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Lecture 1

Differential Geometry and Poisson Structures

We will start with a *general question: What does it mean that a system of ODEs (ordinary differential equations) or PDEs (partial differential equations) is a hamiltonian one?*

In 19'th century people thought that such a system should possess some Lagrangian formulation. But recent developments in dynamical systems (including soliton theory and the theory of integrable systems) leads us to the conclusion that there exist hamiltonian systems without natural underlying Lagrangian formalism.

For example the following problem is open:

Are the Navier-Stokes equations for compressive liquid hamiltonian or not?

Normally people think that these equations are non-hamiltonian. However this point of view is based on some confusion:

It is known that the noncompressive limit of this system is non-hamiltonian but the Navier-Stokes system for the compressive fluid is very similar to a hamiltonian system.

As a simple example consider the heat equation

$$u_t = u_{xx}.$$

This system has only one conservation law

$$\int u(x)dx.$$

If it is hamiltonian it should have at least two conservation laws corresponding to time and space shifts so it is non-hamiltonian.

But the compressible Navier-Stokes system has a right number of conservation laws, so the question is open.

This problem has been posed after the discussion with F.Goltz and C.Bardos in Paris (June 1993).

The hamiltonian formalism even for the Euler equations was constructed rather late. In 1940 L.Landau tried to quantize the Euler equations. He wrote formulas for the quantum commutators in 1940 which in fact coincide with the Poisson Brackets of basic fields. But the notion of nontrivial (non-canonical) Poisson structures did not exist till 70-ies and his results were forgotten for a long period.

About 1965 Arnold constructed the hamiltonian formalism for the Euler equations in terms of symplectic structures on the so-called 'Coadjoint orbits' on the dual space to the algebra of all vector fields with zero divergence (this algebra is the Lie algebra of the Lie group of all volume-preserving diffeomorphisms.)

Before doing anything for PDEs we would like to recall the definition of Poisson structure on a finite-dimensional manifold. In all mathematical textbooks written after Arnolds work not the Poisson structure but the symplectic structure was treated as the basic object. But in fact the Poisson structure is a more fundamental object than the symplectic one. This fact has been realised by A.Lichnerowitz in early 70-ies who suggested the exact notion of Poisson Structure.

For example for many important systems the Poisson bracket is well-defined but degenerate and it is rather nontrivial to construct the inverse object (we have to restrict Poisson structure to some invariant submanifolds and so on.) In the field theory normally the Poisson structure is local (given by differential operators), but the symplectic structure is nonlocal and complicated. It is normally degenerate. So we work with Poisson structures, not with symplectic ones.

Let M be a finite-dimensional or an infinite-dimensional manifold (to avoid technical problems we may assume now that M is finite-dimensional).

Let $<,>$ be a bilinear form on *covectors*

$$< \nabla f, \nabla g >= h^{ij} \nabla_i f \nabla_j g. \tag{1.1}$$

Note the h^{ij} is a tensor field with upper indices in contrast with the standard metric tensor which is a tensor field with lower indices. Hamiltonian systems can be written in the following form:

$$\dot{x}^i = h^{ij}\frac{\partial H}{\partial x_j}.$$

h^{ij} is like a metric but it is skew-symmetric.

Definition: A tensor field with upper indices h^{ij} is a Poisson structure if the bilinear form on the gradients defined by the formula (1.1) possesses the following properties:

Let $\{f,g\} = < \nabla f, \nabla g >$. Then

1) $\{f,g\} = -\{g,f\}$, i.e. $<,>$ is skew-symmetric.

2)$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$

3) $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

Property 2) is called Jacobi identity. Property 3) may be treated as something like Leibniz rule.

The definition of a Hamiltonian system is the following. Let us have a function H on our manifold. Then for any function f we have:

$$\dot{f} = \{f,H\} = < \nabla f, \nabla H > .$$

If the matrix h^{ij} is nondegenerate, i.e. $\det h^{ij} \neq 0$ then there exists a local coordinate system in which the Poisson structure reads as:

$$h^{ij} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}.$$

(Darboux's theorem). Consider a 2-form

$$\Omega = h_{ij} dx^i \wedge dx^j,$$

where h_{ij} is the inverse matrix to the matrix h^{ij}, i.e.

$$h^{ij} h_{jk} = \delta^i_k.$$

(We always sum over repeating indexes if the opposite is not stated explicitly). If the matrix h^{ij} is non-degenerate then the Jacobi identity is equivalent to

$$d\Omega = 0.$$

In some important cases $\det h^{ij} = 0$. If the rank of the matrix h^{ij} is constant in a neighbourhood of some point then there exists a local coordinate system $x = x(y)$ such that

$$h^{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we would like to give some examples of Poisson structures. (Let us recall that if we have a Poisson structure and a Hamiltonian function then a Hamiltonian system is well-defined).

Example 1.

$$h^{ij} = \begin{bmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{bmatrix}, \quad \dim M = 2k.$$

Example 2. Assume that we have a charged particle in magnetic field. Then we have a $2k$ - dimensional phase space with the coordinates $(p_1, \dots, p_k, q^1, \dots, q^k)$.

The magnetic field reads as

$$B = B_{\alpha\beta} dq^\alpha \wedge dq^\beta, \quad dB = 0.$$

Let us consider the following Poisson structure

$$h^{ij} = \begin{bmatrix} -eB_{\alpha\beta} & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{bmatrix}. \tag{1.2}$$

Formula (1.2) gives us a new Poisson structure because $dB = 0$. The new Poisson structure generates a new Hamiltonian system.

For example

Standard tops in the gravity field may be written in such a form with magnetic field of Dirac monopole type on the 2-sphere S^2 after the proper factorization of Hamiltonian formalism using the so-called 'Area integral' - see the paper of Novikov published in Russian Math Surveys, 1982, vol. 36, iss 4.

Definition. A function f belongs to the annihilator of the Poisson bracket $\{,\}$ if $\{f, g\} = 0$ for any function g. Functions belonging to the annihilator of the Poisson bracket are also called Casimirs.

In symplectic geometry we have only trivial annihilators – constant functions. In field theory we usually have nontrivial annihilators.

Example 3. Lie-Poisson (Kirillov-Kostant-Beresin) brackets.

In some important cases h^{ij} are nonconstant. The simplest case is when the functions h^{ij} are linear functions of the coordinates

$$h^{ij} = c^{ij}_k x^k. \tag{1.3}$$

Using skew-symmetry and Jacobi identity it is easy to check that (1.3) generates Poisson structure if and only if c^{ij}_k are structure constants of some Lie algebra.

From the point of view of symplectic geometry these brackets are rather complicated because they usually have nontrivial Casimirs.

Another important object are quadratic Poisson brackets connected with the Yang-Baxter equation.

Now we would like to consider infinite-dimensional phase spaces. We shall consider not general abstract infinite-dimensional manifolds but only specific ones - manifolds of functions.

A general Poisson brackets on the space of functions read as

$$\{\varphi^i(x), \varphi^j(y)\} = h^{ij}(x,y) = h^{(ix)(jy)},$$

where $h^{ij}(x,y)$ are some distributions. (Of course skew-symmetry and Jacobi identity pose some restrictions on $h^{ij}(x,y)$). In this theory x is like an index, summation over continuous index x will be replaced by integration.

Definition. An infinite-dimensional Poisson bracket on the space of functions is called local if it reads as:

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k=0}^{K} B^{ij}_k(x, \vec{\varphi}(x), \vec{\varphi}'(x), \dots, \vec{\varphi}^{(n)}(x))\partial^k_x \delta(x-y). \tag{1.4}$$

In this case the corresponding hamiltonian system reads as

$$\varphi^i_t = A^{ij}\frac{\delta H}{\delta \varphi^j}$$

where A^{ij} is some differential operator known as hamiltonian operator.

The function $\varphi(x)$ is a point of M so each point of our manifold is a function of x.

In most of the important examples Poisson brackets are local (in 99 % cases). Locality means that the functions $h^{ij}(x,y)$ are different from 0 only in an infinitely small (infinitesimal) neighbourhood of the diagonal $x-y=0$.

In the infinite-dimensional case it is important which of the structures is local - Poisson structure or the symplectic one. (Inversion of a local operator is usually nonlocal). As a rule Poisson structures are local and the corresponding symplectic structures are nonlocal (V.Sokolov and I. Dorfman found some important examples when the symplectic structure is local and the Poisson structure is nonlocal but such situations are very rare).

The Poisson bracket (1.4) can be written as

$$\{I[\varphi],J[\varphi]\}=\int dx\frac{\delta I}{\delta\varphi^i(x)}A^{ij}\frac{\delta J}{\delta\varphi^j(x)}. \tag{1.5}$$

General Poisson brackets (without locality assumption) read as

$$\{I[\varphi],J[\varphi]\}=\int dxdy\frac{\delta I}{\delta\varphi^i(x)}h^{ij}(x,y)\frac{\delta J}{\delta\varphi^j(y)}.$$

In the case of local brackets we integrate over only one variable x because $h^{ij}(x,y)$ contains $\delta(x-y)$.

Let us consider some examples.

1). Ultralocal Poisson brackets

$$h^{ij}[\varphi](x,y)=c^{ij}(\varphi)\delta(x-y).$$

Here the functions $\varphi(x)$ are maps from x-space to some finite-dimensional manifold M and $c^{ij}(\varphi)$ are functions on this manifold. Such brackets are well-known in the field theory.

2). Gardner-Zakharov-Faddeev bracket for Korteveg- de Vries (KdV) system. In this case x is one-dimensional, we have only one field $\varphi(x)$ and

$$\{\varphi(x),\varphi(y)\}=\delta'(x-y). \tag{1.6}$$

From the classical point of view this bracket is nontrivial because it is not ultralocal (it contains the derivative of the δ-function).

