BEAM IMPEDANCE

Olav Berrig / CERN

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Outline

1. What is beam impedance?
2. Beam impedance is modelled as a lumped impedance
3. New formula for longitudinal beam impedance
4. Panofsky-Wenzel theorem and transverse impedance
5. Lab measurements of beam impedance

What is beam impedance?

- Beam impedance is just a normal impedance.

- However, it is very difficult to understand beam impedance because it is not a lumped impedance but measured over a length.

- In addition it is defined as the difference in impedance between an accelerator equipment and a straight vacuum chamber. The straight vacuum chamber must have constant cross-section; have the same length as the accelerator equipment and have walls that are superconducting (also called perfectly conducting PEC).

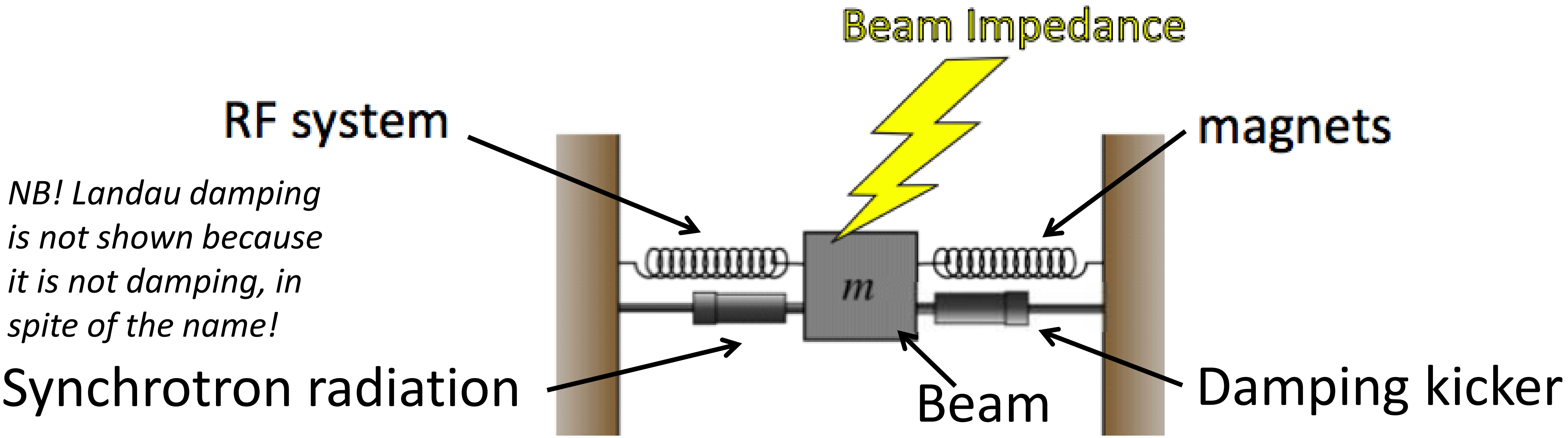
- A particle moving in a straight vacuum chamber with constant cross-section and superconducting walls have no beam impedance.

What is beam impedance?

An accelerator without beam impedance does not have instabilities. Beam impedance is **not** our friend!

Beam impedance gives the beam a **kick** i.e. a disturbing force acting on the beam. The beam impedance forces will make the beam oscillate, just like a mass suspended between springs:

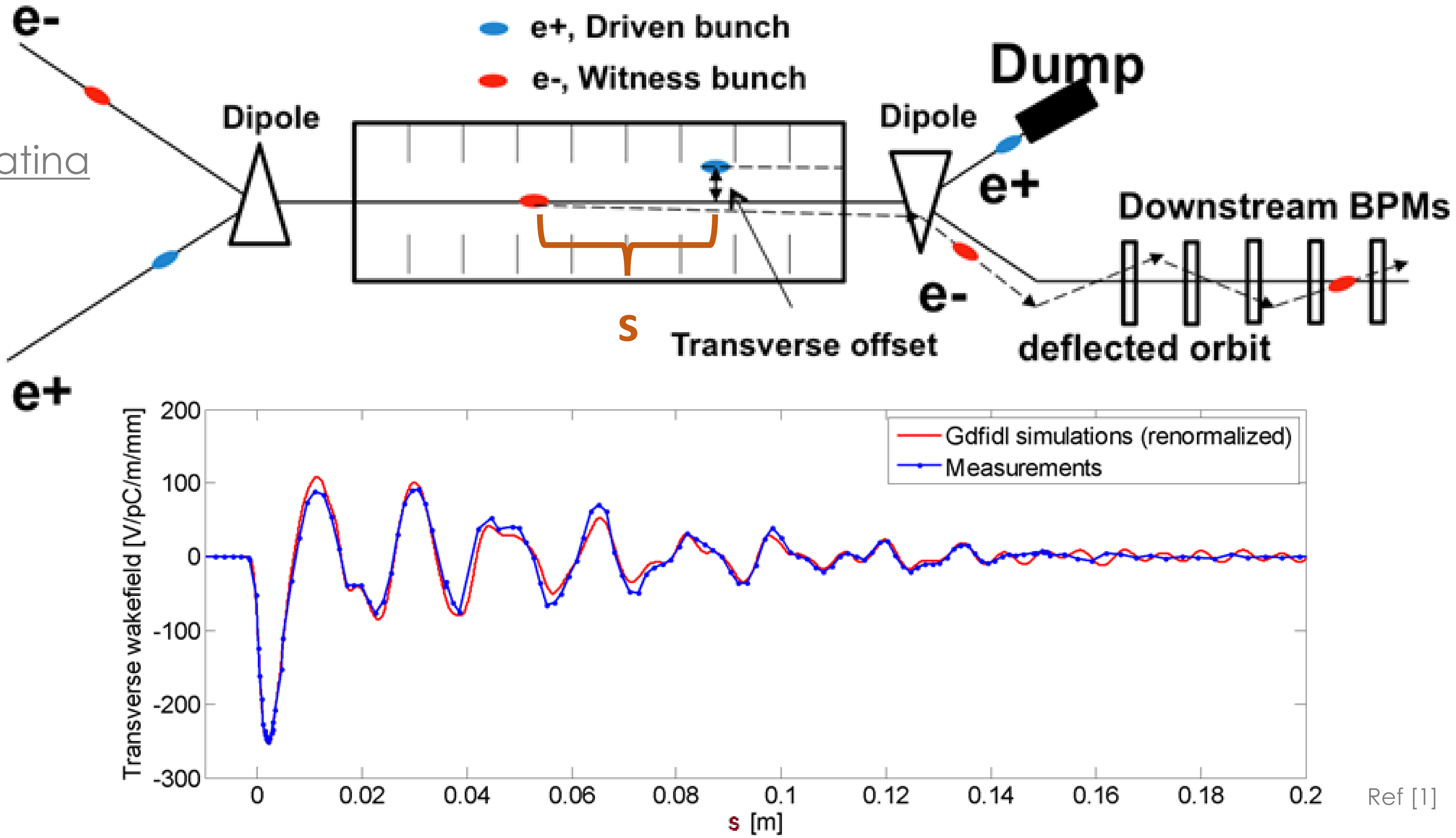
NB! Landau damping is not shown because it is not damping, in spite of the name!



What is beam impedance?

An example of transverse impedance, that gives the beam a transverse kick! Here measured with the beam

Andrea Latina
Hao Zha

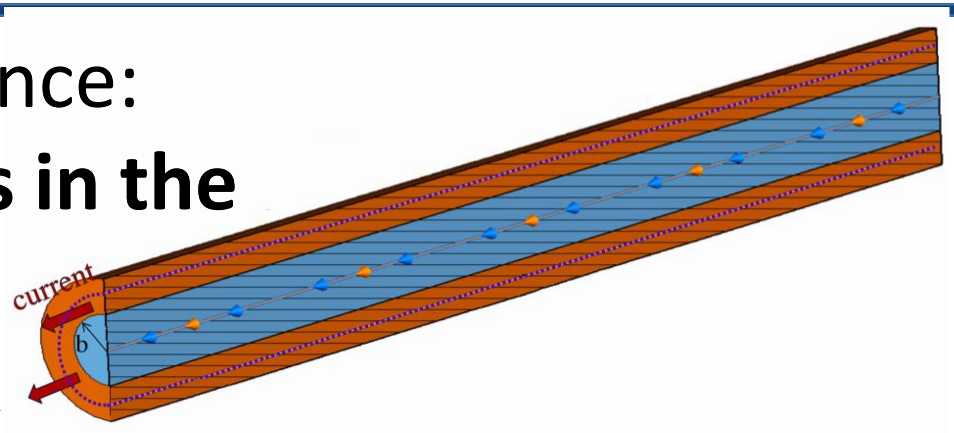


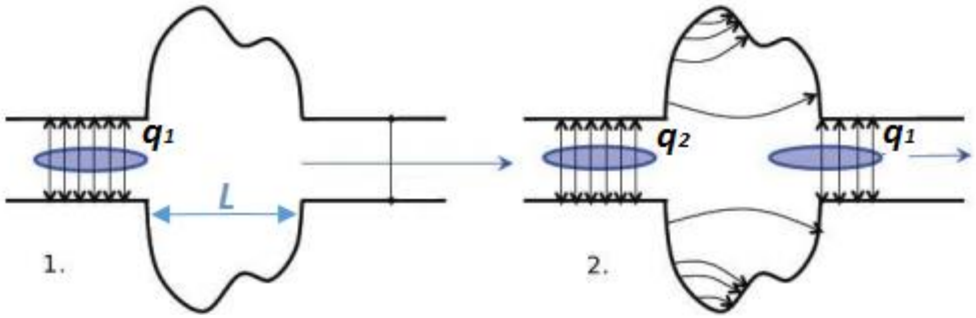
Ref [1]

What is beam impedance?

There are many types of beam impedance:

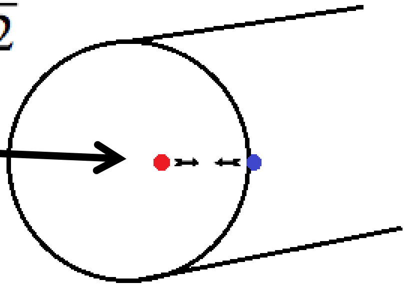
- Beam impedance from the **currents in the walls** of accelerator equipment (beam coupling impedance):
  1) Resistive wall impedance
  2) Geometric impedance





- **Space charge** beam impedance
  1) Direct space charge impedance
  2) Indirect space charge impedance

$$Z_{sc}(w) \propto \frac{1}{\gamma^2}$$



- Damping kicker impedance, Electron cloud , impedance, …

What is beam impedance?

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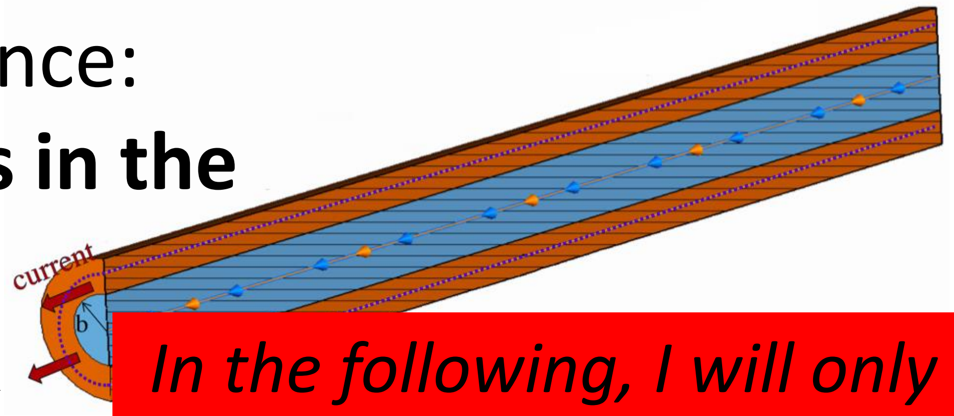
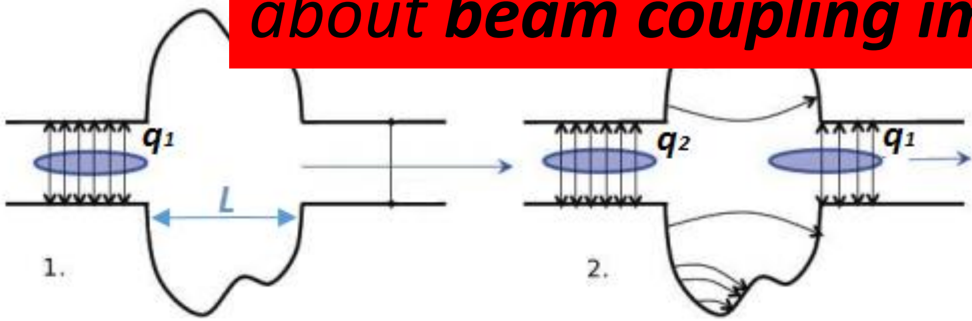


*In the following, I will only talk about **beam coupling impedance***



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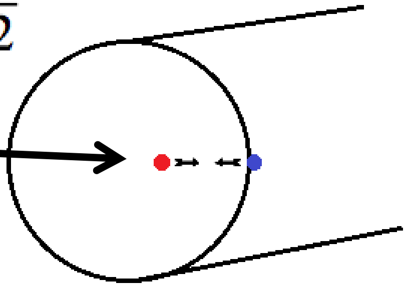
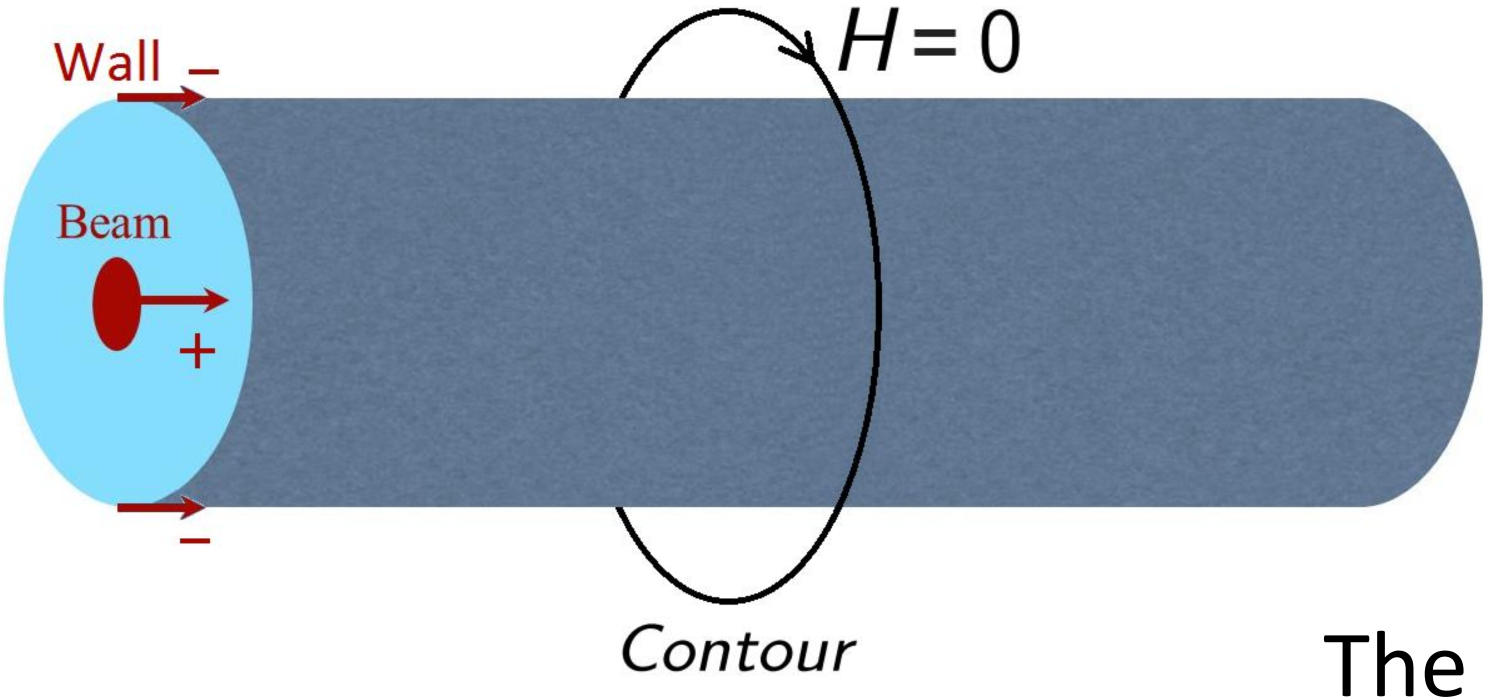


- Damping kicker impedance, Electron cloud , impedance, …

What is beam impedance?

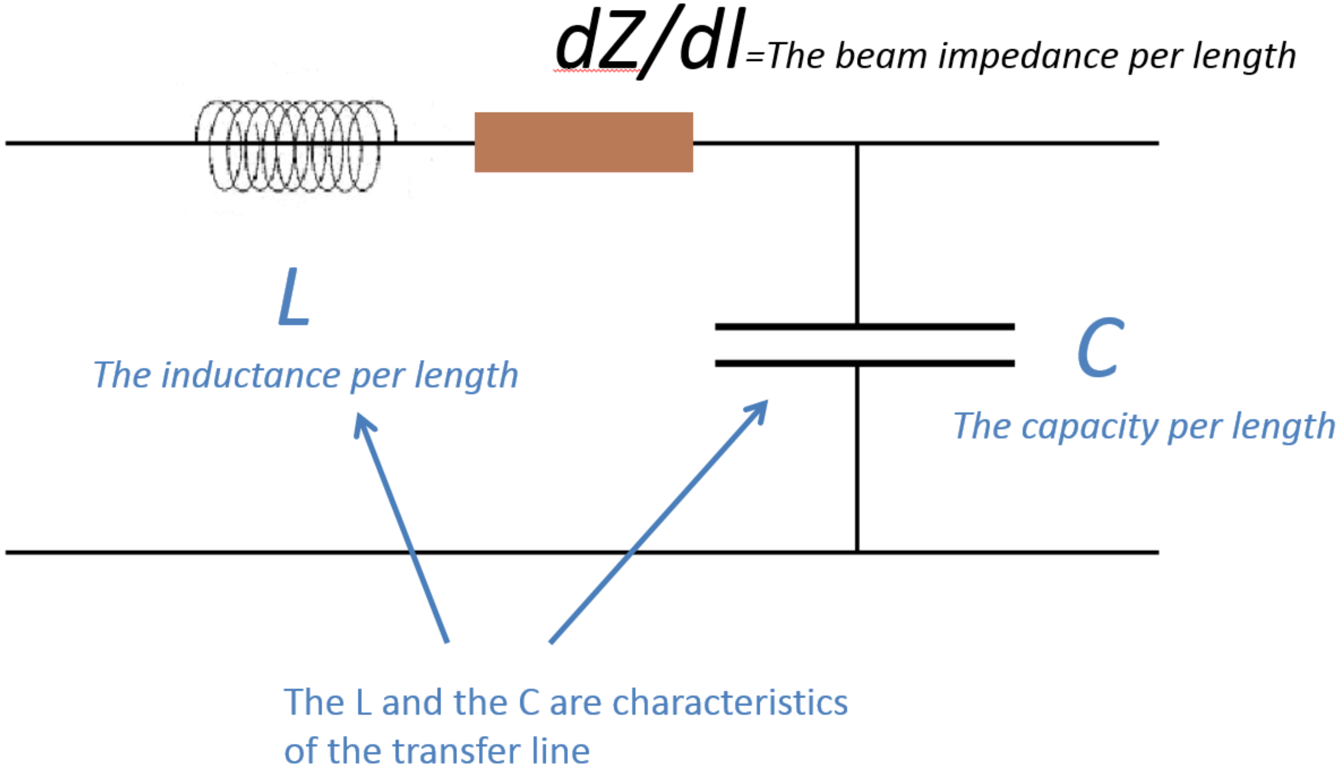
Vacuum chamber = Faraday cage

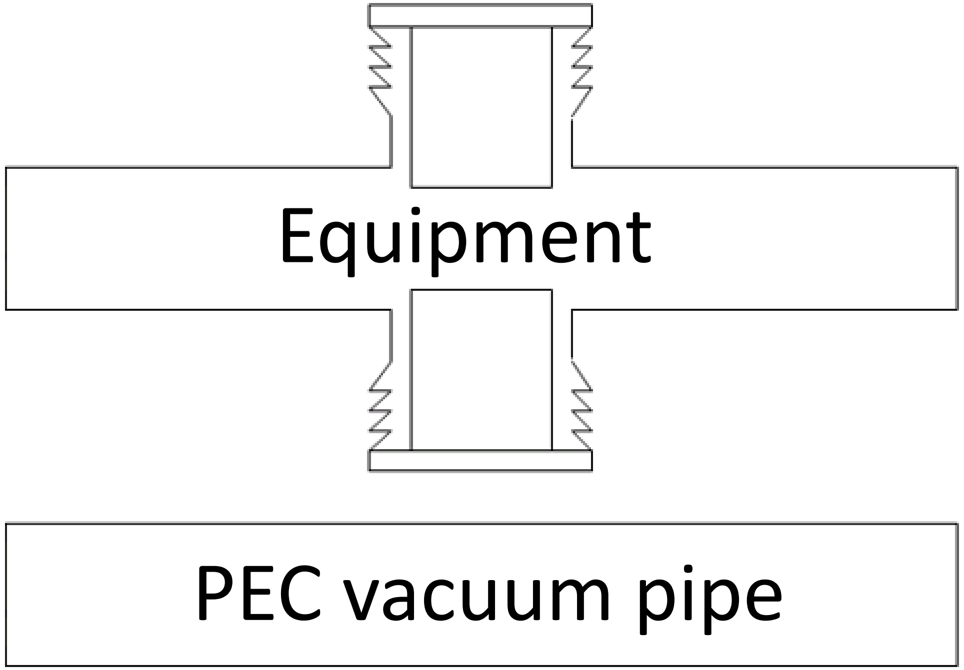


$$\oint_{Contour} H\,dl = I_{Beam} + I_{Wall} = 0$$

The wall currents must oppose the beam current, so that the fields outside the vacuum chamber are zero

What is beam impedance?



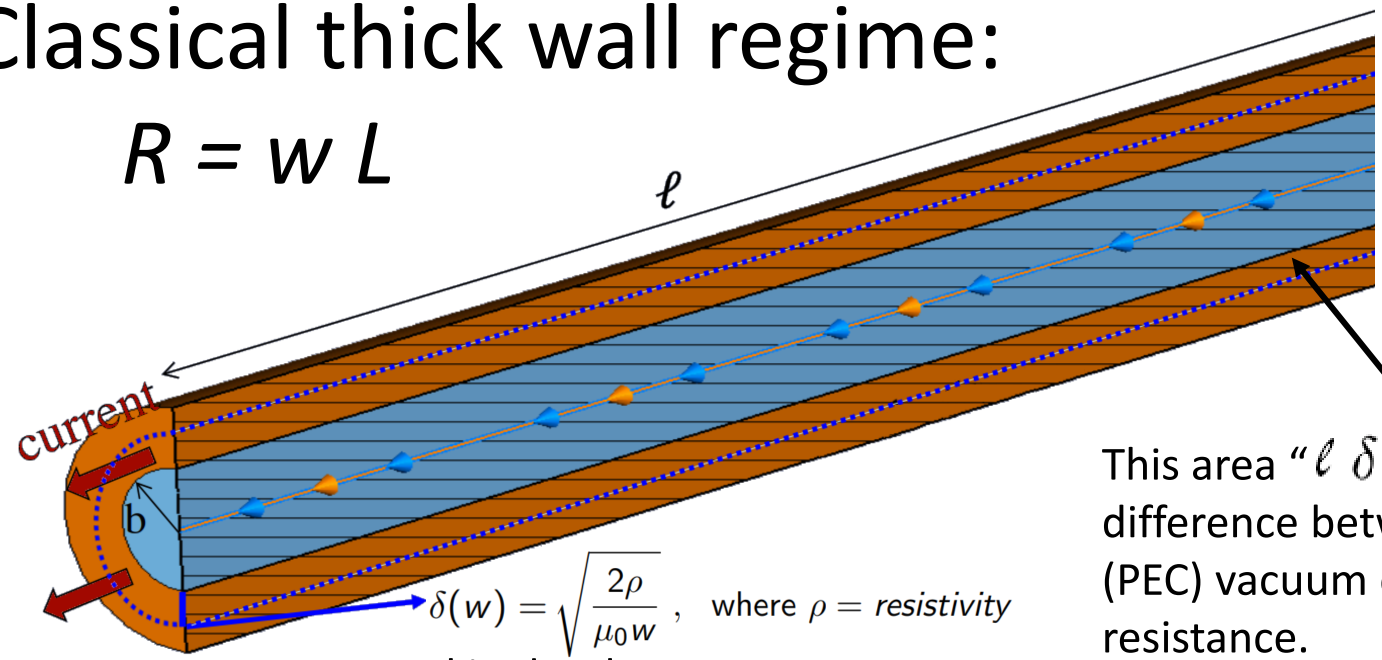


When we calculate the beam impedance for an equipment, we compare the equipment to a perfectly conducting (PEC) vacuum chamber with the same dimensions at start and end.

What is beam impedance?

