^4 \qquad m'_\tau \sim m_\tau \lambda^6 \, ,$$

where v_u and v_d are the two vacuum expectation values giving mass to the up- and down-type fermions. The masses have been parametrised *à la* Wolfenstein [39] so that the order of magnitude of the various elements becomes manifest.

In order to determine the unitary transformation matrices $U_{u,d,e}$, anomalous and hence the structure of the original Higgs-Yukawa interaction sector, simplifying assumptions are needed. Since the guiding principle has always been to look for symmetries and to be able to make predictions the requirement of looking for *Ansätze* with a maximal number of zeros compatible with the non-singularity of the Y_i's seems to be a reasonable approach to this problem [12],[4],[11],[7]. In order to limit the possible choices any further it has been also assumed that the Yukawa matrices should be hermitian, so that $U^L = U^R \equiv U$. This would be the case if the Higgs-Yukawa interactions were symmetric in generation space. Analysing all possible choices which satisfy these requirements and are in agreement with the present experimental data, RRR [7] found a set of five distinct classes, each characterised by a particular structure for Y_u and Y_d at M_G. For convenience we write the Yukawa matrix of the up-type quarks in the following way:

$$Y_u = \begin{pmatrix} 0 & \alpha\lambda^6 & \delta\lambda^4 \\ \alpha\lambda^6 & \beta\lambda^4 & \gamma\lambda^2 \\ \delta\lambda^4 & \gamma\lambda^2 & 1 \end{pmatrix} , \tag{45}$$

where the parameters α, β, γ and δ help to classify the different cases according to table (1). It is interesting to note that in the solutions (I), (II) and (IV) found by RRR the up-quark mass matrices are of the (generalized) Fritzsch-type [12], while in the solutions (III) and (V) they are of a different type, first proposed by Giudice [11]. It is basically the parameter δ that distinguishes between the two type of models, being equal to zero in the first case and different from zero in the latter. In contrast the down-quark matrices are always of the Fritzsch-type:

$$Y_d = \begin{pmatrix} 0 & \alpha'\lambda^4 & 0 \\ \alpha'\lambda^4 & \beta'\lambda^3 & \gamma'\lambda^3 \\ 0 & \gamma'\lambda^3 & 1 \end{pmatrix} , \tag{46}$$

with the corresponding values of the parameters α', β', γ' shown also in table (1).

Before turning to the lepton sector of these five classes of models some conceptual clarification is needed. The attentive reader must have namely noticed that so far there has been no ingredient whatsoever in the above classification which could justify them being refered to as GUT models. First they do not represent models but mere classifications of possible models. Second, as long as the different fermion sectors are treated as being apriori independent from each other, the idea of grand unification has not been, strictly speaking, implemented. This will be the case when we relate the lepton to the quark sector. Of course the simplest and most natural realisation of the idea of grand unification would lead to all fermion masses of each generation being equal, a case already ruled out by experiment. Therefore in the construction of phenomenologically viable GUT models a "rich" Higgs sector is needed in order to differentiate between the up- and down-quark masses on one side, and between the quark and lepton sectors on the other side. In the original GUT models based on the $SO(10)$ group this has been *e.g.* achieved by introducing more Higgs fields in the **10**, **16**, the **45** and the **126** representations of the group [40]. For some of the superstring-derived or superstring-inspired GUT models, like those based on the $SU(4) \times SU(2)_L \times SU(2)_R$ and the flipped $SU(5) \times U(1)$ groups [41] - when they are embedded into the $SO(10)$ -, the absence of Higgs bosons in adjoint or any higher representations became a problem and alternative mechanisms have been employed [42]. In any case, and independently of the chosen path, the most economic way in doing so is to keep the two up-type Yukawa sectors and the two down-type Yukawa sectors separately, and up to minor modifications, proportional to each other :

$$Y_u^{ij} \simeq Y_{\nu D}^{ij} \qquad Y_d^{ij} \simeq Y_e^{ij} , \tag{47}$$

where $Y_{\nu D}^{ij}$ are the Dirac-type Yukawa couplings of the neutral lepton sector. In order that the left equality leads to phenomenologically acceptable light neutrino masses, there must also exist a mechanism that generates heavy masses of $\mathcal{O}(R)$ for the right-handed singlet states, so that the seesaw mechanism of equ.(6) becomes effective.

These mass terms can come from different sources, - directly from tree level couplings to Higgs fields or radiatively from loop contributions -, and can a priori lie in any energy range above a few TeV. In grand-unified models that have been inspired from the superstrings they have been linked to nonrenormalisable operators that are abundantly present after string compactification [43]:

$$M_{R_{ij}} = \frac{C}{M_S} \bar{N}^c_{L_i} N_{R_j} < H >< H > , \tag{48}$$

where M_S is the string unification scale, H a Higgs field in the **10** representation of $SO(10)$ developing a vacuum expectation value at M_G, and $C \sim e^{-R_s^2/\alpha_s}$ a scale related to the radius R_s and the string tension α_s of the Calabi-Yau space. For values of $C \simeq 10^{-3} - 1$, the entries in the heavy Majorana matrix will be of the order:

$$M_{R_{ij}} = (10^{11} - 10^{14})\, GeV . \tag{49}$$

In this type of models, but also in general, the structure of M_R is unknown. Only in special cases where *e.g.* the matrices M_R and $M_{u(d)}$ are proportional due to some constraints, one can make definite predictions. Given this fact, the best way to satisfy the "principle of minimality" is to leave M_R as general as possible, *i.e.*, allowing for arbitrary entries R_{ij} $(i, j = 1,2,3)$ as long as the determinant of M_R is nonzero, and disregard any possible phases. The requirement of no extra phases in the lepton sector with respect to the quark sector limits the number of free parameters, thus improving predictibility. We write M_R as follows:

$$M_R = \begin{pmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{pmatrix} . \tag{50}$$