3). Lenard-Magri Poisson bracket for KdV

$$\{\varphi(x),\varphi(y)\}=c\delta'''(x-y)+2\varphi(x)\delta'(x-y)+\varphi'(x)\delta(x-y).$$

The Korteveg-De Vries equation has two hamiltonian representations: in the Gardner-Faddeev-Zakharov bracket with the Hamiltonian

$$H_1 = \int (\frac{\varphi_x^2}{2} + \varphi^3) dx$$

and in the Lenard-Magri bracket with the Hamiltonian

$$H_2 = \int \varphi^2 dx.$$

Gardner-Faddeev-Zakharov bracket has nontrivial annihilator

$$I_{-1} = \int \varphi(x) dx.$$

Existence of two Poisson brackets describing the same system is a nontrivial property, which is observed until now in integrable systems only.

Let us note that any combination of the Lenard-Magri bracket and the Gardner-Faddeev-Zakharov bracket is a Poisson structure. It is a rather nontrivial property, because the Jacobi identity is nonlinear in terms of Hamiltonian operators. (Jacobi identity is linear in terms of inverse objects - symplectic structures). It is a good exercise to check this property.

The level of understanding of these subjects was very low in the scientific community in mid-seventies. For example before that even the simplest Gardner-Faddeev-Zakharov bracket was invented firstly as a nonlocal symplectic structure. Some very good scientists proved in 1975 in a rather complicated way the Jacoby Identity for the Poisson Structures such that the operator A_{ij} has coefficients which depend on x only. In fact it is evident because in this case we have a Poisson structure with 'constant coefficients' on the functional space.

Some important generalizations of Lenard-Magri brackets were obtained by Adler, Gelfand and Dikii.

It is important that the algebra of vector fields on the circle and its central extension – Virasoro algebra – underlines the Lenard-Magri brackets.

In fact Virasoro missed the term δ''' in his calculations. This third- order cocycle was discovered by Gelfand and Fuks as a central extension of the algebra of vector fields on the circle when the cohomology groups for this algebra were studied. They proved that there are no more nontrivial local cocycles on this algebra.

In our examples the Poisson structures depend on the fields themselves and their first derivatives but not on the higher ones.

Now we would like to study from the hamiltonian point of view the Riemann-type equations (we shall call them also hydrodynamic-type equations).

$$\frac{\partial u^i}{\partial t} = v_j^{i,\alpha}(u)\frac{\partial u^j}{\partial x^\alpha}. \qquad (1.7)$$

The problem is: when are these equations hamiltonian?

These equations are of Euler type – not of the Navier-Stokes type.

We will try to write equations (1.7) in the following form:

$$u_t^i = (g^{ij}(u)\frac{\partial}{\partial x} + b_k^{ij}(u)u_x^k)\frac{\delta h}{\delta u^j}. \qquad (1.8)$$

Dubrovin and Novikov proved that representation (1.8) is hamiltonian (i.e. the operator $(g^{ij}(u)\frac{\partial}{\partial x} + b_k^{ij}(u)u_x^k$ is a hamiltonian operator) iff $g^{ij}(u)$ is a flat Riemann metric on the target space and $b_k^{ij}(u) = -g^{is}(u)\Gamma_{sk}^j(u)$, where $\Gamma_{sk}^j(u)$ are Christoffel symbols for this flat metric. Their first paper was published in 1983 (the matrix g^{ij} should be nondegenerate in this theorem; otherwise the classification of Poisson Brackets is complicated and even not yet finished).

A hydrodynamic-type system is hamiltonian if it can be written in the form (1.8).

These equations are connected with the shock wave problem but we do not wand to discuss it now. We shall continue this discussion tomorrow.

Examples of Poisson structures connected with constant curvature metrics can also be constructed. Such structures are related with 2-dimensional topological quantum field theories.

Hydrodynamic-type equations are also connected with nonlinear WKB - approximation for integrable systems.

Literature:

1) Dubrovin B.A., Novikov S.P. Hydrodynamics of weakly deformed soliton lattices, differential geometry and Hamiltonian theor.Russian Math. Surveys 44 (1989), No 6, pp. 35-124.

2) Dubrovin B.A., Novikov S.P. In Soviet Math. Reviews, (1993).

3) Dubrovin B.A. Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models. Comm. Math. Phys. 145 (1992),

No 1, pp. 195-207., Integrable systems in topological field theory. Nucl. Phys. B. 379, (1992), No 3, 627-698.

4) Novikov S.P. Fermi lectures in Pisa, 1992 (to appear soon).

Definition. Hydrodynamic-type equations.

Consider the space of maps from $\mathbf{R}^n \times \mathbf{R}$ to some manifold X, $\dim X = N$ (we shall call X target space). Let $u^1, \dots, u^N$ be local coordinates on X. We call a first-order partial differential equation a hydrodynamic-type equation if it reads as

$$\frac{\partial u^p}{\partial t} = v_q^{p,\alpha}(u) \frac{\partial u^p}{\partial x^\alpha}. \tag{1.9}$$

Definition. A collection of special coordinates in the target space X is called Riemann invariants if all the matrices $v_q^{p,\alpha}$ are diagonal in these coordinates.

Riemann studied the case $n = 1$, $N = 2$. He proved that in this case the general system (1.9) at least locally possesses Riemann (diagonal) coordinates. For $N > 2$ such coordinates for general systems do not exist. If some 3- component system admits Riemann invariants it means serious degeneration (serious restrictions on the matrix $v_q^{p,\alpha}(u)$).

Definition. Hydrodynamic-type functionals. A functional $I[u]$ is called hydrodynamic-type functional if it has the following form:

$$I = \int j(u) dx^i,$$

where the density $j(u)$ depends on the values of $u(x)$ but does not depend on the derivatives of u.

Many important quantities like energy, momentum and so on are of hydrodynamic-type.

Definition. Hydrodynamic-type Poisson brackets are Poisson brackets defined by

$$\{u^p(x), u^q(y)\} = g^{pq,\alpha}(u(x)) \partial_\alpha \delta(x-y) + b_k^{pq,\alpha}(u(x)) u_\alpha^k(x) \delta(x-y). \tag{1.10}$$

In many important cases $\det g^{pq,\alpha} \neq 0$. In this case it can be proved that all $g^{pq,\alpha}(u)$ are flat metrics (with upper indices) and $b_k^{pq,\alpha}(u) = -g^{ps,\alpha}(u) \Gamma_{sk}^{q,\alpha}(u)$, where $\Gamma_{sk}^{q,\alpha}(u)$ are Christoffel symbols for these flat metrics.

The Gardner-Faddeev-Zakharov bracket (1.6) is a simplest example of hydrodynamic-type bracket

$$g^{11} = 1, \quad b_1^{11} = 0, \quad \alpha = 1.$$

(If we have only one α we may omit this index). This bracket can be trivially generalized:

$$\{u^p(x), u^q(y)\} = g^{pq}\delta'(x-y),$$

where g^{pq} is a constant matrix.

Assume that we have a Poisson bracket in some coordinates. We may look how this bracket changes if we change the coordinate system.

In many important cases $\det g^{ij} = 0$ and we have nontrivial foliations.

Simple calculations show that g^{ij} is a tensor in the target space. So in the class of local hydrodynamic-type hamiltonian systems differential geometry appears.

If we consider linear Poisson brackets we have Lie algebras.

Let us consider the one-dimensional case ($n = 1$). If g^{ij} is nondegenerate it is a flat metric and there exists (at least locally) a coordinate system in the target space such that $g^{ij}(u)$ are constant and $b_k^{ij}(u) = 0$. In this case g^{ij} has only one invariant - the number of positive and negative squares or signature.

In the physical systems we usually have a metric of type $(n, n+1)$, so the geometry is a pseudo-Riemann one. We never met Riemann geometry.

Consider the multidimensional Poisson brackets. There exists a canonical form for them.

If $g^{pq,1}$ =constant then for $\alpha \neq 1$ $g^{pq,\alpha} = c_k^{pq,\alpha} u^k + g_0^{pq,\alpha}$ (it can be proved from Jacobi identity).

In physical systems we usually get flat metrics in nontrivial coordinates (we have orthogonal but curvilinear coordinate systems).

Lecture 2

Differential Geometry and Poisson Structures

In the previous lecture we have defined the hydrodynamic-type (local) Poisson brackets

$$\{u^p(x), u^q(y)\} = g^{pq}(u(x))\delta'(x-y) - g^{ps}(u)\Gamma^q_{sk}(u(x))u^k_x(x)\delta(x-y), \quad (2.1)$$

$$\{I[u], J[u]\} = \int dx \frac{\delta I}{\delta u^p(x)} A^{pq} \frac{\delta J}{\delta u^q(x)},$$

where

$$A^{pq} = g^{pq}\partial_x - g^{ps}\Gamma^q_{sk}u^k_x. \quad (2.2)$$

Ferapontov and Mokhov found a beautiful generalization of the Poisson bracket (2.1-2.2) (see 'Functional analysis and Applications', 1992, v.26, No 4 and references therein):

$$A^{pq} = g^{pq}\partial_x - g^{ps}\Gamma^q_{sk}u^k_x + \sum_{\alpha=1}^{m} w^p_{\alpha k}u^k_x\partial_x^{-1}u^l_x w^q_{\alpha l}. \quad (2.3)$$

The first two terms in (2.3) coincide with (2.1) but the curvature of the metric g^{pq} is non-zero.

This Poisson bracket has a nontrivial geometrical interpretation. Let the target space X be imbedded to the Euclidean space R^{N+m}. Assume that we have a basis of unit normals to X $\vec{n}_1, \ldots, \vec{n}_m$. Let $g_{pq}(u)$ be the first fundamental form (the Riemann metric) of X and $h_{\alpha pq}$, $\alpha = 1, \ldots, m$ be the pencil of the second fundamental forms. Then we have a pencil of the Weingarten operators w_α:

$$(w_\alpha)^p_q = w^p_{\alpha q} = g^{ps} h_{\alpha s q}, \quad \alpha = 1, \dots, m.$$

The normal connection to the surface X is flat iff the Weingarten operators form a commutative family:

$$[w_\alpha, w_\beta] = 0 \quad \text{for all} \quad \alpha, \beta. \tag{2.4}$$

For such surfaces Gauss-Peterson-Codazzi equations take the form:

$$g_{ps} w^s_{\alpha q} = g_{qs} w^s_{\alpha p}$$

$$\nabla_p w^s_{\alpha q} = \nabla_q w^s_{\alpha p}$$

$$R^{pq}_{rs} = \sum_{\alpha=1}^{m} w^p_{\alpha r} w^q_{\alpha s}. \tag{2.5}$$

Here R^{pq}_{rs} is the Riemann curvature tensor corresponding to the metric g_{pq}.

