



Modelling and implementation of the “6D” beam-beam interaction

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Keywords: beam-beam, 6D, synchro beam mapping

Abstract

These slides illustrate the numerical modelling of a beam-beam interaction using the “Synchro Beam Mapping” approach. The employed description of the strong beam allows correctly accounting for the hour-glass effect as well as for linear coupling at the interaction point. The implementation of the method within the SixTrack code is reviewed and tested.



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 - Complete multi-slice interaction
- **Handling the denominators**



Goal: review of the 6D beam-beam lens implemented in SixTrack

Tried to answer two main questions:

- **What is the code supposed to do?**
 - Mathematical derivation behind the implemented numerical model
- **Is the code doing what it is supposed to do?**
 - Verify the implementation of the above numerical model



The code simulates a **beam-beam interaction** using the “**Synchro Beam Mapping**” technique in the presence of:

- **Crossing angle** (ϕ)
- Arbitrary **crossing plane** (α)
- Optics at the IP described by a **general 4D correlation matrix** (Σ -matrix)
→ hour glass effect, elliptic beams, alphas, and linear coupling at the IP are included in the modeling

This makes the **mathematical derivation quite heavy**

Implementation in Sixtrack is **largely based on**:

- [1] [*A symplectic beam-beam interaction with energy change*](#), by K. Hirata, H. W. Moshhammer, F. Ruggiero, 1992
- [2] [*Don't be afraid of beam-beam interactions with a large crossing angle*](#), by K. Hirata, 1993
- [3] [*6D Beam-Beam Kick including Coupled Motion*](#), by L.H.A. Leunissen, F. Schmidt, G. Ripken, 2001

... but **important parts** (e.g. inverse boost, “optics de-coupling” including longitudinal derivatives) are **not reported in the papers nor anywhere else**, to our best knowledge...

- Invested some time in **understanding and re-constructing the mathematical treatment** trying to use as little as possible the source code as a reference
 - **Independent reconstruction** of the equations to verify the implementation in Sixtrack and to be used as a basis for a modern implementation (GPU compatible, for example)
 - **Parts not available in literature** (mainly inverse Lorentz boost, and a large fraction of the coupling treatment) **had to be re-derived**
- Prepared a **document** including the full set of equation to enable a possible re-implementation (and avoid that somebody has to redo the same exercise in ten years :-)



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6D beam-beam interaction step-by-step

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Summary

This document describes in detail the numerical method used in different simulation codes for the simulation of beam-beam interactions using the "Synchro Beam Mapping" approach to correctly model the coupling introduced by beam-beam between the longitudinal and the transverse plane. The goal is to provide in a compact, complete and self-consistent manner the set of equations needed for the implementation in a numerical code. The effect of a "crossing angle" in an arbitrary "crossing plane" with respect to the assigned reference frame is taken into account with a suitable coordinate transformation. The employed description of the strong beam allows correctly accounting for the hour-glass effect as well as for linear coupling at the interaction point.



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- We want to simulate a **beam-beam interaction** taking into account the **finite longitudinal size of the two beams**
- We are in the framework on the **weak-strong treatment**: we have a particle (of the weak-beam) that we are tracking. It interacts with a strong beam that is “rigid”, i.e. unaffected by the weak beam



We will use the **“synchro-beam mapping”** approach introduced by Hirata, Moshhammer and Ruggiero [1]. To do so, the following **conditions need do be satisfied**:

- We work in **ultra-relativistic** approximation $v=c$ for both beams
 - The **strong beam is travelling backwards** $s_{\text{strong}}(t) = \sigma_{\text{strong}} + ct$
 - **$P_x = P_y = 0$ for the strong beam**:
 - The transverse electric field can be calculated solving a 2D Poisson problem
 - The **angles of the test particle are small** so that we can assume that it travels at the speed of light along s : $s(t) = \sigma - ct$
-
- In the presence of a **crossing angle** a reference frame satisfying all the conditions above cannot be found by simple rotation in the lab frame, but this can be obtained by applying also a **Lorentz boost in the crossing plane** as shown by Hirata in [2]



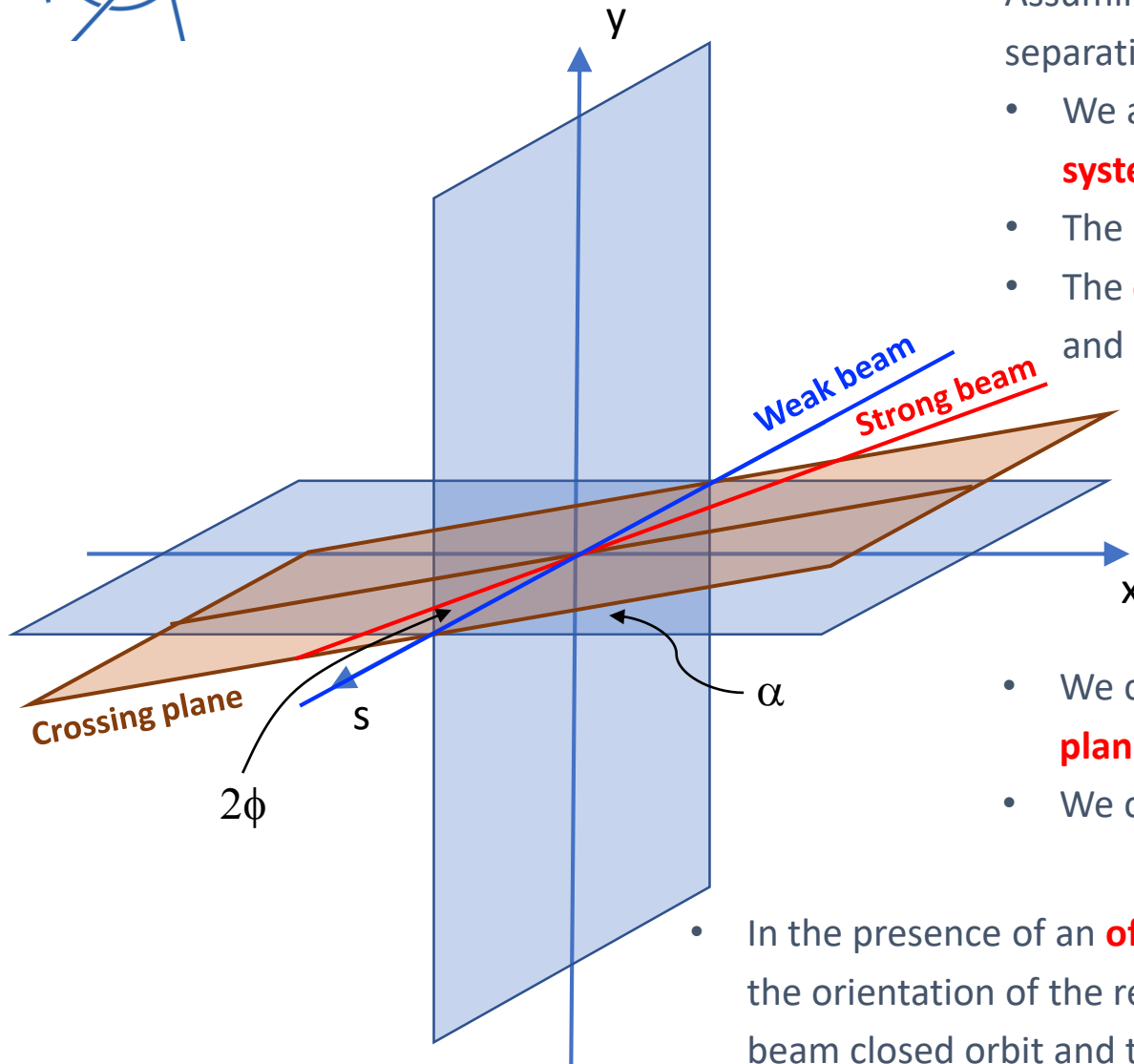
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A dance of reference systems

Assuming that the beams are colliding (no separation):

- We assume that we are in the **reference system of the weak beam**
- The Interaction Point (IP) is at $s=0$
- The **crossing plane** is defined by our s -axis and by the strong beam

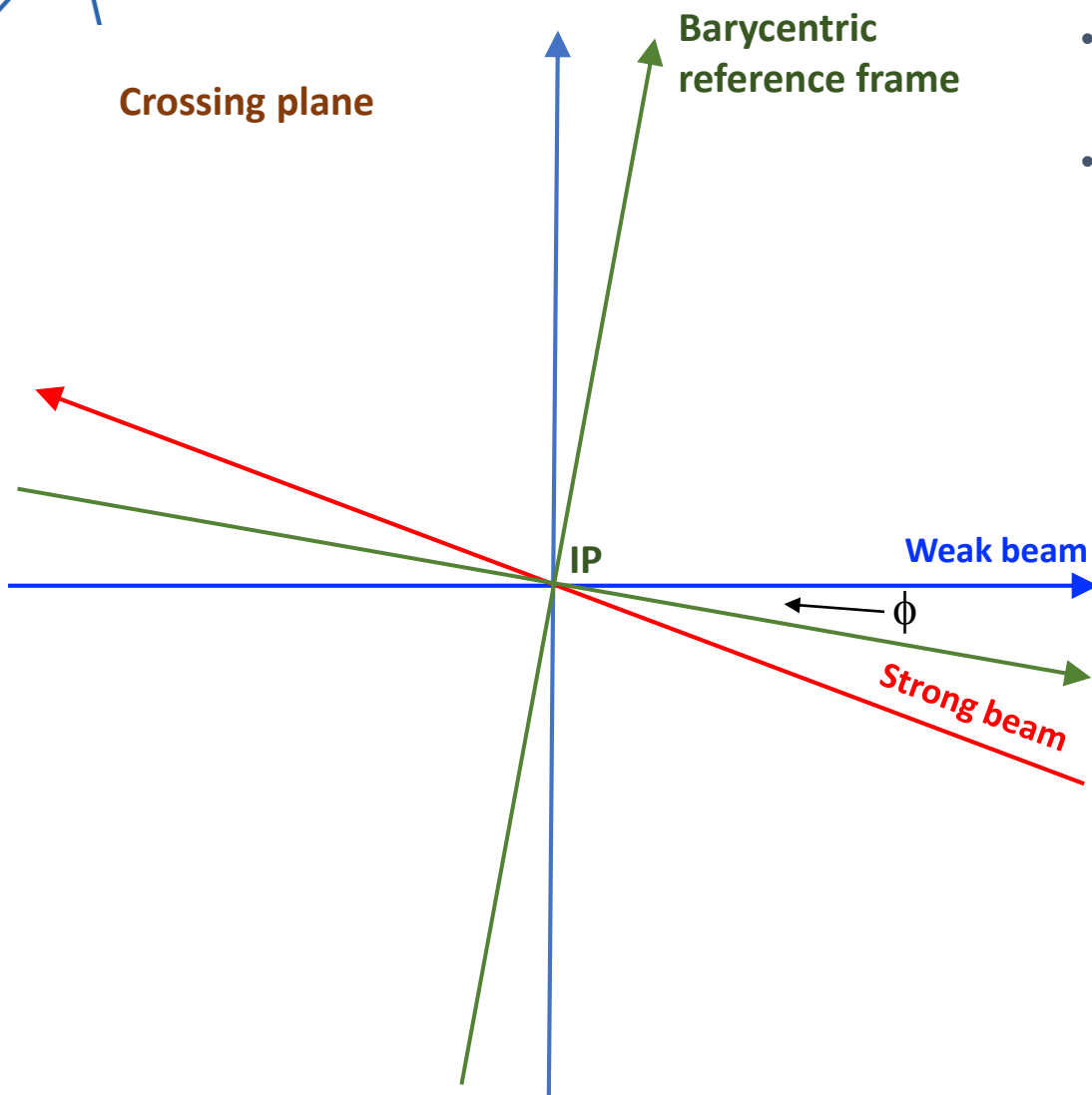


- We call α the **angle between the crossing plane and the x-s plane**
- We call ϕ the **half crossing angle**

- In the presence of an **offset between the beams (separation)**, the orientation of the reference system is defined by the weak beam closed orbit and the system is centered at the IP location as defined for the strong beam \rightarrow the strong beam passes always through the origin of the reference frame



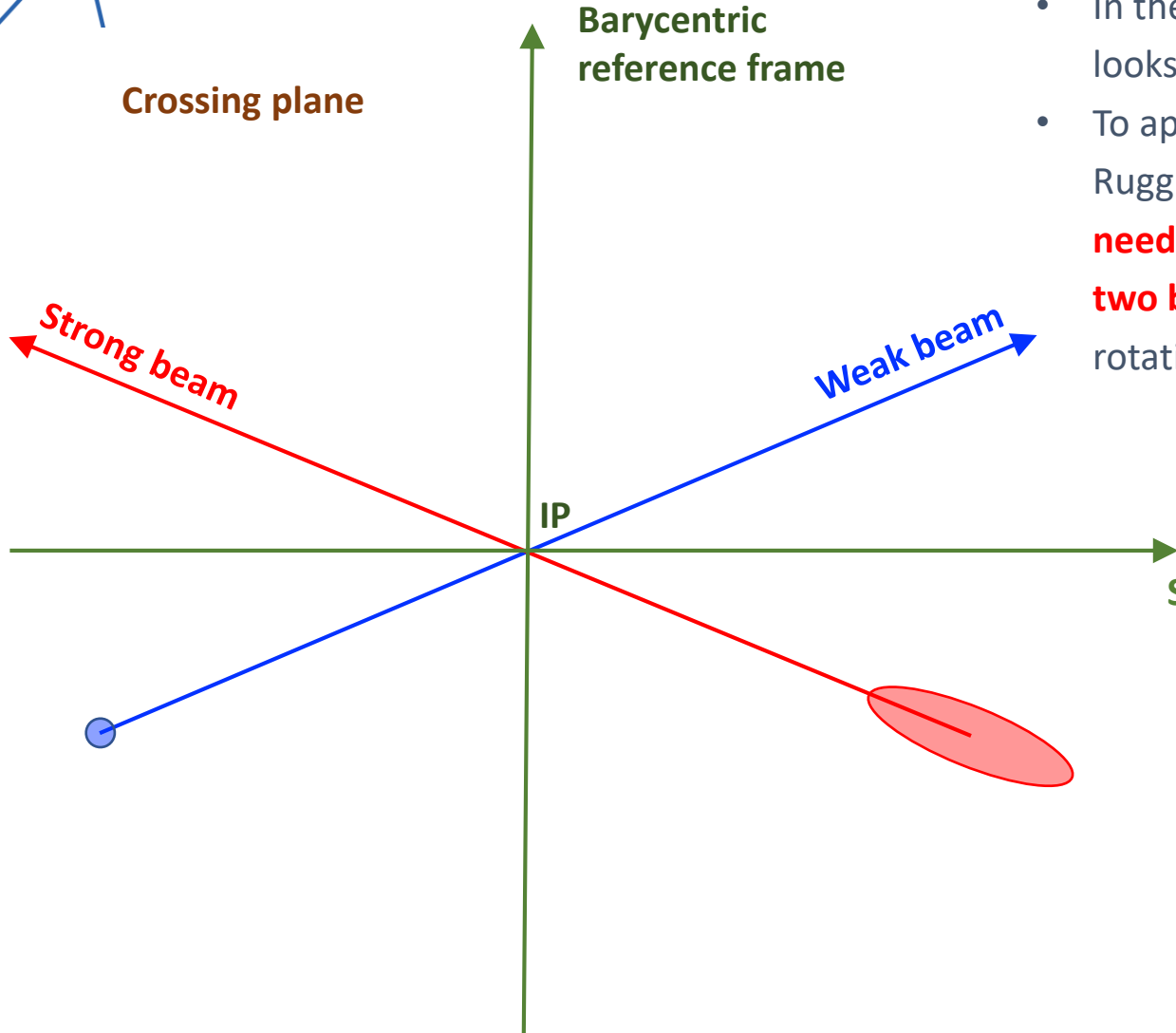
A dance of reference systems



- We look at the problem in the crossing plane
- We introduce move to the **“barycentric” reference system** in which the weak and the strong beam are at $+\phi$ and $-\phi$ respectively