Classical thick wall regime:

$R = w\,L$



$\delta(w) = \sqrt{\dfrac{2\rho}{\mu_0 w}}$, where $\rho = resistivity$

skin depth

This area “$\ell\ \delta(w)$ ” represents the difference between superconducting (PEC) vacuum chamber and one with resistance.

$$Z(w) = R(w) + iwL(w)$$

$$R(w) = w \cdot L(w) = \frac{\ell \cdot \rho}{2\pi b \cdot \delta(w)}$$

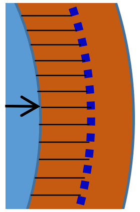
$$I = J \cdot 2\pi b \cdot \delta(w)$$

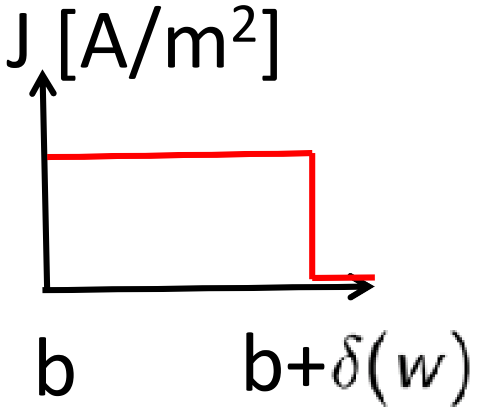
$$\phi = \int_b^{b+\delta(w)} B(r)\,\ell\,dr$$

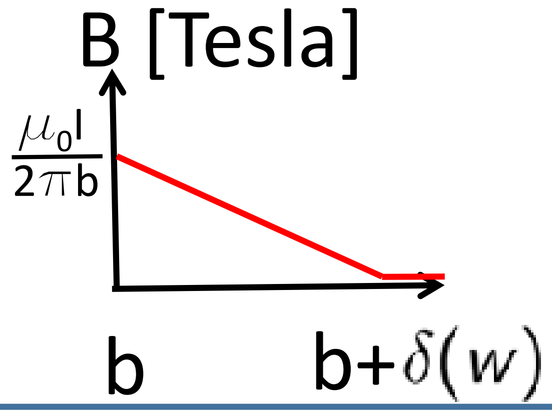
$$= \frac{1}{2}\frac{\mu_0 I}{2\pi b}\ell\,\delta(w)$$

$$L = \frac{\phi}{I} = \frac{\mu_0\delta(w)}{4\pi b}\ell$$

Current density estimation

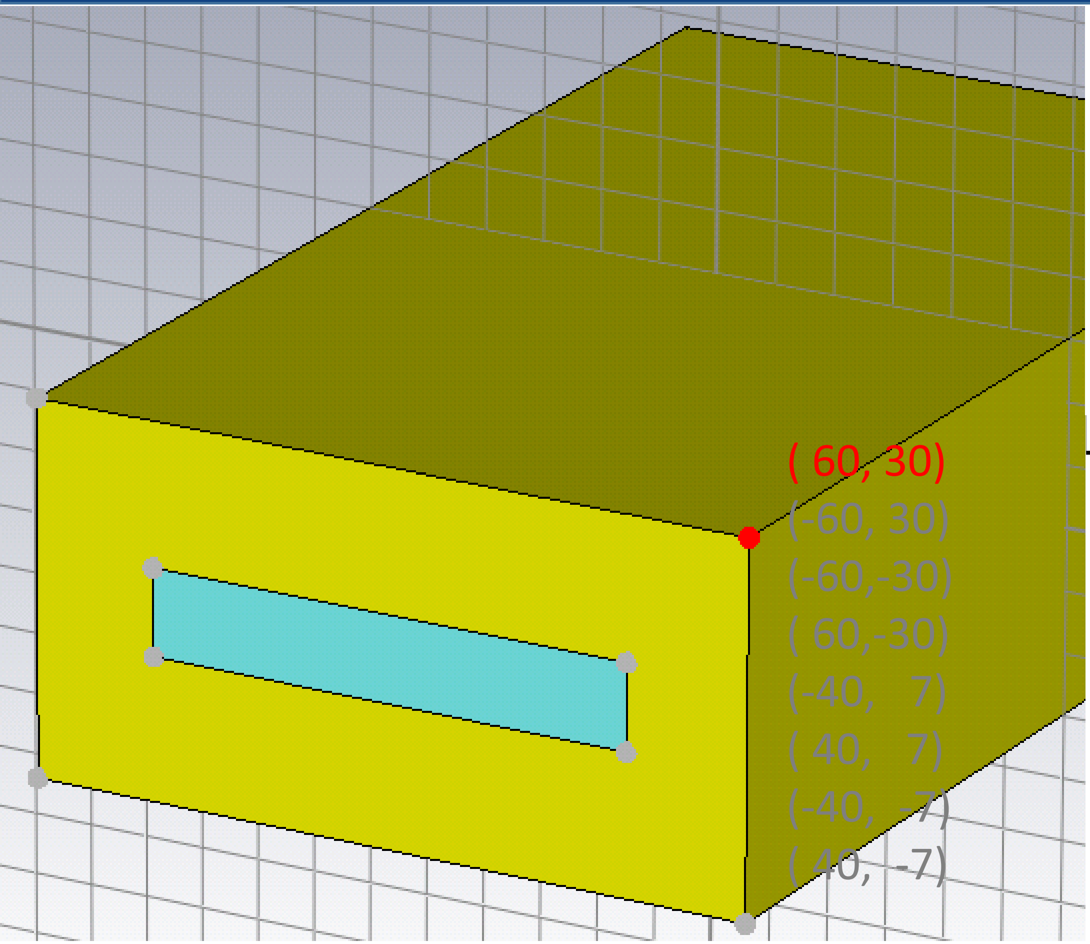






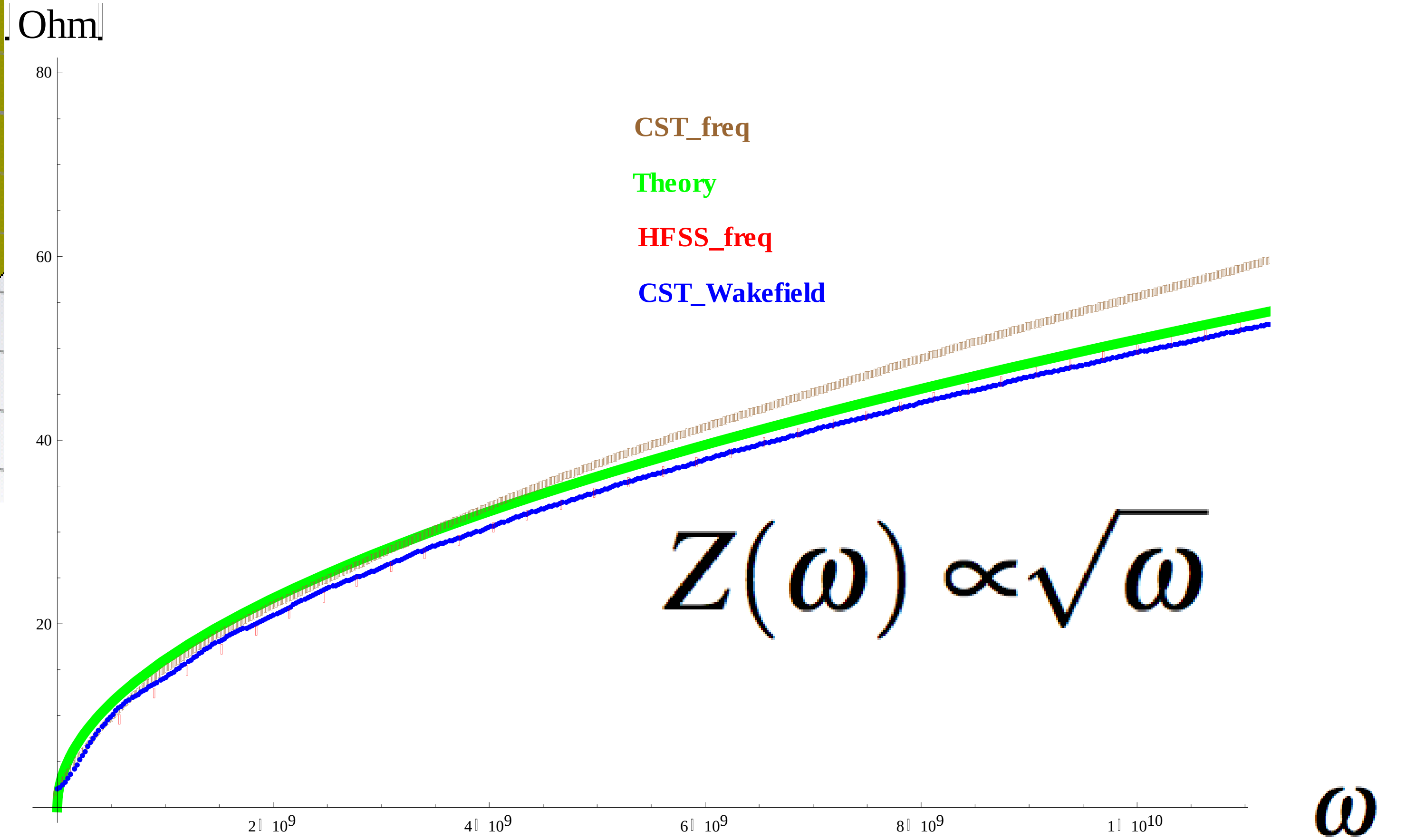
Curtesy of M.Migliorati

What is beam impedance?



$$ZTheo[w] = \frac{(1 + i) * L}{2\pi (g/2)\, \sigma\, \delta[w]}$$

Skin depth: $\delta[w] = \sqrt{\frac{2}{w\, \mu\, \sigma}}$



$$Z(\omega) \propto \sqrt{\omega}$$

Collimator:
Length: 200 mm
Width: 120 mm
Height: 60 mm
Electrical conductivity of jaws: σ = 100 S/m

Beam impedance: R + j w L versus R – i w L

Circuit definition
R + j w L

$$
\begin{aligned}
\text{Impedance:}\quad Z(w) &= R + j\omega L \\
\text{Voltage:}\quad V(t) &= V_0 \cdot Cos(w_0\, t) \\
\text{Add imaginary part:}\quad V(t) &= V_0 \cdot (Cos(w_0\, t) + j \cdot Sin(w_0\, t)) \\
&= V_0 \cdot e^{j w_0 t} \\
\text{Voltage for analysis:}\quad V(\omega) &= V_0 \\
\text{Current for analysis:}\quad I(\omega) &= I_0\, e^{-j\phi} \\
\text{Circuit equation:}\quad V(\omega) &= (R + j w_0 L) \cdot I(\omega) \\
\text{Solution for I:}\quad I(\omega) &= \frac{V(\omega)}{R + j w_0 L} \\
&= \frac{e^{-j\phi}}{\sqrt{R^2 + (w_0 L)^2}} \cdot V(\omega) \\
\text{Convert to time domain:}\quad I(t) &= \frac{e^{-j\phi}}{\sqrt{R^2 + (w_0 L)^2}} \cdot V_0 \cdot e^{j w_0 t} \\
\text{Remove imaginary part:}\quad I(t) &= \frac{V_0}{\sqrt{R^2 + (w_0 L)^2}} \cdot Cos(w_0 t - \phi) \\
&\text{where } \phi = ArcTan(\tfrac{w_0 L}{R})
\end{aligned}
$$

"American" Fourier
R + j w L

$$
\begin{aligned}
\text{Voltage:}\quad V(t) &= V_0 \cdot Cos(w_0\, t) \\
\text{Circuit equation:}\quad V(t) &= R + L\frac{dI(t)}{dt} \\
\text{Fourier Transform:}\quad V(\omega) &= R \cdot I(\omega) + j\omega \cdot I(\omega) \\
V(\omega) &= (R + j\omega L) \cdot I(\omega) \\
\text{Solution for I:}\quad I(\omega) &= \frac{V(\omega)}{R + j\omega L} \\
\text{Inv.Fourier Transform:}\quad I(t) &= \frac{V_0 (R\, Cos(w_0\, t) + w_0 L\, Sin(w_0\, t))}{R^2 + w_0^2 L^2} \\
I(t) &= \frac{V_0}{\sqrt{R^2 + (w_0 L)^2}} \cdot Cos(w_0 t - \phi) \\
&\text{where } \phi = ArcTan(\tfrac{w_0 L}{R})
\end{aligned}
$$

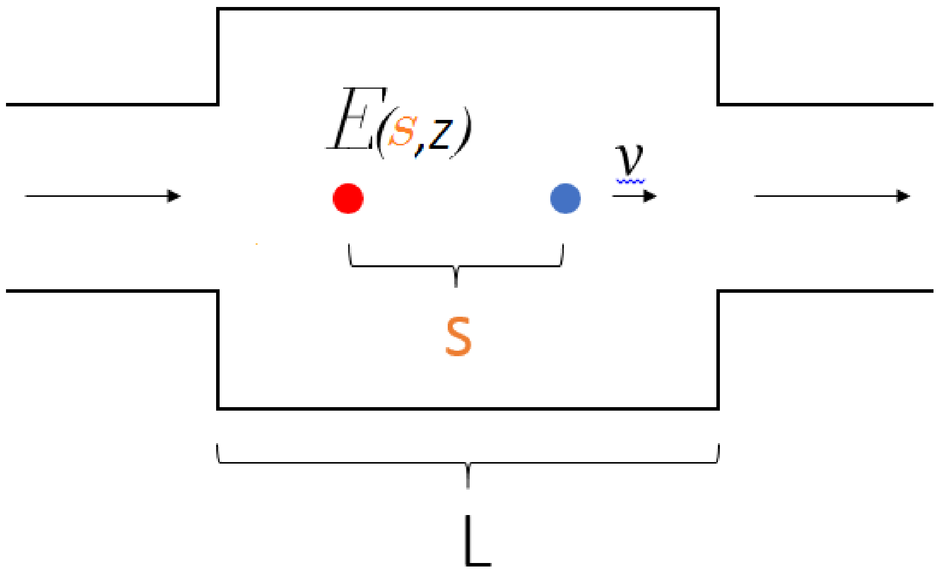
Chinese and European Fourier
R – i w L

$$
\begin{aligned}
\text{Voltage:}\quad V(t) &= V_0 \cdot Cos(w_0\, t) \\
\text{Circuit equation:}\quad V(t) &= R + L\frac{dI(t)}{dt} \\
\text{Fourier Transform:}\quad V(\omega) &= R \cdot I(\omega) - i\omega \cdot I(\omega) \\
V(\omega) &= (R - i\omega L) \cdot I(\omega) \\
\text{Solution for I:}\quad I(\omega) &= \frac{V(\omega)}{R - i\omega L} \\
\text{Inv.Fourier Transform:}\quad I(t) &= \frac{V_0 (R\, Cos(w_0\, t) + w_0 L\, Sin(w_0\, t))}{R^2 + w_0^2 L^2} \\
I(t) &= \frac{V_0}{\sqrt{R^2 + (w_0 L)^2}} \cdot Cos(w_0 t - \phi) \\
&\text{where } \phi = ArcTan(\tfrac{w_0 L}{R})
\end{aligned}
$$

In my experience, accelerator components have only resistive and inductive coupling impedance.

Beam impedance modelled as lumped impedance

Definition of beam impedance:



$$V(s) = -\int_0^L E(s,z)\,dz$$ = Voltage over equipment

$$V(t) = V(s)\text{ , where } s = v \cdot t$$

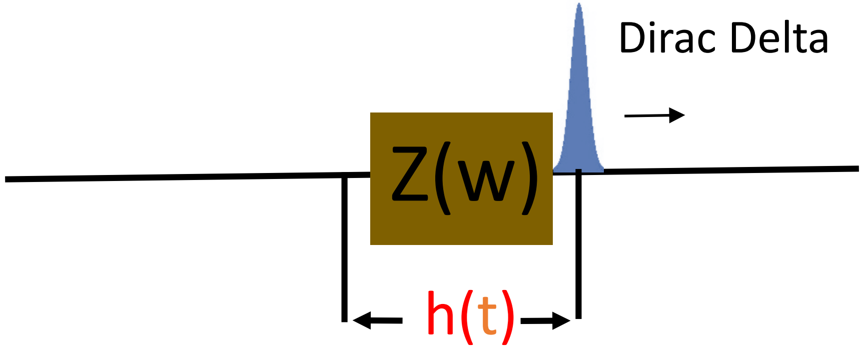
$$I_d(t) = q_d \cdot \delta(t)$$ Drive particle act as a current.

(It's a Dirac delta function)

$$Z(w) = \frac{V(w)}{I_d(w)} = \frac{\mathscr{F}(V(t))}{\mathscr{F}(I_d(t))} = \frac{\int_{-\infty}^{\infty} V(t)\,\mathrm{e}^{-iwt}\,dt}{q_d}$$

$$Z(w) = \int_{-\infty}^{\infty} W_{||}(t)\,\mathrm{e}^{-iwt}\,dt\text{ , where }\quad W_{||}(t) = \frac{V(t)}{q_d}$$

Definition of lumped impedance:

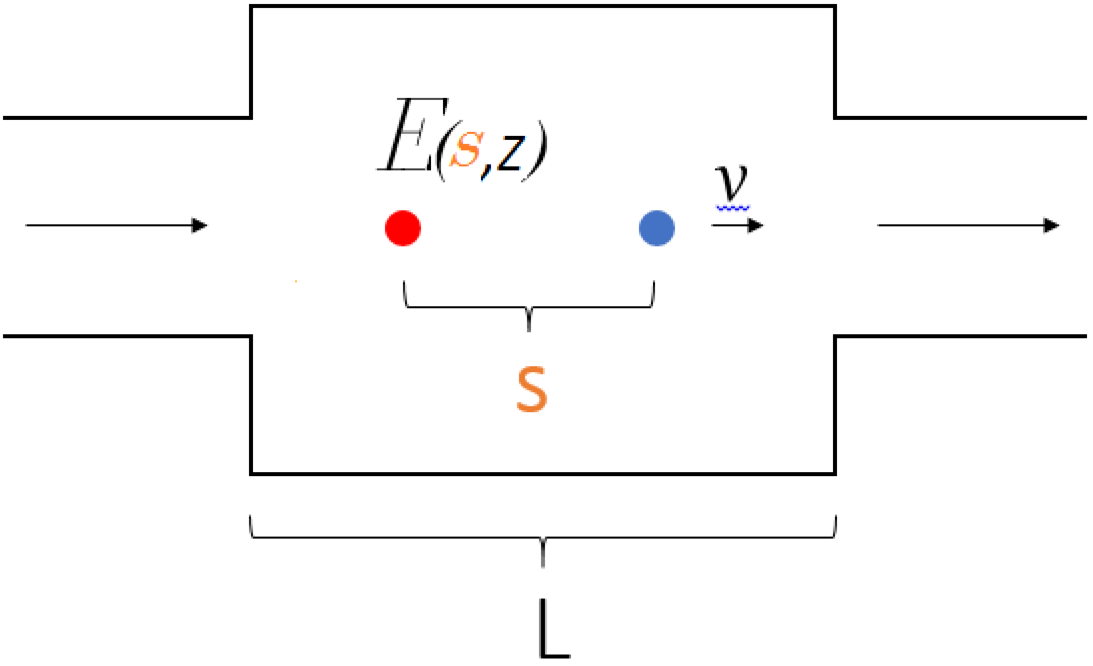


h(t) = impulse response

$$Z(w) = \frac{V(w)}{I(w)} = \frac{\mathscr{F}(V(t))}{\mathscr{F}(I(t))} = \frac{\int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-iwt}\,dt}{\int_{-\infty}^{\infty} \delta(t)\,\mathrm{e}^{-iwt}\,dt}$$

$$Z(w) = \int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-iwt}\,dt$$

Beam impedance modelled by lumped impedance

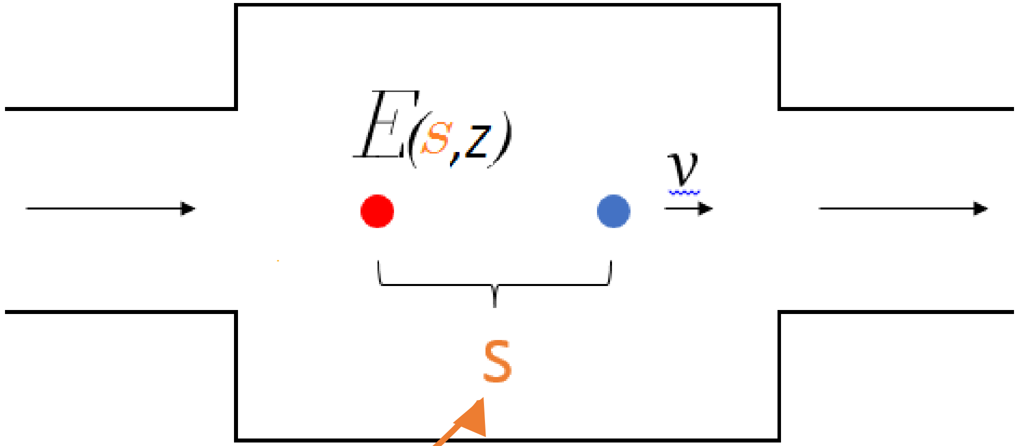


The wake function $W_{||}(t)$ is the equipment response function, i.e. the response to a Dirac delta function. The impedance is, according to normal theory, just the Fourier transform of the response function:

$$W_{||}(t) = \frac{V(t)}{q_d} = \frac{V(s)}{q_d} = -\frac{\int_0^L E(s,z)\,dz}{q_d}, \text{ where } s = v \cdot t$$

$$Z(w) = \int_{-\infty}^{\infty} W_{||}(t)\, e^{-iwt}\, dt$$

Beam impedance modelled by lumped impedance



$$Z(w) = \int_{-\infty}^{\infty} W_{||}(t)\, e^{-iwt}\, dt$$

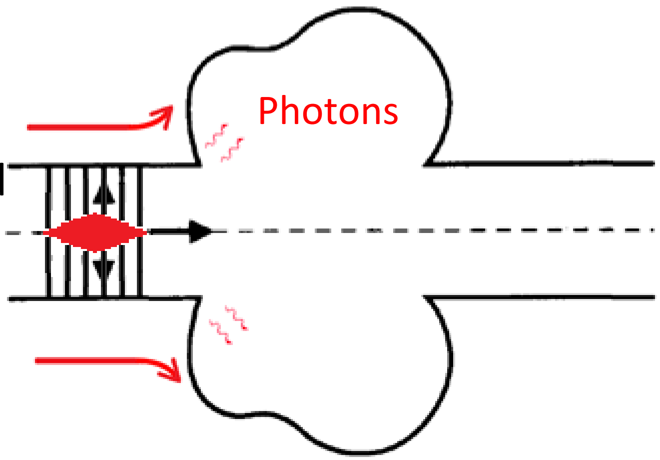
$s = v \cdot t$ $\quad$ $z = c \cdot t$

In other texts (See e.g. Ref. [6]) one will often find this definition:

$$Z_{||}(\omega) = \int_{-\infty}^{\infty} W_{||}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

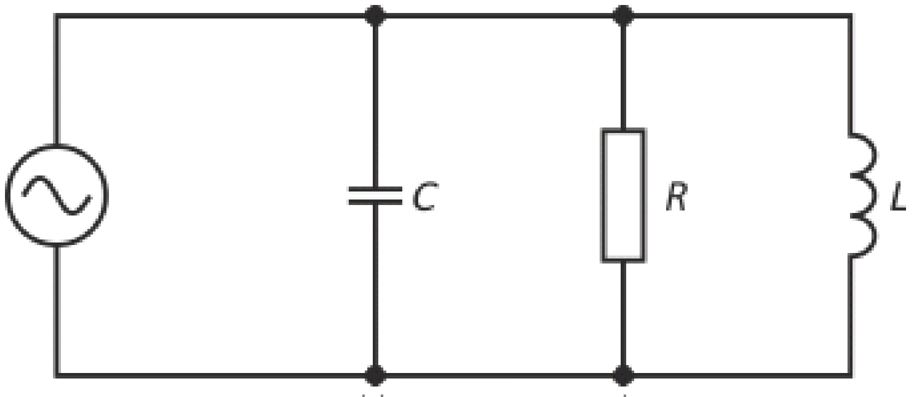
Beam impedance modelled by lumped impedance

Wall currents generate electro-magnetic fields i.e. photons when bend along the cavity walls.



The electro-magnetic fields stays in the cavity and generates a **resonance**, which will disturb i.e. kick the following bunch.

A **resonance** is modeled as a RLC-circuit:



RLC-circuit definition used for resonance ("American" Fourier)

$$Z_{\parallel}(\omega) = \int_0^{\infty} W_{\parallel}(t)\, e^{j\omega t}\, dt$$

$$k_{loss} = \frac{1}{2\pi} \int_0^{\infty} \Re\{Z_{\parallel}(\omega)\}\, d\omega$$

NB! This definition of the loss factor is only valid for a bunch that is a dirac delta function. The more general definition will be given later.

$$\omega_0 = \frac{1}{\sqrt{LC}}; \; Q = R\sqrt{\frac{C}{L}}$$

$$Z_{\parallel}(\omega) = \frac{R}{1 + jQ(\omega/\omega_0 - \omega_0/\omega)}$$

$$k_{loss} = \frac{\omega_0}{4}\frac{R}{Q}$$

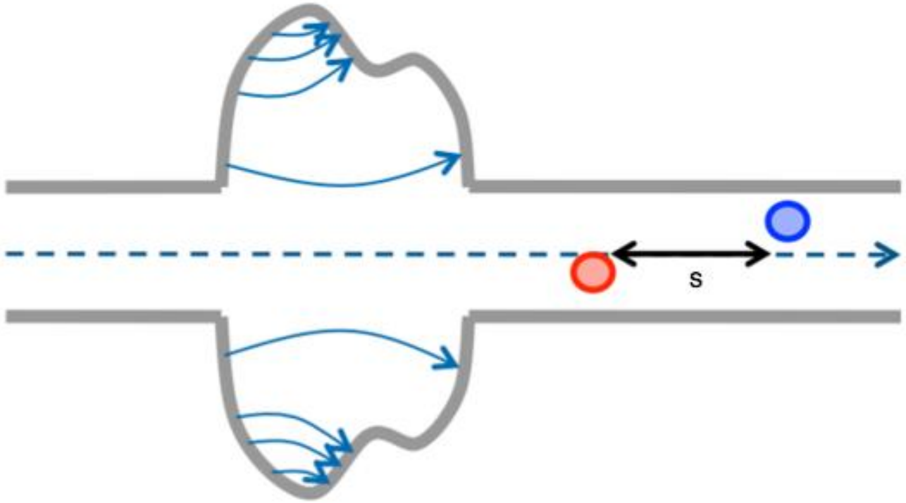
The bigger R/Q the bigger the energy loss.

The energy lost, is equal to the loss factor "k_{loss}" multiplied with the square of the charge of the bunch:

$$E_{loss} = k_{loss} \cdot q_{bunch}^2$$

New formula for longitudinal beam impedance

The Longitudinal beam impedance is a function of the transverse position of the drive and test particles i.e. 4 variables. It can therefore be decomposed into 15 parameters ($Z0$, $Z1_{xd}$, $Z1_{xt}$, etc..) that represent all combinations of the 4 variables:



$$
\begin{aligned}
Z[xd,xt,yd,yt] = Z0 \\
&+ Z1_{xd} \cdot xd + Z1_{xt} \cdot xt + Z1_{yd} \cdot yd + Z1_{yt} \cdot yt \\
&+ Z2_{xdxd} \cdot xdxd + Z2_{xtxt} \cdot xtxt + Z2_{ydyd} \cdot ydyd + Z2_{ytyt} \cdot ytyt \\
&+ Z2_{xdxt} \cdot xdxt + Z2_{xdyd} \cdot xdyd + Z2_{xdyt} \cdot xdyt \\
&+ Z2_{xtyd} \cdot xtyd + Z2_{xtyt} \cdot xtyt + Z2_{ydyt} \cdot ydyt
\end{aligned}
$$

New formula for longitudinal beam impedance

Holomorphic decomposition:

Any two dimensional field, and very importantly a field that can really exist (so not an artificially constructed field), can be decomposed into multipolar components. This is the same idea used in Fourier transforms. The holomorphic decomposition expands the field into normal and skew multipolar functions:

$$f(x,y) = \underbrace{a_0}_{\text{Zero order}} + \underbrace{a_{1,normal} \cdot x + a_{1,skew} \cdot y}_{\text{first order}} + \underbrace{a_{2,normal} \cdot \left(\frac{x^2}{2} - \frac{y^2}{2}\right) + a_{2,skew} \cdot (xy)}_{\text{second order}} + \ldots$$

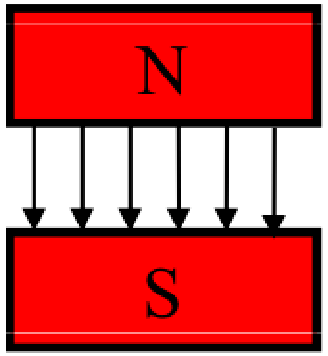
NB! Notice that the coefficients for x squared and y squared are same numerical value but opposite signs

New formula for longitudinal beam impedance

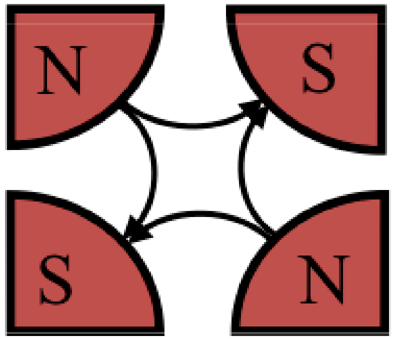
The normal and skew multipolar functions are well known from accelerator magnets:

$$f(x,y) = \underbrace{a_0}_{\text{Zero order}} + \underbrace{a_{1,normal}\cdot x + a_{1,skew}\cdot y}_{\text{first order}} + \underbrace{a_{2,normal}\cdot\left(\frac{x^2}{2}-\frac{y^2}{2}\right) + a_{2,skew}\cdot(xy)}_{\text{second order}} + \ldots$$

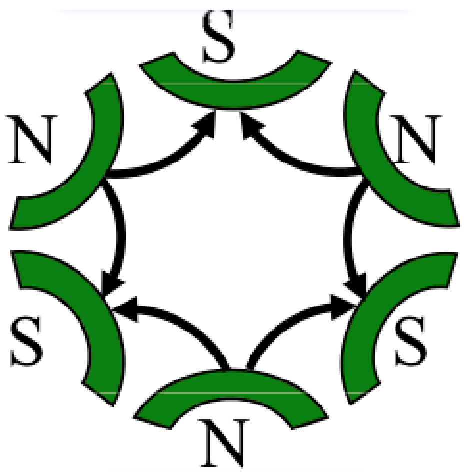
Dipole



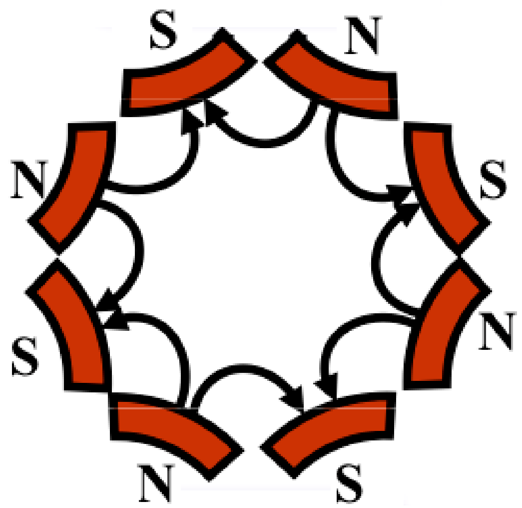
Quadrupole



Sextupole

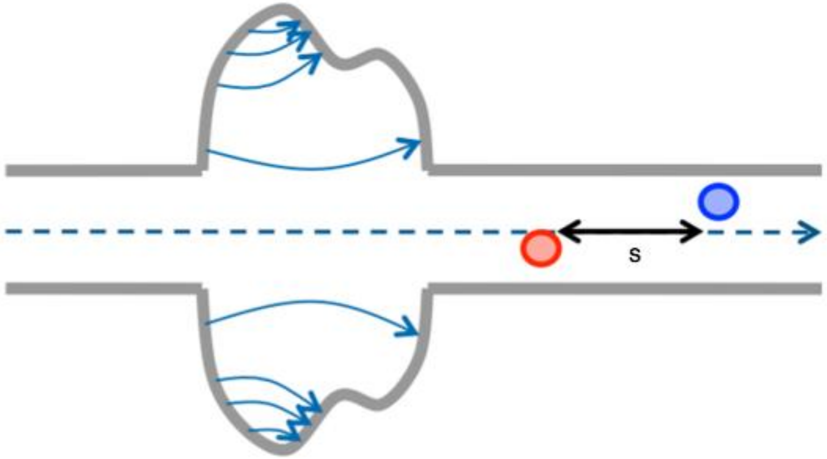


Octupole



New formula for longitudinal beam impedance

Using the holomorphic decomposition for both the drive and test particles , knowing that the coefficients for the squared values of xd & yd and xt & yt must be of opposite sign, the formula can be reduced to 13 terms:



$$
\begin{aligned}
Z[xd, xt, yd, yt] = Z0 \\
&+ Z1_{xd} \cdot xd + Z1_{xt} \cdot xt + Z1_{yd} \cdot yd + Z1_{yt} \cdot yt \\
&+ Z2_{drive} \cdot (xdxd - ydyd) + Z2_{test} \cdot (xtxt - ytyt) \\
&+ Z2_{xdxt} \cdot xdxt + Z2_{xdyd} \cdot xdyd + Z2_{xdyt} \cdot xdyt \\
&+ Z2_{xtyd} \cdot xtyd + Z2_{xtyt} \cdot xtyt + Z2_{ydyt} \cdot ydyt
\end{aligned}
$$

New formula for longitudinal beam impedance

Using a property, called the Lorentz reciprocity principle, which says that if we exchange the positions of the drive and test particles, the beam impedance stays unchanged, i.e. $Z[xd, xt, yd, yt] = Z[xt, xd, yt, yd]$.