If the heavy Majorana mass matrix M_R is not singular, then all the heavy mass eigenstates of $\mathcal{O}(R)$ will decouple at energies below their mass scale, leaving behind (after block diagonalisation) an effective mass matrix for the light Majorana states:

$$M_\nu^{eff} \simeq M_u M_R^{-1} M_u^\dagger . \tag{51}$$

Denoting the inverse of the heavy Majorana matrix as:

$$M_{R_{ij}}^{-1} = \frac{r_{ij}}{\Delta} \tag{52}$$

with:

$$\begin{aligned} r_1 \equiv r_{11} &= R_2 R_3 - R_6^2 & r_4 \equiv r_{12} = r_{21} &= R_5 R_6 - R_3 R_4 \\ r_2 \equiv r_{22} &= R_1 R_3 - R_5^2 & r_5 \equiv r_{13} = r_{31} &= R_4 R_6 - R_2 R_5 \\ r_3 \equiv r_{33} &= R_1 R_2 - R_4^2 & r_6 \equiv r_{23} = r_{32} &= R_4 R_5 - R_1 R_6 , \end{aligned} \tag{53}$$

and $\Delta = det M_R$, the effective light-neutrino mass matrix is given by:

$$M_{\nu_{ij}}^{eff} = \frac{m_t^2}{\Delta} m_{ij} . \tag{54}$$

with:

$$\begin{aligned} m_{11} &= \delta^2 r_3 z^4 + \alpha^2 r_2 z^6 \\ m_{12} &= m_{21} = \gamma\delta r_3 z^3 + (\beta\delta + \alpha\gamma) r_6 z^4 + \alpha\beta r_2 z^5 + \alpha^2 r_4 z^6 \\ m_{13} &= m_{31} = \delta r_3 z^2 + (\alpha + \gamma\delta) r_6 z^3 + (\alpha\gamma r_2 + \delta^2 r_5) z^4 \\ m_{22} &= \gamma^2 r_3 z^2 + 2\beta\gamma r_6 z^3 + \beta^2 r_2 z^4 + 2\alpha\gamma r_5 z^4 + 2\alpha\beta r_4 z^5 + \alpha^2 r_1 z^6 \\ m_{23} &= m_{32} = \gamma r_3 z + (\gamma^2 + \beta) r_6 z^2 + (\beta\gamma r_2 + \alpha r_5 + \gamma\delta r_5) z^3 + (\beta\delta + \alpha\gamma) r_4 z^4 \\ m_{33} &= r_3 + 2\gamma r_6 z + (\gamma^2 r_2 + 2\delta r_5) z^2 + 2\gamma\delta r_4 z^3 + \delta^2 r_1 z^4 , \end{aligned} \tag{55}$$

and where we have set $\lambda^2 = z \simeq 0.05$. The matrix elements of M_ν^{eff} are polynomials in the small parameter z with the minors of the matrix M_R, the r_{ij}'s, as coefficients.

We can now start with the discussion on possible classification schemes of the neutrino sector of maximally predictive GUT models. The first thing to notice is that when

$$r_3 \neq 0 \qquad \text{and} \qquad r_3 \geq r_{ij}\,, \tag{56}$$

it sets, independently of the model, the scale for m_{33}, which being of zero power in z is anyway the largest entry, and for the entire matrix. This condition is in particular satisfied when all the entries of the matrix M_R are of the same order of magnitude and there are no cancellations among them so that none of its invariants becomes singular.

The first case (i) that we will therefore consider is defined through the conditions:

$$\begin{gathered} M_{R_{ij}} \sim \mathcal{O}(\mathcal{R}) \\ r_{ij} \neq 0\,, \end{gathered} \tag{57}$$

which imply that the heavy Majorana sector contains one mass scale only and no extra symmetries. Under these two assumptions the overall scale of the effective light-neutrino matrix is given approximately by:

$$m_0 = m_t^2 \frac{r_3}{\Delta} \simeq \frac{m_t^2}{R}\,. \tag{58}$$

The structure of M_ν^{eff} to lowest order in z is shown for the five different classes of maximally predictive GUT models (from ref.([7])) in table (2) and is written in terms of the ratios:

$$a_1 \equiv \frac{r_2}{r_3} \quad a_2 \equiv \frac{r_6}{r_3} \quad a_3 \equiv \frac{r_4}{r_3} \quad a_4 \equiv \frac{r_5}{r_3} \quad a_5 \equiv \frac{r_1}{r_3}\,. \tag{59}$$

There are two things that one notices immediately. First, there is a strong hierarchy of the Fritzsch-type among the entries of the second and third generation ($m_{22} \sim m_{23}^2 \ll m_{33}$) in all five classes, given by λ^2 or even by λ^4. As far as the mixing with the first generation is concerned, there is a breaking of the usual pattern of only nearest-neighbour generation couplings, known from the quark sector, where in particular $m_{13} \leq m_{22}$. Depending on the class, the m_{13} element can be larger than m_{22} (classes (I) and (III)), while it is always larger than m_{12}. This is an interesting example of a nontrivial case where the heavy Majorana matrix was not chosen to be proportional to the up-quark mass matrix so that one does also expect the proportionality between the mixing matrices V_l and V_{CKM} to be broken, a situation leading to interesting phenomenological consequences. For the same reason a small m_{11} entry appears in the effective light-neutrino matrix .

The structure of the effective neutrino matrix is significantly altered in some cases when either of the two conditions given by equ.(57) or both are not satisfied. We study the different possibilities separately, and start with the case where some of the r_i's are zero, except for r_3, meaning that the matrix M_R has some extra symmetries. In this case **(ii)**, defined through:

$$\begin{aligned} M_{R_{ij}} &\sim \mathcal{O}(R) \qquad orzero \\ r_i &= 0 \qquad i \subset \{1,2,4,5,6\}\,, \end{aligned} \tag{60}$$

the hierarchy among the entries of M_ν^{eff} may be altered with respect to the previous case, but not as drastically as to lead to a complete reversal of the natural-hierarchy pattern.

The situation may differ for the case **(iii)**:

$$\begin{aligned} \frac{R_j}{R_k} &\simeq \mathcal{O}(\mathbf{z^n}) \\ \frac{r_i}{r_l} &\simeq \mathcal{O}(\mathbf{1}) - \mathcal{O}(\mathbf{z^{2n}})\,, \end{aligned} \tag{61}$$

where the matrix M_R is having more scales, characterised by a strong hierarchy, so that some of the r_i minors can be considerably enhanced with respect to others, thus changing completely the structure of M_ν^{eff} - to leading order in z - that is shown in table (2). In some special cases the distortion of the perturbative structure of M_ν^{eff} could go as far as to render all the matrix entries comparable to each other, or even reverse their hierarchical order. This possibility is limited though by the fact that the same r_{ij} minors enter as coefficients to different powers of z in the various m_{ij} elements, thus protecting to a certain degree the inate hierarchy of the matrices. Another interesting possibility is the enhancement of r_3 and the matrix scale from m_0 to:

$$m_0' = p \times m_0 \tag{62}$$

by a factor p considerably larger than one, so that, if m_0' would be the scale of the heaviest neutrino as the HDM candidate, that would correspond to a scale:

$$R' \simeq p \times 10^{12}\, GeV\,, \tag{63}$$

that can nicely fit within the range expected from residual nonrenormalisable terms from the Planck scale, equ.(49).