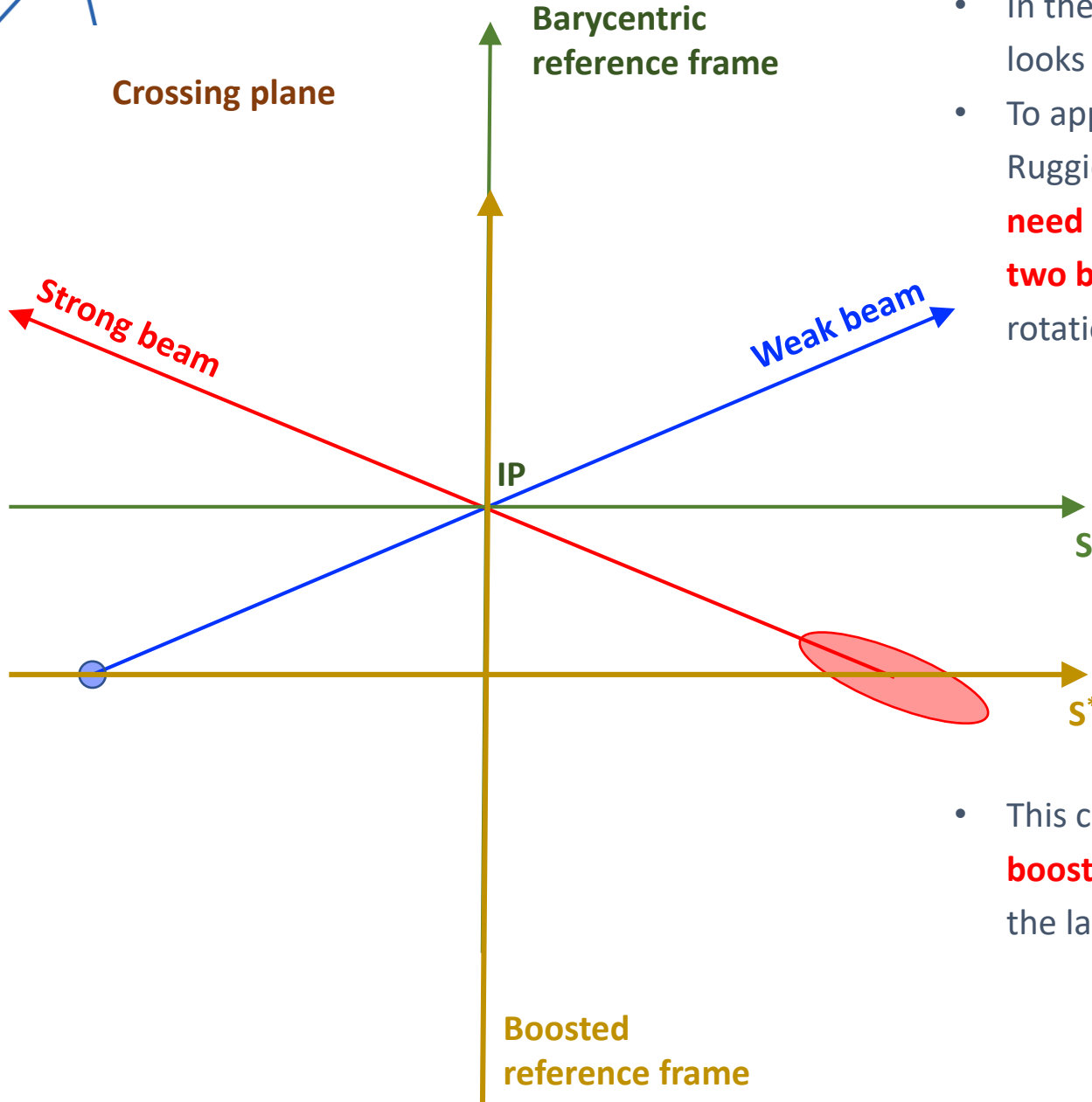
A dance of reference systems



- In the crossing plane the interaction looks like this...
- To apply the Hirata, Moshhammer, Ruggiero treatment we practically **need to suppress the angle for the two beams** (impossible by simple rotation)



A dance of reference systems



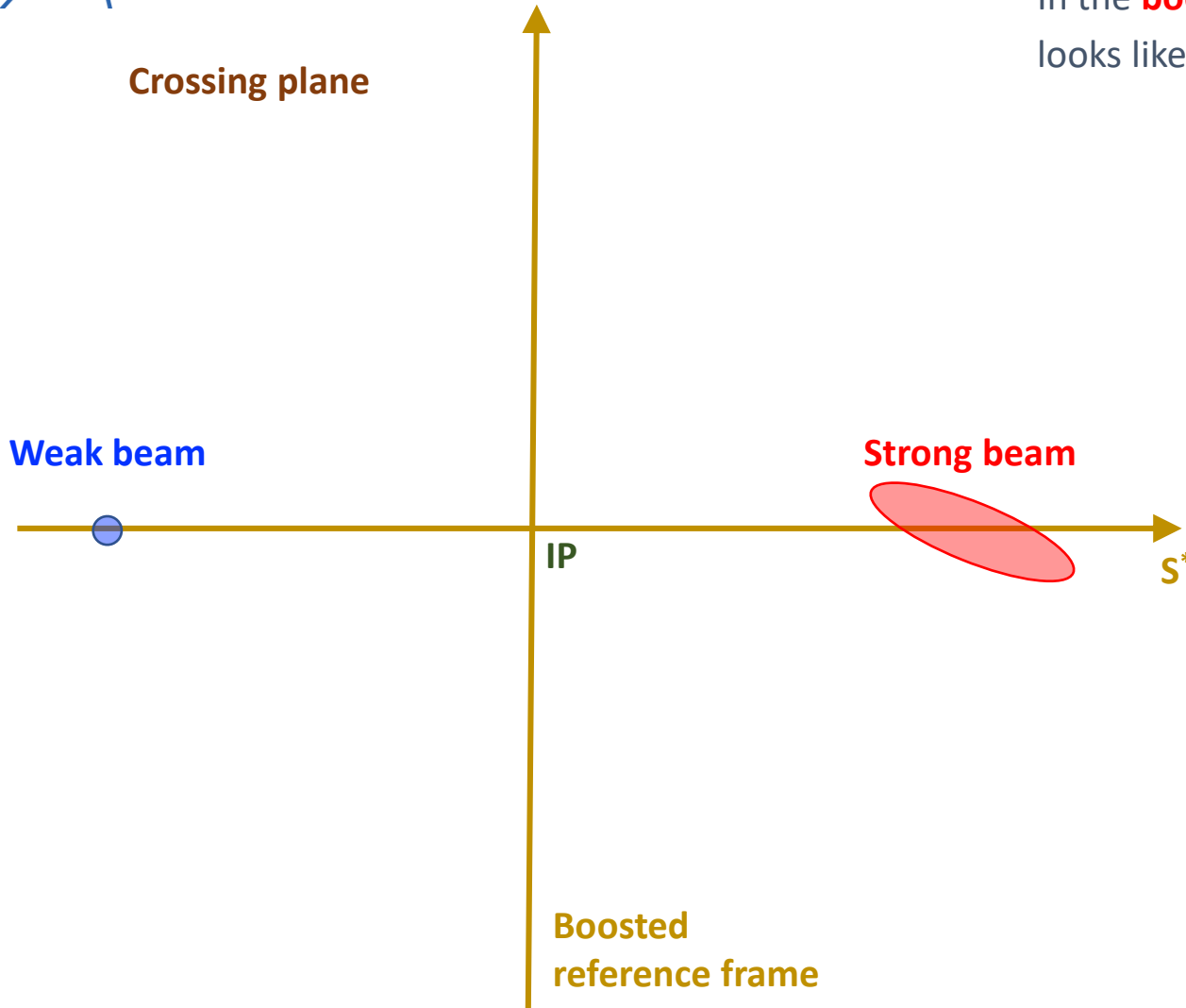
- In the crossing plane the interaction looks like this...
- To apply the Hirata, Moshhammer, Ruggiero treatment we practically **need to suppress the angle for the two beams** (impossible by simple rotation)

- This can be achieved by using a **boosted frame** that is moving w.r.t. the lab



A dance of reference systems

In the **boosted frame** the interaction looks like this





“Boost transformation” in formulas

This transformation is applied for positions:

$$\begin{pmatrix} \sigma^* \\ x^* \\ s^* \\ y^* \end{pmatrix} = A^{-1} R_{CP}^{-1} L_{\text{boost}} R_{CA} R_{CP} A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix}$$

- A is the matrix transforming the accelerator coordinates (Courant-Snyder) to Cartesian coordinates:

$$\begin{pmatrix} ct \\ X \\ Z \\ Y \end{pmatrix} = A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix}$$

- R_{CP} is the rotation matrix bringing the crossing plane in the X-Z plane:
- R_{CA} is the rotation matrix moving to the barycentric frame:

$$R_{CA} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{CP} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \cos \alpha & 0 & \sin \alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

- L_{boost} is the Lorentz boost in the direction of the rotated X-axis:

$$L_{\text{boost}} = \begin{pmatrix} 1/\cos \phi & -\tan \phi & 0 & 0 \\ -\tan \phi & 1/\cos \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



“Boost transformation” in formulas

This transformation is applied for momenta:

$$\begin{pmatrix} \delta^* \\ p_x^* \\ h^* \\ p_y^* \end{pmatrix} = B^{-1} R_{CP}^{-1} L_{\text{boost}} R_{CA} R_{CP} B \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix}$$

- B is the matrix transforming the accelerator coordinates (Courant-Snyder) to Cartesian coordinates:

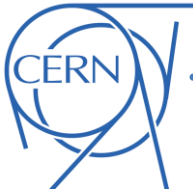
$$\begin{pmatrix} E/c - p_0 \\ P_x \\ P_z - p_0 \\ P_y \end{pmatrix} = p_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix}$$

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“Boost transformation” in formulas

Not all particles with $s=0$ are fixed points of the transformation:

→ **A drift back to $s=0$** needs to be performed as we are tracking w.r.t. s and not w.r.t. time

We compute the angles:

$$p_z^* = \sqrt{(1 + \delta^*)^2 - p_x^{*2} - p_y^{*2}}$$

$$h_x^* = \frac{\partial h^*}{\partial p_x^*} = \frac{p_x^*}{p_z^*}$$

$$h_y^* = \frac{\partial h^*}{\partial p_y^*} = \frac{p_y^*}{p_z^*}$$

$$h_\sigma^* = \frac{\partial h^*}{\partial \delta} = 1 - \frac{\delta^* + 1}{p_z^*}$$

We drift the particles to $s = 0$:

$$x^*(s^* = 0) = x^*(s) - h_x^* s$$

$$y^*(s^* = 0) = y^*(s) - h_y^* s$$

$$\delta^*(s^* = 0) = \delta^*(s) - h_\delta^* s$$

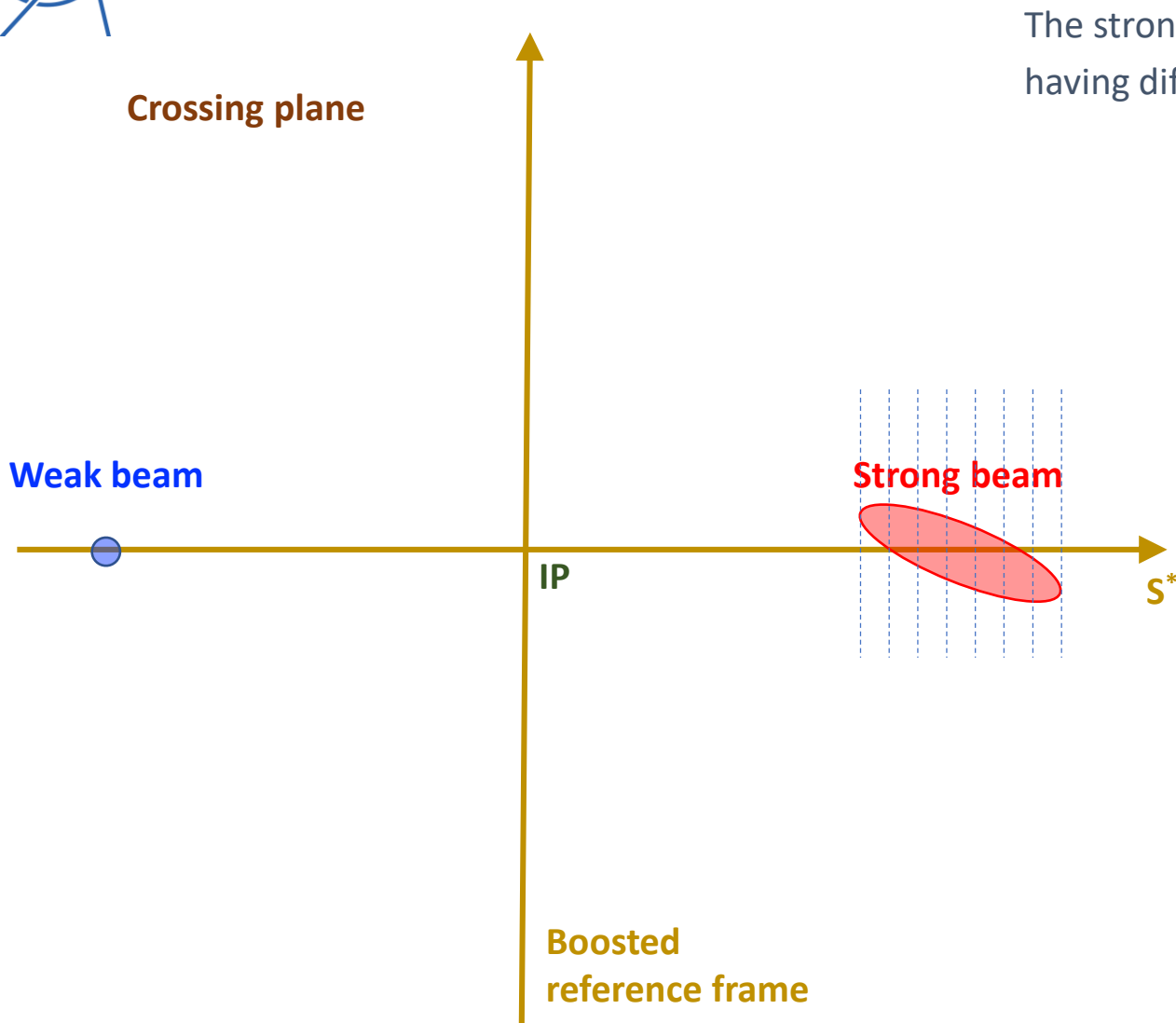
The entire procedure needs to be reverted after the interaction, see note.