This leads to 5 equalities:

$$Z1_{xd} = Z1_{xt}, \quad Z1_{yd} = Z1_{yt}, \quad Z2_{drive} = Z2_{test}, \quad Z2_{xdyd} = Z2_{xtyt}, \quad Z2_{xdyt} = Z2_{xtyd}$$

The new formula for longitudinal beam impedance finally has only 8 terms:

$$\begin{aligned} Z[xd, xt, yd, yt] = Z0 & \\ & + Z1_x \cdot (xd + xt) + Z1_y \cdot (yd + yt) \\ & + Z2_A \cdot (xdxd - ydyd + xtxt - ytyt) \\ & + Z2_B \cdot (xdyd + xtyt) + Z2_C \cdot (xdyt + xtyd) \\ & + Z2_D \cdot xdxt + Z2_E \cdot ydyt \end{aligned}$$

New formula for longitudinal beam impedance

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Quadrupolar term → (third line, $Z2_A$ term)

New formula for longitudinal beam impedance

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This leads to 5 equalities:

$$Z1_{xd} = Z1_{xt}, \quad Z1_{yd} = Z1_{yt}, \quad Z2_{drive} = Z2_{test}, \quad Z2_{xdyd} = Z2_{xtyt}, \quad Z2_{xdyt} = Z2_{xtyd}$$

The new formula for longitudinal beam impedance finally has only 8 terms:

$$Z[xd, xt, yd, yt] = Z0$$

$$+ Z1_x \cdot (xd + xt) + Z1_y \cdot (yd + yt)$$

Quadrupolar term → $+ Z2_A \cdot (xdxd - ydyd + xtxt - ytyt)$

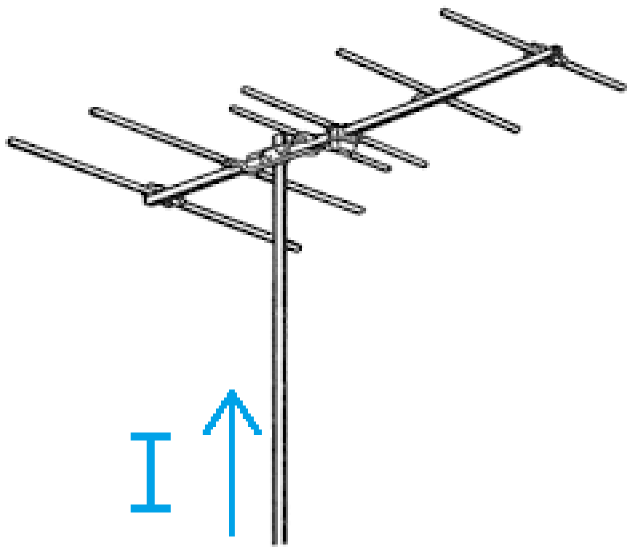
$$+ Z2_B \cdot (xdyd + xtyt) + Z2_C \cdot (xdyt + xtyd)$$

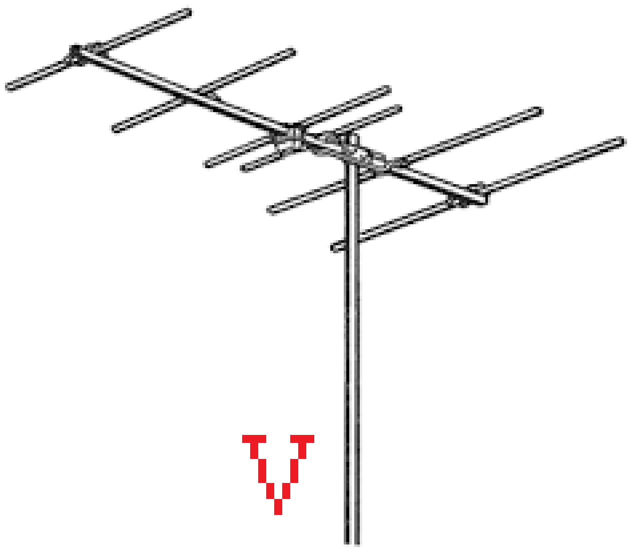
Dipolar terms H & V → $+ Z2_D \cdot xdxt + Z2_E \cdot ydyt$

The longitudinal beam impedance have 8 parameters

Interchanging the drive and test particles,
will give the same beam impedance.

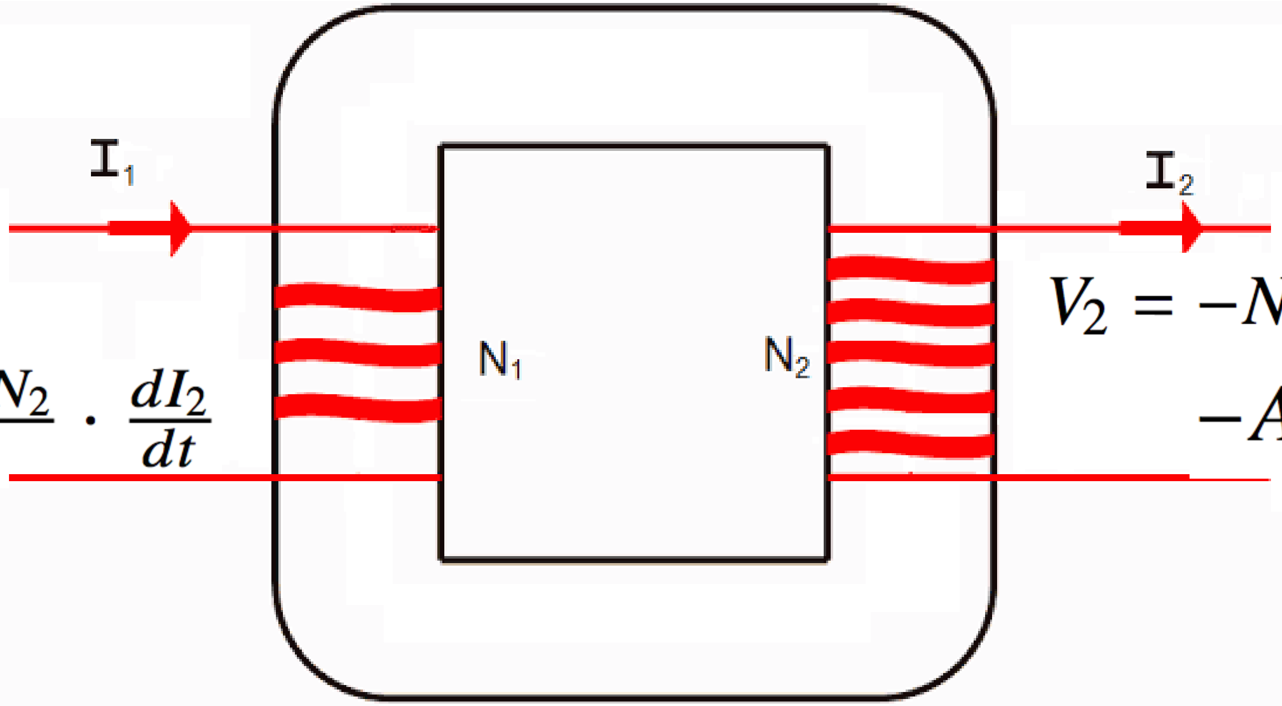
It is caused by the **Lorentz reciprocity theorem**
(well known to RF people as the identity S21≡S12):





The longitudinal beam impedance have 8 parameters

The **Lorentz reciprocity theorem** is responsible for coupling primary and secondary windings in a transformer:



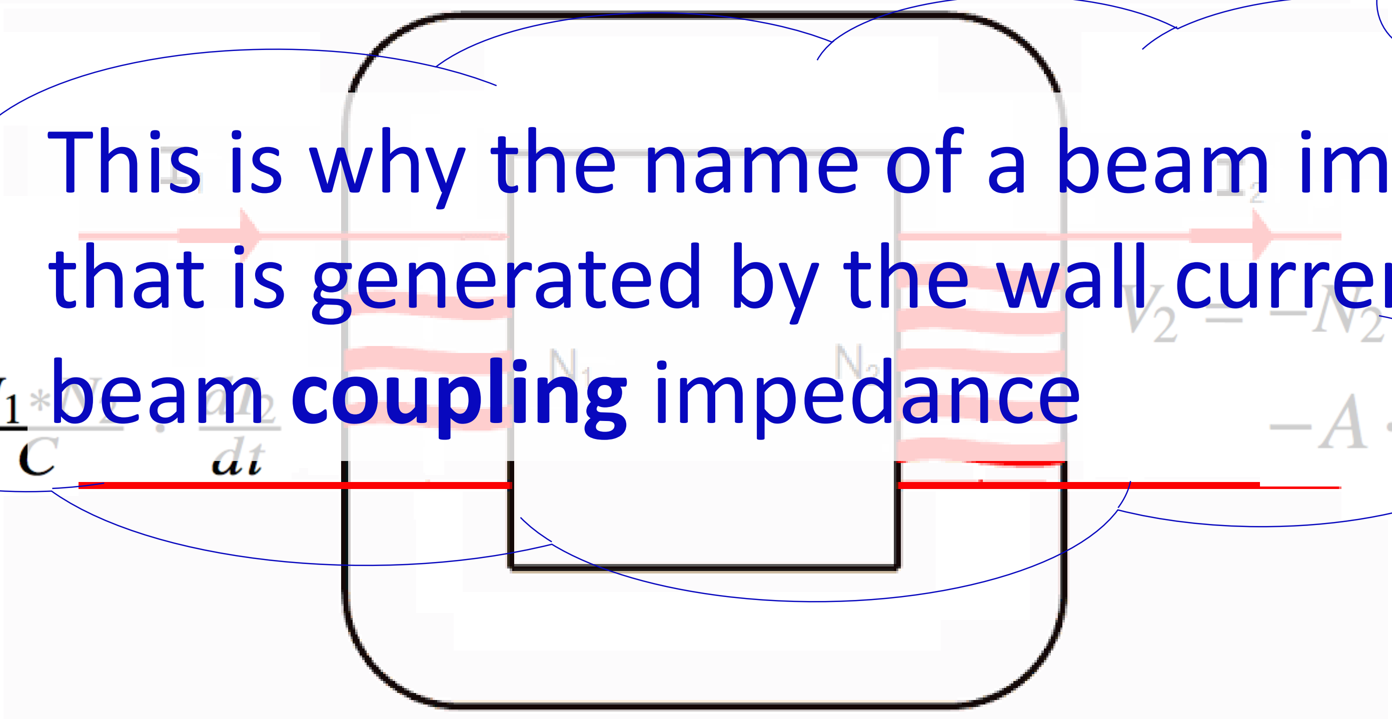
$$V_1 = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_2}{dt}$$

$$V_2 = -N_2 \cdot \frac{d\phi}{dt} - A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_1}{dt}$$

$$\left.\begin{aligned} \phi &= B \cdot A \\ \oint_C B\, dl &= \mu_0 \mu_r I_1 \cdot N_1 \end{aligned}\right\} \Rightarrow \phi = A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \cdot I_1$$

The longitudinal beam impedance have 8 parameters

The **Lorentz reciprocity theorem** is responsible for coupling primary and secondary windings in a transformer:



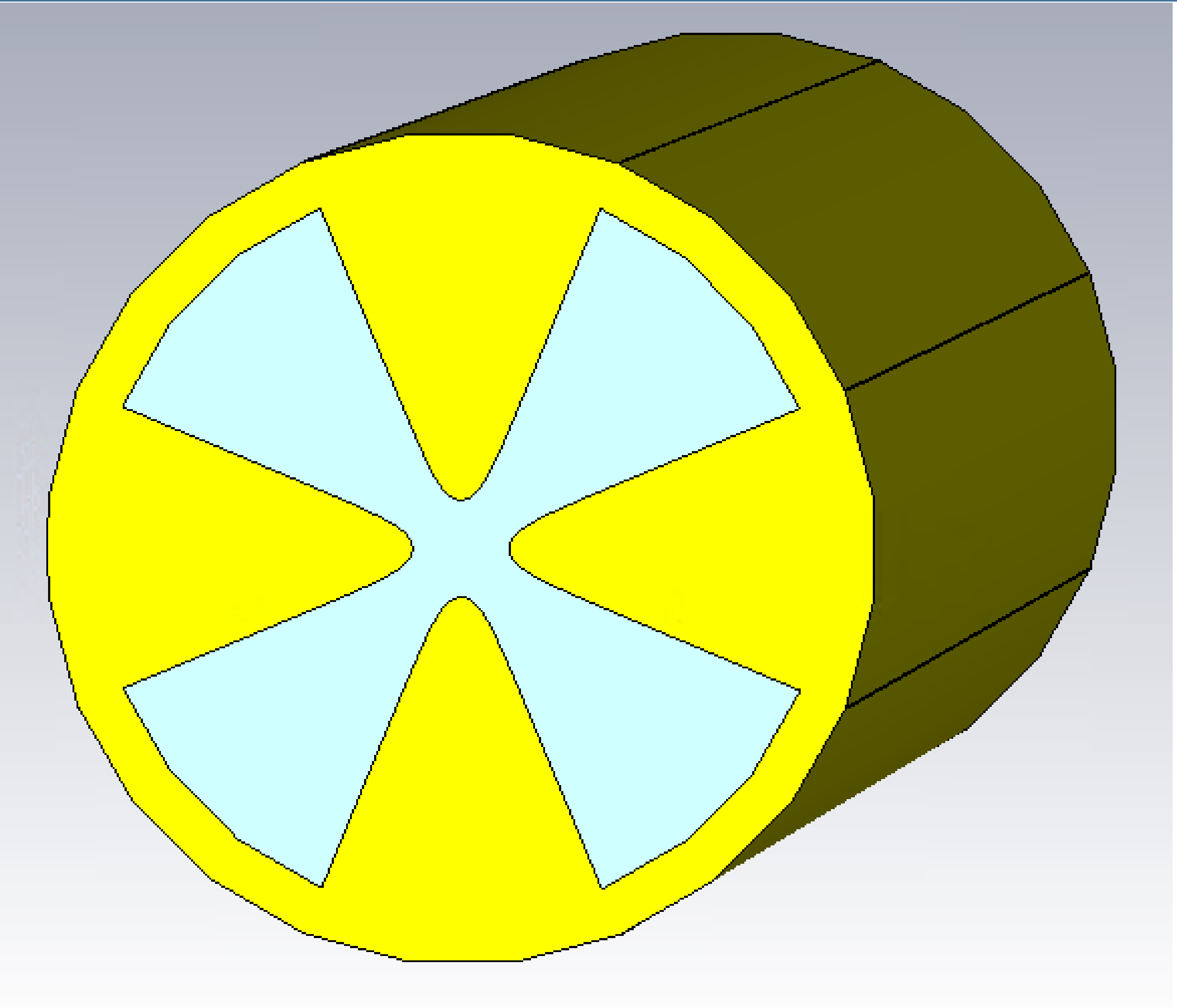
This is why the name of a beam impedance that is generated by the wall currents is a beam **coupling** impedance

$$V_1 = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_2}{dt}$$

$$V_2 = -N_2 \cdot \frac{d\phi}{dt} = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_1}{dt}$$

$$\left.\begin{aligned} \phi &= B \cdot A \\ \oint_C B\,dl &= \mu_0 \mu_r I_1 \cdot N_1 \end{aligned}\right\} \Longrightarrow \phi = A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \cdot I_1$$

The longitudinal beam impedance have 8 parameters



The new formula shows that 90 degree symmetrical structures only have dipolar impedance and that this impedance is the same in all directions

New formula for longitudinal beam impedance

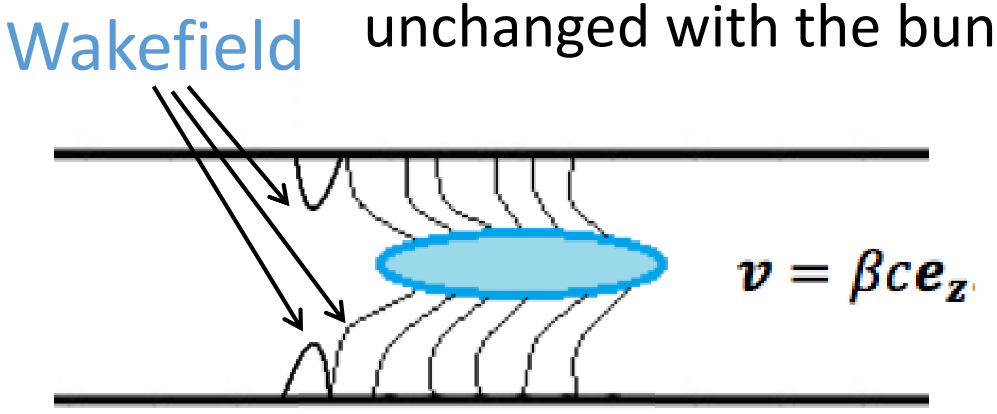
This new formula is not valid for resonances nor for non-relativistic beams $\beta < 1$, because both are spread out in 3D.

The formula is practically valid for beams with $\beta \approx 1$, even though theoretically there will always be other terms, but these terms are proportional to $\frac{1}{\gamma^2}$, so will not be important in practice:

$$
\begin{aligned}
Z[xd, xt, yd, yt] = Z0 \\
&+ Z1_x \cdot (xd + xt) + Z1_y \cdot (yd + yt) \\
&+ Z2_A \cdot (xdxd - ydyd + xtxt - ytyt) \\
&+ Z2_B \cdot (xdyd + xtyt) + Z2_C \cdot (xdyt + xtyd) \\
&+ Z2_D \cdot xdxt + Z2_E \cdot ydyt
\end{aligned}
$$

Panofsky-Wenzel and transverse impedance

The rigid bunch approximation states that the beam motion is little affected during the passage through the structure. So the beam shape is rigid and it always moves unchanged with the bunch.



The force acting on the test particle:

$$\boldsymbol{F}(x,y,z,t) = q(\boldsymbol{E}(x,y,z,t) + \boldsymbol{v}\times\boldsymbol{B}(x,y,z,t))$$

$$\nabla\times\boldsymbol{F} = \nabla\times q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Using Maxwell's equations:

$$\nabla\times\boldsymbol{E}(x,y,z,t) = -\frac{\partial \boldsymbol{B}(x,y,z,t)}{\partial t}$$

$$\nabla\cdot\boldsymbol{B}(x,y,z,t) = 0$$

$$\nabla\times\boldsymbol{F} = q\left[-\frac{\partial \boldsymbol{B}}{\partial t} - \beta c\frac{\partial \boldsymbol{B}}{\partial z}\right]$$

Panofsky-Wenzel and transverse impedance

The force acting on the test particle:

$$\nabla \times \boldsymbol{F} = q\left[-\frac{\partial \boldsymbol{B}}{\partial t} - \beta c \frac{\partial \boldsymbol{B}}{\partial z}\right]$$

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$\nabla_\perp F_z = \frac{\partial \boldsymbol{F}_\perp}{\partial z} - q\left[-\frac{\partial B_y}{\partial t}\hat{\boldsymbol{y}} - \beta c \frac{\partial B_y}{\partial z}\hat{\boldsymbol{y}} + \frac{\partial B_x}{\partial t}\hat{\boldsymbol{x}} + \beta c \frac{\partial B_x}{\partial z}\hat{\boldsymbol{x}}\right]$$

$$\nabla_\perp F_z(x,y,z,\tau) = -\frac{\partial \boldsymbol{F}_\perp(x,y,z,\tau)}{\partial s}$$

Very important:
Because the wakefield is only a function of "s" then: B(s)
This leads to

Position of drive particle: $v \cdot t$

Position of the test particle: z

$$z = vt - s \Rightarrow \begin{cases} \frac{\partial s}{\partial t} = v \\ \frac{\partial s}{\partial z} = -1 \end{cases}$$

When inserting the partial differentials on the right, the terms in the bracket cancels out and gives zero.

Panofsky-Wenzel and transverse impedance

The force acting on the test particle:

$$\nabla\times\boldsymbol{F} = q\left[-\frac{\partial\boldsymbol{B}}{\partial t} - \beta c\frac{\partial\boldsymbol{B}}{\partial z}\right]$$

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$\nabla_\perp F_z = \frac{\partial \boldsymbol{F}_\perp}{\partial z} - q\left[-\frac{\partial B_y}{\partial t}\hat{\boldsymbol{y}} - \beta c\frac{\partial B_y}{\partial z}\hat{\boldsymbol{y}} + \frac{\partial B_x}{\partial t}\hat{\boldsymbol{x}} + \beta c\frac{\partial B_x}{\partial z}\hat{\boldsymbol{x}}\right]$$

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$$\boxed{\nabla_\perp F_z(x,y,z,\tau) = -\frac{\partial \boldsymbol{F}_\perp(x,y,z,\tau)}{\partial s}}$$

Panofsky Wenzel theorem

Panofsky-Wenzel and transverse impedance

$$\nabla_\perp F_z(x,y,z,\tau) = -\frac{\partial \boldsymbol{F}_\perp(x,y,z,\tau)}{\partial s}$$

To obtain the theorem in terms of impedance, one can simply start from the wake function form:

$$\nabla_\perp w_{||}(x,y,z,\tau) = \frac{\partial \boldsymbol{w}_\perp}{\partial s}(x,y,z,\tau)$$

Then change the s derivative with the time derivative. Use $\partial s = v\partial\tau = \beta c\,\partial\tau$:

$$\nabla_\perp w_{||}(x,y,z,\tau) = \frac{1}{\beta c}\frac{\partial \boldsymbol{w}_\perp}{\partial \tau}(x,y,z,\tau)$$

Finally take the Fourier transform on both sides:

$$\nabla_\perp \int_{-\infty}^{+\infty} e^{-i\omega\tau} w_z(x,y,z,\tau)\,d\tau = \frac{1}{\beta c}\int_{-\infty}^{+\infty} e^{-i\omega\tau}\frac{\partial \boldsymbol{w}_\perp(x,y,z,\tau)}{\partial\tau}d\tau$$

$$\boxed{\nabla_\perp Z_{||}(x,y,z,\omega) = \frac{\omega}{\beta c}\boldsymbol{Z}_\perp(x,y,z,\omega)}$$

Panofsky Wenzel theorem

NB! The transverse impedance is defined with a complex i factor:

$$\boldsymbol{Z}_\perp(x,y,z,\omega) = i\int_{-\infty}^{+\infty} e^{-i\omega\tau}\boldsymbol{w}_\perp(x,y,z,\tau)\,d\tau$$

Panofsky-Wenzel and transverse impedance

$$\nabla_\perp Z_{||}(x,y,z,\omega) = \frac{\omega}{\beta c}\mathbf{Z}_\perp(x,y,z,\omega)$$

$Z_{||} = Z_z$

Using the following definitions: $\nabla_\perp Z_{||} = \frac{\partial Z_{||}}{\partial xt}\hat{\mathbf{x}} + \frac{\partial Z_{||}}{\partial yt}\hat{\mathbf{y}}$ and $\mathbf{Z}_\perp = Z_x\hat{\mathbf{x}} + Z_y\hat{\mathbf{y}}$

$$Z_{\perp,x} = \frac{\beta c}{w}\cdot\frac{\partial Z_{||}}{\partial xt}$$

$$Z_{\perp,y} = \frac{\beta c}{w}\cdot\frac{\partial Z_{||}}{\partial yt}$$

$$Z_{\perp,x}(\omega) = Z1_x + 2Z2_A\cdot xt + Z2_B\cdot yt + Z2_C\cdot yd + Z2_D\cdot xd$$

$$Z_{\perp,y}(\omega) = Z1_y - 2Z2_A\cdot yt + Z2_B\cdot xt + Z2_C\cdot xd + Z2_E\cdot yd$$

$$\begin{aligned} Z[xd,xt,yd,yt] = Z0 & \\ & + Z1_x\cdot(xd+xt) + Z1_y\cdot(yd+yt) \\ & + Z2_A\cdot(xdxd - ydyd + xtxt - ytyt) \\ & + Z2_B\cdot(xdyd + xtyt) + Z2_C\cdot(xdyt + xtyd) \\ & + Z2_D\cdot xdxt + Z2_E\cdot ydyt \end{aligned}$$

Panofsky-Wenzel and transverse impedance

$$\nabla_\perp Z_{||}(x,y,z,\omega) = \frac{\omega}{\beta c}\mathbf{Z}_\perp(x,y,z,\omega)$$

$$Z_{||} = Z_z$$

Using the following definitions: $\nabla_\perp Z_{||} = \frac{\partial Z_{||}}{\partial xt}\hat{\boldsymbol{x}} + \frac{\partial Z_{||}}{\partial yt}\hat{\boldsymbol{y}}$ and $\mathbf{Z}_\perp = Z_x\hat{\boldsymbol{x}} + Z_y\hat{\boldsymbol{y}}$

$$Z_{\perp,x} = \frac{\beta c}{w}\cdot\frac{\partial Z_{||}}{\partial xt}$$

$$Z_{\perp,y} = \frac{\beta c}{w}\cdot\frac{\partial Z_{||}}{\partial yt}$$

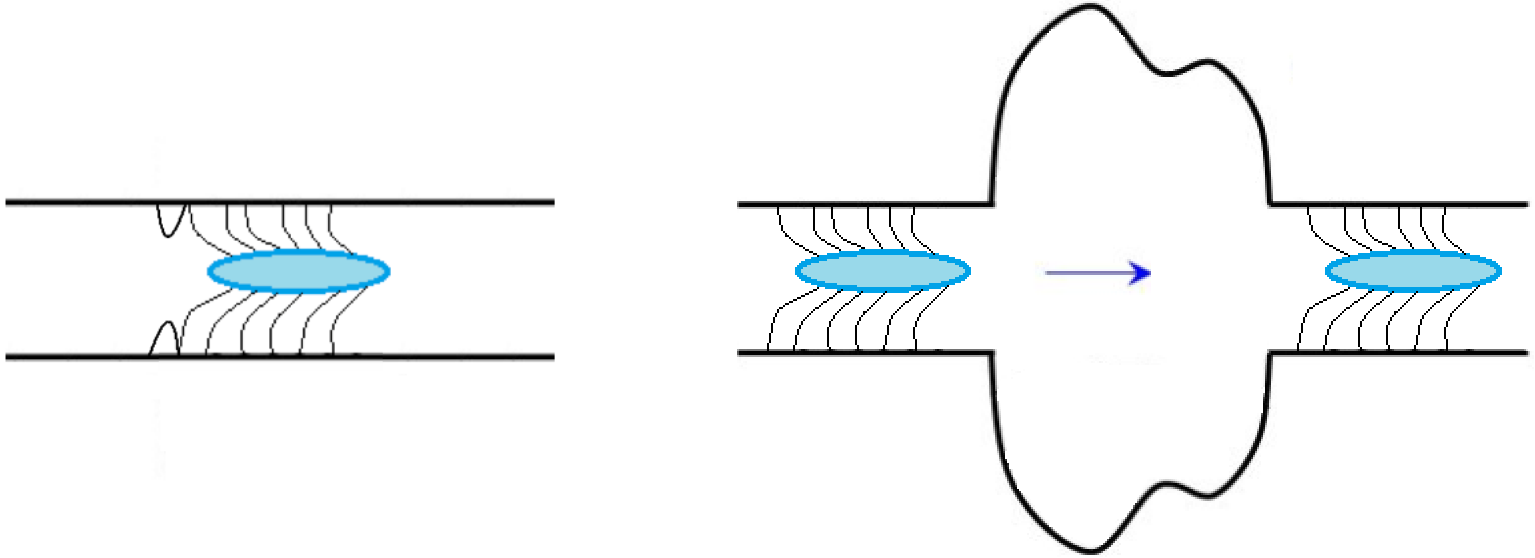
Panofsky Wenzel theorem
In differential form

$$Z_{\perp,x}(\omega) = Z1_x + 2Z2_A\cdot xt + Z2_B\cdot yt + Z2_C\cdot yd + Z2_D\cdot xd$$

$$Z_{\perp,y}(\omega) = Z1_y - 2Z2_A\cdot yt + Z2_B\cdot xt + Z2_C\cdot xd + Z2_E\cdot yd$$

$$\begin{aligned} Z[xd,xt,yd,yt] = Z0 & \\ & + Z1_x\cdot(xd+xt) + Z1_y\cdot(yd+yt) \\ & + Z2_A\cdot(xdxd - ydyd + xtxt - ytyt) \\ & + Z2_B\cdot(xdyd + xtyt) + Z2_C\cdot(xdyt + xtyd) \\ & + Z2_D\cdot xdxt + Z2_E\cdot ydyt \end{aligned}$$

Panofsky-Wenzel and transverse impedance



Because of the rigid bunch approximation, which states that the beam motion is little affected during the passage through a structure, the wake field is the same before and after the passage of an equipment.