Ferapontov proved the following theorem:

The Jacobi identity for the operator (2.3) is equivalent to the equations (2.4), (2.5) on the coefficients $w^p_{\alpha q}$. So the Poisson brackets (2.3) are in one to one correspondence with submanifolds in Euclidean space with flat normal connections.

The simplest case $m = 1$ was studied by Mokhov and Ferapontov in 1990. In this case the metric g_{pq} has constant curvature and the hamiltonian operator reads as

$$A^{pq} = g^{pq} \partial_x - g^{ps} \Gamma^q_{sk} u^k_x + \sum_{\alpha=1}^{m} u^p_x \partial_x^{-1} u^q_x.$$

One good mathematician from the Faddeev's group (Leningrad) tried to calculate Dirac reduction of the constant bracket in the Euclidean space E

$$\delta^{pq} \delta'(x - y)$$

to some submanifold $X \subset E$ but he failed to get this answer. In fact these reduction exactly coincide with the Ferapontovs brackets above.

Corollary. Let all the eigenvalues of the operator $w^i_{\alpha j}$, $\alpha = 1$ be pairwise distinct. Then there exist 'Riemann invariants' $(u^1, \ldots, u^N)$ such that $g^{pq} = g^{pp}\delta^{pq}$ and $w^p_{\alpha q} = \delta^p_q w^p_\alpha$.

Now let us return to the local Poisson brackets. We would like to introduce some important classes of coordinate systems.

1) Flat coordinates:

$$\Gamma = 0, \quad g^{pq} = \text{const}.$$

2) Riemann invariants: The Riemann invariants are such coordinates that the Riemann metric is diagonal and the velocity tensor is diagonal too:

$$g^{pq} = g^{pp}\delta^{pq}, \quad u^p_t = v^p_q(u)u^q_x, \quad v^p_q(u) = v^p(u)\delta^p_q.$$

In fact if a hamiltonian system is diagonal then the Riemann metric is automatically diagonal too.

3) Physical or Liouville coordinates. We call a coordinate system u^i a Liouville one if there exists a tensor $\gamma^{ij}(u)$ such that

$$g^{pq} = \gamma^{pq} + \gamma^{qp}, \quad b^{pq}_k = -g^{ps}\Gamma^q_{sk} = \frac{\partial \gamma^{pq}}{\partial u^k}.$$

This structure is not general covariant. Of course the flat coordinates are always the Liouville coordinates.

Tsarev in his PhD thesis in 1985 proved the following important Novikov's conjecture :

Any hydrodynamic-type $1+1$ dimensional hamiltonian system admitting the Riemann invariants is completely integrable.

For systems constructed from integrable systems by the nonlinear WKB method the Riemann invariants were known from the beginning of 70's.

Let us recall the Tsarev's procedure (the generalized hodograph transformation). Consider a diagonal hamiltonian system:

$$u^p_t(x,t) = v^p(u)u^p_x(x,t) \tag{2.6}$$

We shall use the following Tsarev's identities:

$$1) \quad \Gamma^i_{ij} = \frac{1}{2}\partial_j \ln(g_{ii}) = \frac{\partial_j v^i}{v^j - v^i}. \tag{2.7}$$

2) The flow $u^p_\tau = w^p(u)u^p_x$ commutes with the flow (2.6) if and only if:

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j w^i}{w^j - w^i}. \tag{2.8}$$

Let $w^i(u)$ be a solution of the system (2.8). Consider the following system of equations:

$$w^i(u^1, \ldots, u^N) = v^i(u^1, \ldots, u^N)t + x. \tag{2.9}$$

We have N equations on N unknown quantities u^1, ..., u^N. Solving this system we gain u^k as some functions of x and t.

Tsarev proved that these functions satisfy (2.6). So any solution of (2.7) gives us a solution of (2.6). The system (2.7) is linear, but it is rather non-trivial to construct their solutions. Explicit version of the Tsarev's procedure based on the algebraic-geometrical methods was suggested by Krichever.

In fact the restriction that a system possesses a hamiltonian representation is sometimes too strong. Most of the important systems arising from physics have a hamiltonian representation but important non-hamiltonian systems are known too (for example such systems arose in the chemical kinetics).

Consider the case $N = 2$ ($n = 1$). Let X be a two-dimensional manifold, the phase space of the system be the space of all maps from the unit circle S^1 to X. Such system reads as

$$u^i_t = v^i_j(u)u^j_x, \quad i, j = 1, 2.$$

For a general 2-component system the Riemann invariants exist and the hodograph transformation $x(u^1, u^2)$, $t(u^1, u^2)$ linearizes this system. (These results are attributed to Riemann.) Let us prove that most of these systems are non-hamiltonian.

To see it let us calculate how many hamiltonian systems do we have. For general system we have N^2 functions of N variables. The flat metrics can be parametrized by N functions of N variables (any such metric can be transformed to constant form by a diffeomorphism) and the Hamiltonian itself is a function of N variables. So we have $N+1$ functions of N variables.

For $N = 2$ the hamiltonian systems can be parametrized by 3 functions, but all 2-component systems are integrable so it is necessary to consider more general systems then the hamiltonian ones.

Let us have a system written in the Riemann invariants. For any hamiltonian system we have the following property:

$$\partial_k\left(\frac{\partial_j v^i}{v^j - v^i}\right) = \partial_j\left(\frac{\partial_k v^i}{v^k - v^i}\right), \quad i \neq j \neq k. \tag{2.10}$$

(It follows immediately from (2.7)).

Definition. A diagonal hydrodynamic-type system (2.6) is called semi-hamiltonian if its velocities $v^i(u)$ satisfy (2.10).

For the semi-hamiltonian systems the integration procedure (the generalized hodograph transformation) is the same as for the hamiltonian ones. But in fact many important semi-hamiltonian systems possess nonlocal hamiltonian formulation of Ferapontov-Mokhov type.

Felix Klein wrote in his book that the hamiltonian formalism is something beautiful but rarely used. However Poincare used it often enough. In the quantum theory the hamiltonian approach is especially important.

Let us consider some concrete examples.

1) Classical gas dynamics ($n = 1$, $N = 3$). Here

$$(u^1, u^2, u^3) = (p, \rho, s),$$

where $p(x)$ is the density of momentum, $\rho(x)$ is the density of mass and $s(x)$ is the density of entropy. The Poisson bracket is given by

$$\{p(x), p(y)\} = 2p(x)\delta'(x-y) + p'(x)\delta(x-y)$$

$$\{p(x), \rho(y)\} = \rho(x)\delta'(x-y),$$

$$\{p(x), s(y)\} = s(x)\delta'(x-y),$$

(all the other brackets are equal to 0). The metric reads as

$$g^{pq} = \begin{pmatrix} 2p & \rho & s \\ \rho & 0 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

The Hamiltonian reads as:

$$H = \int_x \left(\frac{p^2}{\rho} + \varepsilon_0(\rho, s)\right).$$

2) Relativistic fluid ($n = 1$, $N = 2$). Let T be the energy-momentum tensor

$$T^{ij} = \begin{pmatrix} \varepsilon & p \\ p & \varepsilon - 2q \end{pmatrix}, \quad \text{where } 2q = \eta_{ij} T^{ij}, \quad \eta^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

ε is the density of energy and p is the density of momentum. The equations of motion read as

$$\begin{cases} \varepsilon_t + p_x = 0 \\ p_t + (\varepsilon - 2q)_x = 0 \end{cases} \tag{2.11}$$

plus one additional state equation

$$\Phi(\mathcal{E}, \mathcal{P}) = 0, \quad \text{where } \mathcal{E}, -\mathcal{P} \text{ are the eigenvalues of } T^i_j, \quad \mathcal{E} - \mathcal{P} = 2q.$$

Let

$$\gamma^{pq} = \begin{pmatrix} p & \varepsilon \\ \varepsilon - 2q & p \end{pmatrix}, \quad H = \int \varepsilon dx.$$

Symmetrization of the tensor γ^{ij} gives us a flat metric. It is a good exercise to check this fact. This metric is of the type $(1,1)$ but it has no direct connections with the metric of the Minkowski space.

Using a nonlinear analog of the WKB-method we can construct some hydrodynamic- type systems from the integrable hamiltonian partial differential equations. But up to now we could not prove that the hamiltonian formulation for this hydrodynamic-type system can be obtained from the original hamiltonian structure. (See the discussion in the paper by Novikov and Maltsev in Russian Math. Surveys, 1993, No 1.)

Now we shall discuss linear Poisson brackets.

Let

$$g^{pq} = c^{pq}_k u^k + g^{pq}_0, \quad b^{pq}_k = \text{const},$$

$X = \mathbf{R}^N$ be a linear space, $e_1, \ldots, e_N$ be a basis in $\mathbf{R}^N$. Consider an algebra with the basis $e_1, \ldots, e_N$ and with the following multiplication law:

$$e^i \circ e^j = b^{ij}_k e^k. \tag{2.12}$$

It is important for us that the metric with the upper indexes is linear and the normal Riemann metric with the lower indexes has a much more complicated form. The quantities b^{ij}_k are constant and the standard Christoffel symbols are non-constant. In the classical differential geometry metrics with constant Christoffel symbols were studied but our situation is absolutely different.

Let L^B be the algebra of maps from $\mathbf{R}^1$ to B

$$p(x) = p_i(x)e^i \in L^B$$

with the following commutator:

$$[p(x), q(x)] = p' \circ q - q' \circ p, \tag{2.13}$$

where $\circ$ is the product (2.12). So we have defined a commutator of two functions.

