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Deformed oscillator algebras for two–dimensional quantum superintegrable systems [1]

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Abstract: Quantum superintegrable systems in two dimensions are obtained from their classical counterparts, the quantum integrals of motion being obtained from the corresponding classical integrals by a symmetrization procedure. For each quantum superintegrable system a deformed oscillator algebra, characterized by a structure function specific for each system, is constructed, the generators of the algebra being functions of the quantum integrals of motion. The energy eigenvalues corresponding to a state with finite dimensional degeneracy can then be obtained in an economical way from solving a system of two equations satisfied by the structure function, the results being in agreement to the ones obtained from the solution of the relevant Schrödinger equation.

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Deformed oscillator algebras for two-dimensional quantum superintegrable systems [1]

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Abstract: Quantum superintegrable systems in two dimensions are obtained from their classical counterparts, the quantum integrals of motion being obtained from the corresponding classical integrals by a symmetrization procedure. For each quantum superintegrable system a deformed oscillator algebra, characterized by a structure function specific for each system, is constructed, the generators of the algebra being functions of the quantum integrals of motion. The energy eigenvalues corresponding to a state with finite dimensional degeneracy can then be obtained in an economical way from solving a system of two equations satisfied by the structure function, the results being in agreement to the ones obtained from the solution of the relevant Schrödinger equation.

1. Intoduction: Quantum integrable systems and their relations to classical integrable systems are attracting recently much attention [1,2,3,4]. Superintegrable systems in N dimensions have more than N integrals of motion, while maximally superintegrable systems have 2N−1 integrals. The classical superintegrable systems in 2 dimensions have been reviewed in [5], while several examples of classical superintegrable systems in 3 dimensions are given in [6]. Two examples of quantum superintegrable systems. the isotropic harmonic oscillator and the Kepler problem in a space with constant curvature, have been studied in [7,8].

In the present work we are going to demonstrate how quantum algebraic techniques can be used for the study of quantum superintegrable systems. It is known that q-deformed oscillators [9,10] are necessary for constructing boson realizations of quantum algebras (also called quantum groups) [11], which are nonlinear algebras reducing to the corresponding Lie algebras when the deformation parameter q is set equal to 1. We are going to show that the study of quantum superintegrable systems can be greatly simplified through the use of an appropriate generalized deformed oscillator [12]. The validity of the method is demonstrated in the case of several superintegrable systems.

2. Classical superintegrable systems: Let us first consider a classical superintegrable system in 2 dimensions, described by the Hamiltonian

$$H = H(x, y, p_x, p_y). \tag{1}$$

If the system is superintegrable there are two independent additional integrals of motion I and C. such that

$$\{H, I\}_{PB} = \{H, C\}_{PB} = 0, \quad \text{and} \quad \{I, C\}_{PB} = F(H, I, C), \tag{2}$$

where $\{\ ,\ \}_{PB}$ denotes the Poisson bracket and $F = F(H, I, C)$ is a constant of motion which depends on the three independent constants of motion H, I, C.

Superintegrable systems in 2 dimensions are necessarily maximally superintegrable, i.e. they possess the maximum number of independent classical invariants. Therefore any other integral can be expressed as a function of the basic integrals H, I, C. As a result we can in general choose two new integrals of motion

$$L = L(H, I, C), \quad \text{and} \quad A = A(H, I, C),$$

such that

$$\{L, A\}_{PB} = B, \quad \{L, B\}_{PB} = -A. \tag{3}$$

[1] presented by C. Daskaloyannis, e-mail:DASKALOYANNI@olymp.ccf.auth.gr

One can then prove that

$$B^2 + A^2 = G(H, L),$$

where $G(H, L)$ is some function depending only on the integrals of motion H, L, and

$$\{A, B\}_{PB} = \Phi(H, L) = -\frac{1}{2}\frac{\partial G}{\partial L}. \tag{4}$$

The structure of the algebra defined by eqs (3-4) has many similarities to the algebraic structure of the deformed oscillator given in references [12,13,14,15], where L is some kind of number operator. while A, B are like the creation and annihilation operators. Therefore it is quite natural to attempt studying the quantum superintegrable systems in terms of suitable generalized deformed oscillators, allowing for the determination of the energy spectrum through purely algebraic manipulations.