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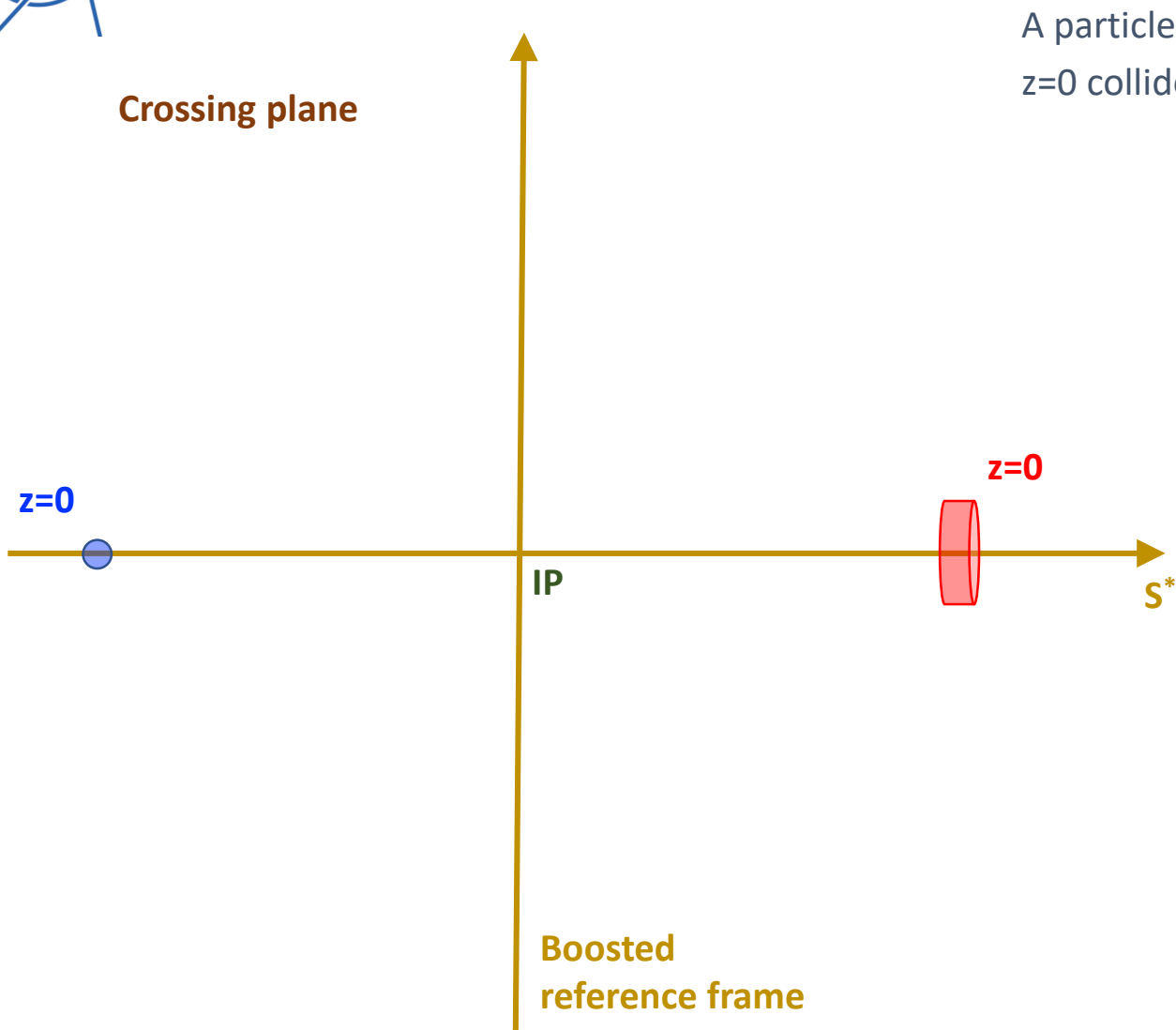
The synchro-beam method: transverse “generalized kicks”



The strong beam is cut in several **slices** having different transverse offset



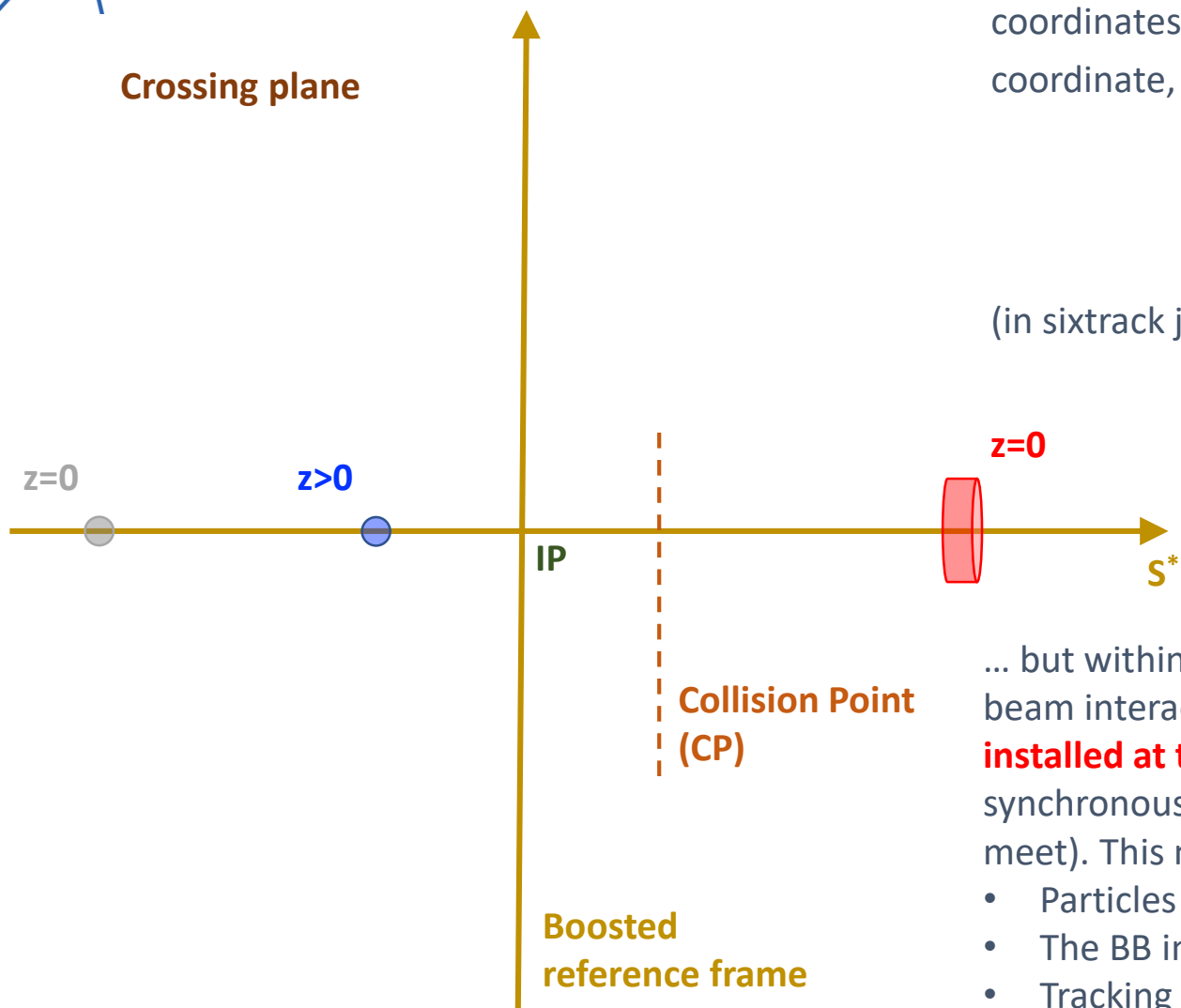
The synchro-beam method: transverse “generalized kicks”



A particle with $z=0$ and a slice having $z=0$ collide at the IP



The synchro-beam method: transverse “generalized kicks”



A particle and a slice with generic z coordinates will collide at a different s coordinate, **Collision Point - CP**, given by:

$$S = \frac{\sigma^* - \sigma_{sl}^*}{2}$$

(in sixtrack jargon z is called σ)

... but within the tracking code, the beam-beam interaction acts as a **thin element installed at the IP** (i.e. the s where the synchronous particles of the two beams meet). This means that:

- Particles are tracked to the IP
- The BB interaction is applied
- Tracking restarts from the IP
- The description of the strong beam is provided at the IP



The synchro-beam method: transverse “generalized kicks”

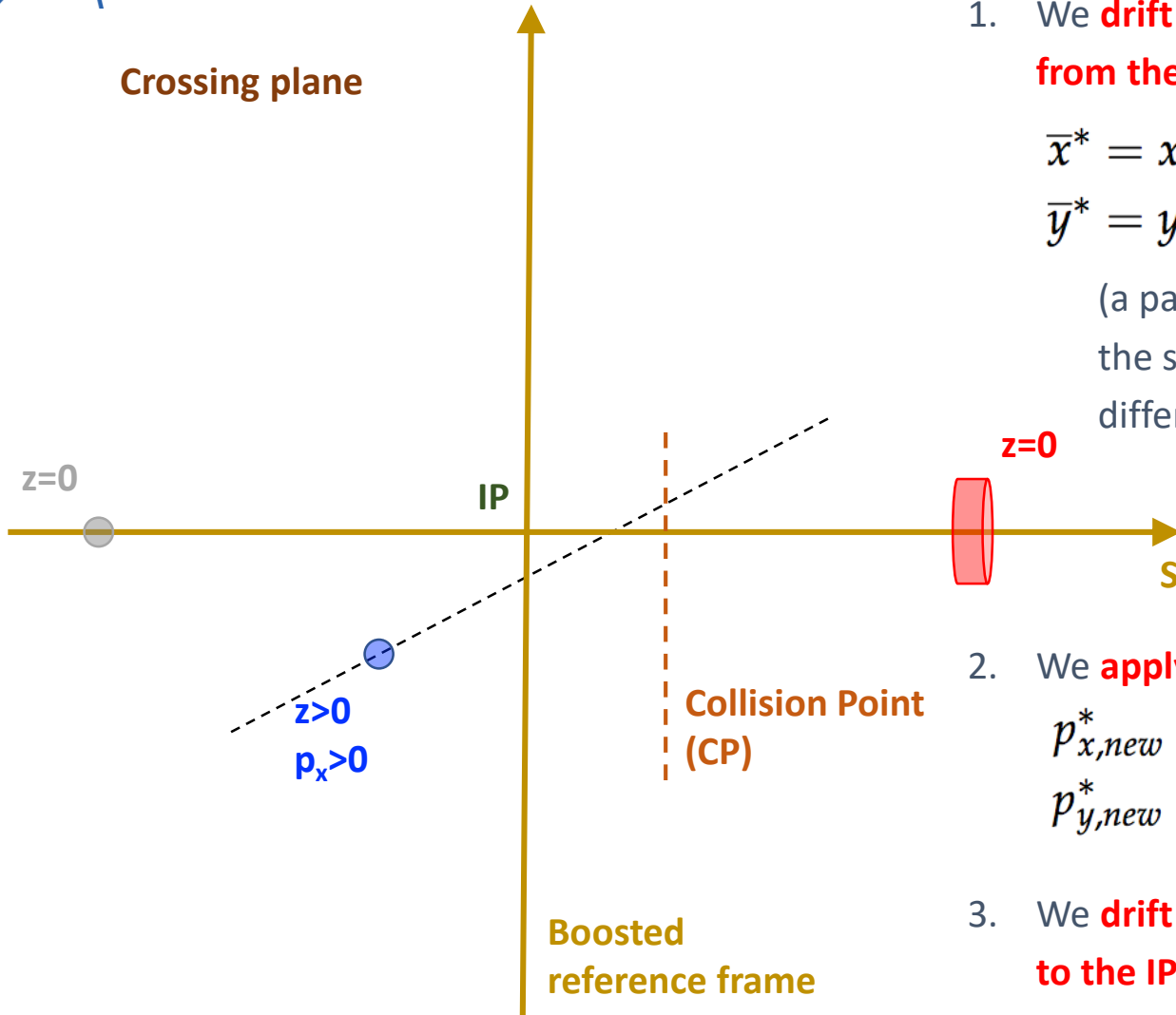
We proceed as follows:

1. We **drift** the slice and the weak particle **from the IP to the CP**

$$\bar{x}^* = x^* + p_x^* S - x_{sl}^* \quad \text{w.r.t. the}$$

$$\bar{y}^* = y^* + p_y^* S - y_{sl}^* \quad \text{slice centroid}$$

(a particle having an angle will probe the strong-beam electric field at a different transverse coordinates)



Transverse kicks need to be computed based on the shape of the strong beam...

2. We **apply the kick** at the CP:

$$p_{x,new}^* = p_x^* + F_x^*$$

$$p_{y,new}^* = p_y^* + F_y^*$$

3. We **drift** the particles **back from the CP to the IP** using the new angles:

$$x_{new}^* = x^* - S F_x^*$$

$$y_{new}^* = y^* - S F_y^*$$



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- The shape of the strong beam is described by **4D correlation matrix (Σ -matrix)**

The **phase space distribution** can be written as:

$$f(\eta) = f_0 e^{-\eta^T \Sigma^{-1} \eta} \quad \text{with} \quad \eta = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}$$

Points having same phase space density lie on hyper-elliptic manifolds defined by the equation:

$$\eta^T \Sigma^{-1} \eta = \text{const.}$$

Σ contains all the information about the beam shape and divergence (including linear coupling) and **can be transported** from the IP to the CP (assuming that we are in a drift):

$$\Sigma_{11}^* = \Sigma_{11}^{*0} + 2\Sigma_{12}^{*0}S + \Sigma_{22}^{*0}S^2$$

$$\Sigma_{33}^* = \Sigma_{33}^{*0} + 2\Sigma_{34}^{*0}S + \Sigma_{44}^{*0}S^2$$

$$\Sigma_{13}^* = \Sigma_{13}^{*0} + (\Sigma_{14}^{*0} + \Sigma_{23}^{*0})S + \Sigma_{24}^{*0}S^2$$

$$\Sigma_{12}^* = \Sigma_{12}^{*0} + \Sigma_{22}^{*0}S$$

$$\Sigma_{14}^* = \Sigma_{14}^{*0} + \Sigma_{24}^{*0}S$$

$$\Sigma_{22}^* = \Sigma_{22}^{*0}$$

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$$\Sigma_{44}^* = \Sigma_{44}^{*0}$$

Convention:

1 \rightarrow x, 2 \rightarrow p_x, 3 \rightarrow y, 4 \rightarrow p_y

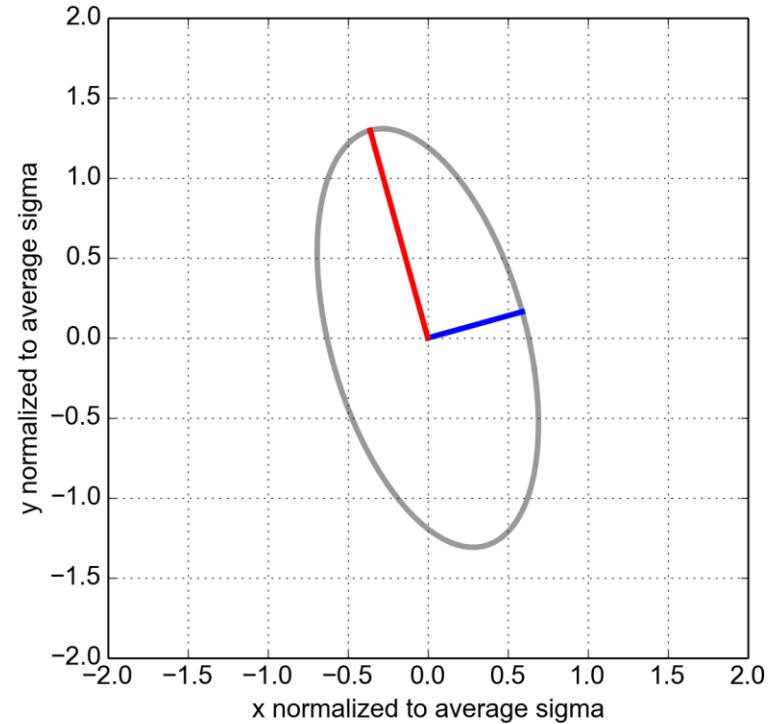
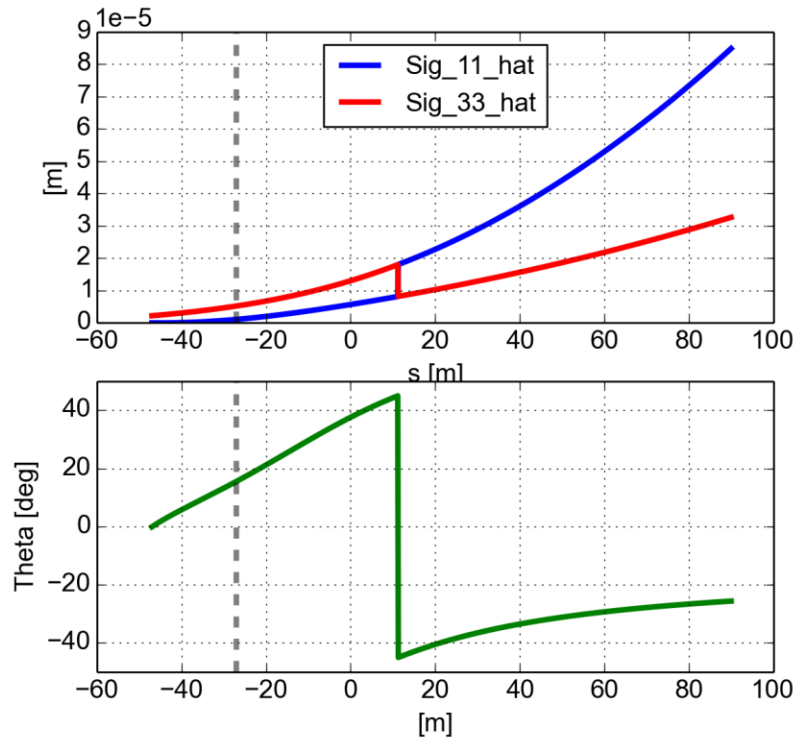


Linear coupling of the strong beam

In general, **linear coupling** of the strong beam can be present:

→ The **coupling angle** and the **beam sizes** in the decoupled frame can be obtained by **diagonalization** of the Σ -matrix