Theorem. L^B is a Lie algebra, i.e. the Jacobi identity holds if and only if:

a) $[L_a, L_b] = 0$, and

b) $[R_b, R_c] = R_{bc-cb}$,

where L_a is the left multiplication and R_b is the right one:

$$L_a(b) = ab = R_b(a).$$

If $1 \in B$ then B is a commutative associative algebra. But B may be neither commutative nor associative if it has no unit.

Definition. The algebra B is called nondegenerate if $\det(b_k^{ij}u^k) \not\equiv 0$, i.e. $\det(b_k^{ij}u^k) \neq 0$ at least in one point.

Definition. The algebra B is called symmetric if $2b_k^{ij} = 2b_k^{ji} = C_k^{ij}$. In the symmetric case the algebra B is commutative and associative.

Definition. A commutative associative algebra is called Frobenius algebra if there exists a nondegenerate inner product $< .,. >$ such that

$$< e^i e^j, e^k > = < e^i, e^j e^k > . \tag{2.14}$$

If B is symmetric and nondegenerate then it is a Frobenius algebra and all the products $< .,. >$ may be obtained in the following way:

$$< a, b > = l(ab),$$

where $l \in B^*$, i.e. l is a linear function on B. (For more details see A.A.Balinskii, S.P.Novikov, 'Poisson brackets of hydrodynamic type and Lie algebras', Sovjet. Math. Dokl, 32 (1985), No 1, 228-231).)

In nontrivial examples coming from the 3-component gas dynamics we have non-commutative associative algebras.

We see that Frobenius algebras are deeply connected with the zero-curvature metrics. For linear Poisson brackets which do not correspond to the Frobenius algebras the problem of finding the flat coordinates is not solved up to now.

The Lie algebras L^B have nontrivial central extensions of the Gelfand-Fuks type. Let us recall that an algebra $\tilde{L}^B$ is called a central extension of L^B if we have the following exact sequence

$$0 \to \mathbf{R} \to \tilde{L}^B \to L^B \to 0,$$

or equivalently, we have a cocycle $\chi(p,q)$ on L^B.

In our situation we have many cocycles of the Gelfand-Fuks type

$$\chi(p,q) = \int_{S^1} (p''', q),$$

where

$$(p,q) = \lambda^{ij} p_i q_j, \quad \lambda^{ij} = b^{ij}_k u^k_0.$$

(Any nondegenerate point of the u-space gives us such cocycle).

Let us discuss how such objects arose in the topological quantum field theory (in the dimension $d = 2$). (See the following papers: Dubrovin B.A. 'Hamiltonian formalism of Whitham-type hieararchies and topological Landau-Ginsburg models', Comm, Math. Phys., 145 (1992), No 1, 195-207; Dubrovin B.A. 'Integrable systems in topological quantum field theory', Nucl. Phys. B, 379 (1992), No 3, 627-689.,and ref. therein).

Topological quantum field theories we suggested by A.S.Schwarz and generalized by Witten to the non-abelian case.

In the topological theories the corellation function of the primary fields does not depend on the positions of the points by definition so all the corellators $< \phi_\alpha, \phi_\beta, \ldots >$ are constants.

Let

$$\eta_{\alpha\beta} = < \phi_\alpha, \phi_\beta >_0, \quad c_{\alpha\beta\gamma} = < \phi_\alpha, \phi_\beta, \phi_\gamma >_0,$$

be the so-called genus zero corellators (i.e. the tree-level corellators). Then all the multipoint corellators can be expressed via the two-point corellators and the three-point ones. We also assume that we have a unit ϕ_1 in the algebra of primary fields. Let:

$$c^{\gamma}_{\alpha\beta} = \eta^{\gamma s} c_{\alpha\beta s}.$$

Lemma. The numbers $c^{\gamma}_{\alpha\beta}$ are structure constants of a commutative associative algebra with a scalar product defined by the tensor $\eta_{\alpha\beta}$. This algebra is a Frobenius algebra, i.e. $< ab, c >=< a, bc >$ for all a, b, c.

Let e_α, e_β be a basis of primary fields, $H^g = \eta^{g\alpha\beta} e_\alpha e_\beta$, where $\eta^g_{\alpha\beta}$ are the genus g two-point corellation functions. Then the genus g corellation function reads as

$$< \phi_\alpha \phi_\beta \ldots >_g = < e_\alpha e_\beta \ldots H^g > .$$

It is more interesting to consider not individual topological theories but families of such theories depending on some extra parameters $(t_1, \ldots, t_N)$. Let the following assumptions hold:

1) All $\eta_{\alpha\beta}$ are constants, i.e. they do not depend on t_k.

2) The unit element in the algebra of the primary fields is constant.

3) There exists a function $F(t_1, \ldots, t_N)$ such that

$$c_{\alpha\beta\gamma}(\vec{t}) = \frac{\partial^3 F(\vec{t})}{\partial t_\alpha \partial t_\beta \partial t_\gamma}, \tag{2.15a}$$

$$\eta_{\alpha\beta} = \frac{\partial^3 F(\vec{t})}{\partial t_1 \partial t_\alpha \partial t_\beta}, \tag{2.15b}$$

where t_1 corresponds to the unit element ϕ_1. Equations (2.15) are known as Witten - Dijkgraaf - Verlinde -Verlinde equations. In fact $F(\vec{t})$ coincides with the logarithm of the partition function.

The dependence of the Frobenius algebras of the parameters t_k is described by some hydrodynamic-type equations which can be obtained as the dispersionless limit of the soliton equations.

Let $\Phi_t = K(\Phi, \Phi_x, \Phi_{xx}, \ldots)$ be a partial differential equation such that Φ=const is a solution. Dispersionless limit means that we consider 'smooth' functions Φ, i.e.$\Phi_x \ll \Phi$.

It is possible to do this consideration more strict introducing grading

$$r(\text{scalar}) = 0, \quad r(\Phi) = 0, \quad r(\Phi^{(n)}) = n, \quad r(fg) = r(f) + r(g),$$

so that e.g. $r((\Phi')^2) = 2$.

Consider a flat diagonal metric

$$ds^2 = \sum \eta_{ii}(u)(du^i)^2, \tag{2.16}$$

such that

$$d\left(\sum_{i=1}^{N} \eta_{ii}(u)du^i\right) = 0, \tag{2.17a}$$

$$\sum_{k=1}^{N} \partial_k \eta_{ii}(u) = 0. \tag{2.17b}$$

Such metrics were studied by Darboux and Egoroff. Darboux suggested to call them Egoroff metrics. For such metrics vanishing of the curvature can be written in the form of the following system on the so-called rotation coefficients:

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}}, \quad i \neq j,$$

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i \neq j \neq k,$$

$$\sum_{k=1}^{N} \partial_k \gamma_{ij} = 0, \quad i \neq j,$$

$$\gamma_{ij} = \gamma_{ji}. \tag{2.18}$$

The system (2.18) is integrable (in fact it coincides with the well-known N-wave systems, see for example Novikov S.P., Manakov S.V., Pitaevskii L.P., Zakharov V.E. 'Theory of solitons',Plenum, 1984.). The zero curvature representation for this system reads as

$$\begin{cases} \partial_j \psi_i = \gamma_{ij}\psi_j, & (i \neq j) \\ \sum_k \partial_k \psi_i = \lambda \psi_i \end{cases}$$

In the massive topological field theories the algebra B is trivial $B = \mathbf{R} \oplus \mathbf{R} \oplus \ldots \oplus \mathbf{R}$ but the original metric is nontrivial – it has Egoroff form:

$$ds^2 = \sum \eta_{ii}(du^i)^2, \quad d\left(\sum \eta_{ii} du^i\right) = 0.$$

Lecture 3

Hopf Algebras and Quantum Groups

A very good physicist from the Landau Institute , Paul Wiegman (who is an expert in the integrable quantum systems and Yang–Baxter equations), once said to me in the late 80-ies: 'I do not know what a quantum group is but it sounds beautiful'.

In fact quantum groups can not be treated as groups of symmetries in this area so this terminology does not seem to be natural.

Functions on topological spaces form commutative algebras. If we consider functions on the Lie groups we have Hopf algebras which are commutative but not cocommutative. So Hopf algebras can be introduced as generalizations of the algebras of the functions on the Lie groups but historically they were discovered in a different way.

Hopf algebras were introduced as an axiomatization of the cohomology algebras of the loop spaces.

1. Hopf algebras in topology.

In 1953 A.Borel introduced the notion of the Hopf algebra as a result of an algebraic analysis of the papers by Heinz Hopf where the cohomology groups of the loop spaces were studied.

In 1957 J.Milnor discovered that the Steenrod algebra of the cohomological operations is a Hopf algebra. This result was connected with the computation of the stable homotopy groups and the cobordisms.

Even richer structures arose in the complex cobordism theory. A number of important results in this direction was obtained in 1966 and later in the

Novikovs seminar by Novikov, Mischenko, Buchstaber, Kasparov, Gusein-Zade, Krichever, Oshanyne and in US by Landweber, Quillen, Morava and other authors.

2. Hopf algebras in the quantum theory.

In 1969 Hopf algebras were introduced as 'supergroups' by F.A.Berezin and G.I.Kac. (G.I.Kac was a mathematician from Kiev and he should not be mixed with Viktor Kac who also was incorporated in the Hopf algebras studies later). In 1970 and later Hopf algebras arose in the boson-fermion symmetry, Lie superalgebras and so on.

In the solvable 2-dimensional models of the quantum field theory arose the Yang-Baxter equation, which is connected with braid groups representations and knots.

In 1986 the objects which are known now as the quantum groups were introduced in the papers by Sklyanin, Drinfeld and Jimbo analysing the algebraic achievements of Physicists who developed a theory of the solvable models of the 2-dimensional quantum field theory. Drinfeld introduced the notion of a quantum group as a special Hopf algebra and suggested an important construction of a 'quantum double' of a Hopf algebra.

In fact the important results in the 2-dimensional quantum field theory and topology are based on the Yang-Baxter equation solutions only (i.e. special representations of the Braid Groups); They do not use the Hopf algebras. There is no need to use a language of the Hopf algebras in these applications: the representations of the braid groups is what people actually used. But the categories of representations of the special Hopf algebras give a beautiful systematisation of this stuff.