3. Quantum superintegrable systems: Let us now consider a two-dimensional quantum system described by a hamiltonian H. H and all relevant operators are generated by nonlinear combinations of the basic algebra of generators x, p_x, y, p_y satisfying the usual commutation relations

$$[x, p_x] = [y, p_y] = i, \quad \text{other commutators} = 0.$$

The system is called *superintegrable*, by analogy to the classical definitions, if there are two operators. I and C, linearly independent from H and from each other, which commute with H but **not** with each other

$$[H, I] = 0, \quad [H, C] = 0, \quad [I, C] \neq 0.$$

We propose the following working hypothesis: Consider the superintegrable systems for which one can construct an associative algebra

$$\mathcal{N} = \mathcal{N}(H, I, C), \quad \mathcal{N}^+ = \mathcal{N}, \quad \mathcal{A} = \mathcal{A}(H, I, C),$$

$$[\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^+\mathcal{A} = \Phi(H, \mathcal{N}), \quad [\mathcal{A}^+\mathcal{A}, \mathcal{A}\mathcal{A}^+] = 0, \tag{5}$$

where $\Phi(E, x)$ is a real positive function definite for $x \geq 0$ and

$$\Phi(E, 0) = 0. \tag{6}$$

From the above equations one can then prove that

$$[\mathcal{N}, \mathcal{A}^+] = \mathcal{A}^+, \quad \mathcal{A}\mathcal{A}^+ = \Phi(H, \mathcal{N} + 1).$$

If this construction is possible one can then define the Fock space for each energy eigenvalue

$$H|E, n >= E|E, n >, \quad \mathcal{N}|E, n >= n|E, n >, \ n = 0, 1 \ldots, \quad \mathcal{A}|E, 0 >= 0,$$

$$|E, n >= \left(\frac{1}{\sqrt{[n]!}}\right) (\mathcal{A}^+)^n \, |E, 0 >,$$

where $[0]! = 1, \quad [n]! = \Phi(E, n)[n-1]!$. In the case of a system with discrete energy eigenvalues, for every energy eigenvalue E there is some degeneracy of dimension N_d+1. Therefore the dimensionality of the Fock space corresponding to that energy eigenfunction should be equal to $N_d + 1$. This is equivalent to the condition:

$$\Phi(E, N_d + 1) = 0. \tag{7}$$

The two conditions (6) and (7), and the positiveness of the structure function $\Phi(E, x)$ suffice in order to determine the energy spectrum of the quantum maximally superintegrable systems.

4. Examples of quantum superintegrable systems

4a. The Harmonic oscillator on a curved space: The harmonic oscillator in a space with constant curvature is defined in ref. [7] by the Hamiltonian:

$$H = \frac{1}{2}\left(\pi_x^2 + \pi_y^2 + \lambda L^2\right) + \frac{\omega^2}{2}\left(x^2 + y^2\right), \tag{8}$$

where

$$L = xp_y - yp_x,$$

$$\pi_x = p_x + \tfrac{\lambda}{2}\left(x\left(xp_x + yp_y\right) + \left(xp_x + yp_y\right)x\right), \quad \pi_y = p_y + \tfrac{\lambda}{2}\left(y\left(xp_x + yp_y\right) + \left(xp_x + yp_y\right)y\right) \tag{9}$$

As in [7] one can define the Fradkin operators

$$B = S_{xx} - S_{yy} = \left(\pi_x^2 + \omega^2 x^2\right) - \left(\pi_y^2 + \omega^2 y^2\right), \quad S_{xy} = \frac{1}{2}\left(\pi_x\pi_y + \pi_y\pi_x\right) + \omega^2 xy. \tag{10}$$

These operators satisfy the following commutation relations

$$[H, L] = [H, B] = 0, \quad [L, B] = 4iS_{xy}, \quad [L, S_{xy}] = -iB, \tag{11}$$

i.e. the operators L, B do not commute, but they form a closed algebra with the operator S_{xy}.