Finally the last and most interesting case **(iv)** arises when:

$$r_3 = 0\,, \tag{64}$$

because this may imply a very light tau neutrino and therefore a realisation of the inverse hierarchy scenario of eqs.(20,21). This is in particular the case for the first

RRR class of models for which $m_{33} = 0$, while $m_{23} \sim \mathcal{O}(z^2)$ and $m_{22} \sim \mathcal{O}(z^3)$. Another case of this type arises for models belonging to class (III), because the leading order behaviour of the corresponding m_{ij} entries contains different r_{ij} coefficients. For example, if $r_5 = 0$, but r_6 or r_2 are nonzero, then $m_{23} \gg m_{33}$. Notice that for models belonging to any of the other RRR classes (I), (II), (IV) and (V), an "anomalous" ordering: $m_{33} \ll m_{23}, m_{22}$ is not possible as long as $m_{33} \neq 0$. On the other hand, for all classes except for class (V), m_{33} can, due to extra symmetries in M_R, be zero without all the other entries being necessarily zero. The breaking of the natural ordering among the generations appears as an interesting possibility, common to four out of five classes of maximally-predictive GUT models, that will be discussed separately. In contrast, for those cases where m_{33} maintains its role as the predominant entry, the overall scale of the matrix is reduced from m_0 to:

$$m_0^\star = m_t^2 \frac{r_{ij} z^n}{\Delta} \simeq \frac{m_t^2}{R} z^n \,, \tag{65}$$

where r_{ij} and the power of z are model dependent, and the second equality holds in the special case of no hierachy in M_R. This obviously modifies the range where R should lie if $m_0^\star$ were to be the scale of the heaviest neutrino that would correspond to a HDM candidate. Instead of equ.(30) one would then obtain a smaller intermediate scale:

$$R^\star \sim z^n \times 10^{12}\, GeV \,. \tag{66}$$

4 The spectrum of light neutrinos in different classes of maximally-predictive grand-unified models.

After the classification of the heavy Majorana sector into four classes, cases (**i** - **iv**), with respect to the structure of the effective light-neutrino mass matrix, we turn to the determination of the masses and mixings of the latter. The mass eigenvalues of the matrix M_ν^{eff} can be found from a perturbative determination of the roots of its characteristic polynomial, written as:

$$P = x^3 - r_3 f x^2 + z^4 r_1^\star g x - \Delta_\nu \,, \tag{67}$$

where by Δ_ν we have denoted the determinant of M_ν^{eff}, by $r_1^\star = r_2 r_3 - r_6^2$ the corresponding minor of the matrix r_{ij} and f and g are polynomials in z:

$$\begin{aligned} f &= 1 + 2\gamma a_2 z + [(1+a_1)\gamma^2 + 2\delta a_4] z^2 + 2\gamma(\delta a_3 + \beta a_2) z^3 \\ &\quad + [(1+a_5)\delta^2 + \beta^2 a_1 + 2\alpha\gamma a_4] z^4 + 2\alpha(\beta a_3 + \delta a_2) z^5 \\ &\quad + (a_1 + a_5)\alpha^2 z^6 \end{aligned} \tag{68}$$

$$
\begin{aligned}
g &= 1 - 2a_1^\star(\alpha\gamma^2 - \alpha\beta + \beta\gamma\delta - 2\gamma^3)z \\
&\quad + \mathcal{O}(z^2) + \ldots + \mathcal{O}(z^8),
\end{aligned}
$$

with $a_1^\star = (r_3 r_4 - r_5 r_6)/r_1^\star$.

4.1 Case (i): Neutrino spectra with a "natural" mass hierarchy.

We start our discussion with the first case (i) of a single scale R and no symmetry for M_R so that f and g are now functions of order one. Redefining next $x \to x/R^2$ and using the fact that:

$$
\begin{aligned}
\Delta_\nu &\simeq \kappa^2 z^{12} \cdot \det M_R^{-1} \\
\kappa &= \alpha^2 + \beta\delta^2 - 2\alpha\gamma\delta \simeq \mathcal{O}(1),
\end{aligned}
\tag{69}
$$

equ.(67) reduces to:

$$
P_0 \simeq x^3 - x^2 + z^4 x - z^{12}, \tag{70}
$$

which is the same for all five RRR classes of models. Therefore under the minimality condition of no hierarchy and no symmetry for the heavy Majorana sector, one obtains an entirely model-independent neutrino-mass spectrum:

$$
\begin{aligned}
m_{\nu_1} &\simeq \frac{m_t^2}{R} z^8 \\
m_{\nu_2} &\simeq \frac{m_t^2}{R} z^4 \\
m_{\nu_3} &\simeq \frac{m_t^2}{R}.
\end{aligned}
\tag{71}
$$

The hierarchy implied by eqs.(71) is of the quadratic-seesaw type:

$$
\frac{m_{\nu_1}}{m_{\nu_2}} \simeq \frac{m_u^2}{m_c^2} \simeq z^4 \tag{72}
$$

$$
\frac{m_{\nu_2}}{m_{\nu_3}} \simeq \frac{m_c^2}{m_t^2} \simeq z^4. \tag{73}
$$

This result strictly holds at the unification scale M_G and approximately at the scale M_X at which the heavy states decouple. For deriving the right-hand side we have used the fact that the running of the quark-mass ratios from the electroweak scale M_Z to M_G, when the top-Yukawa coupling is assumed to be constant and threshold effects are neglected, is controlled by the parameter $\chi = (M_G/M_Z)^{-h_t^2/16\pi^2} \simeq 0.7$ [7] :

$$
\frac{m_{u,c}}{m_t}(M_G) \simeq \chi^3 \frac{m_{u,c}}{m_t}(M_Z) \tag{74}
$$

$$
\frac{m_{d,s}}{m_b}(M_G) \simeq \chi \frac{m_{d,s}}{m_b}(M_Z). \tag{75}
$$

We turn next to the determination of the lepton-mixing matrix V_l. Motivated by the successes of the GJ Ansatz, the texture structure of the charged-lepton Yukawa matrix Y_e at M_G will be chosen to be the same as for Y_d except for the (2,2) entry which will be multiplied by a factor of minus three. For the same minimality reasons we mentioned before we will assume no extra CP-violating phases. Defining by:

$$U_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix} \quad . \tag{76}$$

the matrix relating the basis where M_ν^{eff} is diagonal to the basis where M_e is real,

$$V_l = U_\nu \, U_P \, U_e^{-1} \, , \tag{77}$$

where U_ν and U_e are the matrices diagonalising M_ν^{eff} and M_e respectively. Written in powers of λ and to lowest order,

$$U_e = \begin{pmatrix} 1-\lambda^2/18 & -\lambda/3 & \gamma\lambda^4/3 \\ \lambda/3 & 1-\lambda^2/18 & -\gamma\lambda^3 \\ 0 & \gamma\lambda^3 & 1 \end{pmatrix} \quad . \tag{78}$$