Why did people in the 30-ies and 40-ies never introduce Hopf algebras? The reason (as I believe) is that the definition of a Hopf algebra looks too long; in this period people preferred short definitions as the starting points of their best (most fashionable) theories.

Let us recall the definition of a Hopf algebra. X is a Hopf algebra if:

1). X is an associative algebra with a unit 1 equipped by a homomorphism $\epsilon : X \to k$ called 'counit', $\epsilon(1) = 1$, $\epsilon(xy) = \epsilon(x)\epsilon(y)$. Usually k is a field or the ring of integers $\mathbf{Z}$. The multiplication in this algebra can be treated as a map $\psi : X \otimes X \to X$, $xy = \psi(x \otimes y)$.

2). X is a coalgebra, i.e. a homomorphism $\Delta : X \to X \otimes X$ is given

$$\Delta(x) = \sum_i x'_i \otimes x''_i, \tag{3.1}$$

such that

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \sum_i \epsilon(x_i')x_i'' = \sum_i x_i'\epsilon(x_i'') = x. \tag{3.2}$$

This homomorphism is called a comultiplication. In the old literature the words 'specification' and 'diagonal' were used instead of 'counit' and 'comultiplication' respectively.

3). There exists a map $s : X \to X$ called an antipode such that

$$\sum_i s(x_i')x_i'' = \sum_i x_i's(x_i'') = \epsilon(x)\cdot 1, \quad s(xy) = s(y)s(x). \tag{3.3}$$

Some examples of finite-dimensional Hopf algebras are known but the Hopf algebras we need for the 2-dimensional quantum field theory and topology are infinite-dimensional.

We will assume for simplicity that X has a basis $\{e_i\}$ such that all the sums above are finite for all operations so we have no problems of convergence and we need no functional analysis.

Let X^* be the space of k-linear forms on X, i.e. the dual space to X. Then X^* is also a Hopf algebra with a basis e_*^j dual to e_i, $(e_*^j, e_i) = \delta_i^j$ with multiplication Δ^*, comultiplication ψ^* and antipode s^*. So the conjugation maps the multiplication to the comultiplication and vice versa. The unit in X^* is ϵ^* where $\epsilon : X \to k$ and the counit in X^* is dual to the unit in X, $1 : k \to X$. A requirement should be posed that both X and X^* are associative.

Let us consider some examples.

1). The group rings. Let G be a finite group with elements e_i. Then $X = k[G]$, i.e. all the elements of X read as

$$x = \sum k_i e_i, \quad k_i \in k.$$

The elements e_i form a basis in X, the multiplication law e_ie_j is the same as in G, the comultiplication Δ reads as $\Delta e_i = e_i \otimes e_i$, the antipode is defined by $s(e_i) = e_i^{-1}$, the counit is $\epsilon(e_i) = 1$. Dual Hopf algebra X^* is formed by functions on G.

2). The enveloping algebra of a Lie algebra $X = U(L)$. Let L be a Lie algebra with an ordered basis $\{x_j\}$. Then the basis in $U(L)$ is formed by all ordered polynomials:

$$\{1; x_{i_1}; x_{i_1}x_{i_2}; \ldots; x_{i_1}\ldots x_{i_n}; \ldots\}, \quad i_1 \le i_2 \le \ldots \le i_n.$$

Commutation relations in $U(L)$ are generated by the commutation relations in L

$$x_i x_j - x_j x_i = C_{ij}^k x_k = [x_i, x_j],$$

where $[,]$ is the commutator in L. The diagonal in $U(L)$ is generated by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ and $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(x_{i_1} \cdot \ldots \cdot x_{i_k}) = 0$ for $k \geq 1$.

The dual algebra to $U(L)$ is formed by ordered polynomials in the variables x_j^*.

3). The enveloping algebra of a Lie superalgebra $U(L)$. A Lie superalgebra is a $\mathbf{Z}_2$ graded algebra, i.e. as a linear space $L = L_0 \oplus L_1$. Let $\sigma(x) = 0$ if $x \in L_0$ and $\sigma(x) = 1$ if $x \in L_1$ (otherwise $\sigma(x)$ is undefined). Let $\{x_j\}$ be a graded (ordered) basis in X (it means that $\sigma(x_i) = 0, 1$ for all i).

The basis in $U(L)$ is formed by ordered polynomials in x_i:

$$\{1; x_{i_1}; x_{i_1} x_{i_2}; \ldots; x_{i_1} \ldots x_{i_n}; \ldots\}, \quad i_1 \leq i_2 \leq \ldots \leq i_n.$$

In the case of superalgebras the commutator should be replaced to a graded commutator. The commutation relations in $U(L)$ are generated by:

$$x_i x_j - (-1)^{\sigma(x_i)\sigma(x_j)} x_j x_i = C_{ij}^k x_k = [x_i, x_j]_{(\text{super})},$$

$$\sigma(x_i x_j) = \sigma(x_j x_i) = \sigma(x_i) + \sigma(x_j) \text{ (MOD 2)}.$$

4). Quantum deformations of $U(L)$. Let $L = sl(2)$ with the standard basis X, Y, H. The algebra $U(L)$ has no nontrivial deformations as an algebra, but it has nontrivial deformations as a Hopf algebra:

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = \frac{\text{sh}(hH/2)}{\text{sh}(h/2)}, \tag{3.4a}$$

where $h \to 0$ is 'Planck parameter', $q = e^{h/2}$, the diagonal reads as

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(X) = X \otimes \exp[hH/4] + \exp[-hH/4] \otimes X,$$
$$\Delta(Y) = Y \otimes \exp[-hH/4] + \exp[hH/4] \otimes Y, \tag{3.4b}$$

the antipode is

$$s(H) = -H, \ \ s(X) = -qX, \ \ s(Y) = q^{-1}Y.$$

(This deformation was found by Sklyanin in 1985). In this example $s^2 \neq 1$.

Drinfeld said that quantum groups are mainly non-commutative non-cocommutative Hopf algebras.

There was proved moreover that for a finite-dimensional Hopf algebra some power of s is equal to 1. The deformations of $U(sl(2))$ are infinite-dimensional and for general q it is not so. But if q is a root of unity then some power of s is equal to 1.

It is interesting that after deformation the enveloping algebra is more symmetric than before deformation.

References.

1. Drinfeld V. 1. Proc. Int. Cong. Math. (1986-Berkeley), Vol 1 AMS Publ. (1987), pp. 798-820.

2. Survey: Madjid Sh., Int. Journal. of Modern Physics A., Vol 5, No 1 (1990), pp. 1-91.

S.P.Novikov. Various doubling of Hopf algebras. Operator algebras on quantum groups, complex cobordisms. Russian Math. Surveys, 1992, No 5, 198-199.

Most of the Hopf algebras important for applications possess an additional structure - the Yang-Baxter structure. Let R be an element in the tensor square of X:

$$R \in X \otimes X.$$

(Later we will have a non-standard multiplication in the tensor product). Let the diagonal operator be

$$\Delta(x) = \sum_i x_i' \otimes x_i''.$$

(In real examples we may have some problems connected with infinite sums by let us assume that we have no such problems).

A transformed diagonal can be defined as

$$\Delta^t(x) = \sum_i x_i'' \otimes x_i'.$$

(The new diagonal should satisfy (3.3) so if we transpose the comultiplication then we have to replace the antipode $s(x)$ to a new one: $s^t(x) = s^{-1}(x)$.

Drinfeld suggested to pose the following requirements on R

$$R\Delta(x) = \Delta^t(x)R,$$

$$\Delta_1(R) = R_{13}R_{23},$$
$$\Delta_2(R) = R_{13}R_{12}. \tag{3.5}$$

Here we use the standard notation. Let

$$R = \sum_i r_i' \otimes r_i''.$$

Then R_{12}, R_{13}, R_{13} are elements of $X \otimes X \otimes X$ defined by:

$$R_{12} = \sum_i r_i' \otimes r_i'' \otimes 1, \quad R_{13} = \sum_i r_i' \otimes 1 \otimes r_i'', \quad R_{23} = \sum_i 1 \otimes r_i' \otimes r_i''.$$

$$\Delta_1(R) = \sum_i \Delta(r_i') \otimes r_i'', \quad \Delta_2(R) = \sum_i r_i' \otimes \Delta(r_i'').$$

From (3.5) is it easy to deduce that R satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.6}$$

Drinfeld calls Hopf algebras with an R-matrix almost cocommutative Hopf algebras. For $sl(2)$ the element R (the universal R-matrix) was constructed by Drinfeld.

Let X be a Hopf algebra, M be an associative algebra with 1. Assume that M is an X-module, compatible with the comultiplication in X, i.e. we have a representation ρ_x of X in M such that

$$\rho_x(uv) = \sum_i \rho_{x_i'}(u)\rho_{x_i''}(v), \tag{3.7}$$

where

$$\Delta(x) = \sum_i x_i' \otimes x_i''. \tag{3.8}$$

This property is similar to the Leibniz rule.

We will call such modules the 'Milnor modules' after J.Milnor who discovered the Hopf property of the Steenrod algebra of cohomological operations.

In fact the property (3.7) is the basic algebraic property of the differential operators.

Consider the following algebra A of operators on M. Let $A = M \otimes X$ as a linear space with the commutation relations:

$$xu = \sum_i \rho_{x_i'}(u)x_i'', \quad x \in X, \; u \in M. \tag{3.9}$$

The action of A on M is defined by

$$ux[w] = u\rho_x(w), \quad ux \in A, \quad w \in M. \tag{3.10}$$

The algebra A contains a subalgebra of the left multiplication operators

$$u \to L_u, \quad L_u(v) = uv.$$

Right multiplication operators can be defined by

$$R_u(v) = vu.$$

Important examples of such algebras arose in topology in the second part of the 60-ies. One of them was the algebra A^U of the so-called cohomological operations in the complex cobordisms computed by the author in 1966. In this case $X = S$ is the so-called Landweber-Novikov algebra and $M = S^*$ as an X-module over Q coincides with representations R_* below (it is not so over the ring Z and this difference is very important) It was finally established by Novikov and Buhstaber recently, on the basis of the paper by Buhstaber and Shokurov, written in 1978).