Therefore, it is as if B is only a function of “s”. <u>A criterion for the Panofsky-Wenzel theorem is therefore that the vacuum chamber has to have the same cross-section before and after the equipment</u> – otherwise the B-field is not the same.

Lab measurements of beam impedance. Wire #1

We can measure the beam impedance with wire measurements

This is based on the assumption that a bunch interacts with an equipment in exactly the same way as a coaxial cable (i.e. a wire inside the equipment):

Ultra-relativistic beam field

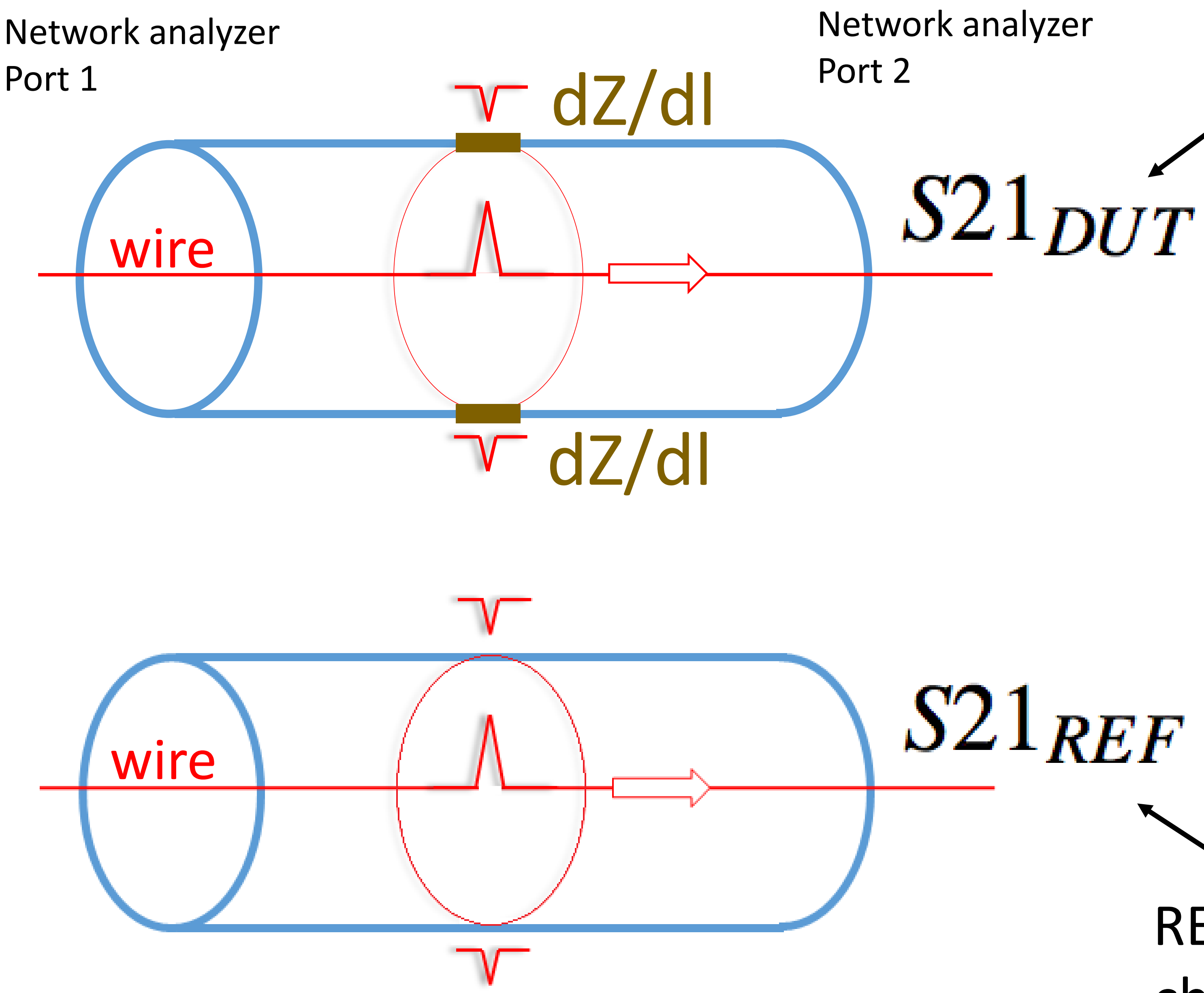
$$E_r(r,\omega) = Z_0 H_\varphi(r,\omega) = \frac{Z_0 q}{2\pi r} \exp\left(-j\frac{\omega}{c}z\right)$$

TEM mode coax waveguide

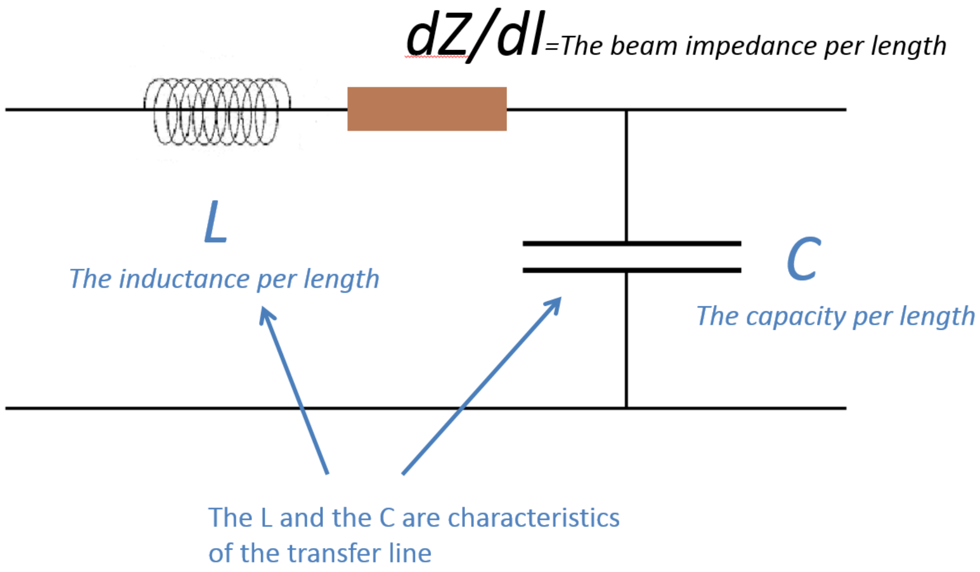
$$E_r(r,\omega) = Z_0 H_\varphi(r,\omega) = Z_0 \frac{\mathrm{const}}{r} \exp\left(-j\frac{\omega}{c}z\right)$$

See A.Mostacci: http://pcaen1.ing2.uniroma1.it/mostacci/wire_method/care_impedance.ppt

Lab measurements of beam impedance. Wire #2

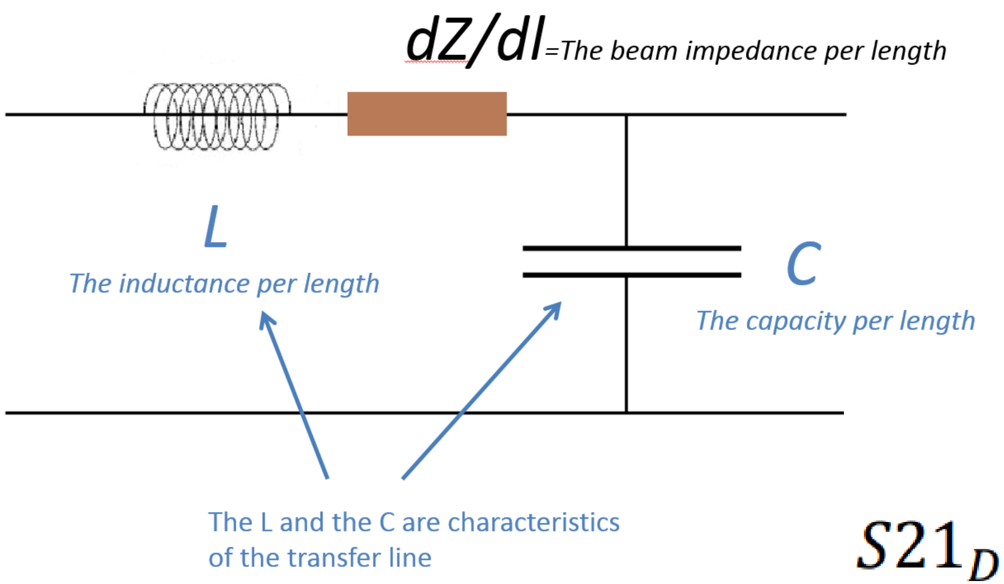


DUT=Device under test



REF = Reference = PEC vacuum chamber – same length as DUT

Lab measurements of beam impedance. Wire #3



$$\theta = l\,\omega\cdot\sqrt{LC} \qquad v = \frac{1}{\sqrt{LC}}$$

$$\left.\begin{aligned} S21_{DUT} &= e^{-i\cdot\omega\cdot\sqrt{C_{REF}\cdot L_{REF}}\cdot\left(\sqrt{\left(1+\frac{dZ/dl}{i\cdot\omega\cdot L_{REF}}\right)}\right)\cdot l} \\ S21_{REF} &= e^{-i\cdot\omega\cdot\sqrt{C_{REF}\cdot L_{REF}}\cdot l} \end{aligned}\right\} \Longrightarrow$$

Beam impedance:

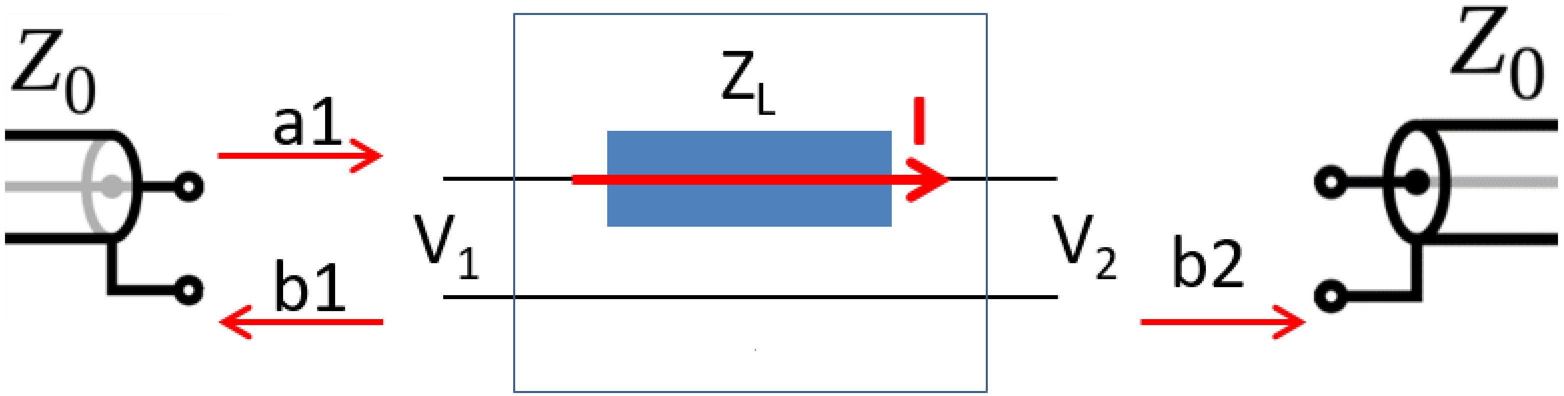
$$Z_{Longitudinal} = {}^{dZ}/_{dl}\cdot l = -2\cdot Z_0\cdot Ln\left[\frac{S21_{DUT}}{S21_{REF}}\right]\cdot\left(1+\frac{i\cdot Ln\left[\frac{S21_{DUT}}{S21_{REF}}\right]}{2\cdot\theta}\right)$$

This is the improved log formula, which is used for wire measurements

Lab measurements of beam impedance. Wire #4

$$S_{21,DUT} = \frac{b_2}{a_1} = \frac{\frac{V_2+Z_0 I}{2\sqrt{Z_0}}}{\frac{V_1+Z_0 I}{2\sqrt{Z_0}}} = \frac{V_2+Z_0 I}{V_1+Z_0 I} = \frac{V_2+Z_0 I}{V_2+Z_L I+Z_0 I} = \frac{Z_0 I+Z_0 I}{Z_0 I+Z_L I+Z_0 I} = \frac{2Z_0}{2Z_0+Z_L}$$

$$S_{21,REF} = \frac{\frac{V_2+Z_0 I}{2\sqrt{Z_0}}}{\frac{V_1+Z_0 I}{2\sqrt{Z_0}}} = 1$$

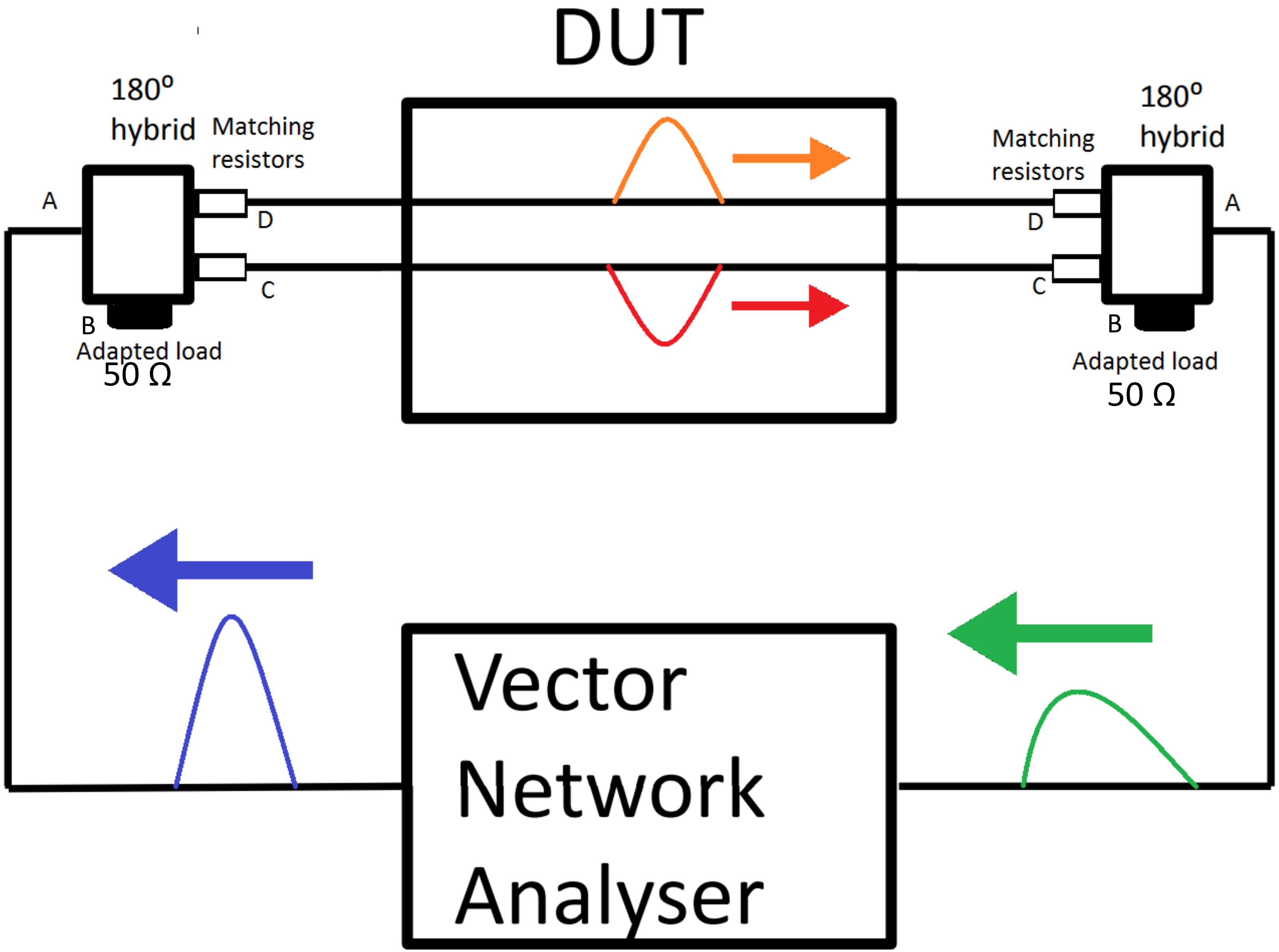


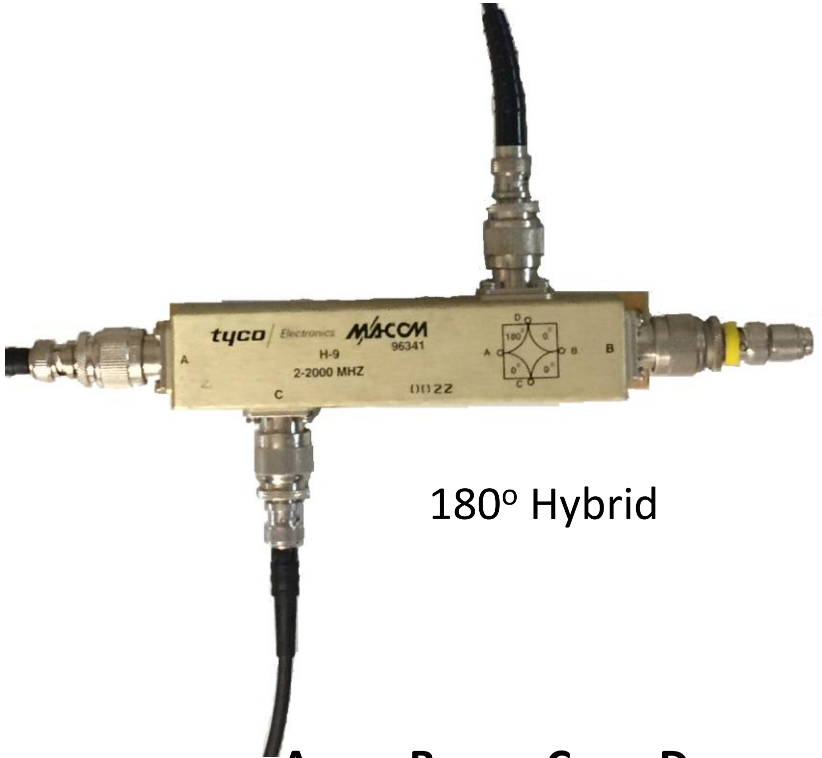
$$V_1 - Z_L \cdot I = V_2$$

$$\frac{S_{21,DUT}}{S_{21,REF}} = \frac{2Z_0}{2Z_0+Z_L}$$

$$\frac{Z_L}{Z_0} = 2\frac{S_{21,REF}}{S_{21,DUT}} - 2$$

Lab measurements of beam impedance. Wire #5

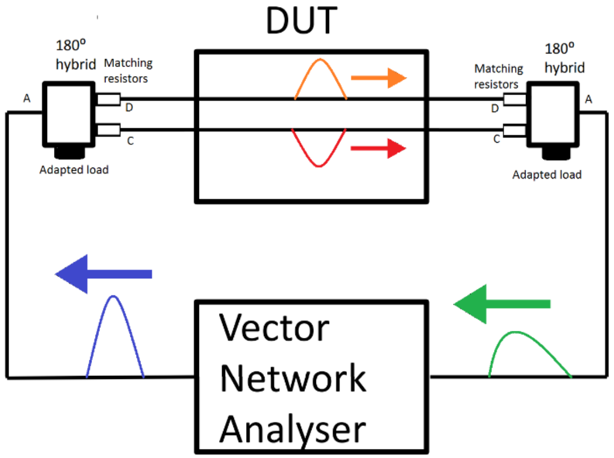




180º Hybrid

$$\mathbf{S}=\frac{1}{\sqrt{2}}\begin{array}{c} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\ \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array}$$

Lab measurements of beam impedance. Wire #6



$$Z_\perp = \frac{\beta c}{\omega \Delta^2} Z_0 \, Ln\left[\frac{S21_{REF}}{S21_{DUT}}\right]\left[1+\frac{Ln[S21_{DUT}]}{Ln[S21_{REF}]}\right]$$

Characteristic impedance Z_0 of two wires, each with diameter "d" and with distance between them "Δ" is (See https://en.wikipedia.org/wiki/Twin-lead):

$$Z_0 \approx \frac{120}{\sqrt{\varepsilon_r}} Ln\left[2\frac{\Delta}{d}\right]$$

Two wire measurement give only the dipolar impedance

Example:

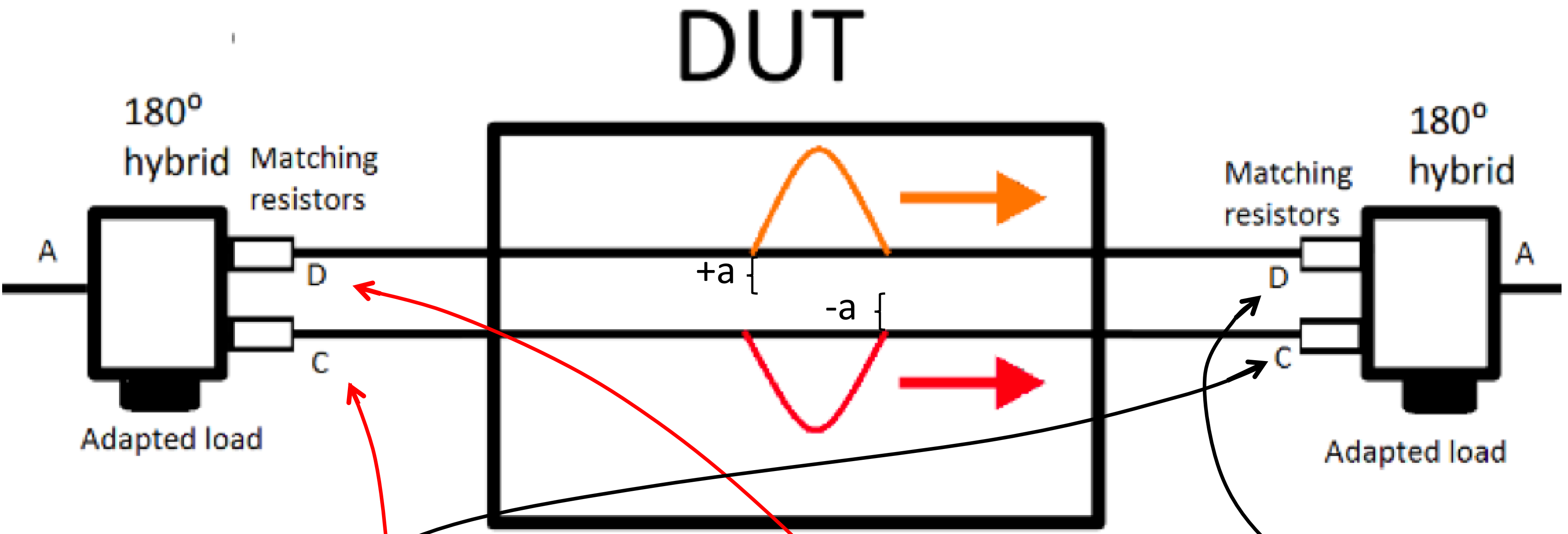
Δ = 10.0 mm Z = 120/1 · ln(40) ~ 450 Ohm

d = 0.5 mm i.e. 225 Ohm per wire

Subtract 50 Ohm, as usual, this gives 175 Ohm per wire. So it is always 175 Ohm per wire – independent of the chamber diameter!