Except for class (V) the structure of the lepton-mixing matrix, again under the assumption of no hierarchy and no symmetry for the heavy Majorana sector, is universal: First, the mixing between the first two generations is always:

$$| \, V^{(I-IV)}_{\nu_1-\mu} \, | \simeq s^l_{12} \simeq \frac{\lambda}{3} \simeq 0.07 \, , \tag{79}$$

where the factor-three reduction with respect to the Cabibbo angle is a direct consequence of the GJ relation. This leads to a $\nu_e - \nu_\mu$ mixing of:

$$sin^2 2\theta_{e-\mu} \simeq 0.02 \, , \tag{80}$$

which falls naturally within the range required by the small-angle MSW solution to the solar neutrino problem, equ.(14). In contrast, in models of the class-(V) type the mixing is two to three times too large.

$$sin^2 2\theta_{e-\mu} \simeq 0.05 \, . \tag{81}$$

The mixing between the second and third generation is:

$$| \, V^{(I,III)}_{\nu_2-\tau} \, | \simeq s^l_{23} \simeq 4\lambda^3 \simeq 0.03 \tag{82}$$

for models belonging to the classes (I) and (III), giving rise to a $\nu_\mu - \nu_\tau$ mixing angle:

$$sin^2 2\theta_{\mu-\tau} \simeq 7. \times 10^{-3}\,, \tag{83}$$

while it is somewhat larger in models of the type (II) and (IV):

$$\mid V^{(II,IV)}_{\nu_2-\tau} \mid \simeq \lambda^2 \qquad sin^2 2\theta_{\mu-\tau} \simeq 9. \times 10^{-3}\,. \tag{84}$$

For the anomalous case (V) the mixing between ν_μ and ν_τ is negligible. Finally the mixing between the first and third generation is:

$$\mid V^{(I,II,IV)}_{\nu_1-\tau} \mid \simeq s^l_{13} \simeq \alpha\lambda^6 \tag{85}$$

for models of the type (I), (II) and (IV), giving rise to a $\nu_e - \nu_\tau$ mixing angle:

$$sin^2 2\theta_{e-\tau} \simeq (0.5 - 1.) \times 10^{-7}\,. \tag{86}$$

In models of the type (III) and in particular of the type (V) the first-to-third generation mixing is considerably enhanced:

$$\mid V^{(III)}_{\nu_1-\tau} \mid \simeq \lambda^4 \qquad sin^2 2\theta_{e-\tau} \simeq 2.2 \times 10^{-5} \tag{87}$$

$$\mid V^{(V)}_{\nu_1-\tau} \mid \simeq \lambda^2 \qquad sin^2 2\theta_{e-\tau} \simeq 9. \times 10^{-3}\,. \tag{88}$$

For the last type of models the value of the $\nu_e - \nu_\tau$ mixing angle is rather representative of what one would expect for $\nu_\mu - \nu_\tau$ mixing.

One can summarise these results by saying that apart from a certain variation in the values of the lepton-mixing angles, the four classes (I - IV) of GUT models, in the presence of a single scale R and the absence of any extra M_R symmetries, lead to the following universal mixing pattern:

$$\mid V_{\nu_1-\mu} \mid \sim \frac{\lambda}{3} \gg \mid V_{\nu_2-\tau} \mid \sim \lambda^2 - 4\lambda^3 \gg \mid V_{\nu_1-\tau} \mid \sim \lambda^4 - \lambda^6\,. \tag{89}$$

The numerical range of the mixing angles is close to the naive seesaw-based estimates in section 2.2 , resulting from a scenario that incorporates the small-angle solution to the solar neutrino problem and the tau neutrino as a candidate for the HDM. They confirm the simple guess that:

$$s^l_{12} < s_{12} \qquad s^l_{23} < s_{23} \qquad s^l_{13} \ll s_{13}\,. \tag{90}$$

However, due to the running of the up-quark masses, the hierarchy of the light neutrino masses at M_G is best described by the quadratic seesaw rather than the low energy mixed-seesaw relations, eqs.(24 - 26).

Only the "anomalous" class (V) models break this pattern. Due to the predictions of a vanishing $\nu_\mu - \nu_\tau$ mixing and a rather large $\nu_e - \nu_\tau$ mixing they represent a quite distinct case.

4.2 Case (ii): Neutrino spectra with a slightly distorted mass hierarchy.

In the preceding section we have examined classes of maximally predictive GUT models with the least number of constraints imposed upon the structure of the heavy-neutrino Majorana mass matrix, namely the case of no hierarchy of scales and no underlying symmetry principle, and we have obtained a universal spectrum of masses and mixing angles for the three light neutrinos. Their hierarchy was considered as the most natural since it corresponds to what was expected from the simplest seesaw scenario. In this and the following section we shall discuss classes of models with a more elaborate structure in what concerns the heavy Majorana-mass sector which give a distorted neutrino spectrum with respect to the previous case (i). It is to be expected that such cases will arise in the presence of a strong (inverse) hierarchy of large mass scales and/or in the presence of extra symmetries.

We will first focus on the possibility that as a result of such symmetries some of the subdeterminants of M_R are zero and therefore certain powers of λ in the effective light-neutrino mass matrix are suppressed. We start with the case **(ii)** where there is a single scale R and $r_3 \neq 0$, but allow some of the ratios a_i to be zero rather than of order one, equ.(60). The characteristic polynomial of the matrix M_ν^{eff} assumes to leading order now the more general form:

$$P_1 \simeq x^3 - x^2 + xz^n - z^{12}, \tag{91}$$

where n is an integer between four and six. Though the overall scale of M_ν^{eff}, set by the heaviest state, the ν_τ, remains unchanged, as given by equ.(58), the splitting between the mass eigenstates can be in general different from the case studied previously:

$$\frac{m_{\nu_2}}{m_{\nu_3}} \simeq z^n \qquad \frac{m_{\nu_1}}{m_{\nu_3}} \simeq z^{12-n} \qquad n = 4, 5, 6 \quad , \tag{92}$$

