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Deformed oscillator algebras for quantum superintegrable systems in two dimensions [1]

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Abstract

A new method for the description of quantum superintegrable systems in two dimensions through the use of quantum algebraic techniques is introduced. It is suggested that such systems can be described in terms of a generalized deformed oscillator, characterized by a structure function specific to the system. The energy eigenvalues corresponding to a state with finite dimensional degeneracy can then be determined directly from the properties of the relevant structure function.

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1. Intoduction

Quantum integrable systems and their relations to classical integrable systems are attracting recently much attention. Superintegrable systems in N dimensions have more than N integrals of motion, while maximally superintegrable systems have 2N−1 integrals.

In the present work we are going to demonstrate how quantum algebraic techniques can be used for the study of quantum superintegrable systems. It is known that q-deformed oscillators are necessary for constructing boson realizations of quantum algebras (also called quantum groups), which are nonlinear algebras reducing to the corresponding Lie algebras when the deformation parameter q is set equal to 1. We are going to show that the study of quantum superintegrable systems can be greatly simplified through the use of an appropriate generalized deformed oscillator [1, 2, 3].

2. Classical superintegrable systems

Let us first consider a classical superintegrable system in 2 dimensions, described by the Hamiltonian

$$H = H(x, y, p_x, p_y). \tag{1}$$

If the system is superintegrable there are two independent additional integrals of motion I and C, such that

$$\{H, I\}_{PB} = \{H, C\}_{PB} = 0, \quad \text{and} \quad \{I, C\}_{PB} = F(H, I, C), \tag{2}$$

[1] Presented by D. Bonatsos

where $\{\ ,\ \}_{PB}$ denotes the Poisson bracket and $F = F(H, I, C)$ is a constant of motion which depends on the three independent constants of motion H, I, C.

Superintegrable systems in 2 dimensions are necessarily maximally superintegrable, i.e. they possess the maximum number of independent classical invariants. Therefore any other integral can be expressed as a function of the basic integrals H, I, C. As a result we can in general choose two new integrals of motion

$$L = L(H, I, C), \quad \text{and} \quad A = A(H, I, C),$$

such that

$$\{L, A\}_{PB} = B, \quad \{L, B\}_{PB} = -A. \tag{3}$$

One can then prove that

$$B^2 + A^2 = G(H, L),$$

where $G(H, L)$ is some function depending only on the integrals of motion H, L, and

$$\{A, B\}_{PB} = \Phi(H, L) = -\frac{1}{2}\frac{\partial G}{\partial L}. \tag{4}$$

The structure of the algebra defined by eqs (3-4) has many similarities to the algebraic structure of a generalized deformed oscillator [1, 2, 3], where L is some kind of number operator, while A, B are like the creation and annihilation operators. Therefore it is quite natural to attempt studying the quantum superintegrable systems in terms of suitable generalized deformed oscillators, allowing for the determination of the energy spectrum through purely algebraic manipulations.

3. Quantum superintegrable systems

Let us now consider a two-dimensional quantum system described by a hamiltonian H. H and all relevant operators are generated by nonlinear combinations of the basic algebra of generators x, p_x, y, p_y satisfying the usual commutation relations

$$[x, p_x] = [y, p_y] = i, \quad \text{other commutators} = 0.$$

The system is called *superintegrable*, by analogy to the classical definitions, if there are two operators, I and C, linearly independent from H and from each other, which commute with H but **not** with each other

$$[H, I] = 0, \quad [H, C] = 0, \quad [I, C] \neq 0.$$

We propose the following working hypothesis: Consider the superintegrable systems for which one can construct an associative algebra

$$\begin{aligned} \mathcal{N} =& \ \mathcal{N}(H, I, C), \\ \mathcal{N}^+ =& \ \mathcal{N}, \\ \mathcal{A} =& \ \mathcal{A}(H, I, C), \\ [\mathcal{N}, \mathcal{A}] =& \ -\mathcal{A}, \\ \mathcal{A}^+\mathcal{A} =& \ \Phi(H, \mathcal{N}), \\ [\mathcal{A}^+\mathcal{A}, \mathcal{A}\mathcal{A}^+] =& \ 0, \end{aligned} \tag{5}$$

where $\Phi(E,x)$ is a real positive function definite for $x \geq 0$ and

$$\Phi(E,0)=0. \tag{6}$$

From the above equations one can then prove that

$$\begin{aligned} [\mathcal{N},\mathcal{A}^+] =& \ \mathcal{A}^+, \\ \mathcal{A}\mathcal{A}^+ =& \ \Phi(H,\mathcal{N}+1). \end{aligned}$$

If this construction is possible one can then define the Fock space for each energy eigenvalue

$$\begin{aligned} H|E,n>=& \ E|E,n>, \\ \mathcal{N}|E,n>=& \ n|E,n>, \ n=0,1\ldots, \\ \mathcal{A}|E,0>=& \ 0, \\ |E,n>=& \ \left(\frac{1}{\sqrt{[n]!}}\right)(\mathcal{A}^+)^n\,|E,0>, \end{aligned}$$

where

$$[0]!=1, \quad [n]!=\Phi(E,n)[n-1]!.$$

In the case of a system with discrete energy eigenvalues, for every energy eigenvalue E there is some degeneracy of dimension N_d+1. Therefore the dimensionality of the Fock space corresponding to that energy eigenfunction should be equal to N_d+1. This is equivalent to the condition:

$$\Phi(E,N_d+1)=0. \tag{7}$$

The two conditions (6) and (7), and the positiveness of the structure function $\Phi(E,x)$ suffice in order to determine the energy spectrum of the quantum maximally superintegrable systems.

4. Discussion

The method presented here is of general applicability. Such a construction can be carried out for the well known examples of the harmonic oscillator [4] and the Kepler system in a two-dimensional space with constant curvature. The method can also be used for constructing the quantum superintegrable versions of well-known classical superintegrable systems in 2 dimensions, such as the Fokas–Lagerstrom potential, the Smorodinsky–Winternitz potential, the Holt potential [4], the Hartmann potential. It can also be extended to quantum superintegrable systems in 3 dimensions. Work in these directions is in progress.

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