Lab measurements of beam impedance. Wire #7

only the dipolar impedance



$$Voltage = -(-I\cdot Z[-a,-a,0,0] + I\cdot Z[a,-a,0,0]) + (I\cdot Z[a,a,0,0] - I\cdot Z[-a,a,0,0])$$

$$= I * (4a^2 Z2_D)$$ = dipolar impedance

The distance between the wires is 2 a:

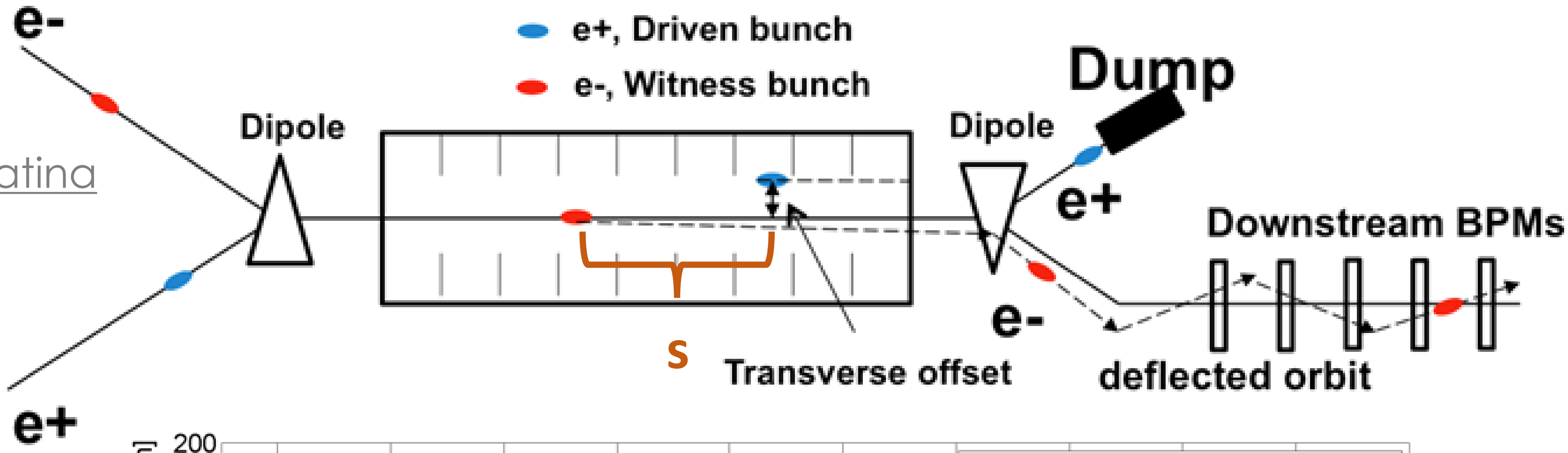
$$\Delta = 2a$$

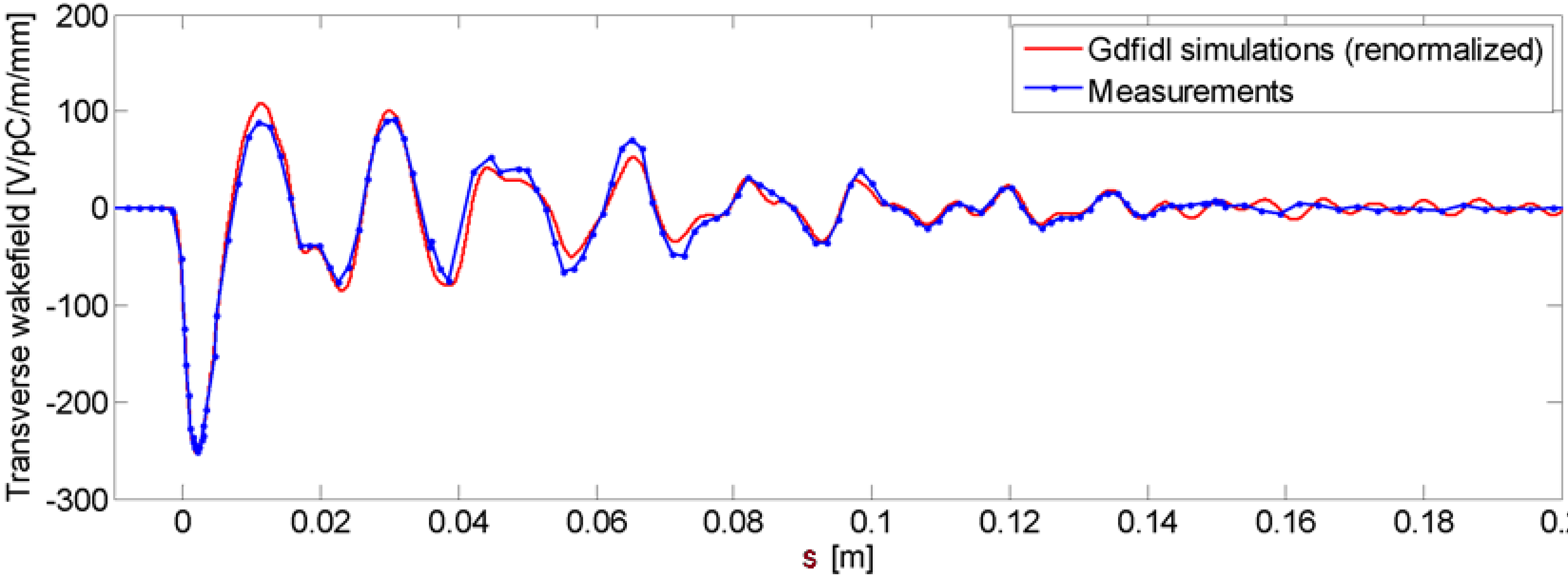
$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{array}{cccc} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{array} \\ \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{array}$$

Another measure of transverse beam impedance!

An example of transverse impedance, that gives the beam a transverse kick! Here measured with the beam

Andrea Latina
Hao Zha



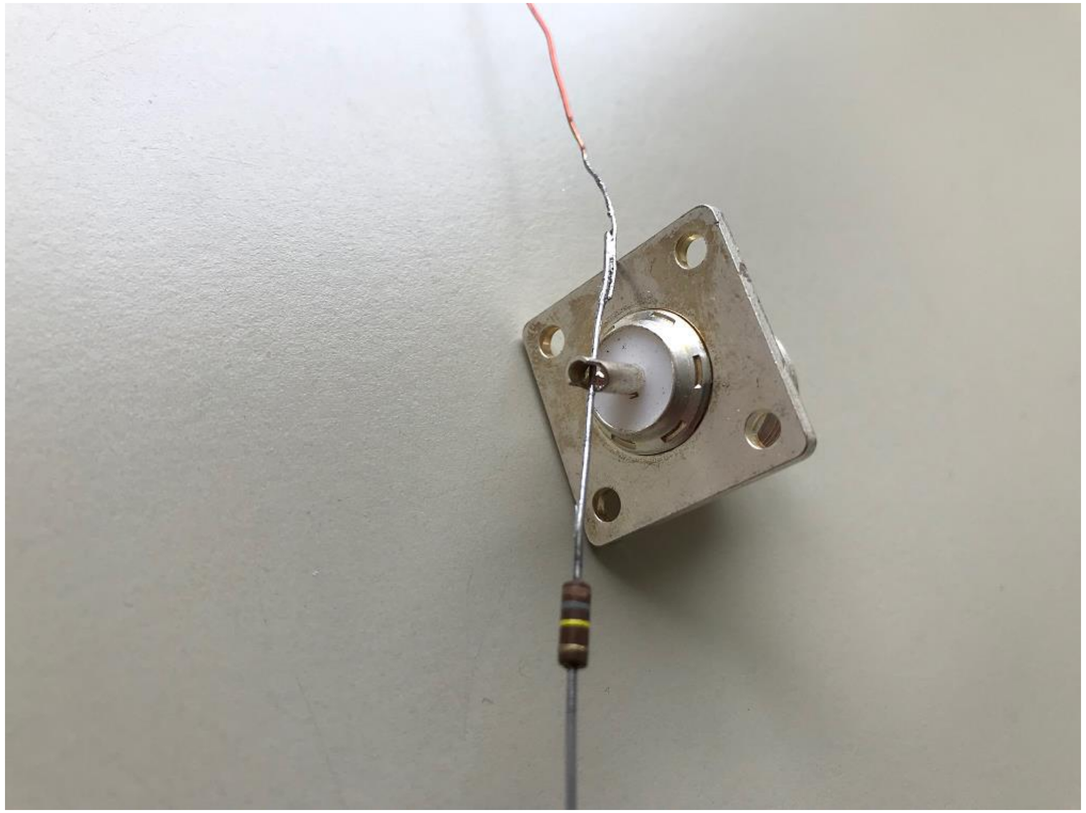


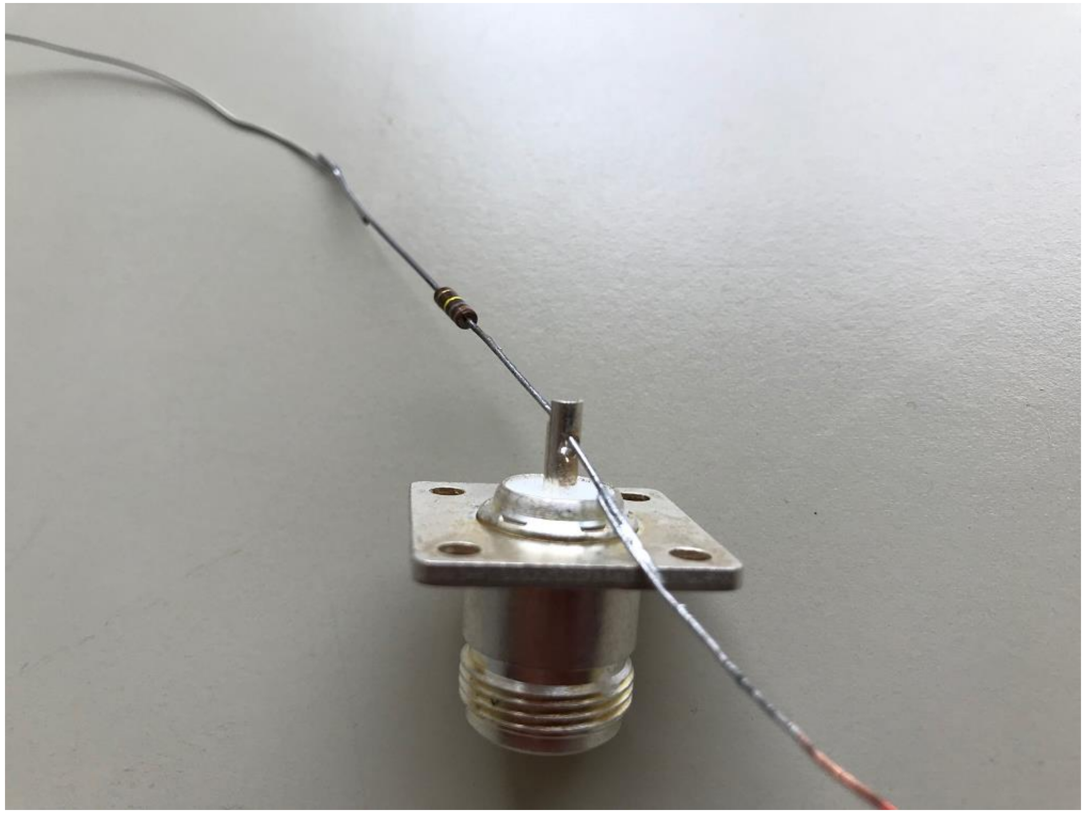
Ref [1]

Lab measurements of beam impedance. Wire #8

Easy method to firmly straighten the wire.

Make hole in connector and solder a thin wire to the resistor.





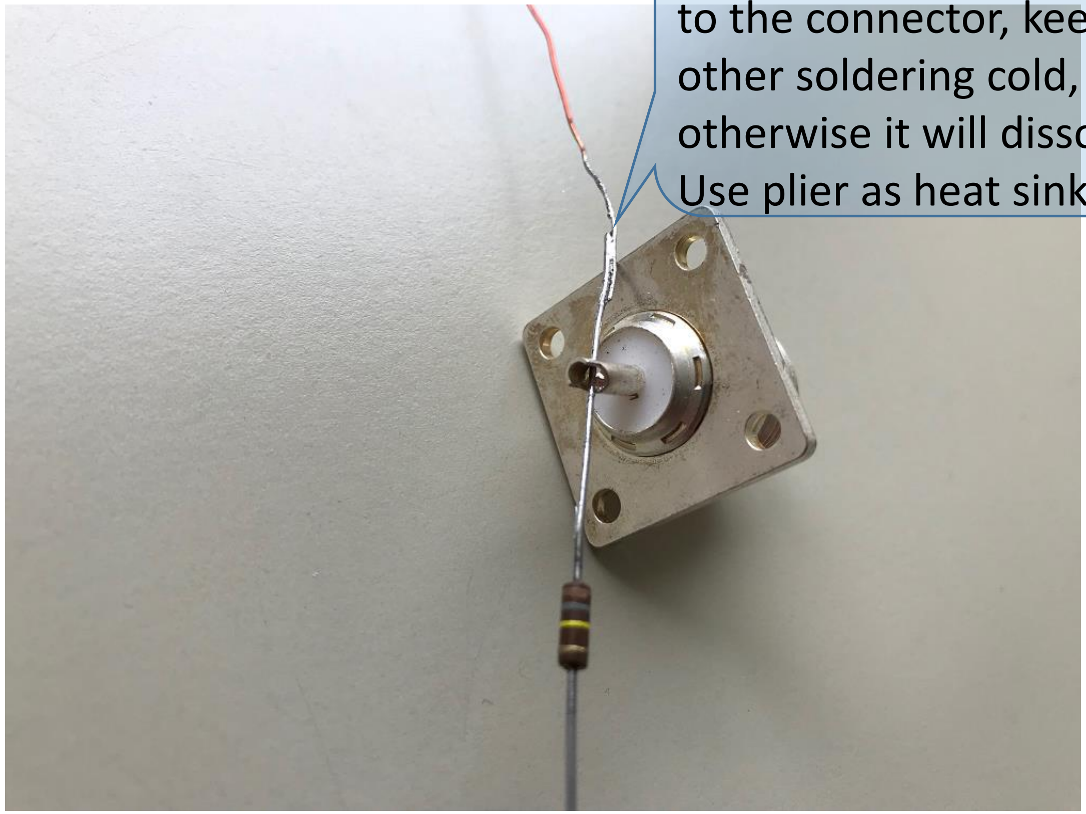
This method was invented by Muzhaffar Hazman

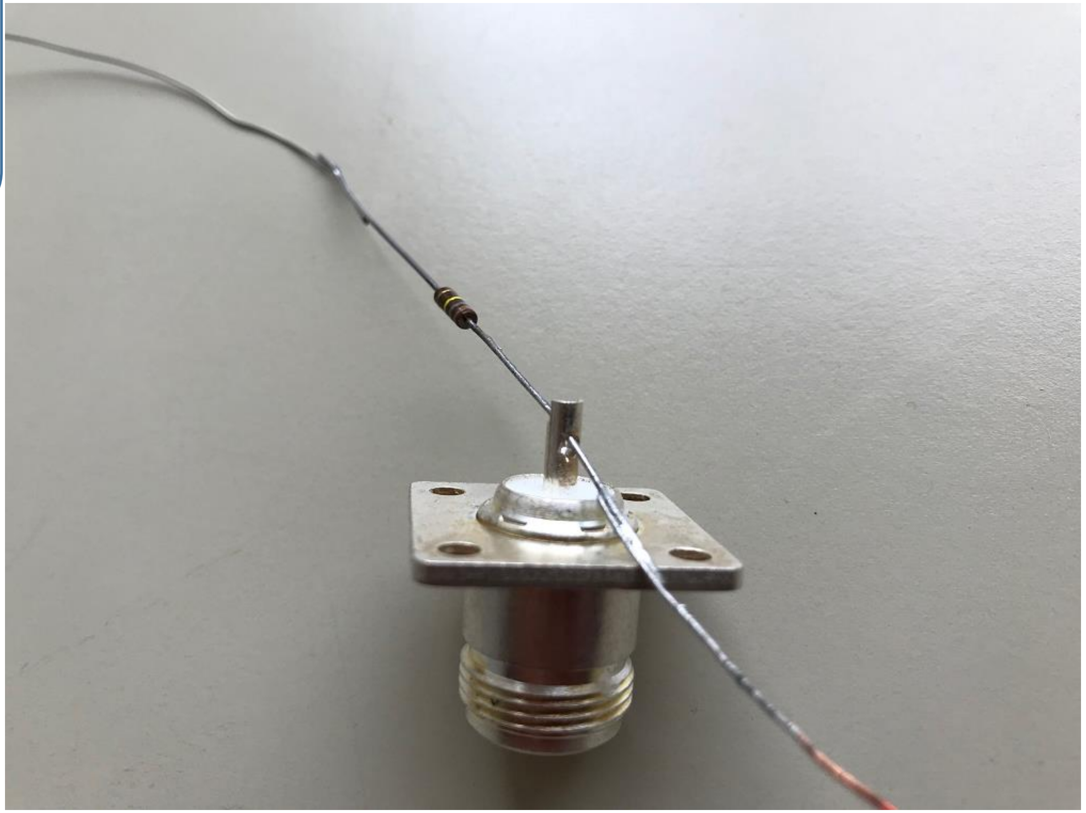
Lab measurements of beam impedance. Wire #8

Easy method to firmly straighten the wire.

Make hole in connector and solder a thin wire to the resistor.

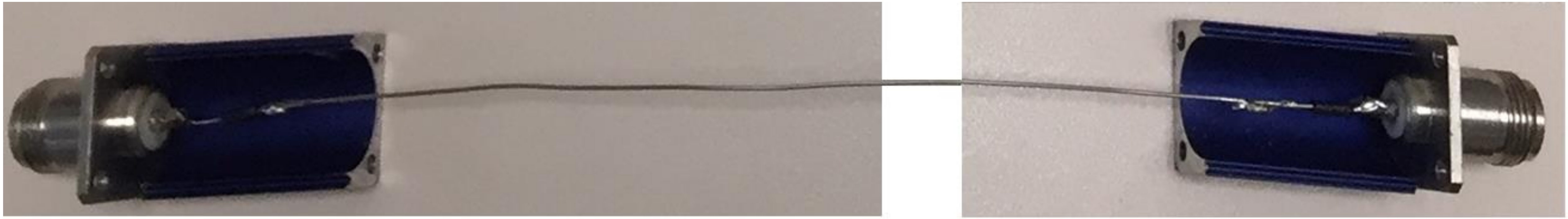
When soldering the resistor to the connector, keep the other soldering cold, otherwise it will dissolve. Use plier as heat sink.



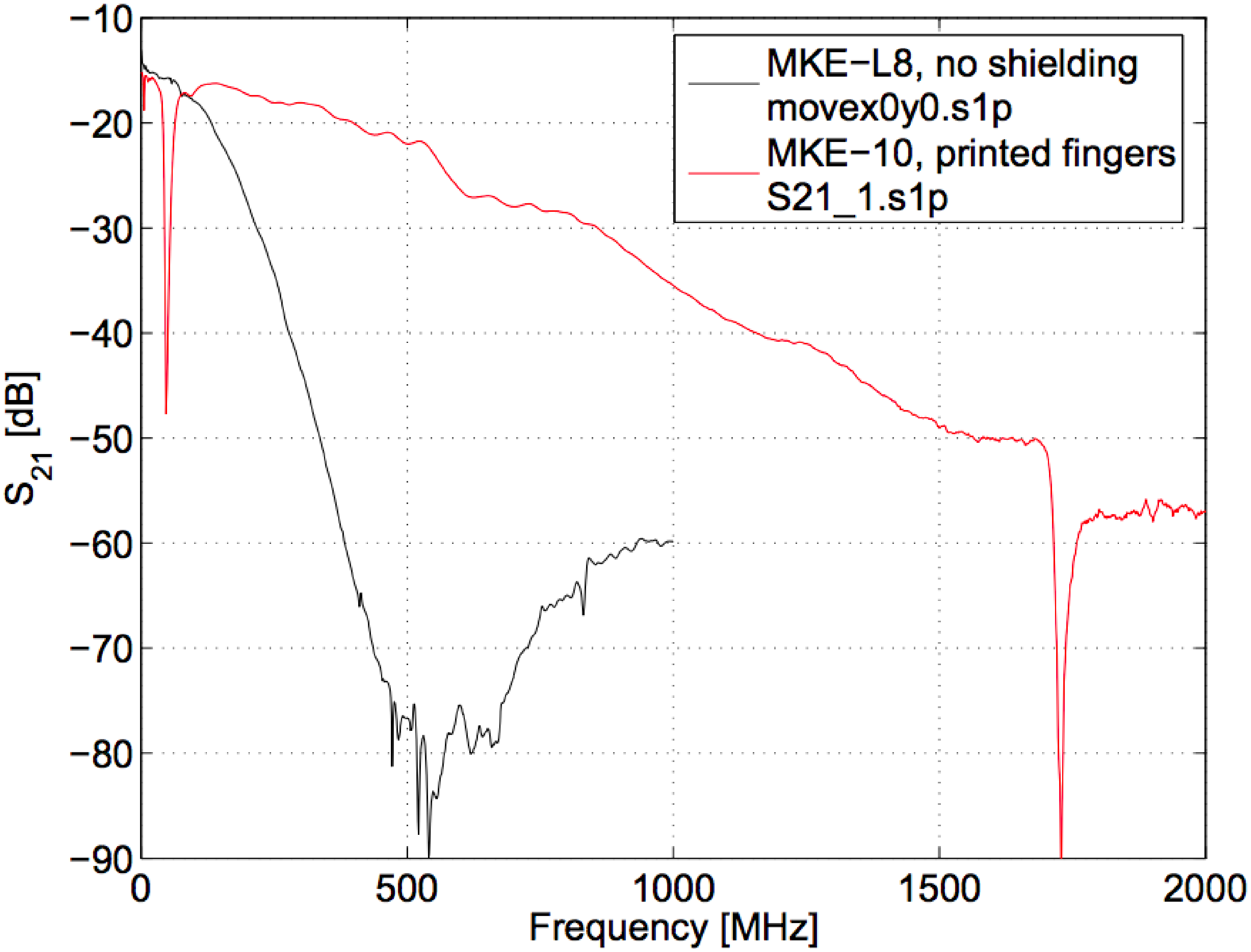


This method was invented by Muzhaffar Hazman

Lab measurements of beam impedance. Wire #9



Lab measurements of beam impedance. Wire #10



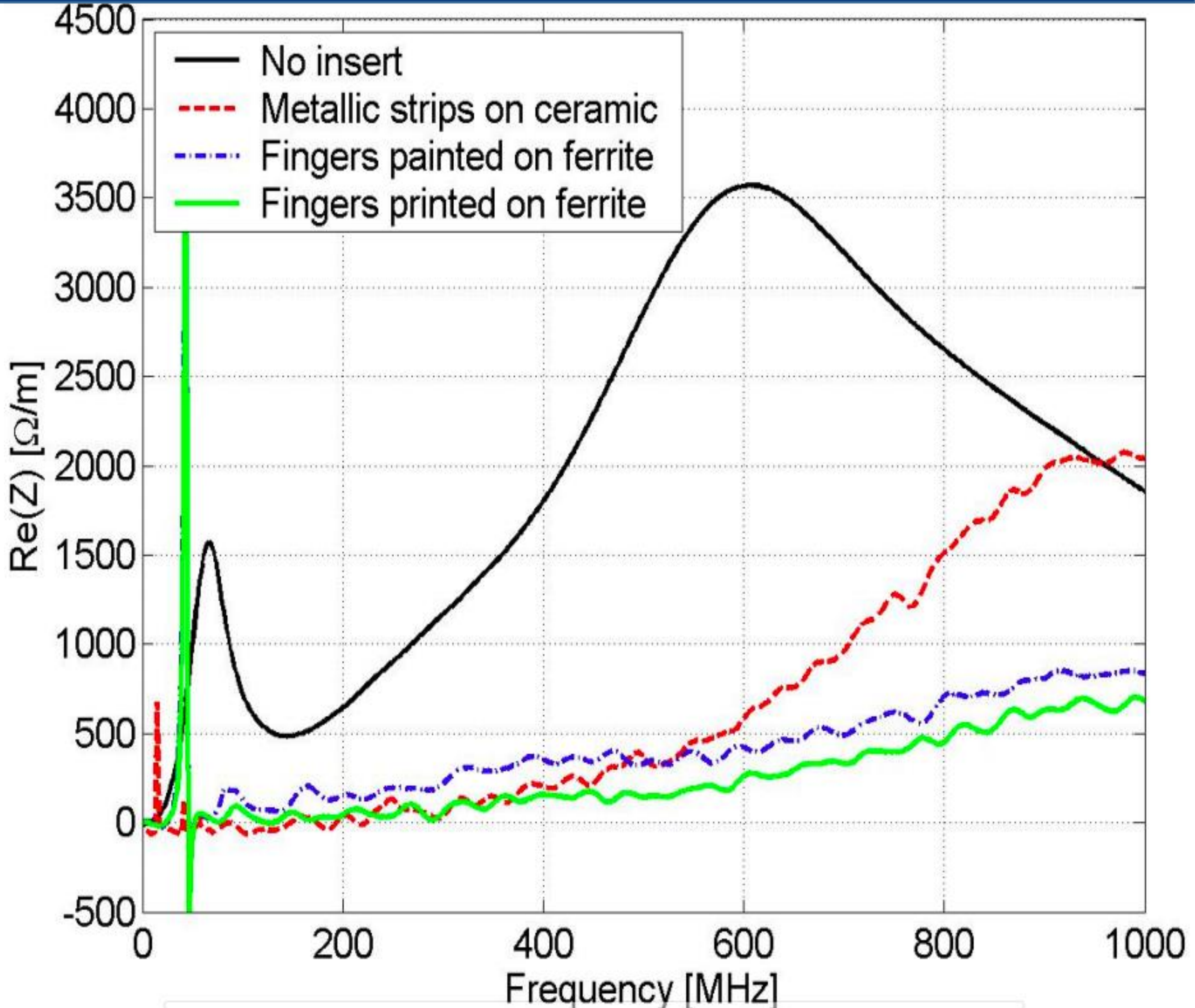
http://cds.cern.ch/record/1035461/files/ab-note-2007-028.pdf

T. Kroyer, F. Caspers, E. Gaxiola

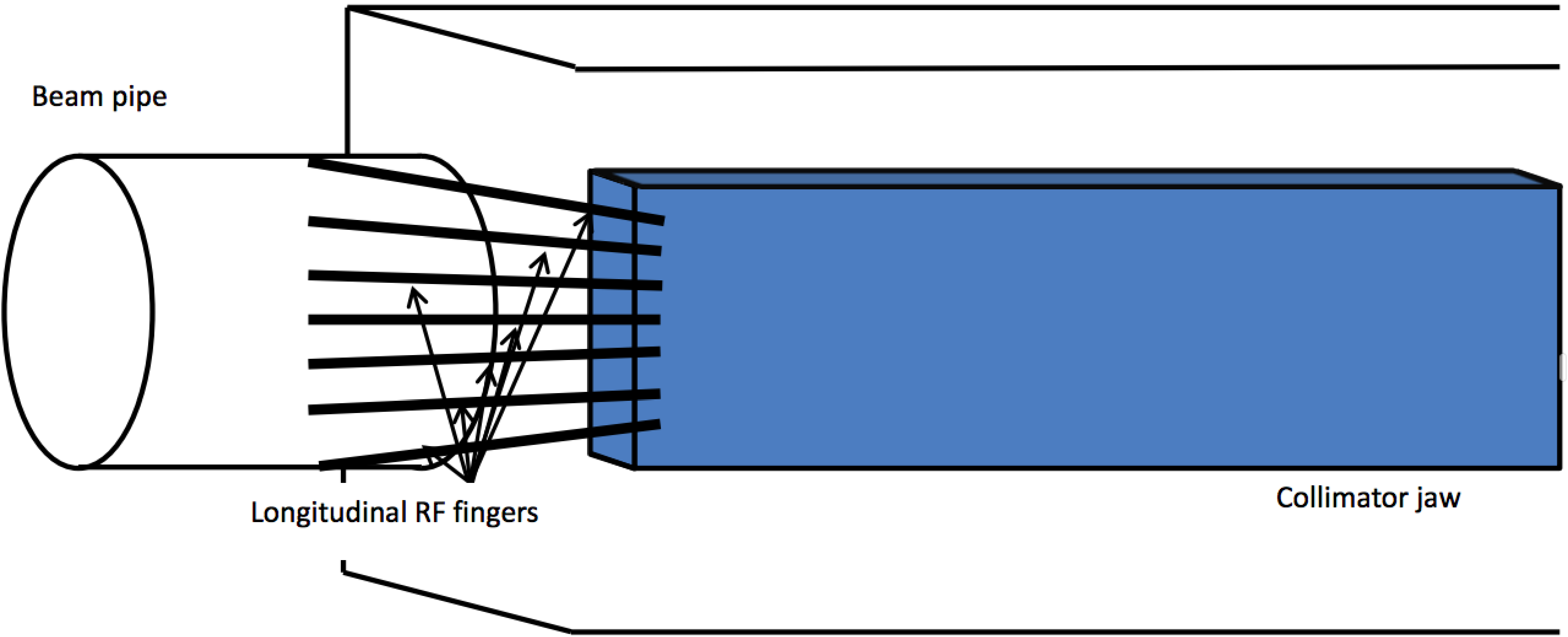
MKE Kicker measurements

Lab measurements of beam impedance. Wire #11

An example of serigraphy in the SPS Extraction Kicker Magnets (SPS-MKE)