Let X be a Hopf algebra. Then we can define a transposed algebra X^t which coincides with X as an algebra but has the opposite comultiplication

$$\Delta^t(x) = \sum_i x_i'' \otimes x_i',$$

(see (3.8)) and the inverse antipode

$$s^t = s^{-1},$$

and an algebra X' with the opposite multiplication

$$u \square v = vu,$$

and the inverse antipode

$$s' = s^{-1},$$

but with the same comultiplication as in X. The algebra which coincides with M as a linear space but has the opposite multiplication will be denoted M' respectively. It is easy to see that $(X^t)^* = (X^*)'$.

Consider some basic examples:

1)
$$M = X^*, \quad \rho_x = R^*_{s^{2k}x}, \quad \tilde{\rho}_x = L^*_{s^{2l+1}x}.$$
The following lemma holds:

Lemma.
$$\rho_x(uv) = \sum \rho_{x_i'}(u)\rho_{x_i''}(v), \quad \tilde{\rho}_x(uv) = \sum \rho_{x_i''}(u)\rho_{x_i'}(v).$$
This Lemma means that ρ is a representation of X and $\tilde{\rho}$ is a representation of the algebra X^t.

A general differential operator can be written as a sum of products of the left multiplications on some functions and the left-invariant differentiations.

2) Let $k = l = 0$, i.e. $M = X^*$, $\rho_x = R^*_x$, $A = X^*X$. Then the commutation relations in A read as
$$xu = \sum R^*_{x_i'}(u)x_i'' = \sum u_j' L^*_{u_j''}(x),$$
where
$$\Delta x = \sum x_i' \otimes x_i'', \quad \Delta u = \sum u_i' \otimes u_i''.$$
We have two representations of A:

Representation 1. $A \to \mathrm{End}X^*$, $u \to L_u$, $x \to R^*_x$,

Representation 2. $A \to \mathrm{End}X$, $u \to L^*_u$, $x \to R_x$.

These representations are the **Fourier dual** ones.

3) Let $M = X^*$, the Hopf algebra $Y = X \otimes X^t$, ρ_x and $\tilde{\rho}_y$ are some representations of X and X^t respectively, $\hat{\rho}_{x\otimes y} = \rho_x \tilde{\rho}_y$, $A = MY$. The algebra A contains an important subalgebra $C = M\Delta X$ where ΔX is the image of the map $\Delta : X \to X \otimes X$. ($X \otimes X$ coincides with $X \otimes X'$ as an algebra.) If $\rho = R^*_x$ and $\tilde{\rho} = L^*_{sx}$ then C coincides with the Drinfeld's 'quantum double'. For general M A is not a Hopf algebra, but for the Drinfeld's quantum double C is a Hopf algebra with the comultiplication $\Delta(ux) = \Delta^t(u)\Delta(x)$.

The quantized $U(L)$ is more symmetric then the nondeformed one. For the semisimple Lie algebras the quantized $U(L)$ for generic q is the Drinfeld's quantum double of its Borel part modulo the trivial part.

As a linear space $C = X^* \otimes X$.

Now we are ready to formulate some theorems:

Theorem 3.1. Let $M = X^*$, $A = X^*X$, $\rho = R^*_x$ and $\psi : A \otimes A \to A$ be the multiplication on A. Consider the conjugate space A^* and the map $\psi^* : A^* \to A^* \otimes A^*$. Then the following formula is true:

$$\psi^*(xu) = \Delta(x)R\Delta(u),$$

where

$$R = \sum e_i \otimes e^i_*,$$

$\{e_i\}$ is some basis in X, $\{e^i_*\}$ is the dual basis in X^*, $< e_i, e^j_* >= \delta^j_i$, and $< u, x >= (\epsilon R^*_x(u))$.

Theorem 3.2. Let the antipode s be invertible. Then

1) The map

$$ux \to s^{-1}(x)u = (ux)^*$$

is a well-defined antiisomorphism of the algebra $A = X^*X$, $\rho = R^*_x$ into the algebra $A' = X^{*\prime}X^t$, $\rho' = R^*_x$. This is the definition of formal adjoint operators.

2) The map

$$ux \to s^{-1}(x)s(u)$$

is a well-defined antiisomorphism of the algebra $A = X^*X$, $\rho = R^*_x$ into the algebra $A^+ = X^*X^t$, $\rho' = L^*_{s^{-1}x}$. This is the definition of Hermitian adjoint operators.

Theorem 3.3. If the antipode s is invertible then the algebras $A^+ = X^*X^t$, $\rho^+ = L^*_{s^{-1}x}$ and $\tilde{A} = XX^*$, $\rho = R^*_u$ are canonically isomorphic.

Some constructions based on ideas similar to the Novikov's ones were obtained independently by L.Faddeev, A.Alexeev and in the form where the Hopf algebras actually were mentioned by M.Semenov-Tyan-Shanskii. His paper was published in 'Theoretical and mathematical physics', v 93 (1992), No 2, 1293-1307.

Lecture 4

String equation

In the papers by Gross-Migdal, Brezin-Kazakov, Douglas-Shenker written in 1989/90 the double scale limit of the matrix model was studied. The matrix model can be treated as a model of zero dimensional quantum field theory. The partition function of the matrix model reads as:

$$Z = \int \dots \int_{H^N} dM \exp\left\{ t_1 \mathrm{Tr} M^2 + \sum_i t_i \mathrm{Tr} M^{2i} \right\}, \tag{4.1}$$

where H^N is the space of all Hermitian $N \times N$ matrices, dM the natural measure on H^N. This integral depends on a set of parameters $\{t_i\}$.

This integral is connected with the soliton theory even for the finite N case because some quantities which arose during the calculation of the partition function satisfy the discrete KdV system. The partition function is connected with some special solutions of this system.

The times $t_1, t_2, \dots$ play the role of the coupling constants in this theory. The continuous limit corresponds to the case when the dimension of the matrices $N \to \infty$. The double scale limit means that simultaneously $N \to \infty$ and the coupling constants are rescaled. The limit depends of the choice of the rescaling. In the simplest case we have the KdV equation. Let us fix the times with sufficiently large numbers to be zero:

$$t_i = 0, \quad i > q. \tag{4.2}$$

The KdV equation reads as

$$\frac{\partial L}{\partial t_i} = [A_i, L], \quad i = 1, \dots, q-1, \tag{4.3}$$

$$L = -\partial_x^2 + u(x), \quad A_k = \partial_x^{2k+1} + \dots .$$

For the matrix models we need a special solution satisfying an extra condition:

$$[L, A_q] = 1 \text{ (String equation).} \tag{4.4}$$

This equation naturally arose in the fixed point of the renormalization group. In this point the matrix model may be treated as a model of the string theory (Migdal with Bulatov, Kostov and Kazakov), because if we write the Feynman diagrams for the matrix model we get a sum over triangulated Riemann surfaces of arbitrary genus. This sum can be treated as a discrete approximation of the integration over all the Riemann surfaces. (see Zuber, Nucl.Phys.B. Proc. Suppl. 18B(1990), pp. 313-326).

So we have the following:

The partition function Z satisfies some discrete integrable equations, namely discrete KdV (it is an exact statement). The calculation of Z is based on the technique of the orthogonal polynomials.

The discrete KdV system was known as a 'Volterra system' before and probably it was Mark Kac who investigated it first. Its integrability was first discovered by S.Manakov in 1974 under the name 'Lengmur Chain' (published in the Journal of Theor and Math Physics in 1974—see the reference in the survey: B.A. Dubrovin, V.B. Matveev, S.P. Novikov, Russian Math. Surveys 31:1(1976), pp. 59-146, where the periodic problem for it has been discussed). For a second time its integrability has been independently found by J.Moser in the paper: J.Moser. Adv. in Math. 16(1975), 354.

The specific heat of the matrix model in the double scale limit is a special solution of the following ordinary differential equation:

$$[L, A_q] = 1. \tag{4.5}$$

In soliton theory the equation $[L, A] = 0$ was studied in details. It is well-known that solutions of the equation $[L, A] = 0$ can be constructed by algebraic geometrical methods. But equation (4.5) is much more complicated.

The problem of constructing solutions of (4.5) is equivalent to finding representations of the Heisenberg algebra in the ring of ordinary differential operators.

The equation (4.5) is known as the *string equation.* It is convenient to introduce a small parameter ε to the equation (4.5)

$$[L, A_q] = \varepsilon \cdot 1. \tag{4.6}$$

(A trivial scaling maps (4.5) to (4.6) and vice versa). This small parameter allows us to use asymptotical methods.

Assume, that L is a second-order operator (only the second-order operators arose in the matrix models like the formula above)

$$L = -\partial_x^2 + u(x), \tag{4.7}$$

and A has an odd order, $2n+1$.

The case $n = 0$ is trivial:

$$u(x) = -\varepsilon x.$$

For $n = 1$ we have the Painlevé-1 equation:

$$u_{xxx} = 6uu_x + \varepsilon, \quad A = -4\partial_x^3 + 3(u\partial_x + \partial_x u). \tag{4.8}$$

Integrating the equation (4.8) we get

$$u_{xx} = 3u^2 + \varepsilon x. \tag{4.9}$$

Before the scaling limit we have the discrete Korteveg-de Vries (KdV) equation. In soliton theory the following discrete Schrodinger operator arose as the auxiliary linear operator for discrete KdV:

$$L\Psi_n = c_{n-1}^{1/2}\Psi_{n-1} + c_n^{1/2}\Psi_{n+1} + v_n^{1/2}\Psi_n = \lambda\Psi_n. \tag{4.10}$$

In spectral theory another discretization corresponding to the choice $c_n = 1$ is used, but it is nonintegrable in contrast to the discretization (4.10). (Of course $c_n = 1$ means some restriction on the scattering data but it is very difficult to characterize these restrictions.)

A-operators with n bigger than 1 can also be considered but now we would like to study the case $n = 1$ of the Painlevé-1 equation (4.9).