depending upon the leading power of the polynomial g, equ.(68). In addition to the spectrum of equ.(71) which corresponds to the case $n = 4$, there are namely two more neutrino-mass spectra:

$$m_1 : m_2 : m_3 = \left\{ \begin{array}{ccccc} z^7 & : & z^5 & : & 1 \\ z^6 & : & z^6 & : & 1, \end{array} \right\} \times m_0 , \tag{93}$$

where the first exhibits a reduced hierarchy with respect to the natural hierarchy and the second an approximate mass degeneracy among the neutrinos of the first- and second-generation. In table (3) we show possible values for the lepton-mixing angles in the five type of models considered before, setting the parameters a_i subsequently equal to zero. We have limited ourselves to those cases, where the perturbative structure of M_ν^{eff} is considerably altered with respect to table (2). In general, we have not found any substantial distortion in the spectra as compared to case (i), except for some special cases where the mixing angle between the first and third generation or the second and third generation are zero. Otherwise, one finds also for the case (ii) a universal lepton-mixing pattern:

$$| V_{\nu_e-\mu} | \sim \frac{\lambda}{3} \gg | V_{\nu_\mu-\tau} | \sim \lambda^3 \gg | V_{\nu_e-\tau} | \sim \lambda^3 - \lambda^5 , \tag{94}$$

which qualitatively resembles the one of case (i). For the models belonging to class (V), for which $r_3 \neq 0$ enters as a coefficient in all m_{ij} elements to leading order, the resulting spectrum is basically the same as in the previous case (i). It should be pointed out that the cases shown in table 3 require the existence of precise relations among the entries of M_R, implying that some of the R_i's are equal to each other and some are not, though all of them are chosen to be be of the same order of magnitude.

4.3 Case (iii): Neutrino spectra in the presence of a strong hierarchy of large Majorana-mass scales.

We consider next the case (iii) where, due to a strong hierarchy among the entries of heavy Majorana matrix M_R, there is an even stronger splitting among the r_{ij} coefficients in M_ν^{eff}, equ.(61). If p and q are enhancement (suppression) factors resulting from such splittings in the polynomials f and g, the characteristic polynomial is given by:

$$P_2 \simeq x^3 - px^2 + qz^4x - z^{12} . \tag{95}$$

Then, for $q^2 \geq 4z^4p$, one obtains the following neutrino mass eigenstates:

$$m_{\nu_1} : m_{\nu_2} : m_{\nu_3} = (\frac{z^8}{q} : \frac{q}{p}z^4 : p) \times m_0 , \tag{96}$$

that explicitly reflect the distortion of the quadratic seesaw spectrum, found in case (i). Notice that, for certain values of p and q, two of the neutrino states can become mass degenerate: $m_{\nu_1} \simeq m_{\nu_3}$ or $m_{\nu_2} \simeq m_{\nu_3}$, while, when p and q are both zero, all three neutrinos would be mass degenerate.

A detailed analysis of the lepton-mixing sector is complicated, by the existence of a too large number of possible hierarchical orderings of the entries of M_R. We will therefore adopt a more schematic approach and concentrate first upon the question of the breaking of the natural ordering for the second and third generation only. Let us consider the following types of a mass-matrix structure that arise naturally in models belonging to classes (I - IV):

$$M_0 = \begin{pmatrix} z^m & z^n \\ z^n & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} z^m & 1 \\ 1 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & z^n \\ z^n & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad . \tag{97}$$

One can easily check that, while for a mass matrix of the type M_0 the mixing between the two neutrino states:

$$\nu_2 \simeq \cos\theta\nu_\mu - \sin\theta\nu_\tau \qquad \nu_3 \simeq \sin\theta\nu_\mu + \cos\theta\nu_\tau \,, \tag{98}$$

is small, $\sin\theta_{(0)} \sim z^n$, for matrices of the types M_1, M_2 and M_3 it is maximal, $\sin\theta_{(1,2,3)} \sim 2^{-1/2}$, as this would be required for a solution to the atmospheric neutrino deficit, if it would persist.

For models belonging to class (I), a large $\nu_\mu - \nu_\tau$ mixing can also come from a matrix structure of the type:

$$M_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & z^n & 1/z \\ 1 & 1/z & 1 \end{pmatrix} \,, \tag{99}$$

or in models belonging to class (III) from:

$$M_5 = \begin{pmatrix} 0 & z^2 & z^2 \\ z^2 & 0 & 1 \\ z^2 & 1 & 1 \end{pmatrix} \,. \tag{100}$$

One finds also cases leading to large $\nu_e - \nu_\tau$ mixing, like *e.g* for models belonging to classes (II) and (IV) having a mass structure of the type:

$$M_6 = \begin{pmatrix} 0 & z & 1 \\ z & z^2 & z \\ 1 & z & 1 \end{pmatrix} \,. \tag{101}$$

These would be good candidates for a scenario that could explain the solar neutrino deficit via the large-angle vacuum-oscillation or MSW solution. In order to also implement a solution to the HDM problem into this type of scenario the muon neutrino should be the heaviest state. Such cases of a completely reversed hierarchy will be discussed next.

4.4 Case (iv): Neutrino spectra with an inverse mass hierarchy.

As mentioned previously, an important role in our classification scheme is attributed to the r_3 subdeterminant of M_R. When it is zero the leading order behaviour of the effective light-neutrino mass matrix changes in all five classes of models. It is zero when, irrespective of the other entries and of fine-tuning, one has one of the following nonsingular textures:

$$M_R = \begin{pmatrix} 0 & 0 & R_5 \\ 0 & R_2 & \star \\ \star & \star & \star \end{pmatrix} \qquad \text{or} \qquad M_R = \begin{pmatrix} R_1 & 0 & \star \\ 0 & 0 & R_6 \\ \star & \star & \star \end{pmatrix}. \tag{102}$$