→ Coupling angle depends on the s-coordinate



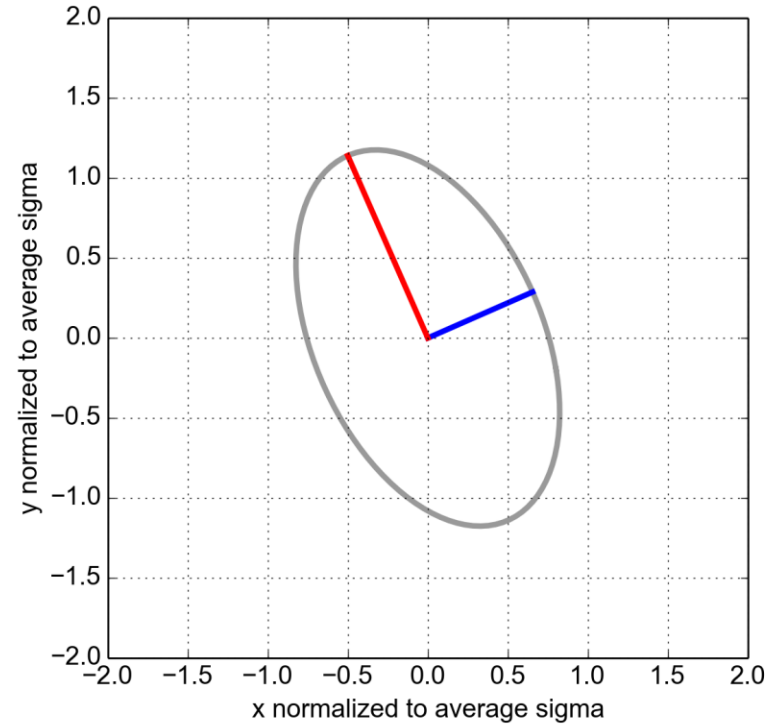
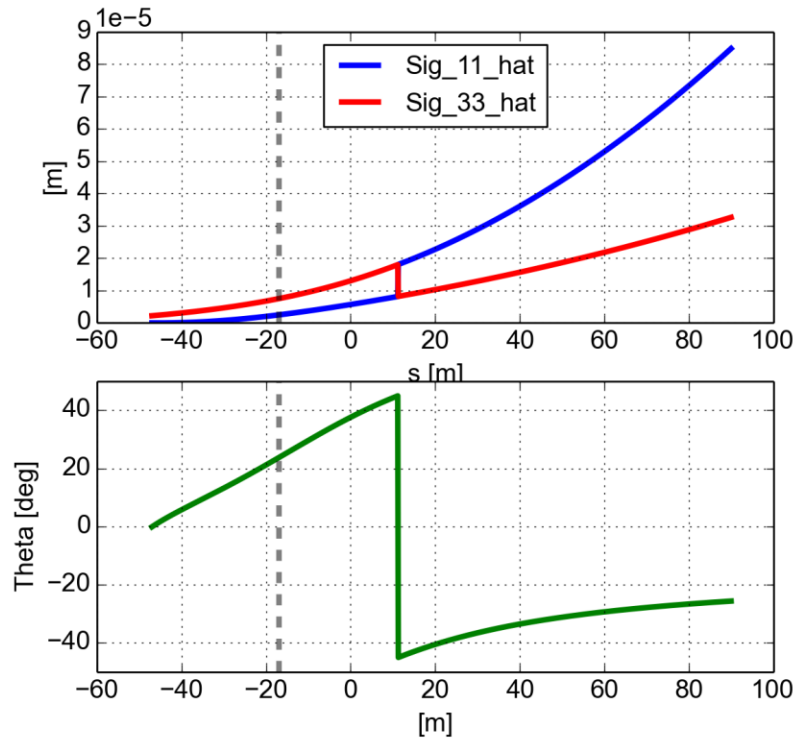


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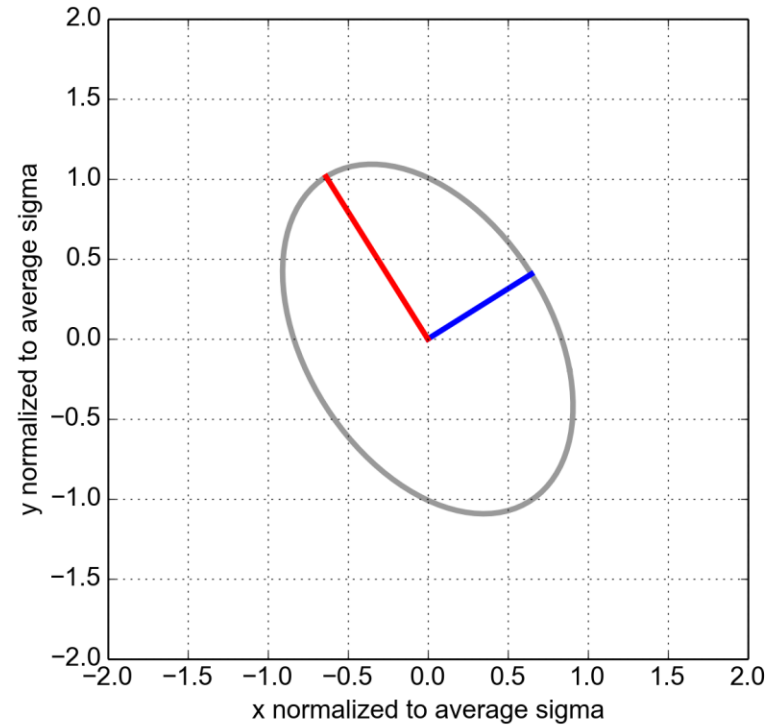
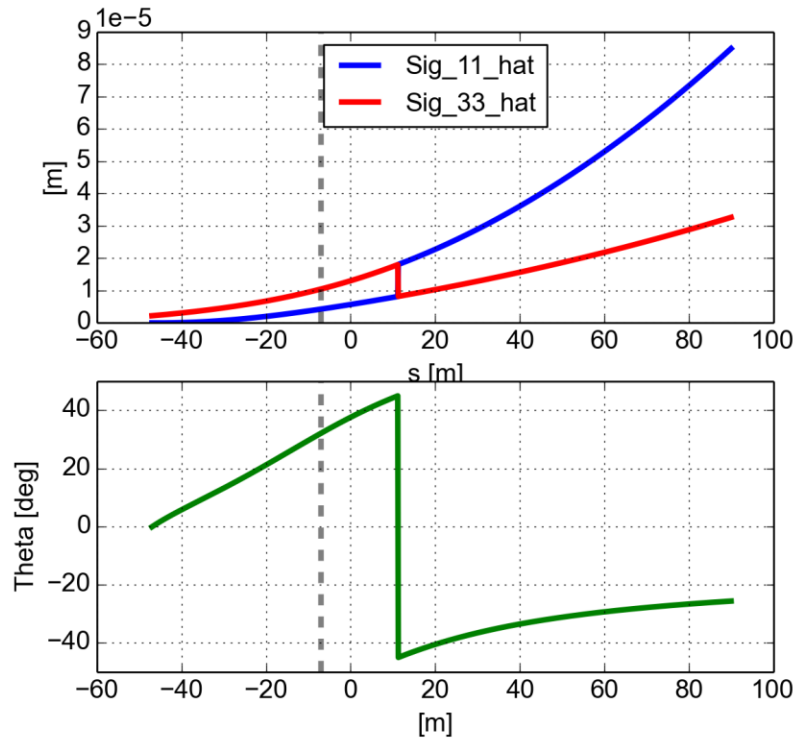


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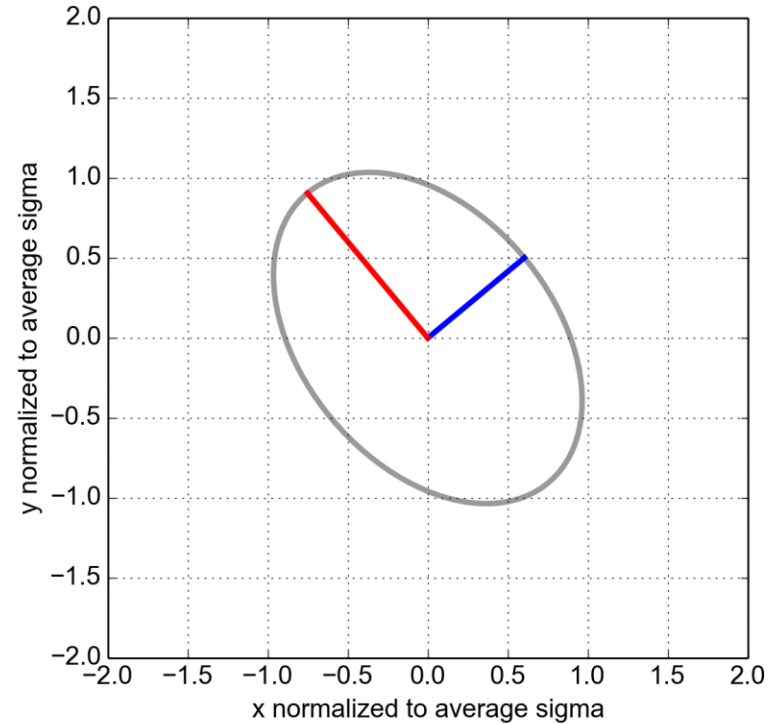
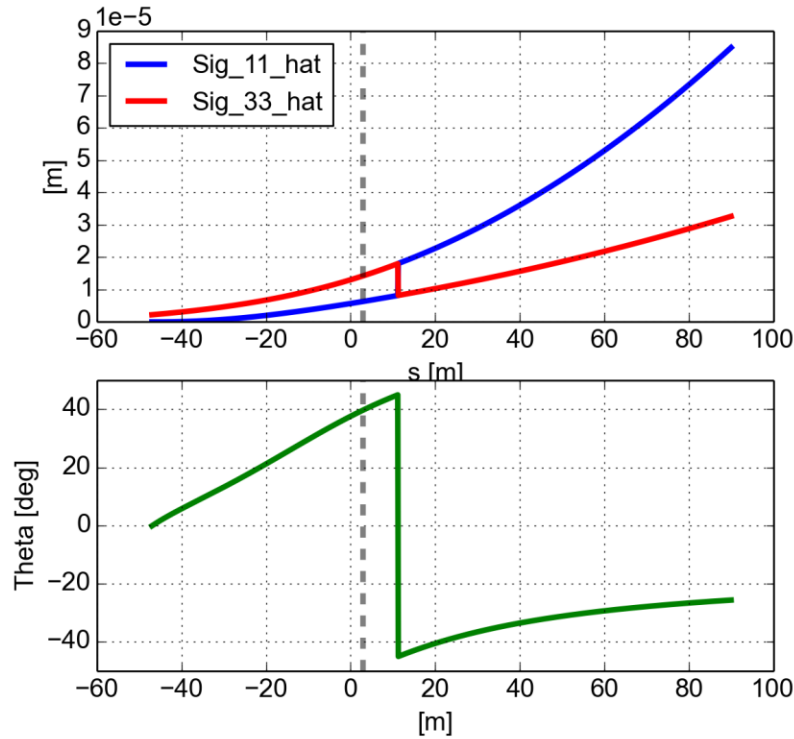


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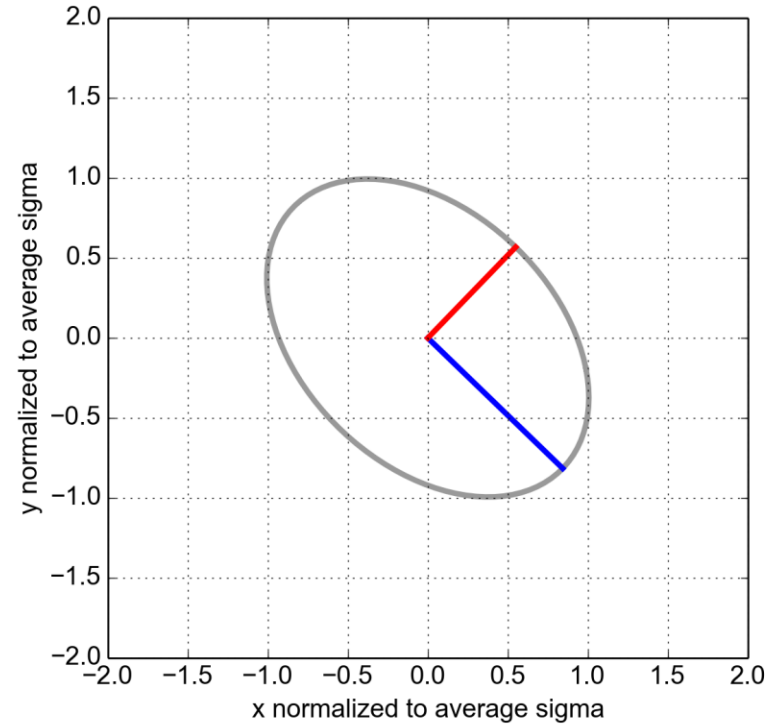
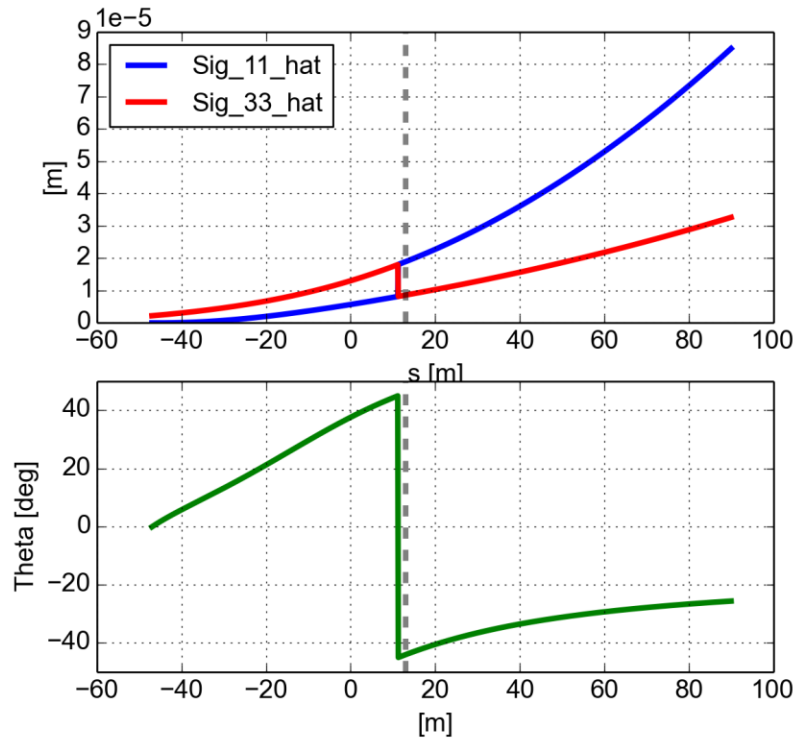