Only such Painlevé-1 solutions which are real-valued as $x \to -\infty$ are important. (In the physical literature the so-called triply truncated solutions are discussed but I was not able to recover what they had in mind.) There are exactly 3 families of real-valued as $x \to -\infty$ solutions:

a) $u_+(x) \sim \sqrt{-\varepsilon x/3}$.

b) $u_-(x) \sim -\sqrt{-\varepsilon x/3}$.

c) $u(x)$ has infinitely many poles as $x \to -\infty$.

The one-parametric family a) is called **Physical** and the two-parametric families b) and c) are non-physical. The families b) and c) are generic (i.e. 2-parametric), but the existence of the family a) is rather nontrivial (it was proved only in 1984 by Holmes and Spence by the methods of qualitative analysis of ordinary differential equations).

For the two families a) and b) the following formal solutions can be constructed:

$$u_+^f(x,\varepsilon) = +\sqrt{-\frac{\varepsilon x}{3}}\left(1+\sum_{i\geq 1}\varepsilon^{2i}a_i^+\tau^{-2i}\right), \quad -\varepsilon x = \tau^{\frac{4}{5}}, \tag{4.11a}$$

$$u_-^f(x,\varepsilon) = -\sqrt{-\frac{\varepsilon x}{3}}\left(1+\sum_{i\geq 1}\varepsilon^{2i}a_i^-\tau^{-2i}\right). \tag{4.11b}$$

Both expansions (4.11a) and (4.11b) diverge but the formal expansion (4.11a) is a true asymptotical expansion for the exact solution, whereas the expansion (4.11b) does not describe the asymptotical behaviour of the exact solutions because the true asymptotics for the solutions from family b) contains oscillating terms and reads as

$$u_-(x,\varepsilon) = -\sqrt{-\frac{\varepsilon x}{3}}\left(1+|x|^{-\frac{1}{8}}\sin(ax+b)+o(|x|^{-\frac{1}{8}})\right),$$

so the formula (4.11b) is wrong in the first nontrivial term.

We would like to recommend the following literature:

Literature:

1) Novikov S.P., Functional. Anal. Appl., 24(1990), No 4, pp. 196-306.

2) Krichever I.M., On the Heisenberg Relations for the Ordinary Linear Differential Operators. Preprint IHES, 1990.

3) Grinevich P.G., Novikov S.P. -in preparation.

4) Kapaev A.A.. Differentzialnye Uravnenya, 24(1988), 1684-1694 (In Russian).

5) Kitaev A.V. Uspekhi. Mat. Nauk. 49(1994), No 1.

6) Moore G. Commun. Math. Phys., 133(1990), pp. 261-304.

G. Moore obtained some results for the higher analogs of Painlevé-1.

In Novikov's paper (1990) a conjecture was formulated how exact Painlevé-1 solutions can be constructed. This conjecture was proved to be wrong.

Another conjecture was formulated in the Krichever paper and it is wrong too.

In an appendix to the Novikov paper written by Dubrovin and Novikov an analog of the nonlinear WKB method for the Painlevé-1 was developed and some asymptotic formulas for the type c) solutions were found. Krichever developed this approach and constructed asymptotical formulas for the family b). (He wrote that he had found asymptotics for the family a) but it was an error).

Kapaev and Kitaev obtained some results about the non-physical solutions using the method of isomonodromy deformations.

We have said that the formal power series approximate only the physical solutions. We hope that the physical solutions have deeper symmetry than the general ones but up to now nobody succeeded to uncover it.

The coefficients of the formal expansion (4.11a) arose from perturbation theory for some quantum field theory. The exact solutions can be treated as something which lies behind the perturbation theory. But the analytic structure of the physical solutions is not clear now.

In the Painlevé-1 theory both the nonlinear quasiclassics and the linear one are useful. We have mentioned the nonlinear quasiclassics or the nonlinear WKB method. Let us discuss the linear one.

In 1974 the following Lax-type representation for KdV was found by Novikov for the needs of studying the periodic problem:

$$\Psi_t = \Lambda\Psi, \quad \Psi_x = Q\Psi, \tag{4.12}$$

where

$$Q = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -u_x & 2u+4\lambda \\ -u_{xx}+2u^2+2\lambda u-4\lambda^2 & u_x \end{pmatrix}. \tag{4.13}$$

Compatibility conditions for the system (4.12) read as

$$\Lambda_x = [Q, \Lambda] + Q_t.$$

Some Lax pairs for the Painlevé equations were found by Jimbo, Miva and their collaborators in early 80-ies. But for Painlevé-1 it is more convenient to use the following natural construction: start from a matrix pair above for KdV, after that replace ∂_t to $\varepsilon\partial_\lambda$ in (4.12)

$$\varepsilon\Psi_\lambda = \Lambda\Psi, \quad \Psi_x = Q\Psi. \tag{4.14}$$

We are coming to a Lax-type (or Zero-Curvature) representation of P-1 equation. Flaschka and Newell were the first(in 1979) to use some special Lax representations for Painleve type ordinary differential equations and the so-called 'isomonodromic methods'), Japanese scientists continued their activity.

Let us try to construct a semiclassical solution of (4.14), i.e. a semiclassical common eigenfunction for the Lax pair. The first step is the diagonalization in the leading order. Consider the transformation

$$\Psi = U^{-1}\tilde{\Psi},$$

where U is chosen so that the matrix

$$U\Lambda U^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

is strictly diagonal. Here

$$\lambda_\pm = \pm\sqrt{R(\lambda)}, \quad R(\lambda) = -\det\Lambda = a^2 + bc = -16\lambda^3 - 4C\lambda - D.$$

After this transformation we have

$$\epsilon\tilde{\Psi}_\lambda = (U\Lambda U^{-1} - \varepsilon U(U^{-1})_\lambda)\tilde{\Psi}$$

$$\Psi = U^{-1}\tilde{\Psi}, \quad U^{-1} = \begin{bmatrix} 1 & 1 \\ \chi_- & \chi_+ \end{bmatrix}, \quad \chi_\pm = \frac{-a \pm \sqrt{R(\lambda)}}{b}.$$

The second step is the following: we construct a formal quasiclassical solution in the form:

$$\tilde{\Psi}_{SC} = (1 + \sum_{j\geq 1} \varepsilon^j A_j)\exp\{\frac{1}{\varepsilon}B_1 + B_0 + \sum_{j\geq 1}\varepsilon^j B_j\}, \tag{4.15}$$

where all the matrices A_j are antidiagonal and B_j are diagonal:

$$A_j = \begin{pmatrix} 0 & (*) \\ (*) & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} (*) & 0 \\ 0 & (*) \end{pmatrix}.$$

Let us consider now the first nontrivial term of the quasiclassical approximation:

$$\Psi^{(1)}_{SC} = \exp\{\frac{1}{\varepsilon}B_1 + B_0\}, \tag{4.16}$$

where

$$B_{-1\lambda} = \sqrt{R}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_{0\lambda} = \left(\frac{a}{b}\right)_\lambda \frac{b}{2\sqrt{R}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \left(\frac{1}{4}(\ln R)_\lambda - \frac{1}{2}(\ln b)_\lambda\right).$$

The analytic properties of this function are very much like the properties of Baker-Akhieser function. We have the following matrix-valued meromorphic differential on the Riemann surface $w^2 = \sqrt{R(\lambda)}$:

$$d_\lambda(\frac{1}{\varepsilon}B_1 + B_0).$$

The matrices B_{-1} and B_0 are diagonal. If we change the sign before the $\sqrt{R(\lambda)}$ then we simultaneously transpose the matrix elements b_{11} and b_{22} so they are the values of a meromorfic scalar differential $d_\lambda b_0$ on different sheets of our Riemann surface. So we have a scalar meromophic differential which we will denote $d_\lambda b_0$. Let:

$$d_\lambda B_0 = \Omega + [\ln(R^{1/4}/b^{1/2})]_\lambda d\lambda. \tag{4.17}$$

(In fact (4.17) is the definition of Ω). The differential Ω has the following properties:

1) Ω has poles only in the points ∞ and $\lambda_\pm = (-u/2, \pm)$.

2) In the point ∞ Ω has the following asymptotic expansion

$$\Omega = \left(\frac{8}{\varepsilon z^6} + \frac{C}{\varepsilon z^2} - \frac{D}{4\epsilon} + O(z^2)\right),\ z^2 = -\lambda \tag{4.18}$$

3) In the points $\lambda_\pm = (-u/2, \pm)$ Ω has first order poles with the residues $\pm 1/2$ respectively.

The last property is equivalent to the Painlevé-1 equation. We can write this equation in the following form:

$$u_x^2 = R(\lambda)|_{\lambda=-u/2},\quad g_2 = -\varepsilon x, g_{3x} = \varepsilon u/2. \tag{4.19}$$

The elements of the diagonal matrix - valued function $\Psi_{SC}^{(1)}$ can also be treated as values of a scalar function Φ on different sheets of our Riemann surface.

Theorem 4.1. The function Φ can be written as

$$\Phi(w) = \frac{\sigma(w+a)}{\sigma(a)\sigma(w)} \exp\left\{-\frac{4}{5\varepsilon}\wp\wp'(w) - \frac{4g_2}{5\varepsilon}\zeta(w) + \left(\frac{6}{5\varepsilon}g_3 - \zeta(a)\right) w\right\}, \tag{4.20}$$

where

$$u = 2\wp(a), \quad g_2 = -\varepsilon x, \quad g_{3x} = -\varepsilon u/2.$$

The Painlevé-1 equation can be written as

$$u_x/2 = \wp'(a), \tag{4.21}$$

and it is equivalent to

$$\left(\frac{\partial \ln \Phi}{\partial w}\right) dw = \Omega + d_w \ln(u+2\lambda)^{1/2}. \tag{4.22}$$

So we have something like a nonlinear Lax representation.

The following problem is very important: how to characterize physical solutions in terms of this scheme?