As a result, the $(3,3)$ entry of M_ν^{eff} will be zero ($e.g$ in models of class (I)) or suppressed by some power of z. Let us concentrate on the first possibility and assume the more general situation where the other entries can, but need not, be zero. Let us further assume an "anomalous" ordering $m_{33} \leq m_{23}, m_{22}$ that is possible for any model belonging to classes (I - IV), and for simplicity concentrate on those cases with a zero-mass neutrino state, being some linear combination of ν_τ with the other two neutrino flavours. This is then equivalent to the matrix M_ν^{eff} being singular. Requiring this to be achieved without fine tuning and through a small number of texture zeros one is led to the following textures for M_ν^{eff}:

$$L_1^{(4)} = \begin{pmatrix} 0 & \star & 0 \\ \star & \star & \star \\ 0 & \star & 0 \end{pmatrix} \quad L_2^{(4)} = \begin{pmatrix} \star & \star & \star \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix},$$

$$L_3^{(5)} = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & \star \\ \star & \star & 0 \end{pmatrix} \quad L_4^{(5)} = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{103}$$

$$L_5^{(5)} = \begin{pmatrix} 0 & \star & 0 \\ \star & 0 & \star \\ 0 & \star & 0 \end{pmatrix} \quad L_6^{(5)} = \begin{pmatrix} 0 & \star & \star \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix}$$

with four or five texture zeros respectively. The star symbol stands for an entry of order one or some power of z, implying strong mixing or weak mixing. Notice first that the texture $L_4^{(5)}$ will always give a zero mass tau neutrino that does not mix with the other species, and a more or less strong mixing between the electron and muon neutrinos:

$$\nu_1 = \nu_\tau \qquad \nu_2 = \cos\theta\nu_e - \sin\theta\nu_\mu \qquad \nu_3 = \sin\theta\nu_e + \cos\theta\nu_\mu \,. \tag{104}$$

On the other hand, the two block diagonal textures $L_1^{(4)}$ and $L_5^{(5)}$ will give a zero-mass eigenstate ν_1 that is a linear combination of the electron and tau neutrinos, and a muon neutrino (with a small admixture of the other two components) as the heaviest state.

$$\nu_1 \simeq \cos\theta\nu_\tau - \sin\theta\nu_e \qquad \nu_2 \sim \sin\theta\nu_e + \cos\theta\nu_\tau \qquad \nu_3 \sim m_{\nu_\mu} \,. \tag{105}$$

The spectra of eqs.(104,105) are examples of an inverse mass hierarchy in the neutrino sector, eqs.(20,21). Of the remaining three matrix textures, $L_2^{(4)}$ and $L_6^{(5)}$ will be never encountered, and $L_3^{(5)}$ will give a zero-mass eigenstate that can be a linear combination of all three neutrinos with varying degrees of mixing.

4.5 Radiative corrections

All our results on neutrino masses and mixing were obtained by assuming an exact tree-level proportionality between the quark and lepton Yukawa matrices at the grand unification scale M_G and by subsequently diagonalising the latter. The resulting mass eigenstates and mixing angles need therefore to be corrected before any attempt to relate them to the low-energy observables. As is well known, corrections to the fermion masses come from two sources, the gauge and the Higgs-Yukawa couplings. Gauge corrections can be applied to individual eigenvalues, because apart from differences in mass thresholds they are "family-blind". Higgs corrections are proportional to the Yukawa couplings, but are practically negligible except for the couplings of the third generation [38]. However the treatment of the running of the light neutrino masses from M_G or M_X down to M_Z or even to the much lower energy scale M_0 where experiments hope to measure them, has been somewhat ambiguous. Some of the authors [17] have chosen to treat ratios of light neutrino masses simply as ratios of up-quark masses and consider radiative corrections only to the latter. Others [16] [8] decided not to consider any radiative corrections to the neutrino masses and mixings, most likely because they found themselves faced with the problem of a conflicting evolution behaviour of the M_u and M_R parts of the full neutrino mass matrix. One is namely faced with the peculiarity of the seesaw that only part of the physical spectrum will give measurable effects at low energies and therefore receive radiative corrections, while the other part effectively decouples already at the scale M_X. The most reasonable approach seems therefore to consist in calculating radiative corrections only to the eigenstates of the effective light-neutrino matrix M_ν^{eff}. It is then obvious that this should not be translated into treating only the nominator of the seesaw masses, since there are no physical up-quark mass eigenstates contained in a neutrino. In particular there is no Yukawa coupling for the light neutrinos at the tree level; this is generated only through radiative corrections. We will therefore adopt the approach used in ref.([5]) and complement the evolution equations for the mass ratios and mixing elements, given by Olechowski and Pokorski [38] for the quarks, to include also the leptons. Following the notation of the latter we write the one-loop RGE for the Yukawa matrices Y_A, where $A = e(\nu)$ stands for the charged leptons (neutrinos), as follows:

$$\frac{d}{dx}Y_A = (c_A\mathbf{1} + \sum_B a_{AB}H_B)Y_A\,, \tag{106}$$

where $H_A = Y_A Y_A^{\dagger}$ is a hermitian matrix, and $x = (1/16\pi^2)ln(M_X/M_0)$. In the MSSM the radiative corrections to the gauge couplings $c_A = G_A - T_A$ are:

$$G_e = 3g_2^2 + \frac{9}{5}g_1^2 \qquad T_e = TrH_e + TrH_d \tag{107}$$

for the charged leptons and:

$$G_\nu = 3g_2^2 + \frac{3}{5}g_1^2 \qquad T_\nu = TrH_u \tag{108}$$

for the neutrinos, where g_1 and g_2 are the electroweak gauge couplings. The corrections to the Yukawa couplings are specified by $a_{ee} = -3$ and $a_{e\nu} = 0$ for the charged leptons, and $a_{\nu e} = -1$, $a_{\nu\nu} = 0$ for the neutrinos.

Using the weak-basis invariants and their relations to the mass and mixing observables, that have been introduced in ref.([38]) and are based upon the usual requirements of unitarity and hermiticity and the presence of hierarchy, which remain valid also for the lepton Yukawa matrices, we obtain the following evolution for the Yukawa ratios and the mixing elements of the latter:

$$\frac{d}{dx}ln(\frac{h_{e_i}}{h_\tau}) = 3(h_\tau^2 - h_{e_i}^2) \simeq 3\,h_\tau^2 \tag{109}$$

$$\frac{d}{dx}ln(\frac{h_{\nu_i}}{h_{\nu_j}}) = -\sum_{k=1}^{3} h_{e_k}^2(\mid V_{ik}^l\mid^2 - \mid V_{jk}^l\mid^2) \tag{110}$$

$$\simeq -h_\tau^2(\mid V_{i3}^l\mid^2 - \mid V_{j3}^l\mid^2) \tag{111}$$

$$\simeq h_\tau^2\,. \tag{112}$$

In going from the first to the third line of the last equation we have made use of the predominance of the third generation Yukawa coupling and the fact that the $\mid V_{\nu_3\tau}\mid$

mixing element is of order one. Neglecting the small corrections to the tau-Higgs Yukawa coupling and defining $\chi_l = (M_X/M_0)^{-h_\tau^2/16\pi^2}$ we obtain:

$$\frac{h_{\nu_{e,\mu}}}{h_{\nu_\tau}}(M_X) \simeq \chi_l \frac{h_{\nu_{e,\mu}}}{h_{\nu_\tau}}(M_0), \tag{113}$$

and:

$$\frac{h_{e,\mu}}{h_\tau}(M_X) \simeq \chi_l^3 \frac{h_{e,\mu}}{h_\tau}(M_0). \tag{114}$$