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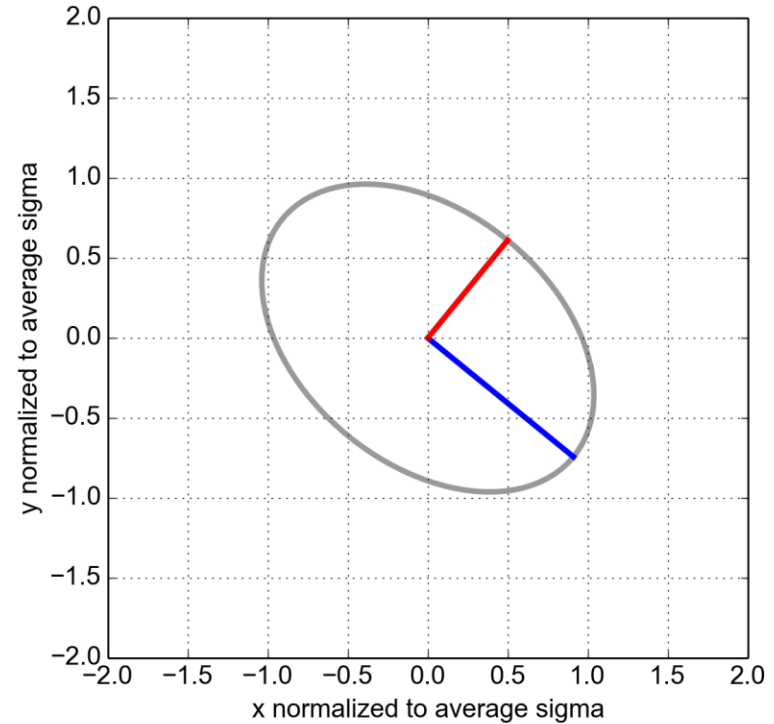
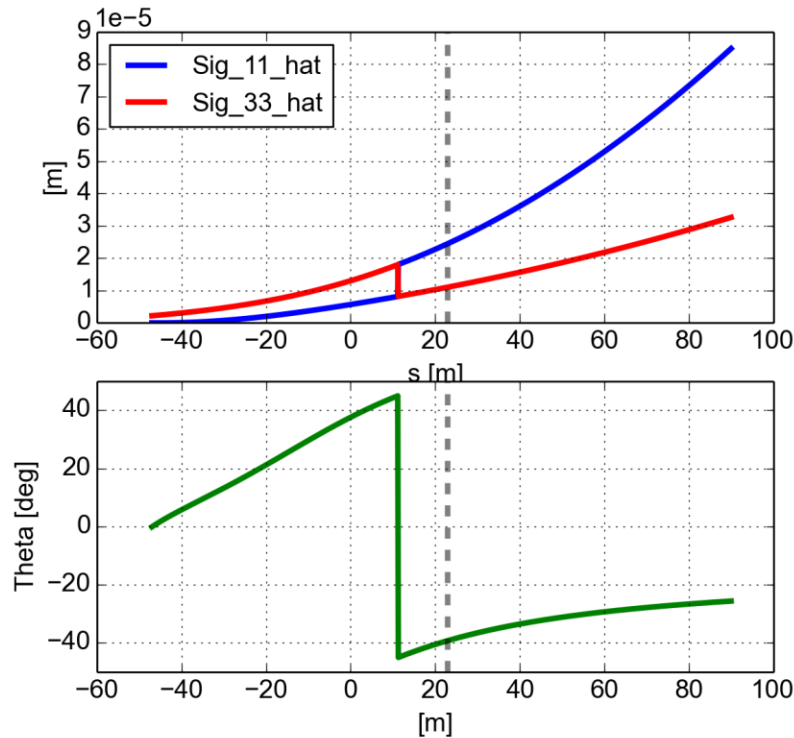


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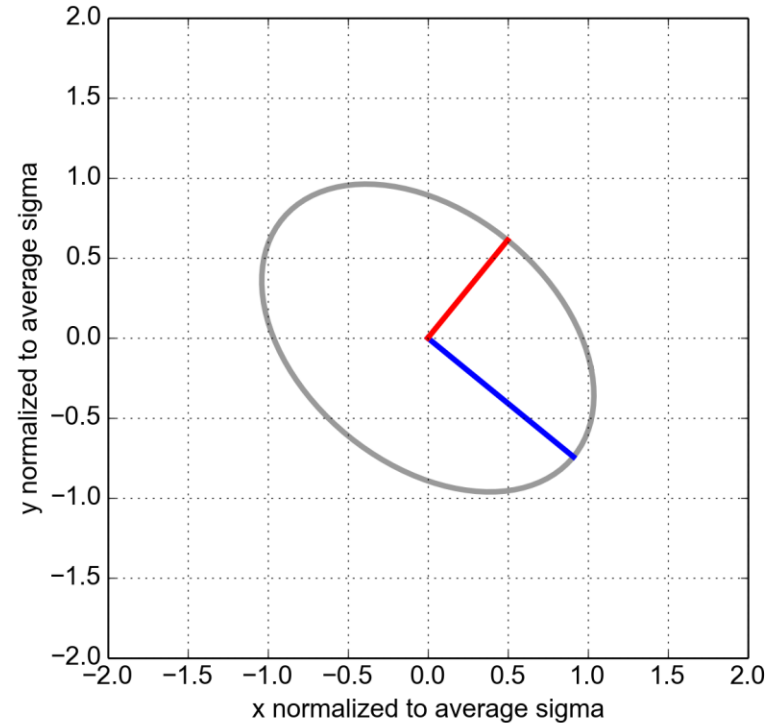
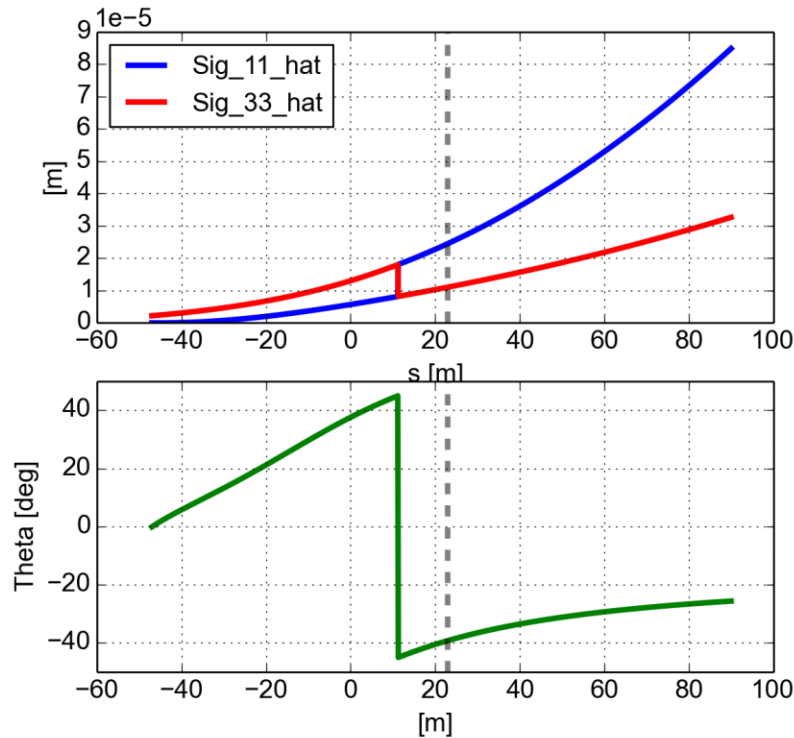


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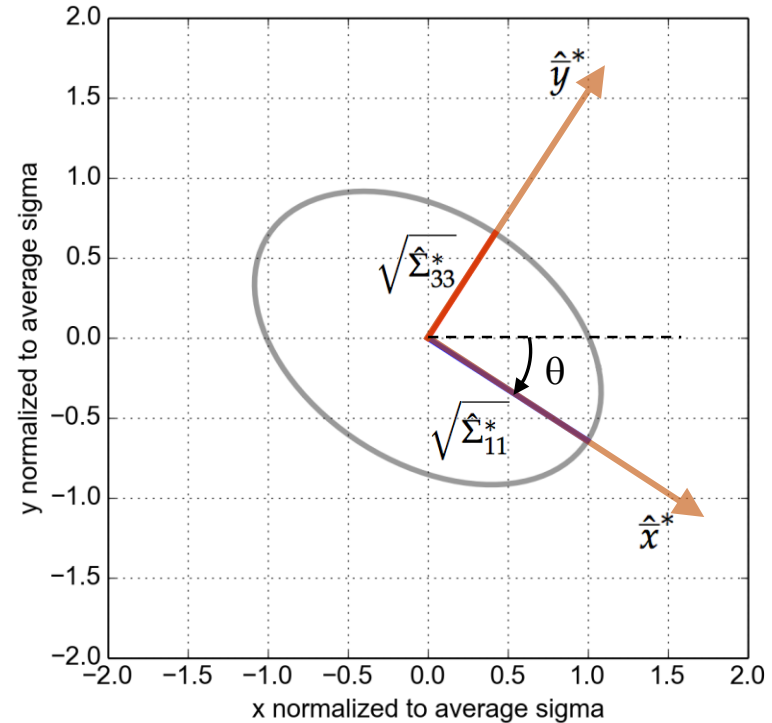
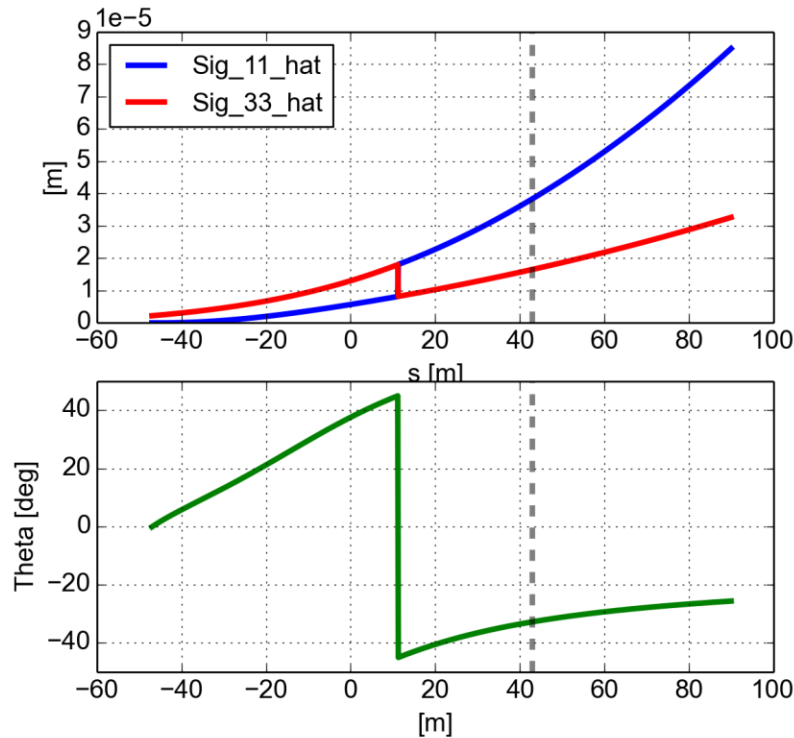


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→ The **coupling angle** and the **beam sizes** in the decoupled frame can be obtained by **diagonalization** of the Σ -matrix

→ Coupling angle depends on the s-coordinate



Worked on simplifying the notation in this part:

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$

$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

Semi-axes in the decoupled frame:

$$\hat{\Sigma}_{11}^* = \frac{1}{2} \left(W + \text{sgn}(R) \sqrt{T} \right)$$

$$\hat{\Sigma}_{33}^* = \frac{1}{2} \left(W - \text{sgn}(R) \sqrt{T} \right)$$

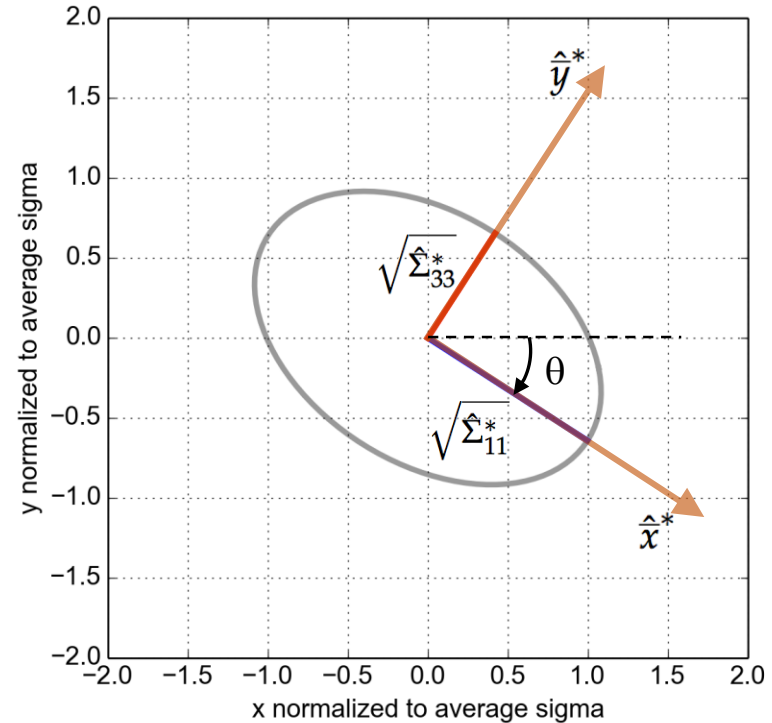
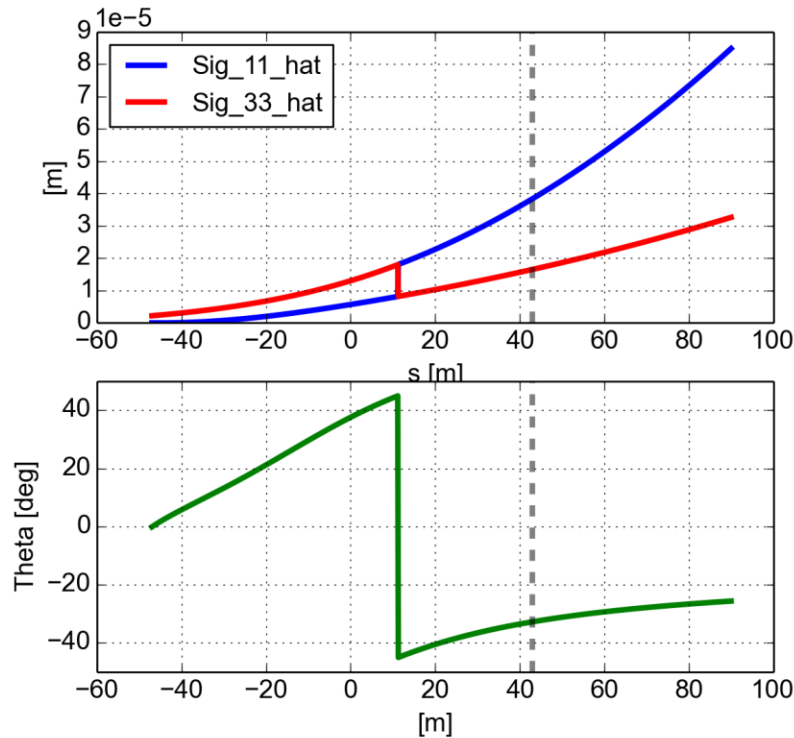


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$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

$$\cos 2\theta = \text{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \text{sgn}(R) \text{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$



Once the coupling angle and the beam sizes in the decoupled plain are known, we proceed as follows:

1. We calculate the particle coordinates in the **decoupled frame** at the **CP**:

$$\hat{x}^* = \bar{x}^* \cos \theta + \bar{y}^* \sin \theta$$

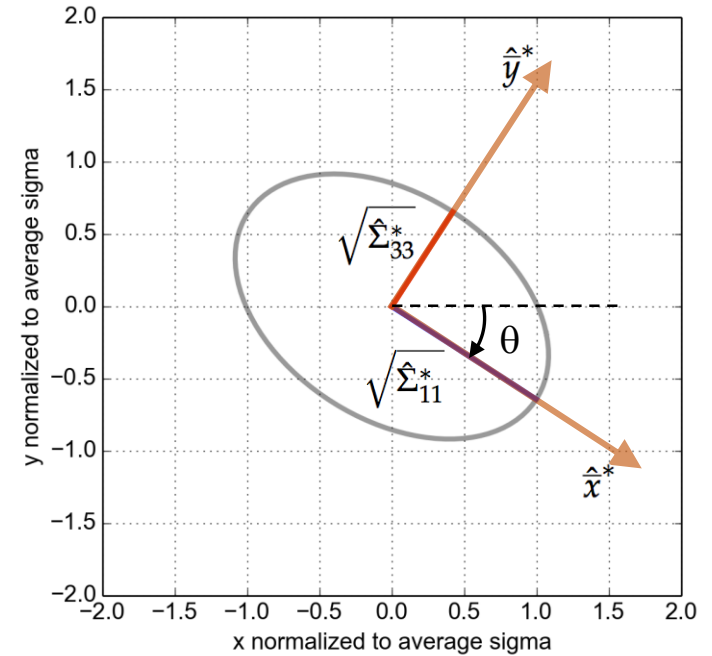
$$\hat{y}^* = -\bar{x}^* \sin \theta + \bar{y}^* \cos \theta$$
2. We calculate the **kick** from the slide in the decoupled reference frame:

$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

where \hat{U}^* is the electric potential

$$K_{sl} = \frac{N_{sl} q_{sl} q_0}{P_0 c}$$



For Gaussian (uncoupled) beams, closed forms exist to evaluate these quantities.

For a bi-Gaussian beam (elliptic) [2]:

Bassetti-Erskine

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \text{Im} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{y}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \text{Re} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$



Linear coupling of the strong beam

Once the coupling angle and the beam sizes in the decoupled plain are known, we proceed as follows:

1. We calculate the particle coordinates in the **decoupled frame** at the **CP**:

$$\hat{x}^* = \bar{x}^* \cos \theta + \bar{y}^* \sin \theta$$

$$\hat{y}^* = -\bar{x}^* \sin \theta + \bar{y}^* \cos \theta$$
2. We calculate the **kick** from the slide in the decoupled reference frame:

$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

where \hat{U}^* is the electric potential

$$K_{sl} = \frac{N_{sl} q_{sl} q_0}{P_0 c}$$

For Gaussian (uncoupled) beams, closed forms exist to evaluate these quantities.