The above relations suggest the possibility of expressing the two-dimensional harmonic oscillator algebra by using the deformed oscillator formulation:

$$\mathcal{N} = \frac{L}{2} - u\mathbf{1}, \quad \mathcal{A}^+ = \frac{B}{2} + iS_{xy}, \quad \mathcal{A} = \frac{B}{2} - iS_{xy}, \tag{12}$$

where u is a constant to be determined and

$$[\mathcal{N}, \mathcal{A}^+] = \mathcal{A}^+, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^+\mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A}\mathcal{A}^+ = \Phi(H, \mathcal{N} + 1). \tag{13}$$

where the function $\Phi(E, x)$ is given by

$$\Phi(E, x) = E^2 - \left(\omega^2 + \frac{\lambda^2}{4} + \lambda E\right)(2x + 2u - 1)^2 + \frac{\lambda^2}{4}(2x + 2u - 1)^4, \tag{14}$$

The existence of a finite dimensional representation of the oscillator algebra is equivalent to the existence of a maximum number $N + 1$ which is a root of the structure function, with N being the dimensionality of the algebra representation, coinciding with the dimensionality of the appropriate Fock space. This restriction, combined with the annihilation of the structure function for $x = 0$, is written as

$$\Phi(E, 0) = 0, \quad \Phi(E, N + 1) = 0. \tag{15}$$

Solving this system of two equations with two unknowns, E and u, one obtains the eigenvalues of the harmonic oscillator in a space with constant curvature

$$E = E_N = \sqrt{\omega^2 + \frac{\lambda^2}{4}}(N + 1) + \frac{\lambda}{2}(N + 1)^2, \tag{16}$$

which coincide with the findings of [7], while the value of the constant $u = -\frac{N}{2}$ determines the angular momentum eigenvalues allowed for each N

$$L = -N, -N + 2, \ldots, N - 2, N,$$

in agreement with [7]. The symmetries of the harmonic oscillator in a curved space was studied in ref. [16] using the notion of the quadratic Racah algebras QR(3).

4b. The Kepler problem in a curved space The case of the Kepler problem in a space with constant curvature has been studied by Higgs [7]. The hamiltonian is given by:

$$H = \frac{1}{2}\left(\pi_x^2 + \pi_y^2 + \lambda L^2\right) - \frac{\mu}{r}, \quad r = \sqrt{x^2 + y^2}, \tag{17}$$

where the angular momentum operator L and the π_x, π_y are defined in eq. (9).

The Runge–Lenz vectors in the curved space can be defined by:

$$R_x = -\frac{1}{2}\{L, \pi_y\} + \mu\frac{x}{r}, \quad R_y = \frac{1}{2}\{L, \pi_x\} + \mu\frac{y}{r}. \tag{18}$$

This system is a quantum superintergrable system in a curved space because:

$$[H, L] = 0, \quad [H, R_x] = 0,$$

and the operators L, R_x, R_y form a closed algebra:

$$[L, R_x] = iR_y, \quad [L, R_y] = -iR_x.$$

Using the same hypothesis as previously we can define the deformed oscillator algebra:

$$\mathcal{N} = L - u, \quad \mathcal{A}^+ = R_x + iR_y, \quad \mathcal{A} = R_x - iR_y, \quad \mathcal{A}^+\mathcal{A} = \Phi(H, \mathcal{N}). \tag{19}$$

The structure function in this case is defined by:

$$\Phi(E, x) = \mu^2 + 2E\,(x + u - 1/2)^2 - \lambda\,(x + u - 1/2)^2\left((x + u - 1/2)^2 - 1/4\right).$$

The solution of eqs (15) is given by:

$$u = -\frac{N}{2}, \quad E_N = -\frac{2\mu^2}{(N+1)^2} + \lambda\frac{N(N+2)}{8}. \tag{20}$$

The symmetries of the Kepler problem are compatible with the existence of angular momenta equal to 0, 1/2, 1, 3/2,.... In physical situations, however, only integer angular momenta appear, which means that $N = 2n$. In this case the spectrum given by eq. (20) is the same to that obtained by Higgs [7]. The symmetries of the Kepler potential in a curved space was studied in ref. [17] using the notion of the quadratic Racah algebras QR(3).