Notice the difference in the evolution of the up-type and down-type lepton-mass ratios to the one in the quark sector. Again in the presence of a predominantly large third-generation Yukawa coupling also the lepton-mixing angles undergo a slow evolution:

$$\frac{d}{dx} ln \mid V_{13} \mid \simeq \frac{d}{dx} ln \mid V_{23} \mid \simeq h_\tau^2, \tag{115}$$

while the evolution of $\mid V_{12} \mid$ is negligible as in the quark sector. Since the numerical value of χ_l is so close to one and given the uncertainty stemming from the unknown Majorana sector, we conclude that radiative corrections to the neutrino masses and mixings are less important in this context and will be neglected.

5 Implications for neutrino-oscillation experiments.

Our previous discussion of the dependence of the light-neutrino mass matrix on the structure of the heavy Majorana sector has revealed at least four distinct classes of neutrino spectra, that span the range of predictions expected from realistic GUT models, that satisfy the principle of economicity. The question now is how to distinguish between models that belong to different classes, due to a different quark-Yukawa or/and heavy Majorana sector, with the help of neutrino-oscillation experiments. For this, one first needs to know the kind of neutrino-flavour transition that is most likely to occur in each case in order to decide on the type of experiments that are best suited.

In ref.([23]), it was shown, that, if there is hierarchy among the neutrino masses the largest transition probability will be into the heaviest neutrino:

$$P(\nu_i \to \nu_j) \simeq \delta_{ij} - 2 \mid V_{\nu_3 - j} \mid^2 (\delta_{ij} - \mid V_{\nu_3 - i} \mid^2) \times [1 - cos(\frac{\Delta m^2 L}{2p})], \tag{116}$$

where p and L are the neutrino momentum and distance from the source to the detector, and $\Delta m^2 \simeq m_{\nu_3}^2 - m_{\nu_1}^2 \simeq m_{\nu_3}^2 - m_{\nu_2}^2 \sim m_3^2$ is the only relevant mass parameter. This implies that if the natural mass-hierarchy scenario of equ.(19) is realised in nature, as predicted by GUT models classified under our cases (i) and (ii), the transition will be preferably into the tau neutrino. On the other hand, if an inverse mass-hierarchy scenario according to eqs.(20,21) is present, as suggested by models obeying the conditions that define case (iv) and partly case (iii), the transition will be predominantly into the muon neutrino.

Another consequence of equ.(116) is the existence of hierarchy in the transition probabilities as a result of the hierarchy of the mixing-matrix elements. Therefore, for models belonging to the classes (I - IV) and fulfilling in general the conditions of case (i) or (ii) - except for the special case where $\mid V_{\nu_e - \tau} \mid$ -, the hierarchy of the mixing elements, eqs.(89,94), implies the following hierarchy of transition probabilities:

$$P^{(i);(ii)}_{(I-IV)}(\nu_e \rightarrow \nu_\mu) \ll P^{(i);(ii)}_{(I-IV)}(\nu_e \rightarrow \nu_\tau) \ll P^{(i);(ii)}_{(I-IV)}(\nu_\mu \rightarrow \nu_\tau)\,. \qquad (117)$$

So the best way to test these classes of models is to look for $\nu_\mu \leftrightarrow \nu_\tau$ oscillations. Given the present experimental sensitivity [44], one can deduce from the fact that no $\nu_\mu \rightarrow \nu_\tau$ oscillations have been observed that the mass of the tau neutrino cannot lie above a few eV. On the other hand, for models belonging to class (V) the transition probabilities will always satisfy the anomalous pattern:

$$P_{(V)}(\nu_e \rightarrow \nu_\mu) \ll P_{(V)}(\nu_\mu \rightarrow \nu_\tau) \ll P_{(V)}(\nu_e \rightarrow \nu_\tau)\,, \qquad (118)$$

and $\nu_e \leftrightarrow \nu_\tau$ oscillation experiments would have been the best place to look for them. Unfortunately, the sensitivity of present and planned experiments [45] is off the range predicted by these models.

What are the experimental prospects for the future? The range of Δm^2 and $sin^2 2\theta$ that could be explored for $\nu_\mu \leftrightarrow \nu_\tau$ oscillations with the two CERN experiments CHORUS and NOMAD that are scheduled for next year is [19]:

$$sin^2 2\theta_{\nu_\mu - \nu_\tau} \geq 2.3 \times 10^{-4} \quad \text{for} \quad \Delta m^2 \geq (7eV)^2\,, \qquad (119)$$

and $\Delta m^2 \simeq 2 \times 10^{-1} eV^2$ for maximal mixing. With respect to GUT models belonging to the classes (I - IV) that predict a natural or an only slightly distorted neutrino-mass hierarchy (our cases (i) and (ii)), these experiments represent a very exciting testing ground. If the tau-neutrino mass is of the order of a few electronvolts, then $\nu_\mu \leftrightarrow \nu_\tau$ oscillations should be measured in accordance with eqs.(79 - 87) and the values given in table (3) respectively. Then the solar neutrino problem would be as well resolved, in terms of matter-enhanced small-angle $\nu_e \leftrightarrow \nu_\mu$ oscillations. This would indeed be the most satisfying scenario, solving simultaneously the two neutrino-deficit problems. It could however well be that the tau-neutrino mass is substantially below the scale that is relevant to the solution of the dark matter problem, in which case coming experiments will be insensitive to such oscillations and the solar neutrino problem will not be resolvable, since the predicted mass hierarchy is too large, eqs.(72,73,93). This of course does not exclude such models. On the contrary, it leaves us with the option that the scale of the heavy Majorana sector is closer to M_G, *i.e.* $10^{14} - 10^{16}$, a possibility, which looks in fact more natural for SUSY GUT models that do not contain an intermediate scale. Even if the sensitivity to this transition could be increased by an order of magnitude with the ICARUS detector, that is planned to settle the dispute on the atmospheric-neutrino deficit and distinguish between the three possible solutions to the solar-neutrino deficit [19], this would not significantly improve the testing of this type of models. On the other hand, there are currently also long-range oscillation experiments with the CERN ν_μ beam send to Gran Sasso and/or Superkamiokande under investigation [19], that could reach $\Delta m^2 \sim 10^{-4} eV^2$ for full mixing in vacuum and push the Majorana scale beyond M_G, bringing GUT models with a single Majorana scale into difficulties.