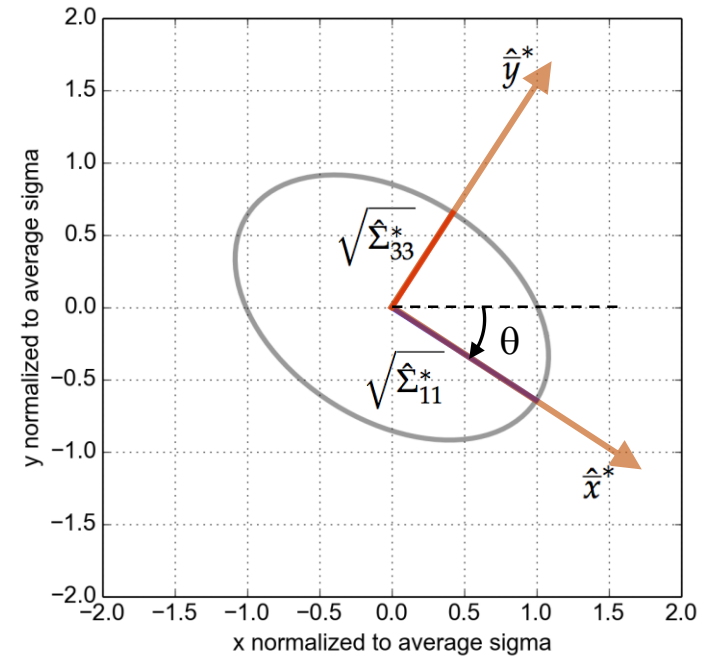
3. We **rotate the kicks** to decoupled reference frame

$$F_x^* = \hat{F}_x^* \cos \theta - \hat{F}_y^* \sin \theta$$

$$F_y^* = \hat{F}_x^* \sin \theta + \hat{F}_y^* \cos \theta$$
4. We **apply the kicks** to the transverse momenta and **drift back** to the **IP** (as explained before)

$$p_{x,new}^* = p_x^* + F_x^* \quad x_{new}^* = x^* - SF_x^*$$

$$p_{y,new}^* = p_y^* + F_y^* \quad y_{new}^* = y^* - SF_y^*$$





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- **Implementation**
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- **Handling the denominators**

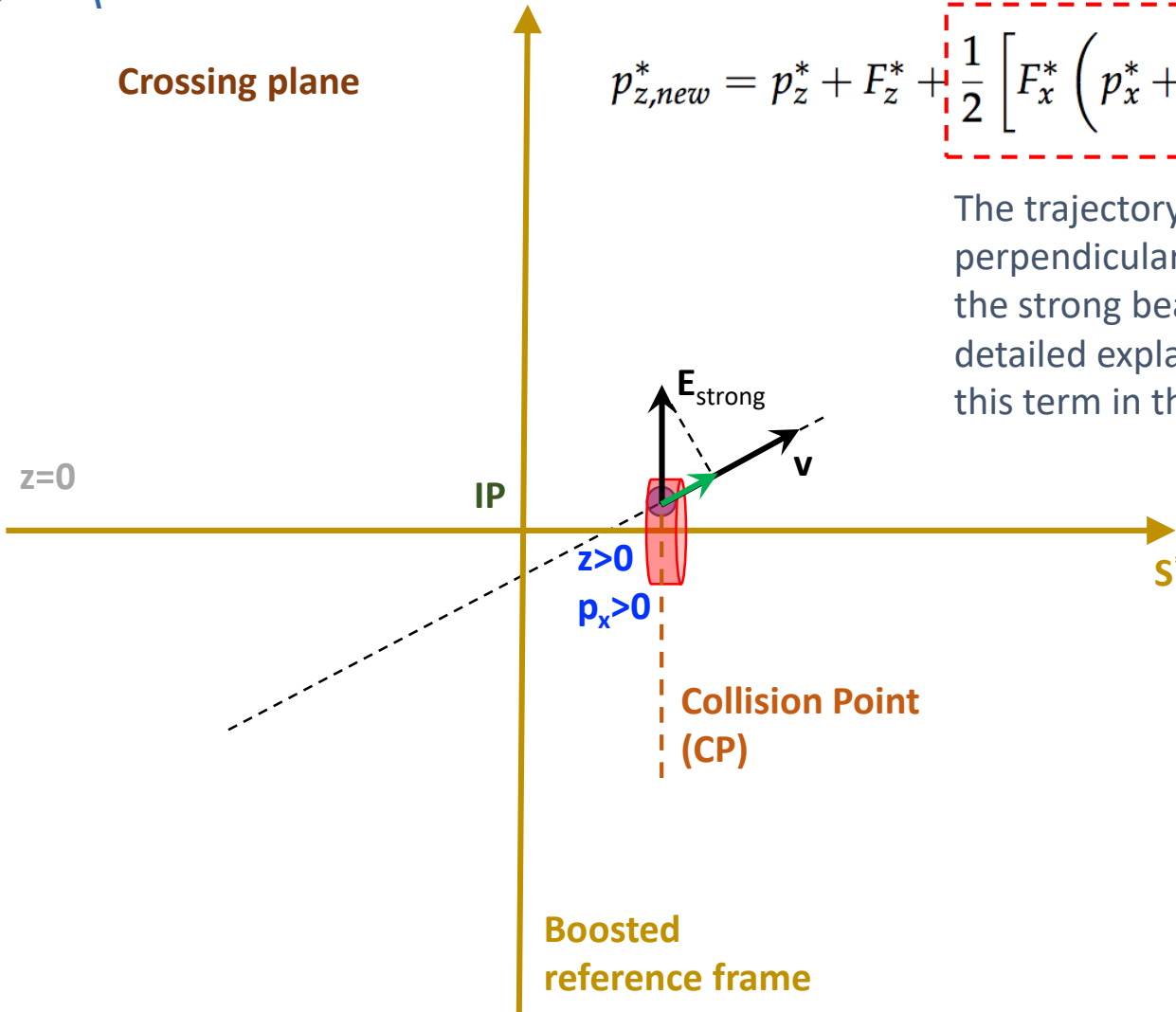


Energy change: effect of the angle

The longitudinal kick has **two components**:

$$p_{z,new}^* = p_z^* + F_z^* + \frac{1}{2} \left[F_x^* \left(p_x^* + \frac{1}{2} F_x^* \right) + F_y^* \left(p_y^* + \frac{1}{2} F_y^* \right) \right]$$

The trajectory is, in general, not perpendicular to the transverse fields of the strong beam (see Hirata [1] for detailed explanation) → this introduces this term in the longitudinal kick





Energy change: grad-phi effect

The longitudinal kick has **two components**:

$$p_{z,new}^* = p_z^* + \boxed{F_z^*} + \frac{1}{2} \left[F_x^* \left(p_x^* + \frac{1}{2} F_x^* \right) + F_y^* \left(p_y^* + \frac{1}{2} F_y^* \right) \right]$$

Another component of the longitudinal kick arises from the fact that the transverse **shape of the strong beam is changing along z** (hour-glass effect, “rotating” coupling angle)

- The electric potential depends on z
- The gradient of the electric potential (i.e. the electric field) has a z component
- There is a z-kick, i.e. again a change in the particle energy

We need to evaluate the **derivative w.r.t. z** (or σ , or small-s) **of the electric potential**

As we have written down most of the involved quantities as a function of the coordinate of the CP (capital-S) we just notice that:

$$S = \frac{\sigma^* - \sigma_{sl}^*}{2} \quad \longrightarrow \quad \frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial S} \quad \longrightarrow \quad F_z^* = \frac{1}{2} \frac{\partial}{\partial S} \left[\hat{U}^* \left(\hat{x}^* (\theta(S)), \hat{y}^* (\theta(S)), \hat{\Sigma}_{11}^*(S), \hat{\Sigma}_{33}^*(S) \right) \right]$$

(in sixtrack jargon
z is called σ)



$$F_z^* = \frac{1}{2} \frac{\partial}{\partial S} \left[\hat{U}^* \left(\hat{x}^* (\theta(S)), \hat{y}^* (\theta(S)), \hat{\Sigma}_{11}^*(S), \hat{\Sigma}_{33}^*(S) \right) \right]$$

Derivative rule for nested functions:

$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} \left[\hat{x}^* (\theta(S)) \right] + \hat{F}_y^* \frac{\partial}{\partial S} \left[\hat{y}^* (\theta(S)) \right] + \hat{G}_x^* \frac{\partial}{\partial S} \left[\hat{\Sigma}_{11}^*(S) \right] + \hat{G}_y^* \frac{\partial}{\partial S} \left[\hat{\Sigma}_{33}^*(S) \right] \right)$$

We need to evaluate these eight terms...

where:

$$\begin{aligned} \hat{F}_x^* &= -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} \left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^* \right) & \hat{G}_x^* &= -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} \left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^* \right) \\ \hat{F}_y^* &= -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} \left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^* \right) & \hat{G}_y^* &= -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} \left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^* \right) \end{aligned}$$



$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} [\hat{x}^*(\theta(S))] + \hat{F}_y^* \frac{\partial}{\partial S} [\hat{y}^*(\theta(S))] + \hat{G}_x^* \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*(S)] + \hat{G}_y^* \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*(S)] \right)$$



$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{G}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{G}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

For these four terms a closed forms exist for transverse Gaussian beams

For a bi-Gaussian beam (elliptic) [2]:

Bassetti-Erskine

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \text{Im} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{y}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \text{Re} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) - 1 \right] \right\}$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}} \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) - 1 \right] \right\}$$

where w is the Faddeeva function.



Energy change: grad-phi effect

$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} [\hat{x}^*(\theta(S))] + \hat{F}_y^* \frac{\partial}{\partial S} [\hat{y}^*(\theta(S))] + \hat{G}_x^* \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*(S)] + \hat{G}_y^* \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*(S)] \right)$$



$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{G}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{G}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

For these four terms a closed forms exist for transverse Gaussian beams

For a round beam, i.e. $\hat{\Sigma}_{11}^* = \hat{\Sigma}_{33}^* = \hat{\Sigma}^*$:

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\pi\epsilon_0} \left[1 - \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right) \right] \frac{x}{(\hat{x}^*)^2 + (\hat{y}^*)^2}$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{y}^*} = \frac{1}{2\pi\epsilon_0} \left[1 - \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right) \right] \frac{y}{(\hat{x}^*)^2 + (\hat{y}^*)^2}$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = \frac{1}{2 [(\hat{x}^*)^2 + (\hat{y}^*)^2]} \left[\hat{y}^* \hat{E}_y^* - \hat{x}^* \hat{E}_x^* + \frac{1}{2\pi\epsilon_0} \frac{(\hat{x}^*)^2}{\hat{\Sigma}^*} \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right) \right]$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2 [(\hat{x}^*)^2 + (\hat{y}^*)^2]} \left[\hat{x}^* \hat{E}_x^* - \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \frac{(\hat{y}^*)^2}{\hat{\Sigma}^*} \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right) \right]$$



Energy change: grad-phi effect

$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} [\hat{x}^* (\theta(S))] + \hat{F}_y^* \frac{\partial}{\partial S} [\hat{y}^* (\theta(S))] + \hat{G}_x^* \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*(S)] + \hat{G}_y^* \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*(S)] \right)$$



$$\hat{x}^* = \bar{x}^* \cos \theta + \bar{y}^* \sin \theta$$

$$\hat{y}^* = -\bar{x}^* \sin \theta + \bar{y}^* \cos \theta$$



$$\frac{\partial}{\partial S} [\hat{x}^* (\theta(S))] = \bar{x}^* \frac{\partial}{\partial S} [\cos \theta] + \bar{y}^* \frac{\partial}{\partial S} [\sin \theta]$$

$$\frac{\partial}{\partial S} [\hat{y}^* (\theta(S))] = -\bar{x}^* \frac{\partial}{\partial S} [\sin \theta] + \bar{y}^* \frac{\partial}{\partial S} [\cos \theta]$$

With some some
goniometric trick

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

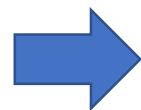
$$\frac{\partial}{\partial S} \sin \theta = -\frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$

We just need
to evaluate

$$\frac{\partial}{\partial S} \cos 2\theta$$

Before we had written:

$$\cos 2\theta = \text{sgn}(R) \frac{R}{\sqrt{T}}$$



$$\frac{\partial}{\partial S} [\cos 2\theta] = \text{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2 (\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

with

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$

$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

where we need to evaluate the
derivatives of R, T and W...



$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} [\hat{x}^*(\theta(S))] + \hat{F}_y^* \frac{\partial}{\partial S} [\hat{y}^*(\theta(S))] + \hat{G}_x^* \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*(S)] + \hat{G}_y^* \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*(S)] \right)$$



Derivatives of R, T and W

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$

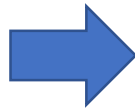
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

$$\Sigma_{11}^* = \Sigma_{11}^{*0} + 2\Sigma_{12}^{*0}S + \Sigma_{22}^{*0}S^2$$

$$\Sigma_{33}^* = \Sigma_{33}^{*0} + 2\Sigma_{34}^{*0}S + \Sigma_{44}^{*0}S^2$$

$$\Sigma_{13}^* = \Sigma_{13}^{*0} + (\Sigma_{14}^{*0} + \Sigma_{23}^{*0})S + \Sigma_{24}^{*0}S^2$$



$$\frac{\partial R}{\partial S} = 2(\Sigma_{12}^0 - \Sigma_{34}^0) + 2S(\Sigma_{22}^0 - \Sigma_{44}^0)$$

$$\frac{\partial W}{\partial S} = 2(\Sigma_{12}^0 + \Sigma_{34}^0) + 2S(\Sigma_{22}^0 + \Sigma_{44}^0)$$

$$\frac{\partial \Sigma_{13}^*}{\partial S} = \Sigma_{14}^0 + \Sigma_{23}^0 + 2\Sigma_{24}^0S$$

$$\frac{\partial T}{\partial S} = 2R \frac{\partial R}{\partial S} + 8\Sigma_{13}^* \frac{\partial \Sigma_{13}^*}{\partial S}$$

With s
gonio

Before we had written: $\cos 2\theta = \text{sgn}(R) \frac{R}{\sqrt{T}}$ $\frac{\partial}{\partial S} [\cos 2\theta] = \text{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$

with

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$

$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

where we need to evaluate the derivatives of R, T and W...



$$F_z^* = \frac{1}{2} \left(\hat{F}_x^* \frac{\partial}{\partial S} [\hat{x}^* (\theta(S))] + \hat{F}_y^* \frac{\partial}{\partial S} [\hat{y}^* (\theta(S))] + \hat{G}_x^* \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*(S)] + \hat{G}_y^* \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*(S)] \right)$$

$$\begin{aligned} \hat{\Sigma}_{11}^* &= \frac{1}{2} (W + \text{sgn}(R)\sqrt{T}) \\ \hat{\Sigma}_{33}^* &= \frac{1}{2} (W - \text{sgn}(R)\sqrt{T}) \end{aligned} \quad \rightarrow \quad \begin{aligned} \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} + \text{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \\ \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} - \text{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \end{aligned}$$

Again what we need to know are the derivatives of R, T and W, which were already shown in the previous slides

Derivatives of R, T and W

$$\begin{aligned} R(S) &= \Sigma_{11}^* - \Sigma_{33}^* \\ W(S) &= \Sigma_{11}^* + \Sigma_{33}^* \\ T(S) &= R^2 + 4\Sigma_{13}^{*2} \end{aligned} \quad \rightarrow \quad \begin{aligned} \frac{\partial R}{\partial S} &= 2 \left(\Sigma_{12}^0 - \Sigma_{34}^0 \right) + 2S \left(\Sigma_{22}^0 - \Sigma_{44}^0 \right) \\ \frac{\partial W}{\partial S} &= 2 \left(\Sigma_{12}^0 + \Sigma_{34}^0 \right) + 2S \left(\Sigma_{22}^0 + \Sigma_{44}^0 \right) \\ \frac{\partial \Sigma_{13}^*}{\partial S} &= \Sigma_{14}^0 + \Sigma_{23}^0 + 2\Sigma_{24}^0 S \\ \frac{\partial T}{\partial S} &= 2R \frac{\partial R}{\partial S} + 8\Sigma_{13}^* \frac{\partial \Sigma_{13}^*}{\partial S} \end{aligned}$$



Handling the denominators

We have all the pieces, but on the way **we introduced some denominators** which can become zero! → we will deal with it later...