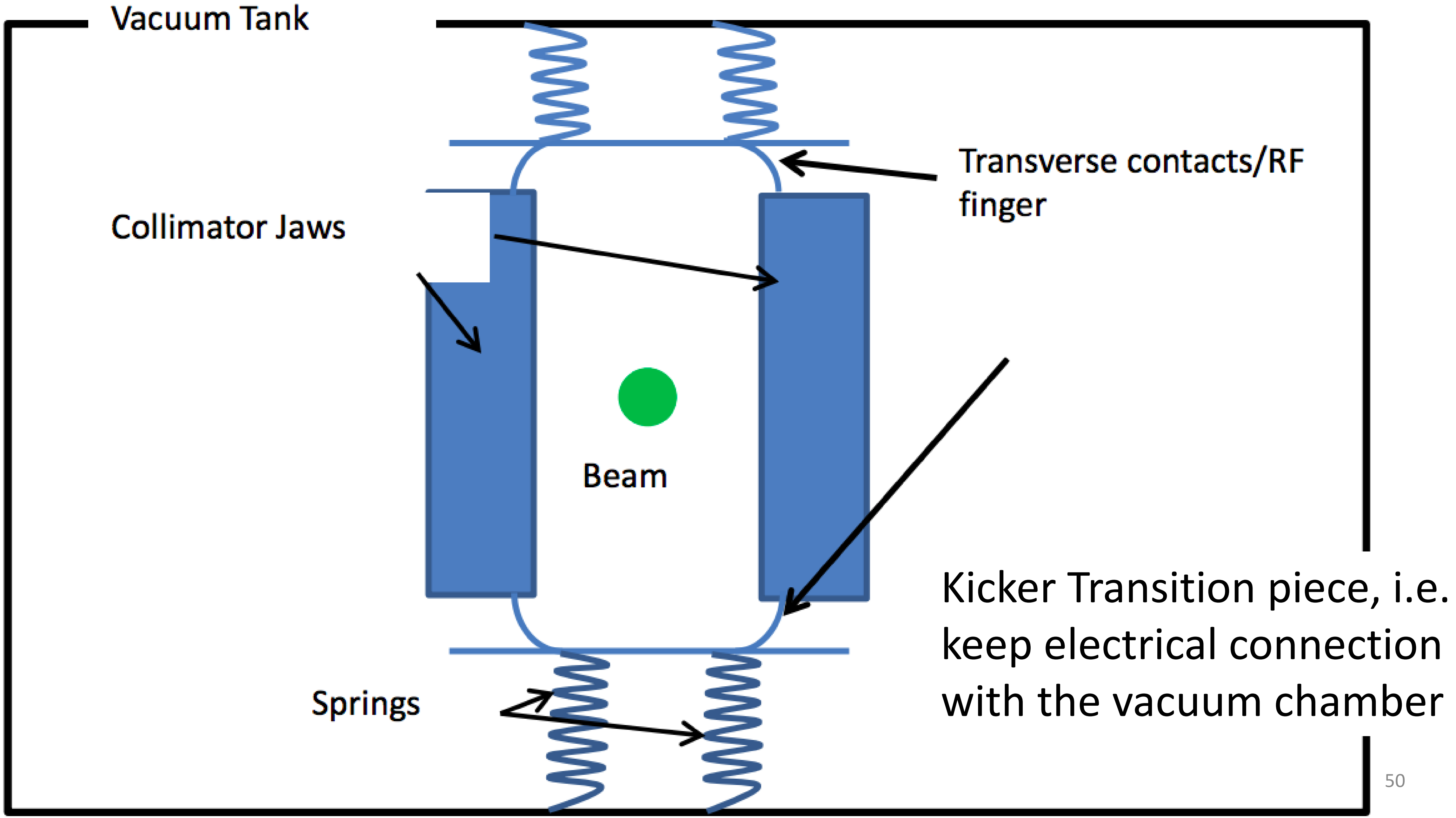


Lab measurements of beam impedance. Wire #12

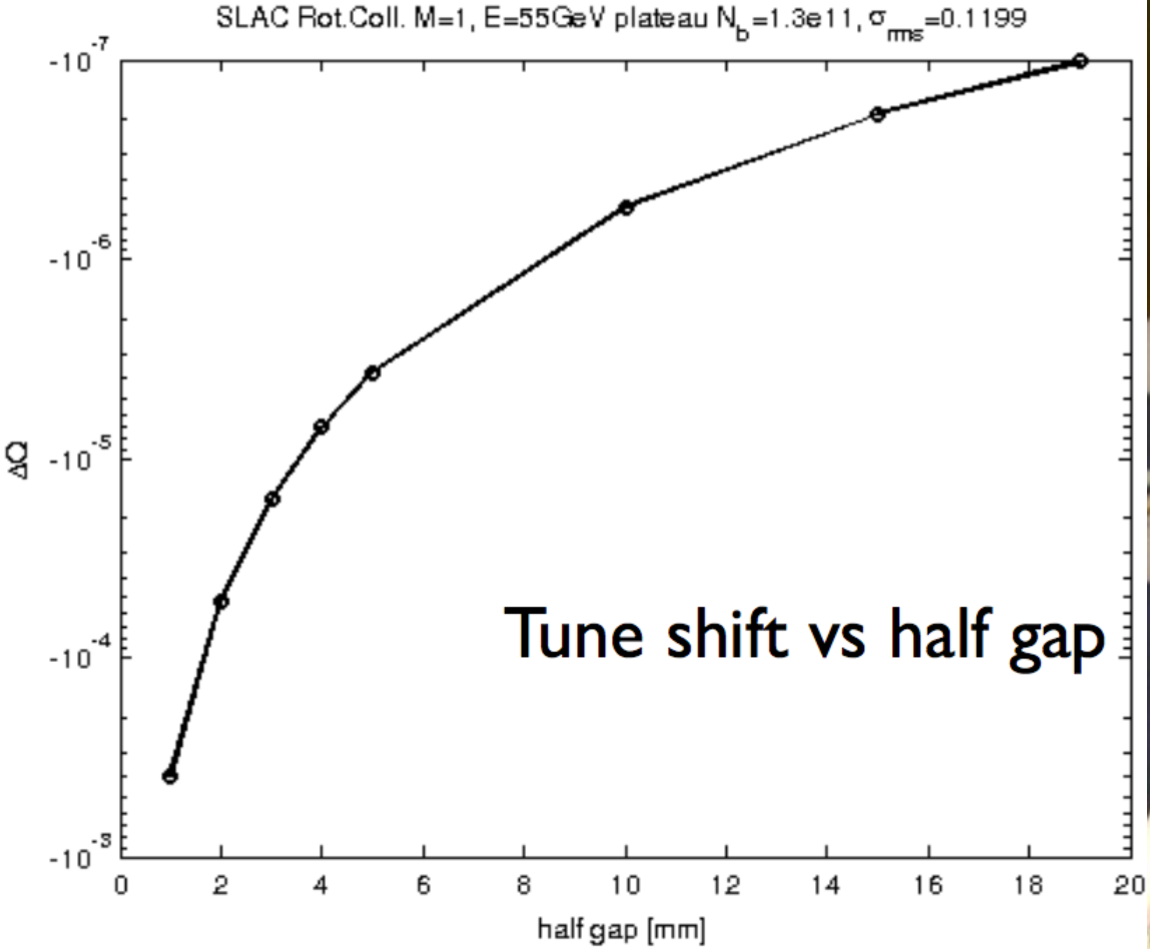


Kicker Transition piece, i.e. keep electrical connection with the vacuum chamber

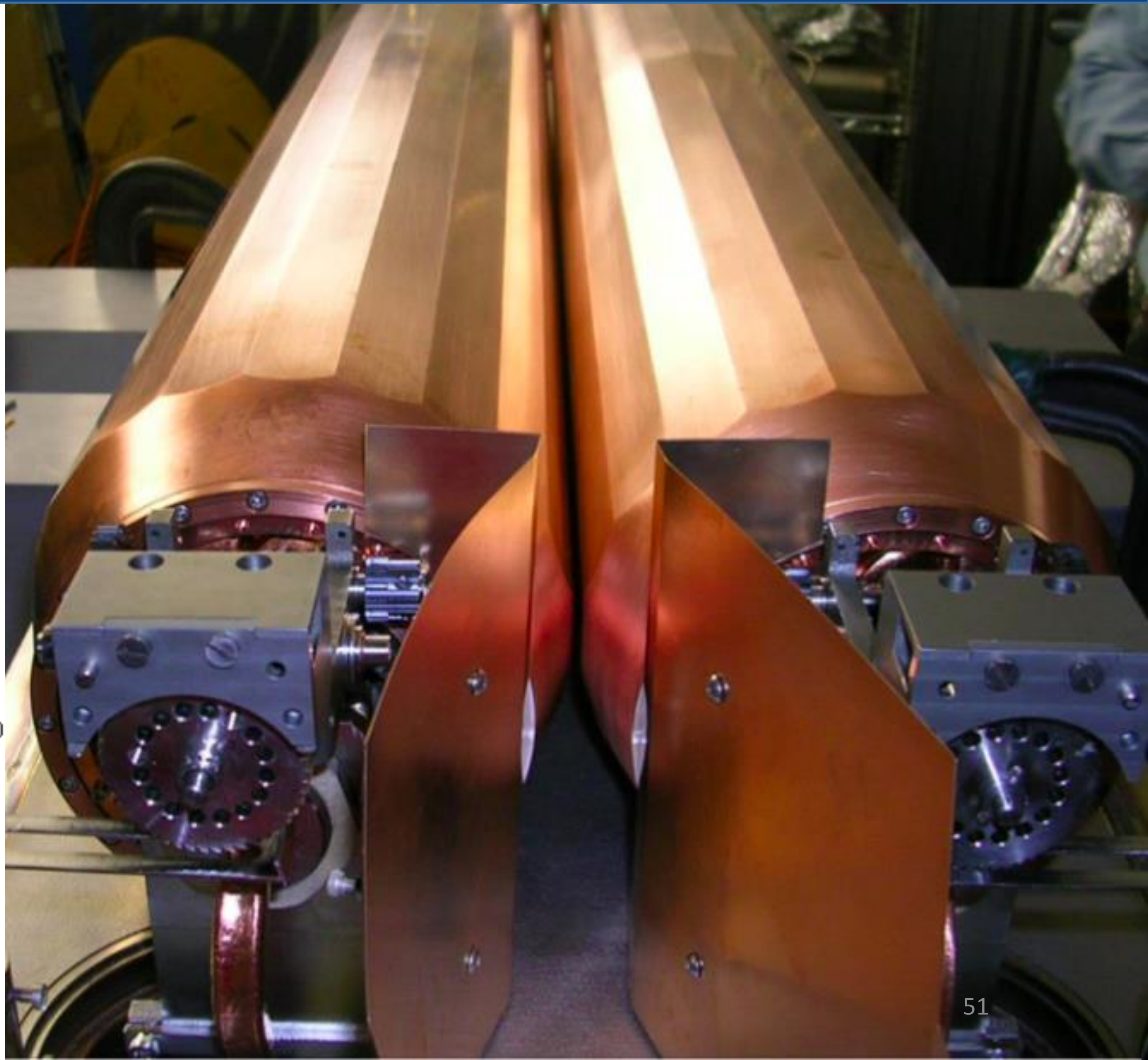
Lab measurements of beam impedance. Wire #13



Lab measurements of beam impedance. Wire #14



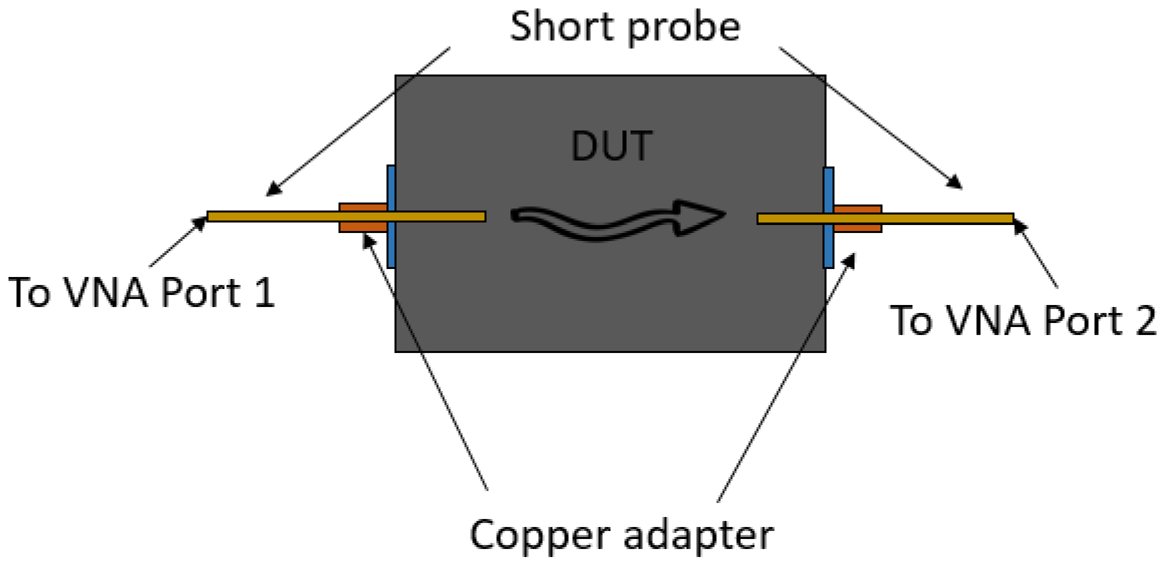
Collimator measurement



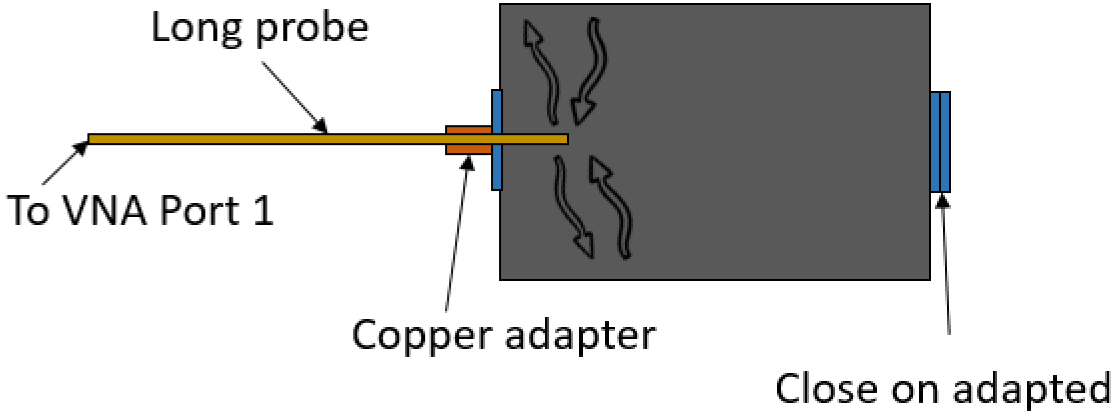
https://indico.cern.ch/event/436682/contributions/1076818/attachments/1140261/1633077/SLAC_RC_SPS_plan.pdf N. Biancacci, P. Gradassi, T. Markiewicz, S. Redaelli, B. Salvant, G. Valentino

Lab measurements of beam impedance. Probe #1

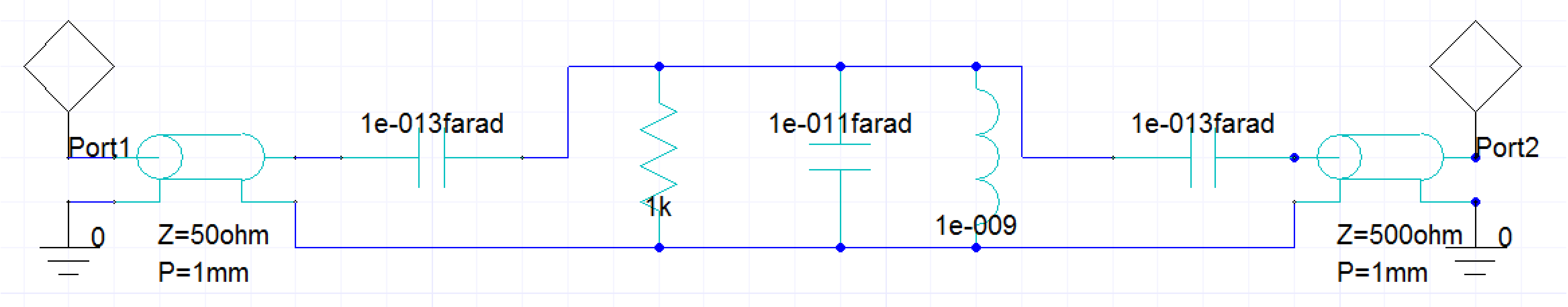
Two probe setup

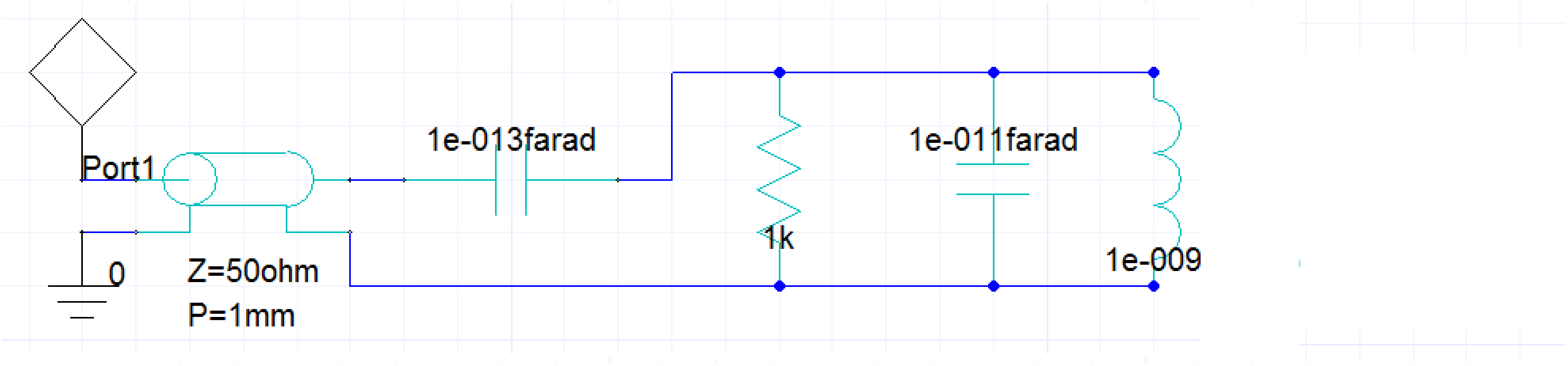


One probe setup

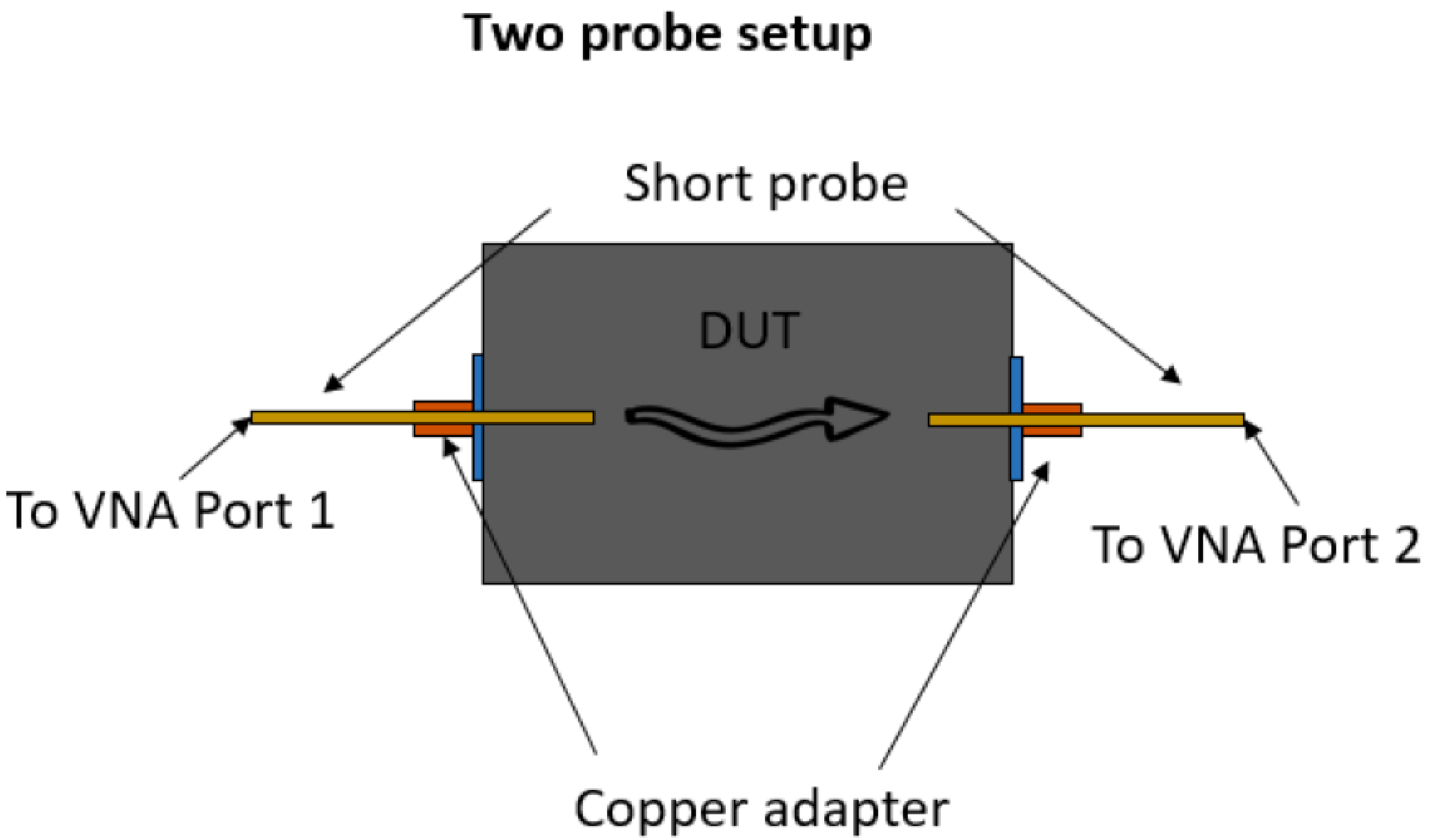


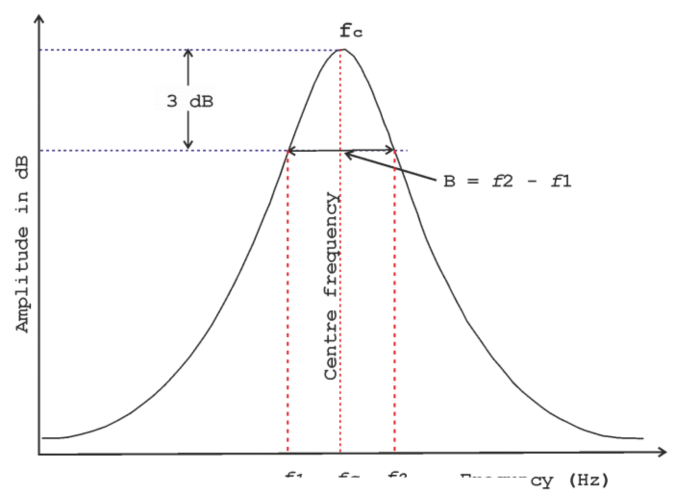
Lab measurements of beam impedance. Probe #2