On the other hand, one can with the sensitivity reached by experiments like the CDHS and CHARM, that have searched for $\nu_\mu \leftrightarrow \nu_\tau$ oscillations, set a limit of $m_{\nu_\mu,\nu_\tau} \leq 2. - 0.5$ eV for $sin^2 2\theta_{\mu\tau} \simeq 0.1 - 1$ for models belonging to case (iii), that predict a strong mixing between the muon and tau neutrino. The corresponding limits for models belonging to case (iv) that predict maximal mixing between $\nu_\mu - \nu_e$, equ.(104), and between $\nu_e - \nu_\tau$, equ.(105), are: $\Delta m^2 \leq 7. \times 10^{-2} eV^2$ and $\Delta m^2 \leq 2. \times 10^{-2} eV^2$ respectively [33], [46]. Of the cited limits the first two exclude the possibility that any combination of the SM neutrinos can resolve the HDM problem, while the third limit leaves us with the option that the muon neutrino is the HDM candidate. Maximal mixing between $\nu_e - \nu_\mu$ or between $\nu_e - \nu_\tau$ could also be the explanation for the observed solar neutrino deficit according to the large-angle MSW or the vacuum oscillation solution. It is interesting to note that the expected sensitivity of future $\bar{\nu}_e \leftrightarrow \bar{\nu}_x$ experiments is getting close to testing the first of these two options. A detailed account of the potential contained in cases (iii) and (iv), as far as theory and experiment is concerned, will be given elsewhere.

6 Conclusions

The new classification scheme of supersymmetric grand-unified models, that has emerged from the requirement of having a most economical quark-Yukawa sector at the unification scale [7], has been extended also to the lepton sector such as to include neutrino masses and lepton mixing through the use of the Georgi-Jarlskog relations and assuming the most general structure for the heavy Majorana sector. The discussion of the latter has revealed yet another classification scheme in terms of four distinct cases, that lead to universal mass ratios and mixings for the three light neutrinos. The universality manifests itself through the fact that models belonging to different classes with respect to the structure of their quark- (and charged-lepton-) Yukawa sectors can give the same neutrino spectrum if the heavy Majorana sector satisfies certain requirements. The first case for example, which is characterised by the presence of only one heavy Majorana scale and the absence of any special symmetries for the heavy Majorana-mass matrix, gives neutrino-mass ratios that are typical for the quadratic seesaw, while in the other cases, that are characterised by a hierarchy of heavy Majorana scales and/or extra symmetries of the Majorana mass matrix, they get more or less distorted up to the point that the natural mass hierarchy among the generations can become reversed. In view of a possible testing of such maximally-predictive GUT models through neutrino-oscillation experiments, a comparision with existing and planned experiments has and could soon throw some more light on the structure of the Yukawa interactions at energies close to the grand-unification scale.

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Table (1): The values of the parameters in equ.(45) and equ.(46) that correspond to the five distinct classes of maximally-predictive GUT models from ref.[7].

	(I)	(II)	(III)	(IV)	(V)
α	$\sqrt{2}$	1	0	$\sqrt{2}$	0
β	1	0	1	$\sqrt{3}$	$\sqrt{2}$
γ	0	1	0	1	$1/\sqrt{2}$
δ	0	0	$\sqrt{2}$	0	1
α'	2	2	2	2	2
β'	2	2	2	2	2
γ'	4	2	4	0	0

Table (2) : The structure of the effective light-neutrino mass matrix to lowest order in z for the five classes of models from ref.[7] in the case M_R fulfills the conditions of equ.(57).

Class	M_ν^{eff}/m_0	M_ν^{eff}/m_0	Class
(I)	$\begin{pmatrix} 2a_1 z^5 & \sqrt{2}a_1 z^5 & \sqrt{2}a_2 z^3 \\ \sqrt{2}a_1 z^5 & a_1 z^4 & a_2 z^2 \\ \sqrt{2}a_2 z^3 & a_2 z^2 & 1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{2}z^4 & \sqrt{2}a_2 z^4 & \sqrt{2}z^2 \\ \sqrt{2}a_2 z^4 & a_1 z^4 & a_2 z^2 \\ \sqrt{2}z^2 & a_2 z^2 & 1 \end{pmatrix}$	(III)
(II) +(IV)	$\begin{pmatrix} \alpha^2 a_1 z^5 & \alpha a_2 z^4 & \alpha a_2 z^3 \\ \alpha a_2 z^4 & z^2 & z \\ \alpha a_2 z^3 & z & 1 \end{pmatrix}$	$\begin{pmatrix} z^4 & z^3/\sqrt{2} & z^2 \\ z^3/\sqrt{2} & z^2/2 & z/\sqrt{2} \\ z^2 & z/\sqrt{2} & 1 \end{pmatrix}$	(V)
$a_1 = \frac{r_2}{r_3}$; $a_2 = \frac{r_5}{r_3}$; $a_3 = \frac{r_4}{r_3}$; $a_4 = \frac{r_5}{r_3}$; $a_5 = \frac{r_1}{r_3}$			

Table (3) : The lepton-mixing elements predicted by maximally-predictive GUT models, belonging to classes (I - IV) from ref.[7], in the case (ii) where , due to the presence of symmetries in M_R, all but one of its subdeterminants r_1, r_2, r_4, r_5 are zero.

	(I) or $a_1 \neq 0$ or $a_2 \neq 0$ or $a_3 \neq 0$	(I) or $a_4 \neq 0$, $a_5 \neq 0$; (II) $a_4 \neq 0$; (IV) $a_4 \neq 0$	(II) $a_2 \neq 0$; (IV) $a_2 \neq 0$	(III) $a_1 \neq 0$ or $\alpha_3 \neq 0$
$\mid V_{\nu_e - \mu} \mid$	$\frac{\lambda}{3}$	$\frac{\lambda}{3}$	$1 - \frac{\lambda^2}{18}$	$\frac{\lambda}{3}$
$\mid V_{\nu_\mu - \tau} \mid$	$\gamma\lambda^3$	$\gamma\lambda^3$	0	$\gamma\lambda^3$
$\mid V_{\nu_e - \tau} \mid$	$\gamma\lambda^5$	0	$2\lambda^3$	λ^4

$a_1 = \frac{r_2}{r_3}$; $a_2 = \frac{r_5}{r_3}$; $a_3 = \frac{r_4}{r_3}$; $a_4 = \frac{r_5}{r_3}$; $a_5 = \frac{r_1}{r_3}$