$$\begin{aligned}
 R(S) &= \Sigma_{11}^* - \Sigma_{33}^* \\
 W(S) &= \Sigma_{11}^* + \Sigma_{33}^* \\
 T(S) &= R^2 + 4\Sigma_{13}^{*2}
 \end{aligned}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}_{11}^* = \frac{1}{2} (W + \operatorname{sgn}(R)\sqrt{T})$$

$$\hat{\Sigma}_{33}^* = \frac{1}{2} (W - \operatorname{sgn}(R)\sqrt{T})$$

$$\begin{aligned}
 \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} + \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \\
 \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} - \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right)
 \end{aligned}$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

$$\frac{\partial}{\partial S} \sin \theta = -\frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$



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Initialization stage:

- Prepare **coefficients** for **Lorentz boost**
- **Slice** strong bunch
 - Compute slice charges and centroid coordinates
- **Boost strong beam** slices
 - Boost centroid coordinates
 - Boost Σ -matrix
- Store all information in a **data block**

Tracking routine:

- **Boost** coordinates of the **weak beam particle**
- Compute S coordinate of the **collision point** (CP)
- **Transport strong beam** optics from the IP to the CP:
 - Transport sigma matrix to the CP
 - Compute coupling angle and beam sizes in the decoupled plane
 - Compute auxiliary quantities for the calculation of the longitudinal kick
- Compute **transverse kicks**
 - Transform coordinates of the weak beam particles to the un-coupled frame
 - Compute transverse forces in the un-coupled frame
 - Transform transverse kicks to the coupled frame
 - Apply transverse kicks in the coupled frame (change p_x, p_y)
 - Transport transverse kick from the CP to the IP and change particle positions (x,y) accordingly
- Compute and apply the **longitudinal kick**
- **Anti-boost** coordinates of the weak beam particles



Very hard to read and to debug, it can be kept alive... but definitely not ideal

```
...
  if (ibbc1.eq.1) then
    dum(8)=two*( (bcu (ibb,4) -bcu (ibb,9) ) +
&(bcu (ibb,6) -bcu (ibb,10) ) *sp
    dum(9)=(bcu (ibb,5) +bcu (ibb,7) ) + (two*bcu (ibb,8) ) *sp
    dum(10)=((dum(4) *dum(8) + (four*dum(3) ) *dum(9) ) /
&dum(5) ) /dum(5) ) /dum(5)
    dum(11)=sfac*(dum(8) /dum(5) -dum(4) *dum(10) )
    dum(12)=(bcu (ibb,4) +bcu (ibb,9) ) + (bcu (ibb,6) +bcu (ibb,10) ) *sp
    dum(13)=(sfac*( (dum(4) *dum(8) ) *half+ (two*dum(3) ) *dum(9) ) ) /dum(5)
    if (abs (costh) .gt. pieni) then
      costhp=(dum(11) /four) /costh
    else
      costhp=zero
    endif
    if (abs (sinth) .gt. pieni) then
      sinthp=( (-1d0*dum(11) ) /four) /sinth
    else
      sinthp=zero
    endif
    track(6,i)=track(6,i) -
&(((bbfx*(costhp*sepx0+sinthp*sepy0) +
&bbfy*(costhp*sepy0-sinthp*sepx0) ) +
&bbgx*(dum(12)+dum(13) ) +bbgy*(dum(12)-dum(13) ) ) /
&cphi) *half
    bbf0=bbfx
    bbfx=bbf0*costh-bbfy*sinth
    bbfy=bbf0*sinth+bbfy*costh
  else
    track(6,i)=track(6,i) -
&(bbgx*(bcu (ibb,4) +bcu (ibb,6) *sp) +
&bbgy*(bcu (ibb,9) +bcu (ibb,10) *sp) ) /cphi
  endif
  track(6,i)=track(6,i) - (bbfx*(track(2,i) -bbfx*half) +
&bbfy*(track(4,i) -bbfy*half) ) *half
  track(1,i)=track(1,i) +s*bbfx
  track(2,i)=track(2,i) -bbfx
  track(3,i)=track(3,i) +s*bbfy
  track(4,i)=track(4,i) -bbfy
```

- Started from **previous work** done by J. Barranco
 - Identified and described the **interface of the main functional blocks**
 - Built tables with the descriptions of the cumbersome **notation** used in the code

TWiki > LHCATHome Web > SixTrack > SixTrackBeamBeam (2017-03-21, Giovanniladarola) Edit Attach PDF

Information on Beam Beam

Overview of what is left to do in this section:

- Explicit description of how the slicing is done in subroutine `steld`
- Explain what `sbcs` is and how it is computed/obtained
- Describe the Synchro-Beam Mapping is performed
- Additional variables needs to be explained (see argument lists for each subroutine)

How a Beam-beam element is defined in fort.2 and 3.

The beam beam element are directly translated from MADX to SixTrack input format. The parameters that define a BB in the **fort.2** lattice are,

Format `_name` type

`name` - May contain up to sixteen characters

`type` - 20

The beam-beam elements definition is now done fully in the BEAM block of fort.3 for both 4D and 6D lens.

4D lens (1 line per element)

`name lbsix Σ_{xx} Σ_{yy} $h\text{-sep}$ $v\text{-sep}$ strength-ratio`

6D lens (3 lines per element)

`name lbsix $x\text{plane}$ $h\text{-sep}$ $v\text{-sep}$`

`Σ_{xx} Σ_{xxpp} Σ_{xpxp} Σ_{yy} Σ_{yyp}`

`Σ_{yypp} Σ_{xy} Σ_{xyp} Σ_{xpy} Σ_{yyp} strength-ratio`

`name` - Name of the beam-beam element.

- Moved to the understanding and testing of the source code...



Library implementation

It quickly became evident that the only viable way of checking the SixTrack code was to build an **independent implementation to compare against**. Done keeping in mind:

- **Readability, modularity**, possibility to **interface with other codes** (PyHEADTAIL, SixTrackLib)
- Compatibility with **GPU**

```
// Boost coordinates of the weak beam
BB6D_boost(&(bb6ddata->parboost), &x_star, &px_star, &y_star, &py_star,
           &sigma_star, &delta_star);

// Synchro beam
for (i_slice=0; i_slice<N_slices; i_slice++)
{
    double sigma_slice_star = sigma_slices_star[i_slice];
    double x_slice_star = x_slices_star[i_slice];
    double y_slice_star = y_slices_star[i_slice];

    //Compute force scaling factor
    double Ksl = N_part_per_slice[i_slice]*bb6ddata->q_part*q0/(p0*C_LIGHT);

    //Identify the Collision Point (CP)
    double S = 0.5*(sigma_star - sigma_slice_star);

    // Propagate sigma matrix
    double Sig_11_hat_star, Sig_33_hat_star, costheta, sintheta;
    double dS_Sig_11_hat_star, dS_Sig_33_hat_star, dS_costheta, dS_sintheta;

    // Get strong beam shape at the CP
    BB6D_propagate_Sigma_matrix(&(bb6ddata->Sigmas_0_star),
                               S, bb6ddata->threshold_singular, 1,
                               &Sig_11_hat_star, &Sig_33_hat_star,
                               &costheta, &sintheta,
                               &dS_Sig_11_hat_star, &dS_Sig_33_hat_star,
                               &dS_costheta, &dS_sintheta);

    // Evaluate transverse coordinates of the weak beam w.r.t. the strong beam centroid
    double x_bar_star = x_star + px_star*S - x_slice_star;
    double y_bar_star = y_star + py_star*S - y_slice_star;

    // Move to the uncoupled reference frame
    double x_bar_hat_star = x_bar_star*costheta + y_bar_star*sintheta;
    double y_bar_hat_star = -x_bar_star*sintheta + y_bar_star*costheta;

    // Compute derivatives of the transformation
    double dS_x_bar_hat_star = x_bar_star*dS_costheta + y_bar_star*dS_sintheta;
    double dS_y_bar_hat_star = -x_bar_star*dS_sintheta + y_bar_star*dS_costheta;

    // Compute derivatives of the transformation
    double dS_x_bar_hat_star = x_bar_star*dS_costheta + y_bar_star*dS_sintheta;
    double dS_y_bar_hat_star = -x_bar_star*dS_sintheta + y_bar_star*dS_costheta;

    // Get transverse fields
    double Ex, Ey, Gx, Gy;
    get_Ex_Ey_Gx_Gy_gauss(x_bar_hat_star, y_bar_hat_star,
                          sqrt(Sig_11_hat_star), sqrt(Sig_33_hat_star), bb6ddata->min_sigma_diff,
                          &Ex, &Ey, &Gx, &Gy);

    // Compute kicks
    double Fx_hat_star = Ksl*Ex;
    double Fy_hat_star = Ksl*Ey;
    double Gx_hat_star = Ksl*Gx;
    double Gy_hat_star = Ksl*Gy;

    // Move kisks to coupled reference frame
    double Fx_star = Fx_hat_star*costheta - Fy_hat_star*sintheta;
    double Fy_star = Fx_hat_star*sintheta + Fy_hat_star*costheta;

    // Compute longitudinal kick
    double Fz_star = 0.5*(Fx_hat_star*dS_x_bar_hat_star + Fy_hat_star*dS_y_bar_hat_star +
                          Gx_hat_star*dS_Sig_11_hat_star + Gy_hat_star*dS_Sig_33_hat_star);

    // Apply the kicks (Hirata's synchro-beam)
    delta_star = delta_star + Fz_star + 0.5*(
        Fx_star*(px_star + 0.5*Fx_star) +
        Fy_star*(py_star + 0.5*Fy_star));
    x_star = x_star - S*Fx_star;
    px_star = px_star + Fx_star;
    y_star = y_star - S*Fy_star;
    py_star = py_star + Fy_star;

}

// Inverse boost on the coordinates of the weak beam
BB6D_inv_boost(&(bb6ddata->parboost), &x_star, &px_star, &y_star, &py_star,
              &sigma_star, &delta_star);
```



- **Introduction**
- **“6D” beam beam treatment**
 - Handling the crossing angles: “the boost”
 - Transverse “generalized kicks”
 - Description of the strong beam (Σ -matrix)
 - Handling linear coupling
 - Longitudinal kick
- **Implementation**
- **Testing:**
 - “Boost” and “Anti-boost”
 - Transverse kicks
 - Other derivatives of the electric potential
 - Σ -matrix propagation with linear coupling
 - Σ -matrix transformation to un-coupled frame
 - Constant charge slicing
 - Complete multi-slice interaction
- **Handling the denominators**



- Very difficult to identify problems by using the full tracking simulations
 - Need to test the single routine **“on the bench”**
- **Procedure** being performed for each functional block
 - Built a **C/python implementation** from the equations in the document
 - Extracted the **corresponding sixtrack source code** and compiled as of a stand-alone python module (f2py)
 - **“Stress test”** performed on the two: consistency checks, comparison against each other



Module	Tests performed	Outcome
Boost/anti-boost	<ul style="list-style-type: none">• Comparison Sixtrack vs C/python routine• Checked that the two cancel each other	<ul style="list-style-type: none">• Bug identified and corrected
Beam-beam forces (with potential derivatives w.r.t. sigmas)	<ul style="list-style-type: none">• Comparison sixtrack vs C/python routine• Force compared against Finite Difference Poisson solver (PyPIC)• Other derivatives compared against numerical integration/derivation	<ul style="list-style-type: none">• All checks passed
Beam shape propagation and coupling treatment	<ul style="list-style-type: none">• Comparison Sixtrack vs C/python routine• Comparison against MAD for a coupled beam line• Crosscheck with numerical derivation	<ul style="list-style-type: none">• Bug identified and corrected• Vanishing denominators not treated correctly → correct treatment developed and implemented in the library, to be ported in SixTrack
Slicing	<ul style="list-style-type: none">• Check against independent implementation	<ul style="list-style-type: none">• Passed but precision is quite poor ($1e-3$)
Computation of the kicks	<ul style="list-style-type: none">• Check against independent implementation	<ul style="list-style-type: none">• All checks passed



- **Introduction**
- **“6D” beam beam treatment**
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 - Constant charge slicing
 - Complete multi-slice interaction
- **Handling the denominators**



- Boost and anti-boost **should cancel each other exactly**
- **“Bench-test” cases:** large crossing angle, test particle very off momentum and large px, py
- **Test passed for the library**
- **Problem identified** in the Sixtrack implementation

Error after boost + anti-boost

Python test routine		SixTrack routine	
x	4.3e-19	x	6.5e-19
px	0.0	px	0.065
y	4.3e-19	y	4.3e-19
py	3.e3-17	py	0.027
sigma	0.0	sigma	0.0
delta	1e-16	delta	2.0e-17



Discrepancy found between in the anti-boost between derived equations and SixTrack source code:

$$p_x = p_x^* \cos \phi + h \cos \alpha \tan \phi \quad (95)$$

$$p_y = p_y^* \cos \phi + h \sin \alpha \tan \phi \quad (96)$$

```
TRACK (2) = (TRACK (2) +CALPHA*SPHI*H1) *CPHI
```

```
TRACK (4) = (TRACK (4) +SALPHA*SPHI*H1) *CPHI
```

The lines should be:

```
TRACK (2) = (TRACK (2) *CPHI+CALPHA*TPHI*H1)
```

```
TRACK (4) = (TRACK (4) *CPHI+SALPHA*TPHI*H1)
```

- Digging a bit we found out that the issue was already present in [Hirata's code](#) from 1996, on which the Sixtrack implementation is based



- **Correction implemented in SixTrack**

Error after boost + anti-boost

Python test routine		SixTrack routine		SixTrack corrected	
x	4.3e-19	x	6.5e-19	x	6.5e-19
px	0.0	px	0.065	px	5.55e-17
y	4.3e-19	y	4.3e-19	y	4.3e-19
py	3.e3-17	py	0.027	py	0.1e-19
sigma	0.0	sigma	0.0	sigma	0.0
delta	1e-16	delta	2.0e-17	delta	2.0e-17



- Problem confirmed by Riccardo simulating a beam-beam interaction with **zero intensity in the strong beam**

Original implementation

Coordinates before interaction

Coordinates after interaction

dump_ip.dat								dump_bb.dat											
#	ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack	#	ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack
1	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03			1	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973250177E-03		
2	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03			2	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973250177E-03		
3	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02			3	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627927274E-02		
4	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02			4	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627927274E-02		
5	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02			5	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058532567E-02		
6	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02			6	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058532567E-02		
7	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02			7	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489140898E-02		
8	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02			8	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489140898E-02		
9	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02			9	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919752266E-02		
10	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02			10	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919752266E-02		
1	2	1	0.00000	1.308501246E-01	8.514045444E-03	-9.961266845E-03	3.153912424E-04			1	2	1	0.00000	1.308501246E-01	8.514045444E-03	-9.961266845E-03	3.153866850E-04		
2	2	1	0.00000	1.308501246E-01	8.514045444E-03	-9.961266845E-03	3.153912424E-04			2	2	1	0.00000	1.308501246E-01	8.514045444E-03	-9.961266845E-03	3.153866850E-04		
3	2	1	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601833146E-03			3	2	1	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601823666E-03		
4	2	1	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601833146E-03			4	2	1	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601823666E-03		
5	2	1	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888831596E-03			5	2	1	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888313646E-03		
6	2	1	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888831596E-03			6	2	1	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888313646E-03		
7	2	1	0.00000	5.084593752E-02	-5.305664373E-02	-2.266176309E-01	7.175036594E-03			7	2	1	0.00000	5.084593752E-02	-5.305664388E-02	-2.266176309E-01	7.174856626E-03		
8	2	1	0.00000	5.084593752E-02	-5.305664373E-02	-2.266176309E-01	7.175036594E-03			8	2	1	0.00000	5.084593752E-02	-5.305664388E-02	-2.266176309E-01	7.174856626E-03		
9	2	1	0.00000	2.417787538E-02	-7.358020677E-02	-2.988386405E-01	9.461798139E-03			9	2	1	0.00000	2.417787538E-02	-7.358020705E-02	-2.988386405E-01	9.461452606E-03		
10	2	1	0.00000	2.417787538E-02	-7.358020677E-02	-2.988386405E-01	9.461798139E-03			10	2	1	0.00000	2.417787538E-02	-7.358020705E-02	-2.988386405E-01	9.461452606E-03		