Lab measurements of beam impedance. Probe #3

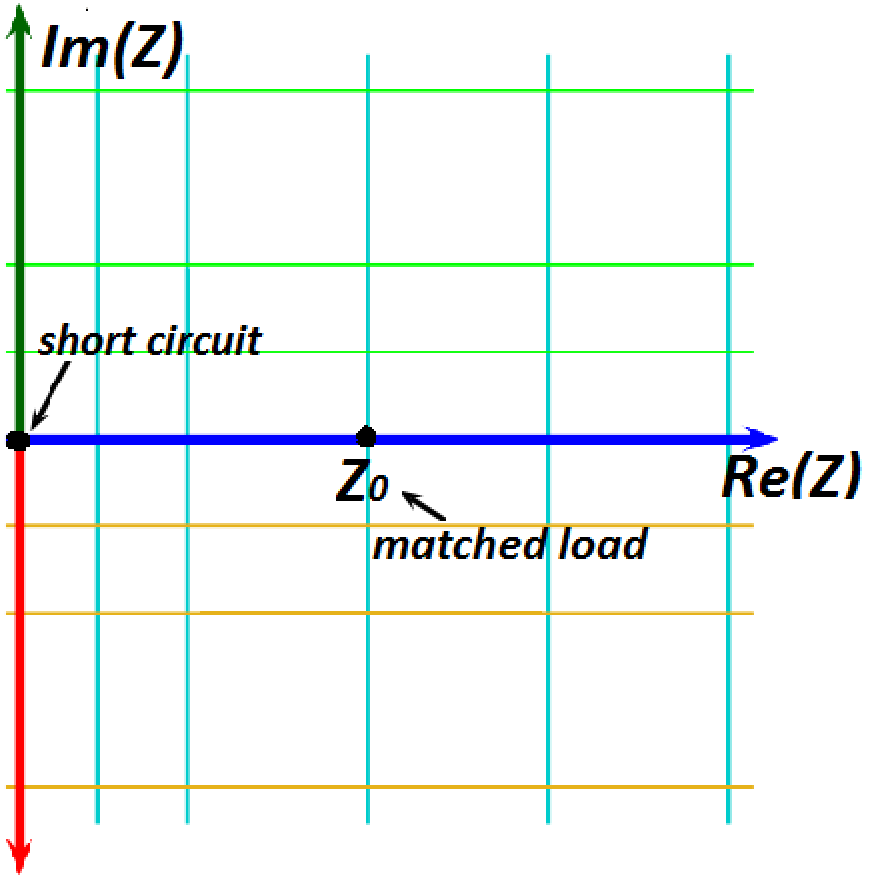




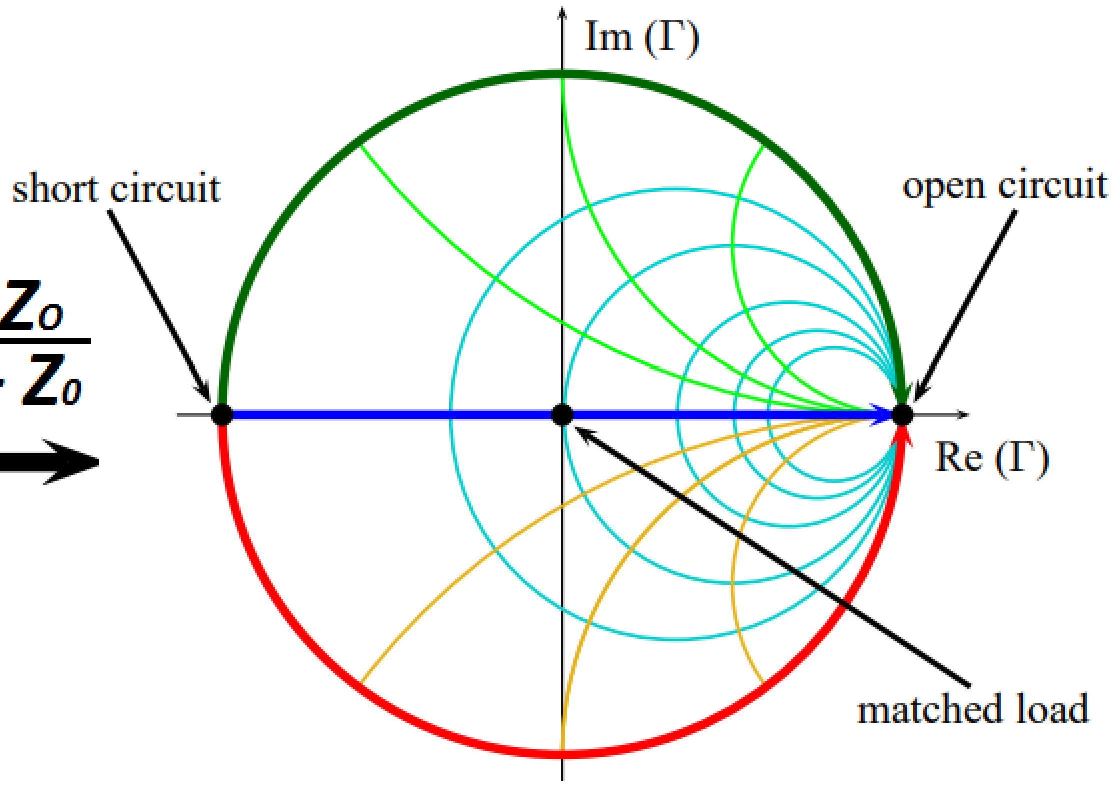
$$Q = \frac{fc}{\Delta f}$$

Lab measurements of beam impedance. Probe #4

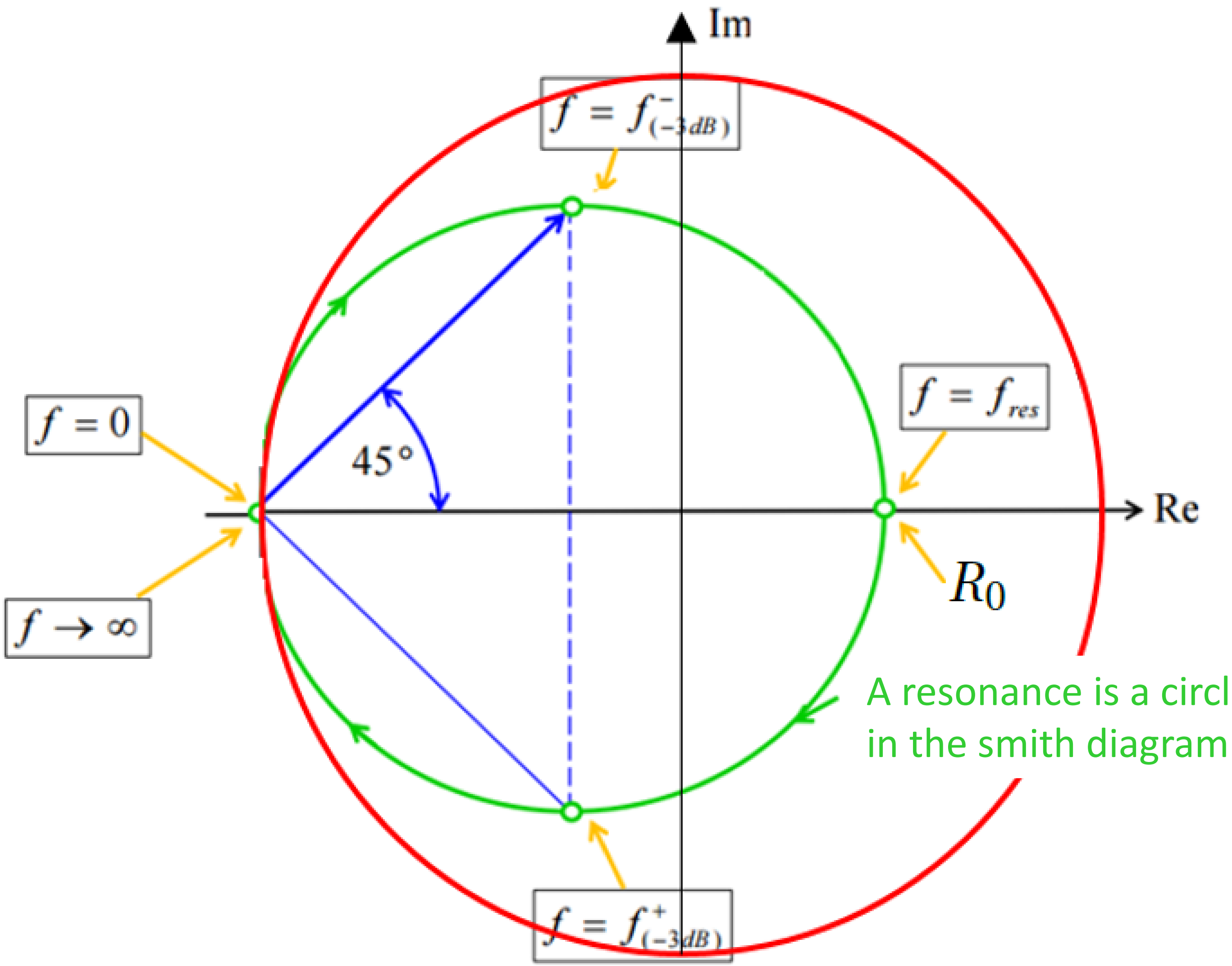
smith chart



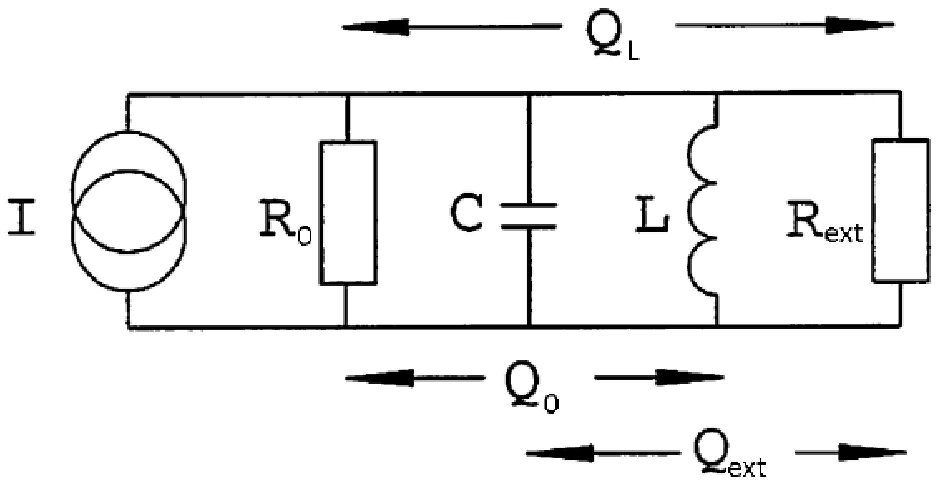
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$



Lab measurements. Measure Q reflection. Probe #5



A resonance is a circle in the smith diagram.



Three different types of *Q*:

1) The loaded *Q* (Q_L)
2) The unloaded *Q* (Q_0)
3) The *Q* of the external world (Q_{ext}).

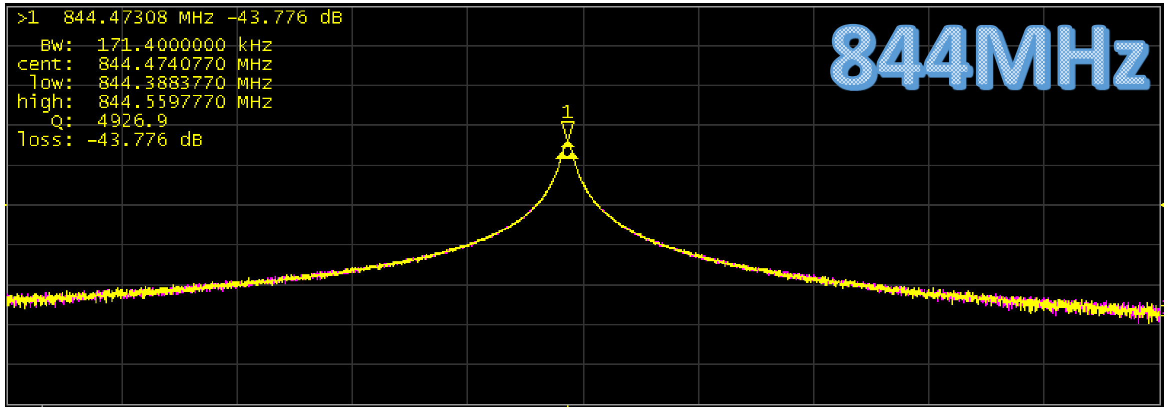
$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

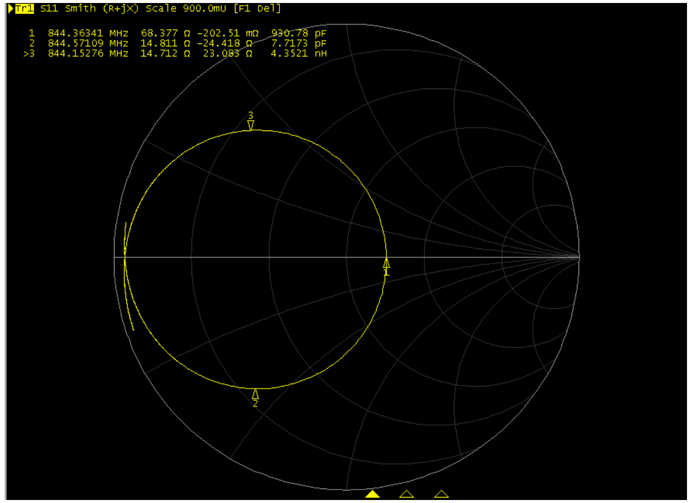
We want Q_0, but we can only measure Q_L and β:

$$Q_L = \frac{f_{res}}{\Delta_f} \qquad \beta = \frac{R_0}{R_{ext}}$$

$$\boxed{Q_0 = Q_L(1+\beta)}$$

Lab measurements. Measure Q reflection. Probe #6

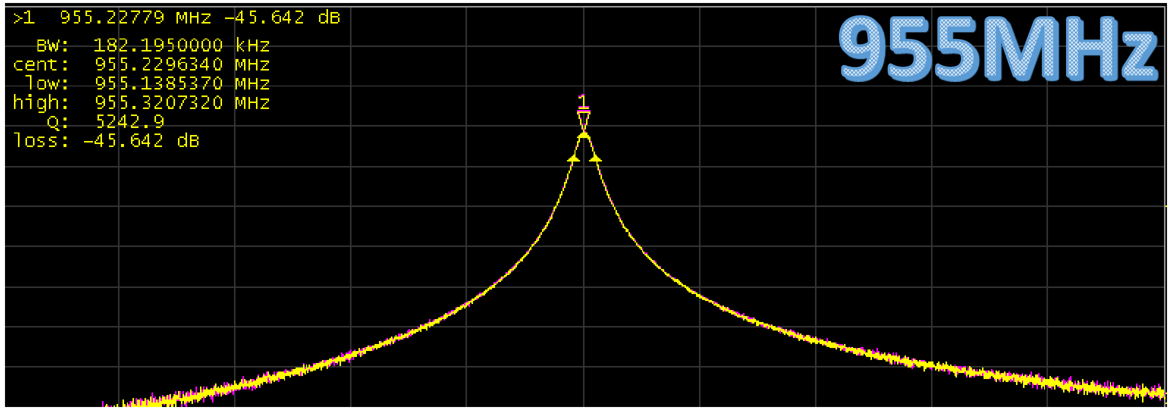


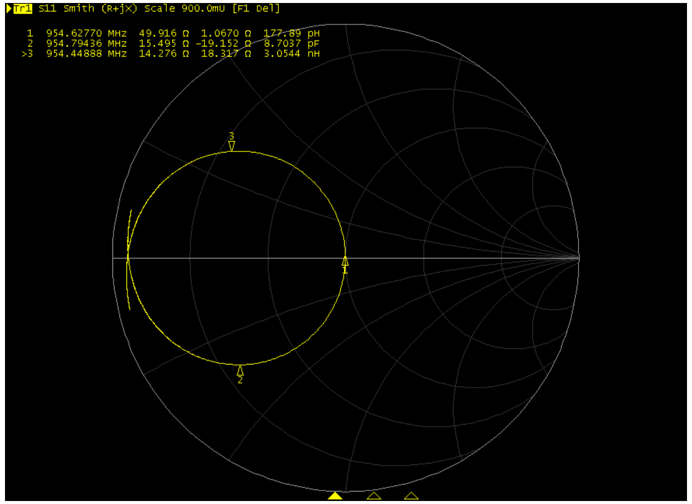


Ql ≈ 2018

1+β ≈2

Q0 ≈4036





Ql ≈ 2763

1+β ≈2

Q0 ≈5526

Courtesy of C.Vollinger and T.Kaltenbacher

感谢您的关注

[1] Measurement of transverse kick in CLIC accelerating structure in FACET Hao Zha, Andrea Latina, Alexej Grudiev
[2] Holomorphic decomposition. John Jowett
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[3] On single wire technique for transverse coupling impedance measurement. H. Tsutsui
http://cds.cern.ch/record/702715/files/sl-note-2002-034.pdf
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[6] G.Rumolo, CAS Advanced Accelerator Physics Trondheim, Norway 18–29 August 2013
https://cds.cern.ch/record/1507631/files/CERN-2014-009.pdf
[7] THE STRETCHED WIRE METHOD: A COMPARATIVE ANALYSIS PERFORMED BY MEANS OF THE MODE MATCHING TECHNIQUE M.R.Masullo, V.G.Vaccaro, M.Panniello
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[7] Two Wire Wakefield Measurements of the DARHT Accelerator Cell. Scott D. Nelson, Michael Vella
https://e-reports-ext.llnl.gov/pdf/236163.pdf
[8] Shunt impedance, RLC-circuit definition, Accelerator definition, Alexej Grudiev
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https://journals.aps.org/prstab/pdf/10.1103/PhysRevSTAB.7.012001
[12] Measurement of coupling impedance of accelerator devices with the wire-method J.G. Wang, S.Y. Zhang

[13] Longitudinal and Transverse Wire Measurements for the Evaluation of Impedance Reduction Measures on the MKE Extraction Kickers. Kroyer, T ; Caspers, Friedhelm ; Gaxiola, E
http://cds.cern.ch/record/1035461/files/ab-note-2007-028.pdf

Energy loss when beam pass through an equipment

Shunt impedance

Accelerator definition: r

$$V = \int_0^L E_z e^{j\frac{\omega z}{c}} dz$$

$$r = \frac{V^2}{P}; Q = \frac{\omega_0 U}{P}; \frac{r}{Q} = \frac{V^2}{\omega_0 U}$$

$$V = 2qk_l; k_l = \frac{U}{q^2}; k_l = \frac{V^2}{4U}$$

$$k_l = \frac{\omega_0}{4}\frac{r}{Q}$$

RLC-circuit definition: R

$$Z_l(\omega) = \int_0^\infty W_l(\tau) e^{j\omega\tau} d\tau$$

$$k_l = \frac{1}{\pi}\int_0^\infty \Re\{Z_l(\omega)\} d\omega$$

$$\omega_0 = \frac{1}{\sqrt{LC}}; Q = R\sqrt{\frac{C}{L}}$$

$$Z_l(\omega) = \frac{R}{1 + jQ(\omega/\omega_0 - \omega_0/\omega)}$$

$$k_l = \frac{\omega_0}{2}\frac{R}{Q}$$

$$\boxed{r = 2R}$$

https://impedance.web.cern.ch/lhc-impedance/Collimators/RLC_050211.ppt
Shunt impedance, RLC-circuit definition, Accelerator definition Alexej Grudiev

The energy lost, when the particle passes a resonance, is equal to: $E_{loss} = k * q^2$

Where k is the loss factor, which is equal to:

$k_{loss\ factor} = \omega_0/2 \cdot R/Q$

And q is the charge of the particle.

As you can see, the bigger R over Q, the bigger the energy l

Wake Loss Factor

The wake loss factor (k) for the longitudinal component is calculated by:

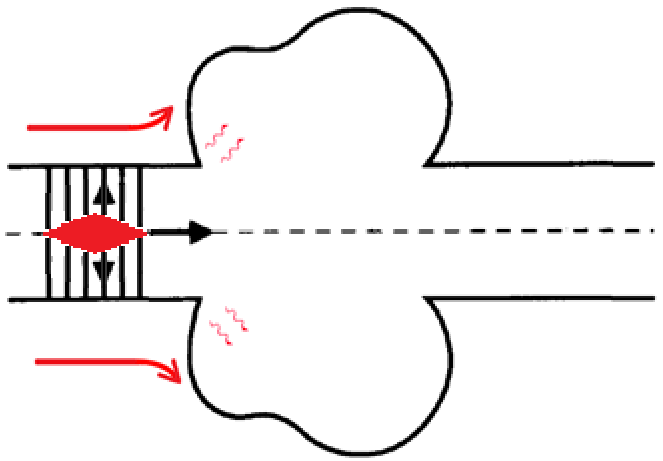
$$k = -\int_{-\infty}^{\infty} \lambda(s) W_{\|}(s)\, ds$$

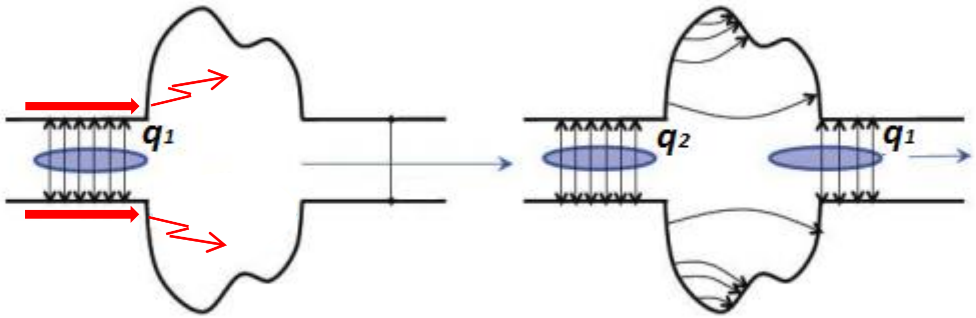
were lambda(s) describes the normed charge distribution function over s (to obt
to multiply this function by q1). It is given in [V / pC].

of the equipment,

Use loss(kick) factor instead of impedance

Beam impedance modelled by lumped impedance





limators/RLC_050211.ppt

:or definition, Alexej Grudiev

The energy lost, when the particle passes a resonance, is equal to: $E_{loss} = k * q^2$

Where k is the loss factor, which is equal to:

$k_{loss\ factor} = \omega_0 /2 \cdot R/Q$

And q is the charge of the particle.

As you can see, the bigger R over Q, the bigger the energy l

The longitudinal beam impedance have 8 parameters

The beam impedance is now decomposed into **13** parameters :

$Z_{||}$[xd, xt, yd, yt] = Z0

+Z1XD xd+Z1XT xt+Z1YD yd+Z1YT yt

+Z2XYDXYD (xd^2-yd^2)+Z2XYTXYT (xt^2- yt^2)

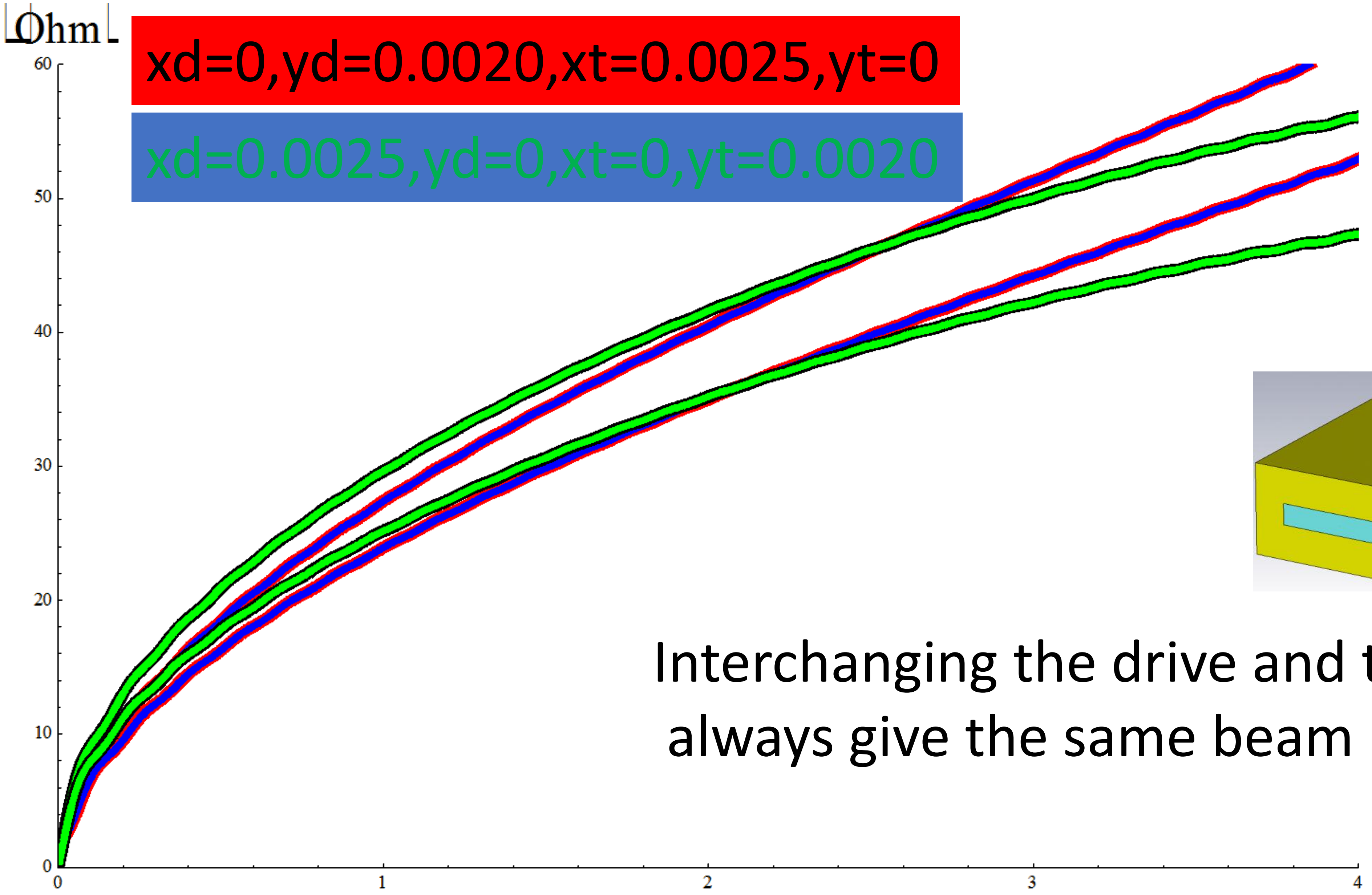
+Z2XDXT xd xt+Z2XDYD xd yd+Z2XDYT xd yt

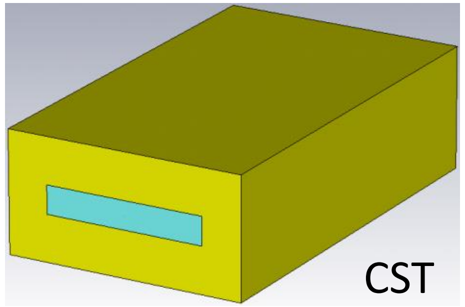
+Z2XTYD xt yd+Z2XTYT xt yt+Z2YDYT yd yt

The new formula is identical to the previous from Tsutsui:

$$
\begin{aligned}
Z \;=\; & Z_{0,0} + (x_1 - jy_1)Z_{1,0} + (x_1 + jy_1)Z_{-1,0} + (x_2 + jy_2)Z_{0,1} + (x_2 - jy_2)Z_{0,-1} \\
& +(x_1 - jy_1)^2 Z_{2,0} + (x_1 - jy_1)(x_2 - jy_2)Z_{1,-1} + (x_2 - jy_2)^2 Z_{0,-2} \\
& +(x_1 - jy_1)(x_2 + jy_2)Z_{1,1} + (x_1 + jy_1)(x_2 - jy_2)Z_{-1,-1} \\
& +(x_1 + jy_1)^2 Z_{-2,0} + (x_1 + jy_1)(x_2 + jy_2)Z_{-1,1} + (x_2 + jy_2)^2 Z_{0,2} \\
& +O((x_1, y_1, x_2, y_2)^3).
\end{aligned}
$$

The longitudinal beam impedance have 8 parameters





Interchanging the drive and test particles always give the same beam impedance.