4c. Fokas-Lagerstrom potential In classical mechanics the superintegrable system described by the Hamiltonian:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{x^2}{2} + \frac{y^2}{18} \tag{21}$$

has been studied by Fokas and Lagerstrom [18]. The quantum version of the hamiltonian (21) corresponds to a quantum superintegrable system with two additional integrals:

$$J = p_x^2 + x^2, \quad \text{and} \quad B = \frac{1}{2}\left\{xp_y - yp_x, p_y^2\right\} + \frac{y^3 p_x}{27} - \frac{\left\{xy^2, p_y\right\}}{6},$$

From the above definitions we can verify that:

$$[H, J] = [H, B] = 0, \quad [J, B] = R, \quad [J, R] = 4B,$$

and

$$[R, B] = 8J^3 - 36J^2H + 48JH^2 - 16H^3 + \frac{56}{9}J - \frac{92}{9}H,$$

$$
\begin{aligned}
R^2-4B^2 = &\; 4J^4-24J^3H+48J^2H^2-32JH^3+\\
&+\tfrac{200}{9}J^2-\tfrac{616}{9}JH+48H^2+\tfrac{20}{9}.
\end{aligned}
$$

From the above closed algebra we can define:

$$
\mathcal{N}=J/2-u,\quad \mathcal{A}^+=B+R/2,\quad \mathcal{A}=B-R/2,
$$

where u is a constant to be determined. These operators correspond to a deformed oscillator algebra (13). The corresponding structure function is defined by:

$$
\begin{aligned}
\Phi(E,x) = &\; \tfrac{1}{9}(2x-1+2u)(2E-2u+1-2x)\\
&(6E-6u+1-6x)(6E-6u+5-6x).
\end{aligned}
$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (15) and it is a positive function. Therefore we can find the possible energy eigenvalues having degeneracy equal to $N+1$:

$$
u=1/2,\quad \text{and}\quad E_N=N+1\quad \text{or}\quad E_N=N+2/3\quad \text{or}\quad E_N=N+4/3,
$$

The Hamiltonian (21) corresponds to the linear combination of two harmonic oscillators with ratio of frequencies 1:3. This example has a special signification, because it is the case of a superintegrable potential, which is not a separable one in two different coordinate systems. The proposed method does not depend on the separation of the variables in two systems.

4d. The Smorodinsky–Winternitz potential: The classical superintegrable Smorodinsky–Winternitz system [19,20,21,22] corresponds to the Hamiltonian:

$$
H=\frac{1}{2}\left(p_x^2+p_y^2\right)+k\left(x^2+y^2\right)+\frac{c}{x^2}. \tag{22}
$$

Evans [22] has proved that the Winternitz–Smorodinsky potential in N dimensions is an example of a superintegrable system. The quantum version of the hamiltonian (22) corresponds to a quantum superintegrable system with two additional integrals:

$$
T=p_y^2+2ky^2,\quad \text{and}\quad B=x^2p_y^2+y^2p_x^2-\{xy,p_xp_y\}+2c\frac{y^2}{x^2}.
$$

From the above definitions we can verify that:

$$
[H,T]=[H,B]=0,\quad [T,B]=R,\quad [T,R]=32kB+8T^2-16HT-16k,
$$

$$
[R,B]=16BT-16BH+32(c-1)T+8R+32H,
$$

$$
\begin{aligned}
R^2= &\; 32kB^2+224kB+32(c-1)T^2+64HT+16RT-16RH-48H^2\\
&+16BT^2-32BTH+192k(c-1).
\end{aligned}
$$

From the above closed non-linear algebra we can define:

$$
\mathcal{N}=\frac{1}{\sqrt{32k}}T+u
$$

$$
\mathcal{A}^+=4kB+\sqrt{\frac{k}{2}}R+T^2-2HT-2k,\quad \mathcal{A}=4kB-\sqrt{\frac{k}{2}}R+T^2-2HT-2k,
$$

where u is a constant to be determined. These operators correspond to a deformed oscillator algebra (13) The corresponding structure function can be factorized as:

$$
\begin{aligned}
\Phi(E,x) = &\; 1024k^2\left(x-\left(u+\tfrac{3}{4}\right)\right)\left(x-\left(u+\tfrac{1}{4}\right)\right)\\
&\left(x-\left(u+\tfrac{1}{2}+\tfrac{E}{\sqrt{8k}}+\tfrac{\sqrt{1+8c}}{4}\right)\right)\left(x-\left(u+\tfrac{1}{2}+\tfrac{E}{\sqrt{8k}}-\tfrac{\sqrt{1+8c}}{4}\right)\right).
\end{aligned}
$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (15). The positiveness of the structure function for every $0 < x \leq \mathcal{N}$ implies:

$$u = -\frac{3}{4},$$

while the energy eigenvalues are given by:

$$E_N = \sqrt{8k}\left(N + \frac{5}{4} + \frac{\sqrt{1+8c}}{4}\right), \quad N = 1, 2, \ldots,$$

with $-\frac{1}{8} \leq c$. If the following restriction is valid:

$$-\frac{1}{8} \leq c \leq \frac{3}{8}, \tag{23}$$

the following energy eigenvalues are also permitted:

$$E_N = \sqrt{8k}\left(N + \frac{5}{4} - \frac{\sqrt{1+8c}}{4}\right), \quad N = 1, 2, \ldots$$

4e. The Holt potential: The classical superintegrable Holt [23] system corresponds to the Hamiltonian:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(x^2 + 4y^2\right) + \frac{\delta}{x^2}. \tag{24}$$

This potential is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. The quantum version of the hamiltonian (24) corresponds to a quantum superintegrable system with two additional integrals:

$$T = p_y^2 + 8y^2, \quad \text{and} \quad B = p_x^2 p_y + 4\{xy, p_x\} - 2x^2 p_y + \frac{2\delta}{x^2} p_y.$$

From the above definitions we can verify that:

$$[H, T] = [H, B] = 0, \quad [T, B] = R, \quad [T, R] = 32B,$$
$$[R, B] = -96 + 256\delta - 64H^2 + 128HT - 48T^2,$$
$$R^2 - 32B^2 = 1024H - 704T + 512\delta T - 128TH^2 + 128T^2H - 32T^3.$$

From the above closed non-linear algebra we can define:

$$\mathcal{N} = \frac{T}{\sqrt{32}} - u, \quad \mathcal{A}^+ = 8B + \sqrt{2}R, \quad \mathcal{A} = 8B - \sqrt{2}R,$$

where u is a constant to be determined. These operators correspond to a deformed oscillator algebra (13). The corresponding structure function is defined by:

$$\Phi(E, x) = 2^{\frac{23}{2}}\left((x+u) - \tfrac{1}{2}\right)\left(\tfrac{E}{\sqrt{8}} - (x+u) + \tfrac{1}{2} + \tfrac{\sqrt{1+8\delta}}{4}\right) \cdot \left(\tfrac{E}{\sqrt{8}} - (x+u) + \tfrac{1}{2} - \tfrac{\sqrt{1+8\delta}}{4}\right).$$

The existence of a finite representation of the algebra for each energy eigenvalue implies that the structure function satisfies eq. (15). Therefore we can find the possible energy eigenvalues having degeneracy equal to $N + 1$:

$$u = \frac{1}{2}, \quad \text{and} \quad E_N = \sqrt{8}\left(N + 1 + \frac{\sqrt{1+8\delta}}{4}\right),$$

where $(1+8\delta) \geq 0$.

In the special case where $-\frac{1}{8} \leq \delta \leq \frac{3}{8}$ there are energy eigenvalues given by:

$$u = \frac{1}{2}, \quad \text{and} \quad E_N = \sqrt{8}\left(N + 1 - \frac{\sqrt{1+8\delta}}{4}\right),$$

The quantum Holt potential was also studied recently by using quadratic algebras by Létourneau and Vinet [24].

5. Discussion: In this paper, starting from classical superintegrable systems, we have shown that the corresponding quantum systems are superintegrable ones, the quantum integrals (quantum constants of motion) being obtained from the classical ones using a symmetrization procedure. Furthermore, the quantum superintegrable systems can be described in terms of a deformed oscillator algebra. The operators of the deformed oscillator algebra are constructed from the quantum integrals. The deformed oscillator algebra is characterized by a structure function $\Phi(E, N)$, which takes a specific form for each superintegrable system. The eigenvalues of the energy and their degeneracies are determined in an economical way directly from equations satisfied by the structure function, the results being in agreement with these coming from the independent solution of the relevant Schrödinger equation.

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