Consider a ring (a differential ring) generated by the symbols u, ∂_x, g_2, ε with the following grading:

$$d(u) = 2,\ d(\partial_x) = 1,\ d(g_2) = 4,\ d(1) = 0,\ d(\varepsilon) = 5, \tag{4.23}$$

and with the following relations

$$\partial_x u = u_x,\ \partial_x u_x = u_{xx}, u_{xx} = 3u^2 - g_2,\ u_{xxx} = 6uu_x + \varepsilon,\ \partial_x g_2 = -\varepsilon. \tag{4.24}$$

Consider the following parameterization of our algebra:

$$u/2 = \wp(a).$$

Then we have

$$u = 2\wp(a), u_x = 2\wp'(a), u_{xx} = 2\wp''(a), u_{xxx} = 2\wp'''(a) + \varepsilon$$

$$u_x^{(4)} = 2\wp^{(4)}(a), u_x^{(5)} = 2\wp^{(5)}(a) + 6u\varepsilon,$$

$$u_x^{(6)} = 2\wp^{(6)}(a) + 24u_x\varepsilon, u_x^{(7)} = 2\wp^{(7)}(a) + (60u_{xx} + 36u^2)\varepsilon, \ldots$$

The function $\wp$ depends on a and on the parameters g_2, g_3. We see that x -derivatives of $\wp$ almost coincide with the a derivtives, but we have some additional terms with ε. In the case $\varepsilon = 0$ the functions g_2 and g_3 are constants so the x derivative and the a-derivative exactly coincide.

Now we would like to discuss the physical solutions.

$$u_+(x) \sim \sqrt{-x/3},$$

(we assume now that $\varepsilon = 1$). It is convenient to introduce a new variable τ such that

$$x = -\tau^{\frac{4}{5}}, \tag{4.25}$$

and to make the following rescaling

$$\lambda = -\frac{\tau^{\frac{2}{5}}}{2\sqrt{3}}\mu, \quad u = \frac{\tau^{\frac{2}{5}}}{\sqrt{3}}\tilde{u}(\tau)$$

$$\psi_1 = \tilde{\psi}_1, \quad \psi_2 = -\frac{5}{4}\tau^{1/5}\tilde{\psi}_2 \tag{4.26}$$

$$\frac{1}{\tau}\tilde{\Psi}_\mu = \tilde{\Lambda}\Psi, \quad \tilde{\Psi}_\tau = \tilde{Q}\Psi, \tag{4.14'}$$

where

$$\tilde{\Lambda} = \begin{bmatrix} \frac{-\tilde{u}(\tau)}{12\tau} - \frac{5\tilde{u}'(\tau)}{24} & \frac{5(-\mu+\tilde{u}(\tau))}{12} \\ \frac{2\left(3-\mu^2-\mu\tilde{u}(\tau)-\tilde{u}(\tau)^2\right)}{5\,3^{\frac{3}{2}}} & \frac{\tilde{u}(\tau)}{12\tau} + \frac{5\tilde{u}'(\tau)}{24} \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} 0 & 1 \\ \frac{16}{25\sqrt{3}}\tilde{u} + \frac{8}{25\sqrt{3}}\mu & -\frac{1}{5\tau} \end{bmatrix} + \frac{2}{5}\mu\tilde{\Lambda} \tag{4.27}$$

In this Lax pair the variable $1/\tau$ plays the role of a small parameter in the μ-equation.

We will construct a quasiclassical approximation for the common eigenfunctions for the system (4.14'). Again we diagonalize this system in the leading order using the following substitution

$$\tilde{\Psi} = U^{-1}\tilde{\tilde{\Psi}}, \quad U = \begin{bmatrix} 1 & 1 \\ -\sqrt{\alpha(\mu+2)} & \sqrt{\alpha(\mu+2)} \end{bmatrix}, \alpha = \frac{8}{25\sqrt{3}} \tag{4.28}$$

After the gauge transformation (4.28) we have

$$\tilde{\tilde{\Lambda}} = \begin{bmatrix} r_+(\mu) & 0 \\ 0 & r_-(\mu) \end{bmatrix} + O(\tau^{-1}), \tilde{\tilde{Q}} = \begin{bmatrix} q_+(\mu) & 0 \\ 0 & q_-(\mu) \end{bmatrix} + O(\tau^{-1}), \tag{4.29}$$

where

$$q_{\pm}(\mu) = \frac{\pm\sqrt{\alpha}}{6}(\mu-3)(\mu+2)^{3/2}, r_{\pm}(\mu) = \frac{dq_{\pm}(\mu)}{d\mu} = \pm\frac{5}{12}\sqrt{\alpha(\mu+2)(\mu-1)}. \tag{4.30}$$

Theorem 4.2. The Lax pair (4.14') has a unique formal quasiclassical solution such, that

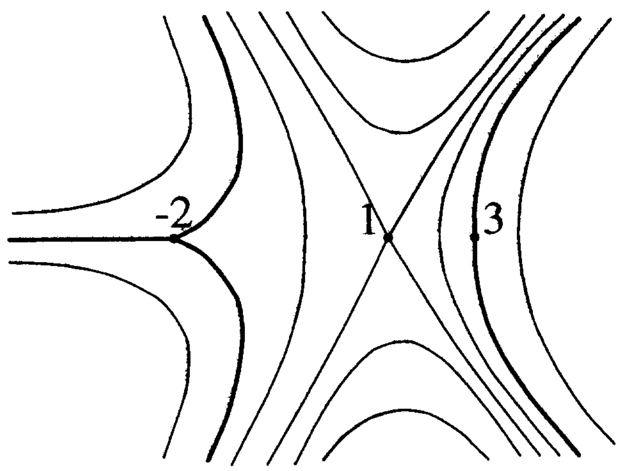
$$\tilde{\Psi} = U^{-1}\left(1+\sum\frac{A_i}{\tau^i}\right)\exp\left\{\tau B_{-1} + B_0 + \sum B_i\tau^{-i}\right\}, \tag{4.31}$$

where all A_i are off-diagonal, all B_i are diagonal, A_i and B_i are algebraic on the following Riemann surface $\Gamma^{(0)}$ of genus zero:

$$y^2 = \mu + 2, \tag{4.31}$$

$$B_1 = \begin{pmatrix} q_+(\mu) & 0 \\ 0 & q_-(\mu) \end{pmatrix}. \tag{4.32}$$

Theorem 4.3. Consider the Stokes sectors on $\Gamma^{(0)}$ bounded by the lines $\mathrm{Re}q_+(\mu) = 0$ (bold lines on the picture).



Then for each Stokes sector there exists a unique up to a constant factor solution of (4.14') decaying as $\tau \to \infty$.

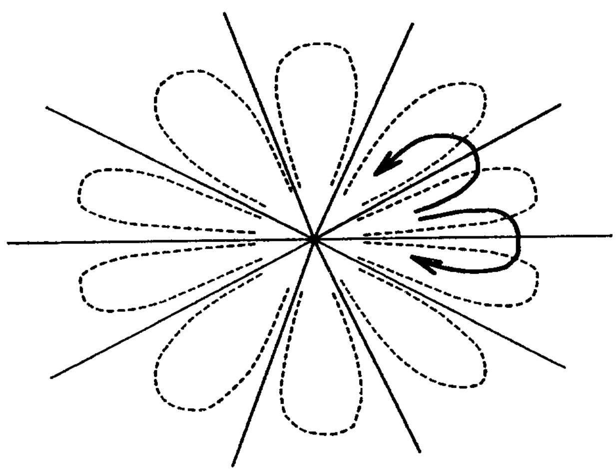
In fact we have a foliation on $\Gamma^{(0)}$ defined by

$$\mathrm{Re}(q_+(\mu)d\mu) = 0. \tag{4.33}$$

(thin lines on the picture). If we have not a Lax pair but only one equation with a small parameter then we can construct quasiclassical solutions only along paths which are transversal to the foliation. In spite of the fact that specialists in the quasiclassical methods and quantum physicists understood it (see, for example the book by Landau and Lifshitz) it is likely that this statement was never formulated in such explicit form in the literature.

We see that if we have only one equation we are unable to connect the points -2 and 3 by a path transversal to the foliation. But from the Lax pair it follows that there exists a solution of the μ equation of (4.14') (after (4.26)) which has the norm 1 in the point 1 and decays exponentially as $\mu \to -2$, $\mu \to 3$. So the energy $E = 0$ is a discrete eigenvalue for the μ-operator on the interval $[-2, 3]$ in (4.14') with the quasiclassical precision. So we have something like quasiclassical eigenvalue 0.

Let us compare this foliation picture with the standard Stokes lines description. In the neighbourhood of the point ∞ we have the following picture: (the foliation is shown by dashed lines, the Stokes sectors are drawn by solid lines)



The quasiclassical formula

$$\Psi_{SC} \sim \exp\left(\int_{\mu_0}^{\mu} q_+(\eta) d\eta\right),$$

defines a good approximation only if the integration path is transversal to our foliation. So we can continue a solution which decays in sector i only to sectors $i+1$ and $i-1$. Let Ψ_{i-1} and Ψ_{i+1} decay in the sectors $i-1$ and $i+1$ respectively, Ψ_i decay in the sector i. Then we can continue Ψ_{i-1} and Ψ_{i+1} to the sector i. In the general case the sectors $i-1$ and $i+1$ could not be connected by a path transversal to the foliation so the difference of the values of these solutions in the sector i is nonzero, but

$$\Psi_{i+1} - \Psi_{i-1} = c_i \Psi_i. \tag{4.34}$$

The numbers c_i are called Stokes multipliers. In the case of the Lax pair these numbers are integrals of motion.

In our case we have an exact definition of the Stokes lines in all finite part of the complex plane (Riemann surface) as zeroes of the real part of the common solution for the both equations in the Lax pair which has exponential decay for $\tau \to \infty$ and well defined inside of each 'Stokes sector. One of these 'Stokes lines' is passing through the point 3. It is closed on the Riemann surface. So the two neighbouring Stokes sectors are in fact connected with each other and one of the Stokes multipliers is equal to 0. The last fact was observed by Kitaev and Kopaev in 1987 but their proof was different and more complicated because they did not use the special semiclassics for the two linear equations in the zero-curvature representation ('Lax pair').