The longitudinal beam impedance have 8 parameters

$Z_{||}[xd, xt, yd, yt] =$

$Z0$

$+Z1X\ (xd+xt)+Z1Y\ (yd+yt)$

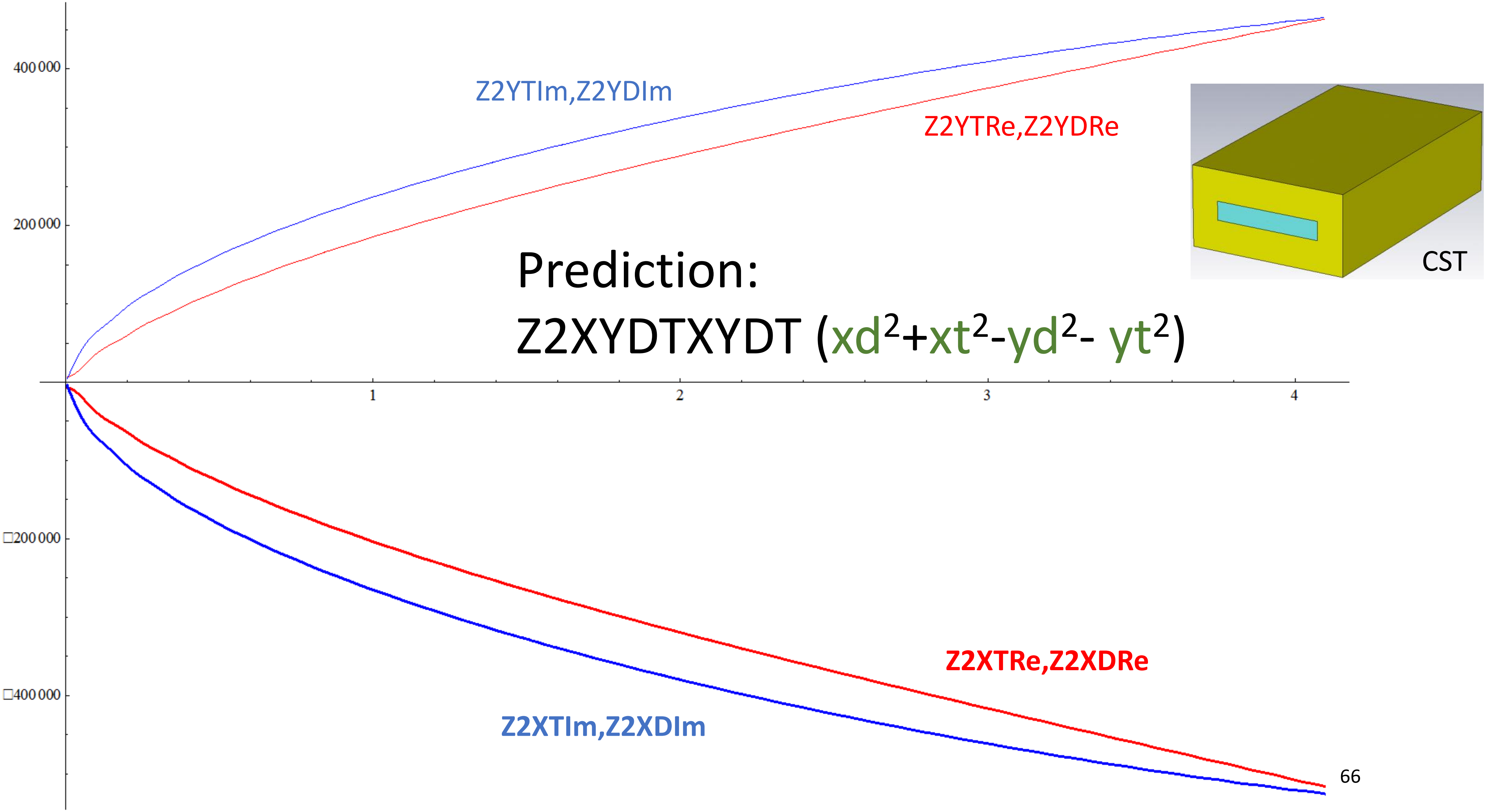
$+Z2XYDTXYDT\ (xd^2+xt^2-yd^2-\ yt^2)$

$+Z2XDTYDT\ (xd\ yd+xt\ yt)$

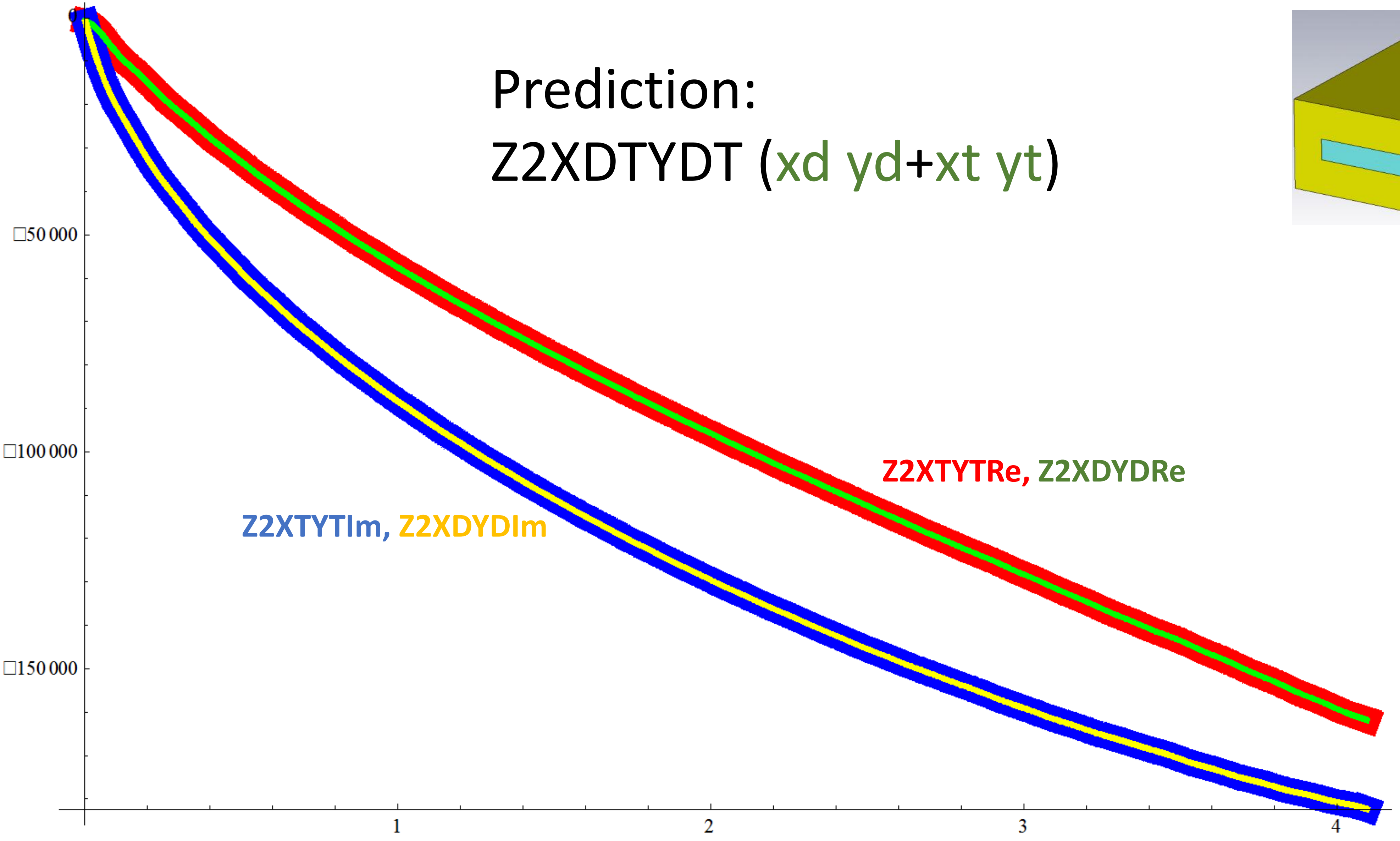
$+Z2XDTYTD\ (xd\ yt\ +xt\ yd)$

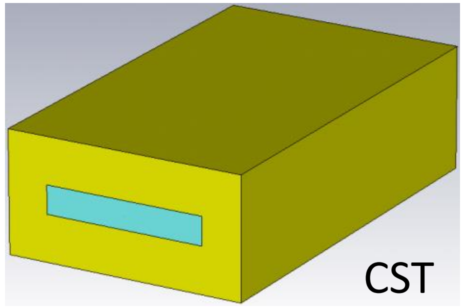
$+Z2XDXT\ xd\ xt+Z2YDYT\ yd\ yt$

CST Wakefield example illustrating the 8 parameters



CST Wakefield example illustrating the 8 parameters

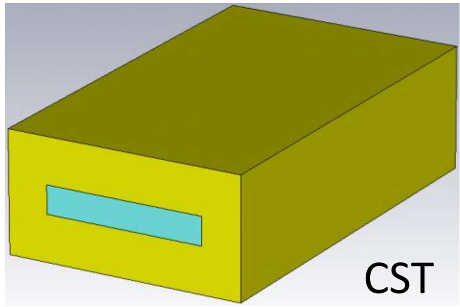




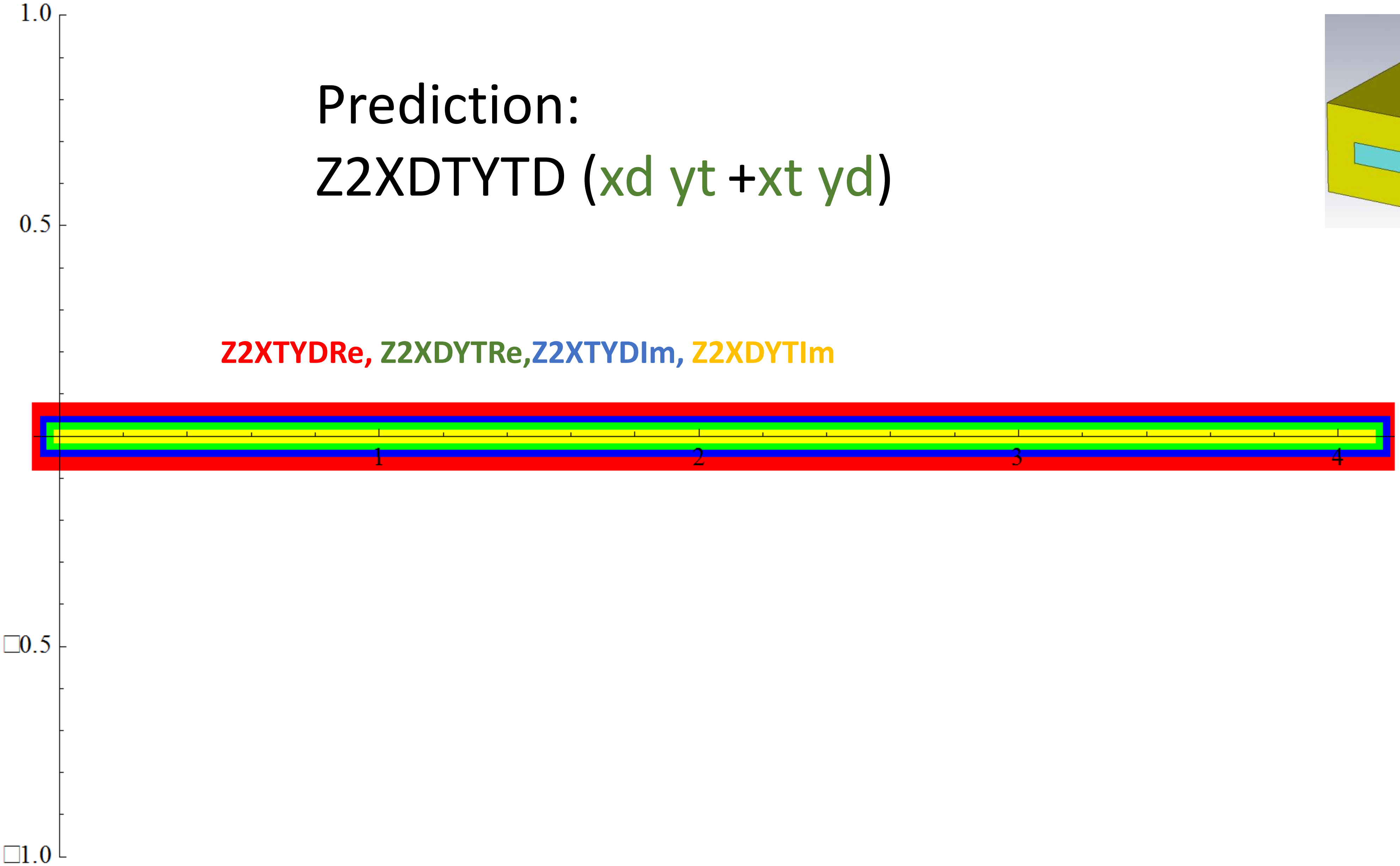
CST Wakefield example illustrating the 8 parameters

Prediction:

Z2XDTYTD (xd yt +xt yd)

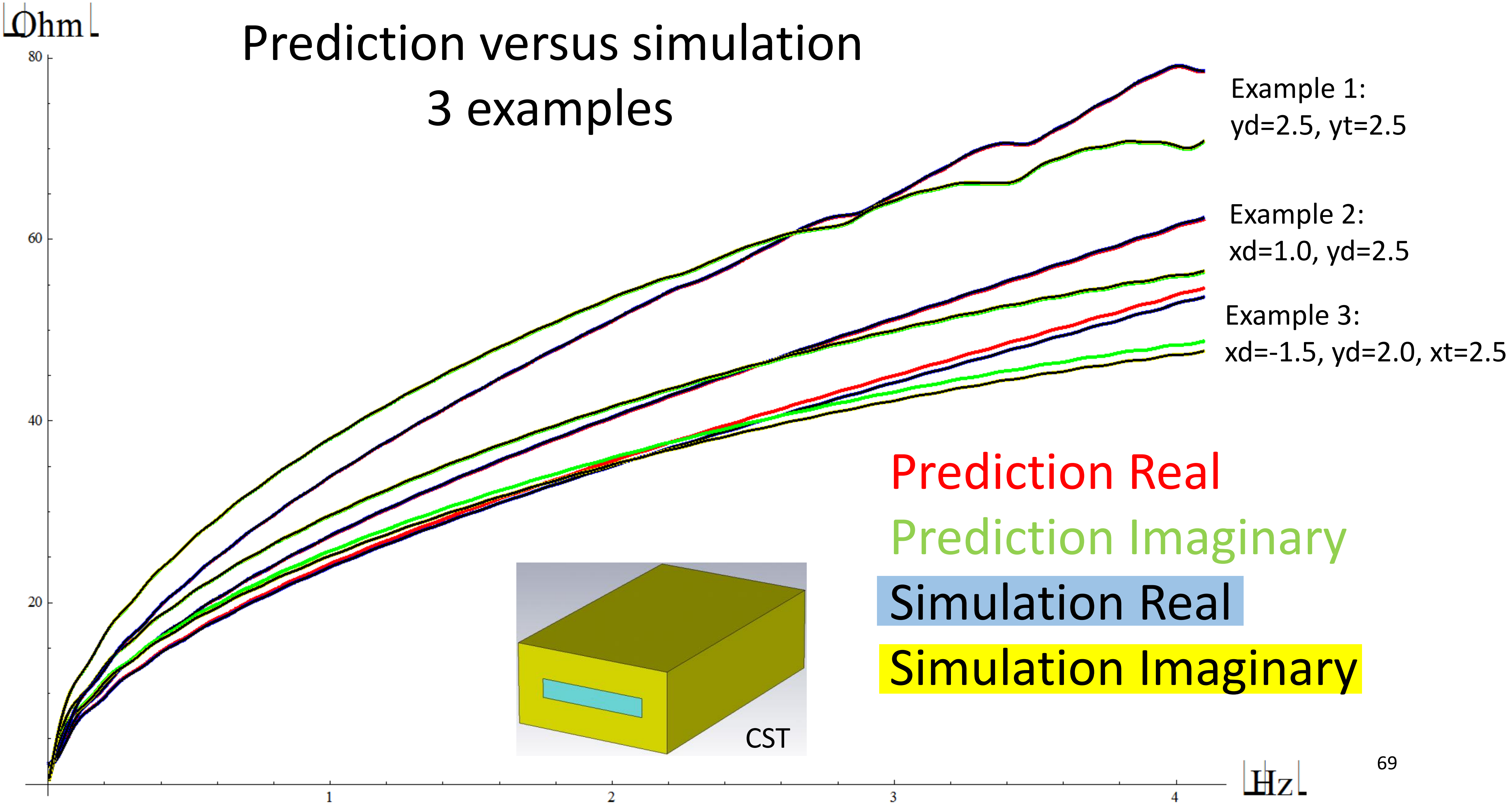


Z2XTYDRe, Z2XDYTRe,Z2XTYDIm, Z2XDYTIm

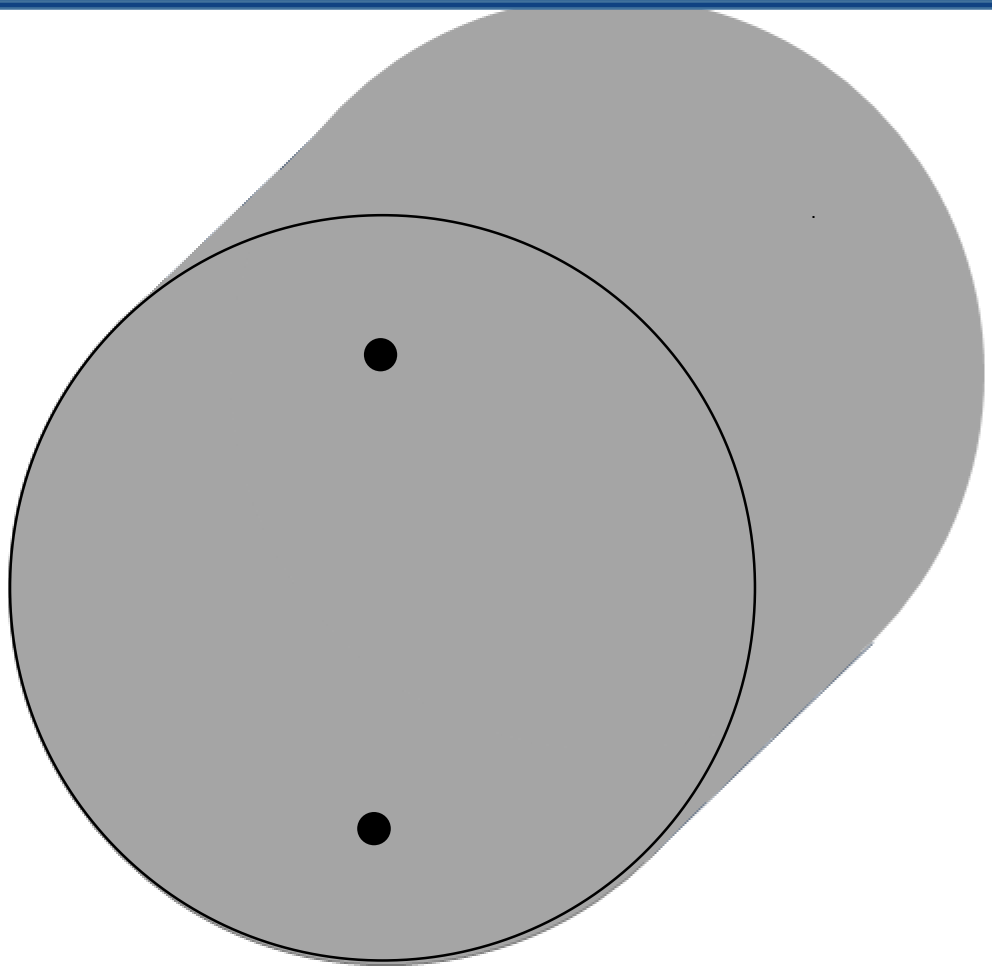


CST Wakefield example illustrating the 8 parameters

Prediction versus simulation
3 examples



The rotating wire method



One wire represents the drive particle and the other wire represents the test particle.

In this measurement, we do not have a positive current in one wire and a negative current in the other

Both wires are measured individually i.e. single-ended

In the additional slide, it is demonstrated how this measurement can derive all 8 parameters

The rotating wire method

Some implications of the new 8 parameter formula:

1) **Transverse impedance**

The offset term is not automatically zero, depends on the shape of the equipment

$$Z_{\perp,x}(\omega) = Z1_x + 2Z2_A \cdot xt + Z2_B \cdot yt + Z2_C \cdot yd + Z2_D \cdot xd$$
$$Z_{\perp,y}(\omega) = Z1_y - 2Z2_A \cdot yt + Z2_B \cdot xt + Z2_C \cdot xd + Z2_E \cdot yd$$

2) **Transverse impedance**

Is it possible to shape a collimator e.g. in three-fold symmetric form so that its transvers impedance is zero up to second order?

3) **Transverse impedance**

The beam oscillates during instability, is it possible to shape equipment in such a way that the drive position works against the instability?

Supporting material for slide

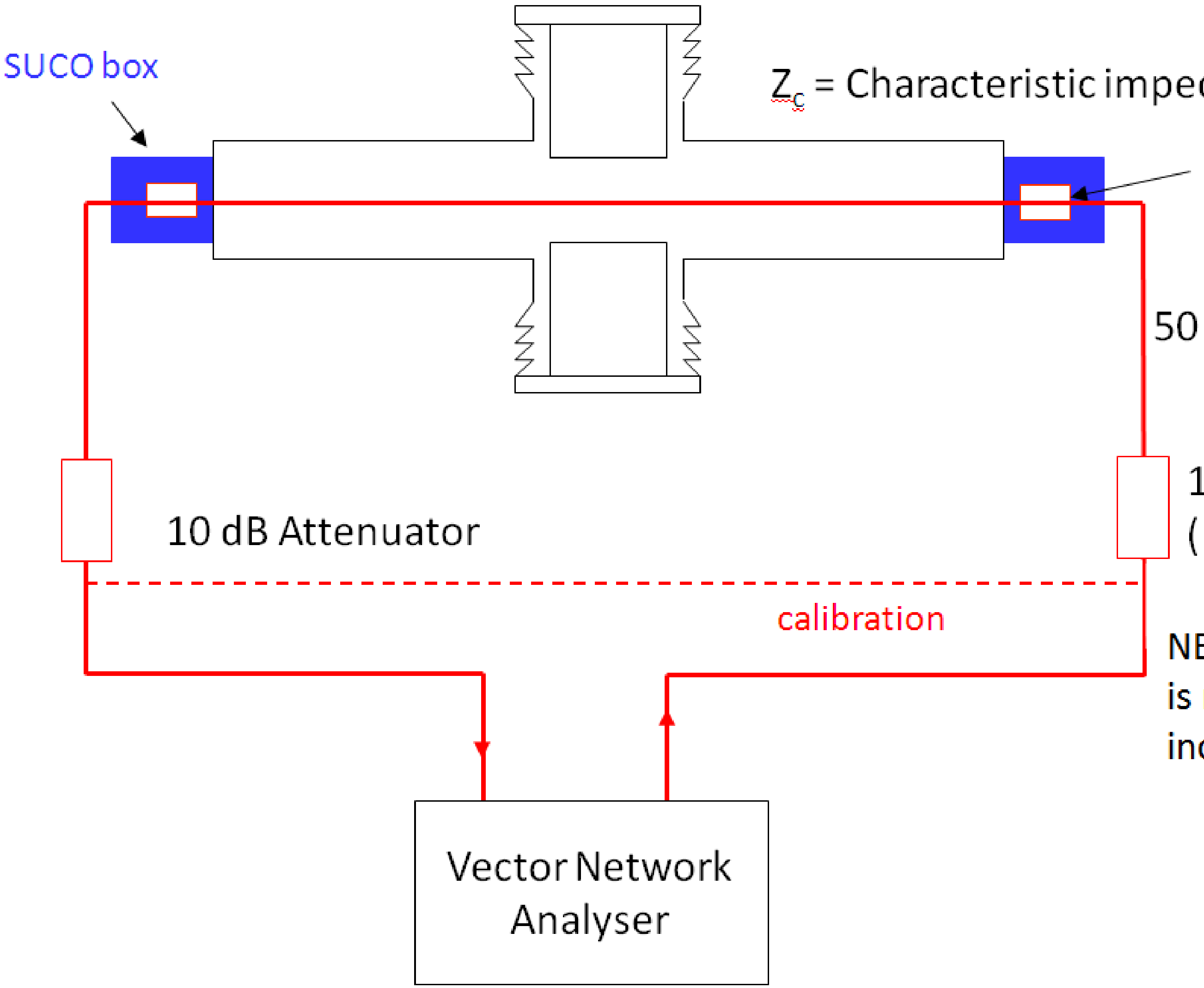
$$\texttt{sol1} = \texttt{Solve}\left[\texttt{Srel} == \texttt{Exp}\left[i * l * w * \sqrt{CC * L} - i * l * w * \sqrt{\left(1 - \frac{i * dZdl}{w * L}\right) * CC * L}\,\right], dZdl\right]$$

$$\left\{\left\{dZdl \to \frac{-2\, l\, \sqrt{CC\, L}\; w\, \text{Log}[\text{Srel}] - i\, \text{Log}[\text{Srel}]^2}{CC\, l^2\, w}\right\}\right\}$$

$$-2\sqrt{\frac{L}{CC}}\;\text{Log}[\text{Srel}]\left(1 + \frac{i\,\text{Log}[\text{Srel}]}{2\, l\,\sqrt{CC\, L}\; w}\right) == \frac{-2\, l\,\sqrt{CC\, L}\; w\,\text{Log}[\text{Srel}] - i\,\text{Log}[\text{Srel}]^2}{CC\, l\, w}$$

True

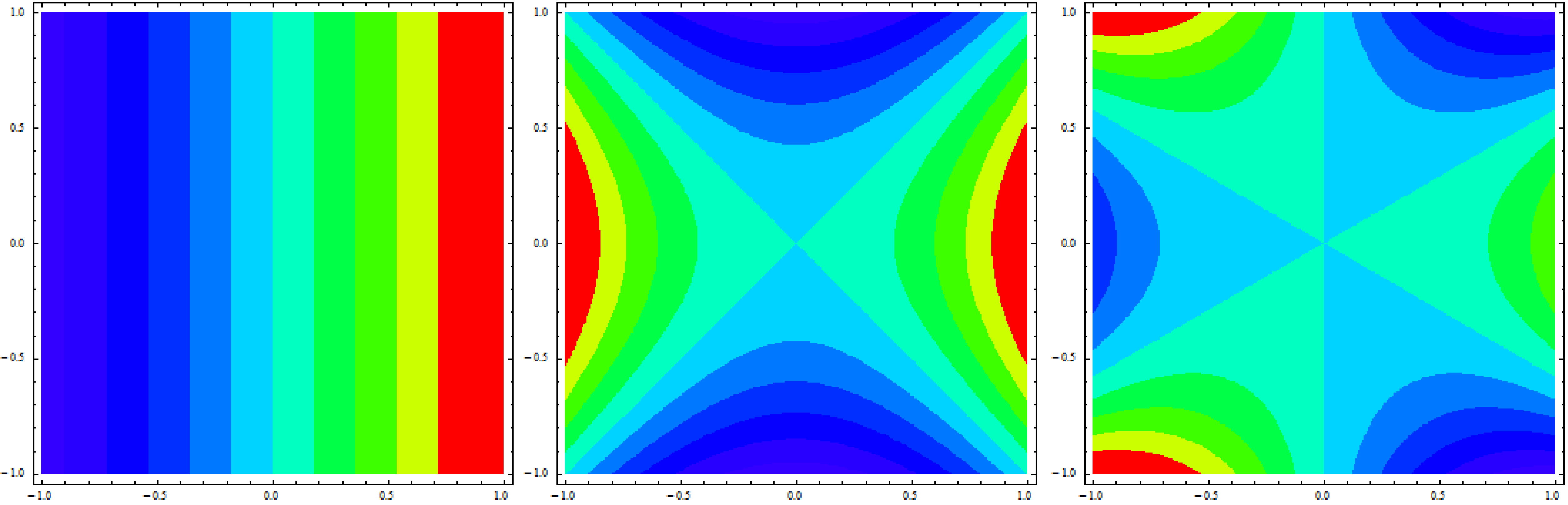
Longitudinal Beam Coupling Impedance of Device Under Test (DUT)



NB! The calibration must be done without the attenuators. It is not yet understood exactly why, but it shown in test that including the attenuators give incorrect calibrations.

Vector network analyzer calibration to remove effects from cables and attenuators.

Measure transmission coefficient S_{21} (=forward transmission) and then calculate impedance.



Dipole Quadrupole Sextupole

Figure 3: **Normal** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:

Dipole: x Quadrupole: $\frac{x^2}{2} - \frac{y^2}{2}$ Sextupole: $\frac{x^3}{3} - xy^2$

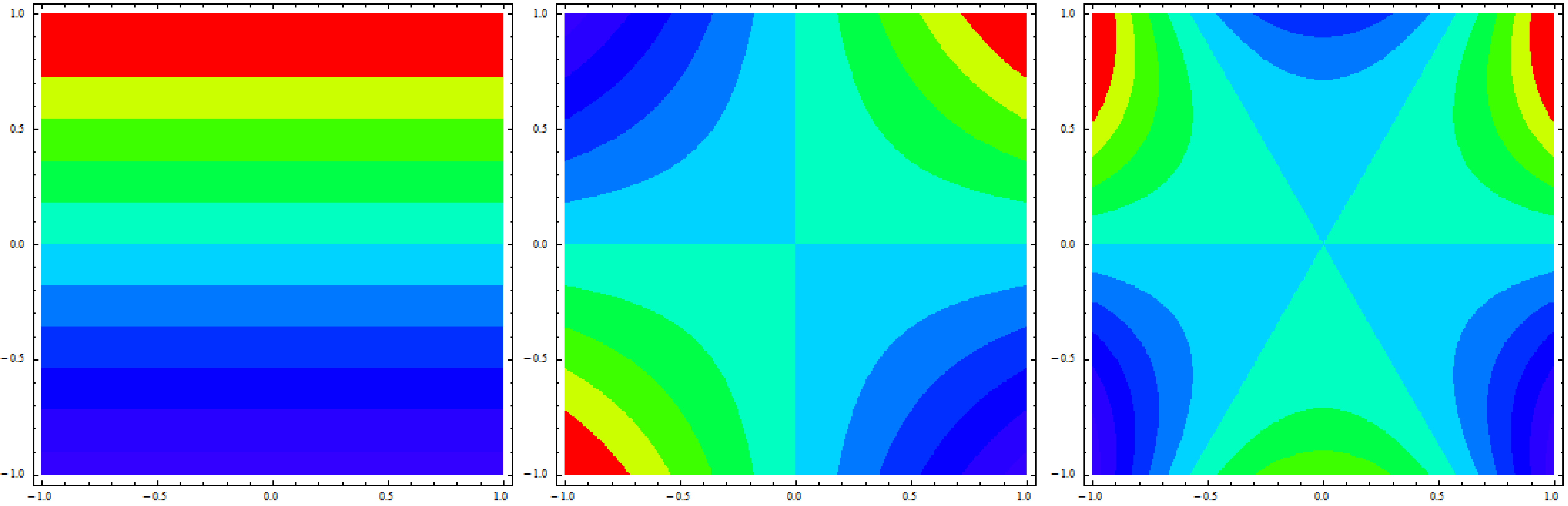


Figure 4: **Skew** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:

Dipole: y Quadrupole: $x \cdot y$ Sextupole: $x^2 y - \frac{y^3}{3}$