Corrected implementation

Coordinates before interaction

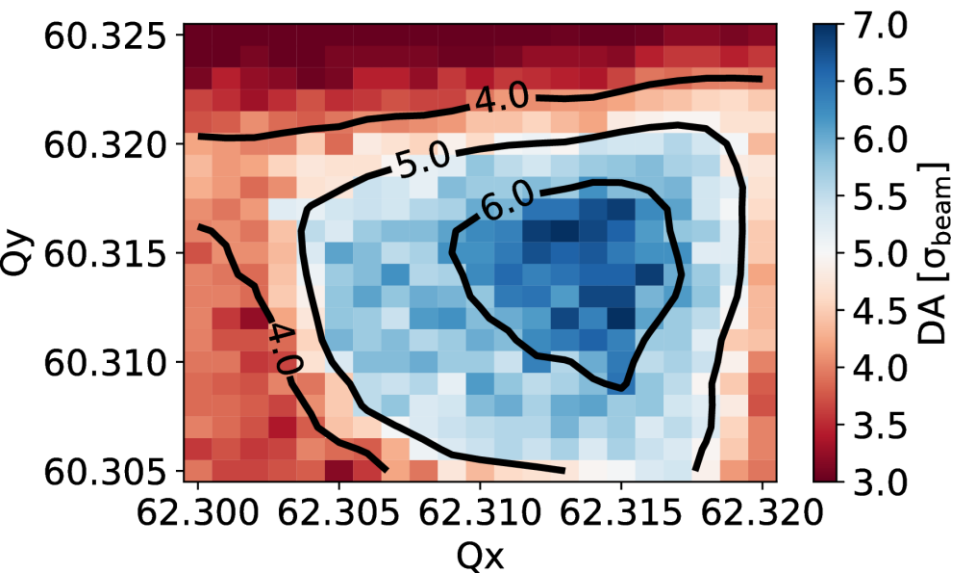
Coordinates after interaction

dump_ip.dat								dump_bb.dat											
ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack	ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack		
1	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03			1	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03		
2	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03			2	1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03		
3	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02			3	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02		
4	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02			4	1	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02		
5	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02			5	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02		
6	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02			6	1	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02		
7	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02			7	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02		
8	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02			8	1	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02		
9	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02			9	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02		
10	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02			10	1	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02		
1	2	1	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04			1	2	1	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04		
2	2	1	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04			2	2	1	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04		
3	2	1	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03			3	2	1	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03		
4	2	1	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03			4	2	1	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03		
5	2	1	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03			5	2	1	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03		
6	2	1	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03			6	2	1	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03		
7	2	1	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03			7	2	1	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03		
8	2	1	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03			8	2	1	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03		
9	2	1	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03			9	2	1	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03		
10	2	1	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03			10	2	1	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03		

- Impact on **realistic simulation study** assessed by Dario
- Tune scans comparison with 2017 ATS optics show no dramatic change, but slightly worse DA

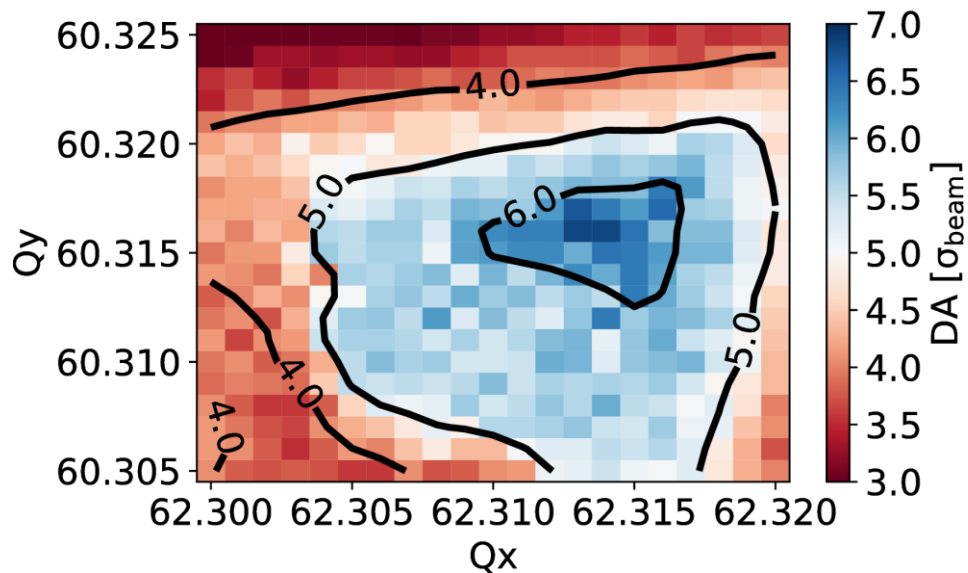
Old version

ATS Optics; $\beta^* = 40$ cm; $Q' = 15$; $I_{MO} = 500$ A;
 $\epsilon = 2.5$ μm ; $I = 1.25 \cdot 10^{11}$ e; $X = 150$ μrad ; Min DA.



Corrected version

ATS Optics; $\beta^* = 40$ cm; $Q' = 15$; $I_{MO} = 500$ A;
 $\epsilon = 2.5$ μm ; $I = 1.25 \cdot 10^{11}$ e; $X = 150$ μrad ; Min DA.





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Transverse kicks for a Gaussian beam

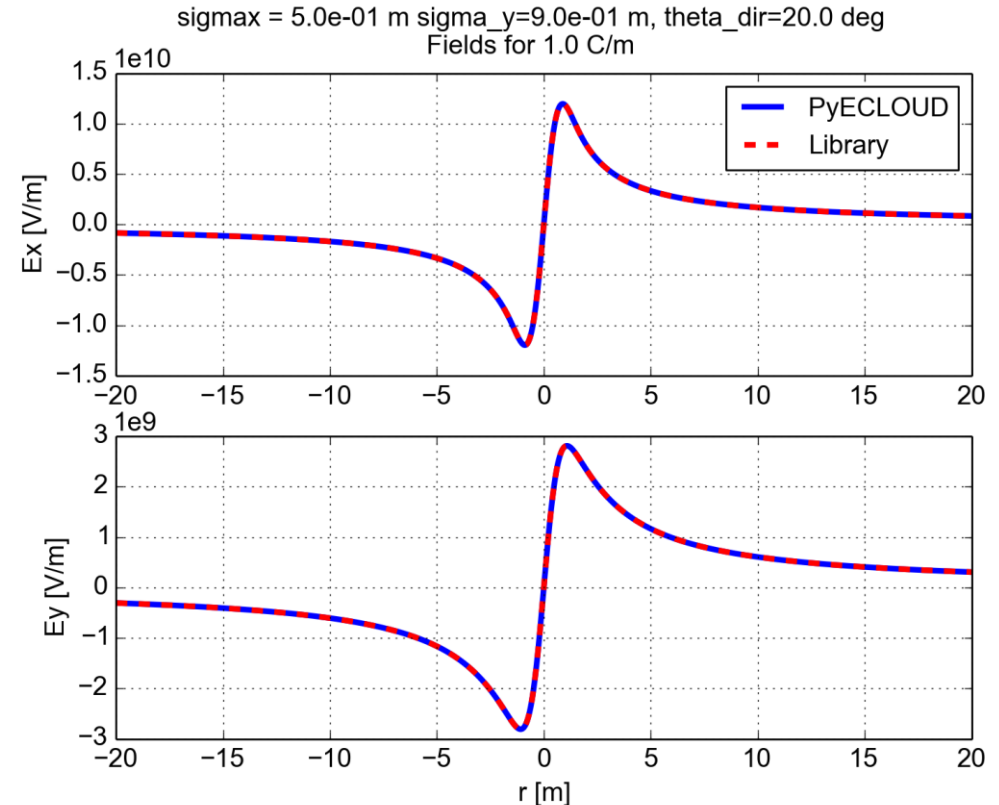
Transverse field for a Gaussian beam (Bassetti-Erskine)

$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \text{Im} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{y}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \text{Re} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2 (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)}} \right) \right]$$

**Library tested against
Poisson solver of PyECLLOUD**

(test repeated for tall, fat and
round beams)





Transverse kicks for a Gaussian beam

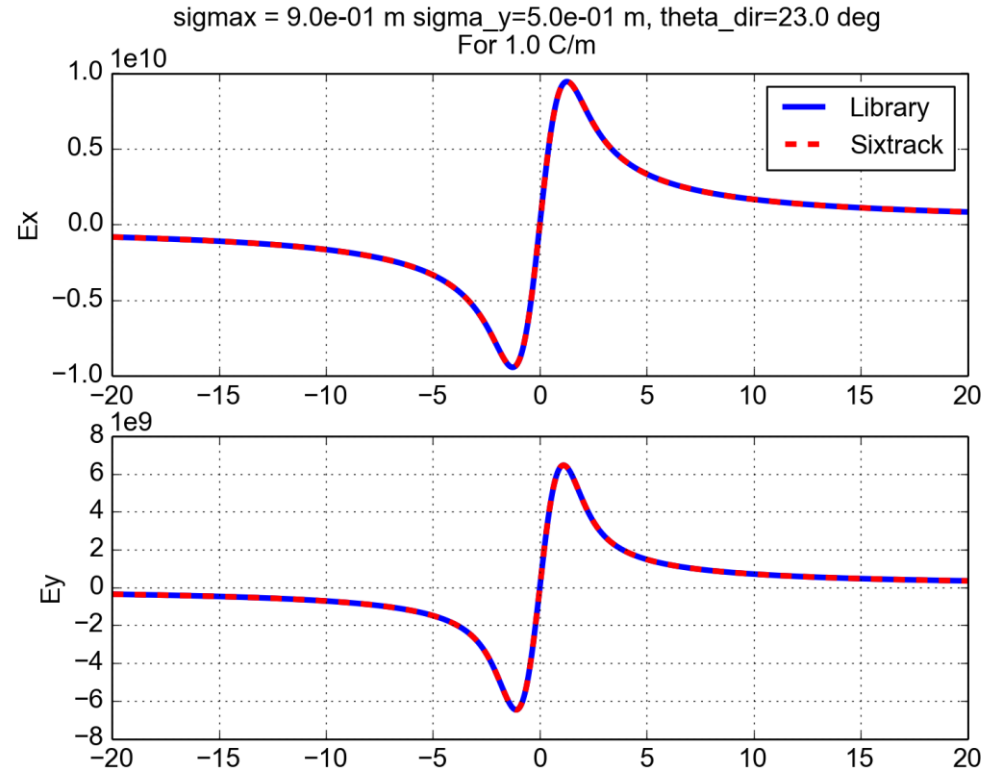
Transverse field for a Gaussian beam (Bassetti-Erskine)

$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \text{Im} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \right) \right]$$

$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{y}^*} = \frac{1}{2\epsilon_0 \sqrt{2\pi} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \text{Re} \left[w \left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \right) - \exp \left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*} \right) w \left(\frac{\hat{x}^* \sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^* \sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2} (\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \right) \right]$$

SixTrack tested against library

(test repeated for tall, fat and round beams)





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Other derivatives of the electric potential

$$\hat{G}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

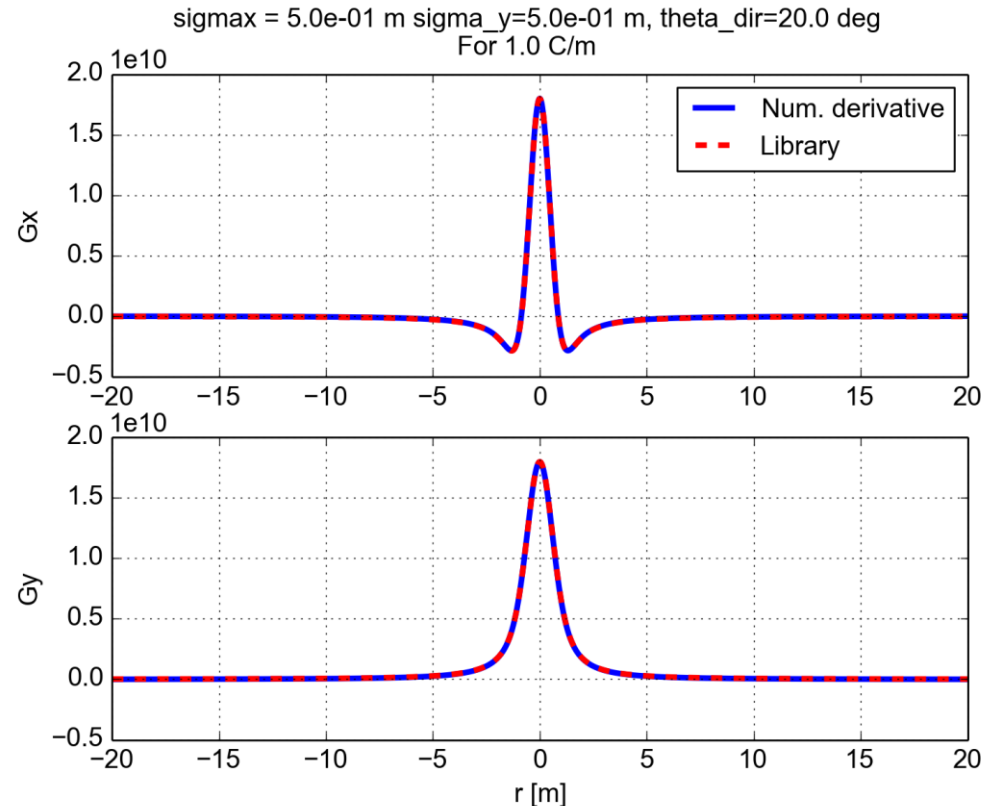
$$\hat{\delta}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1 \right] \right\}$$

$$\hat{G}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*)$$

$$\hat{\delta}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}} \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1 \right] \right\}$$

**Library tested against
numerical derivative**

(test repeated for tall, fat and
round beams)





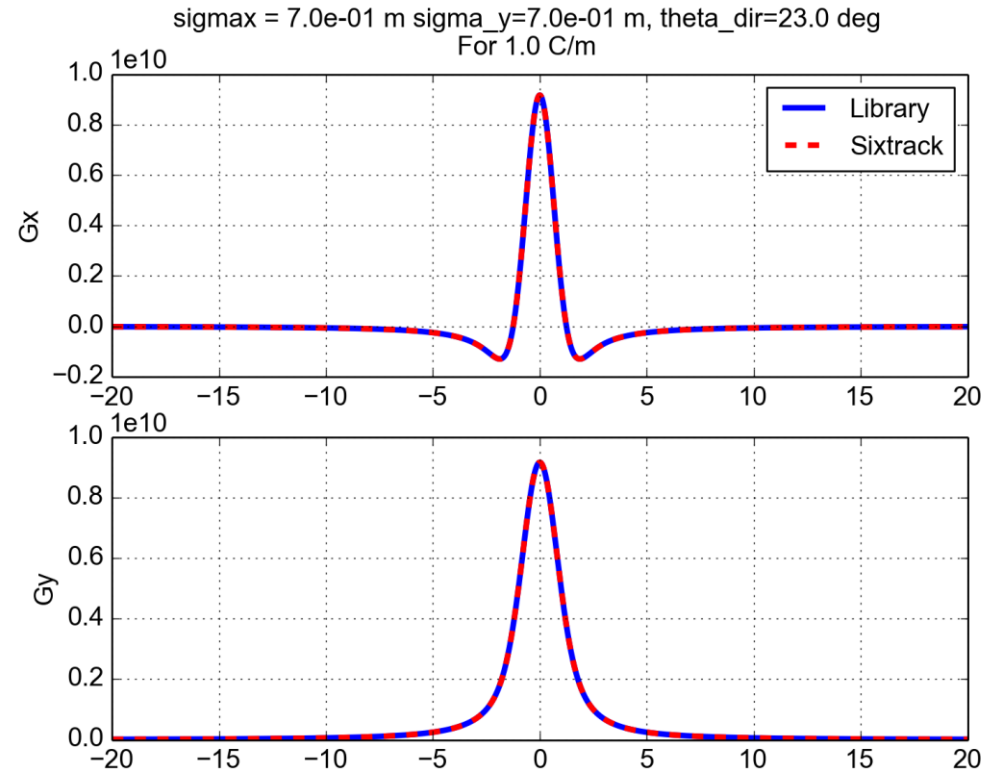
Other derivatives of the electric potential

$$\hat{G}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \delta_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1 \right] \right\}$$

$$\hat{G}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} (\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*) \quad \delta_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*)} \left\{ \hat{x}^* \hat{E}_x^* + \hat{y}^* \hat{E}_y^* + \frac{1}{2\pi\epsilon_0} \left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}} \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1 \right] \right\}$$

SixTrack tested against library

(test repeated for tall, fat and round beams)





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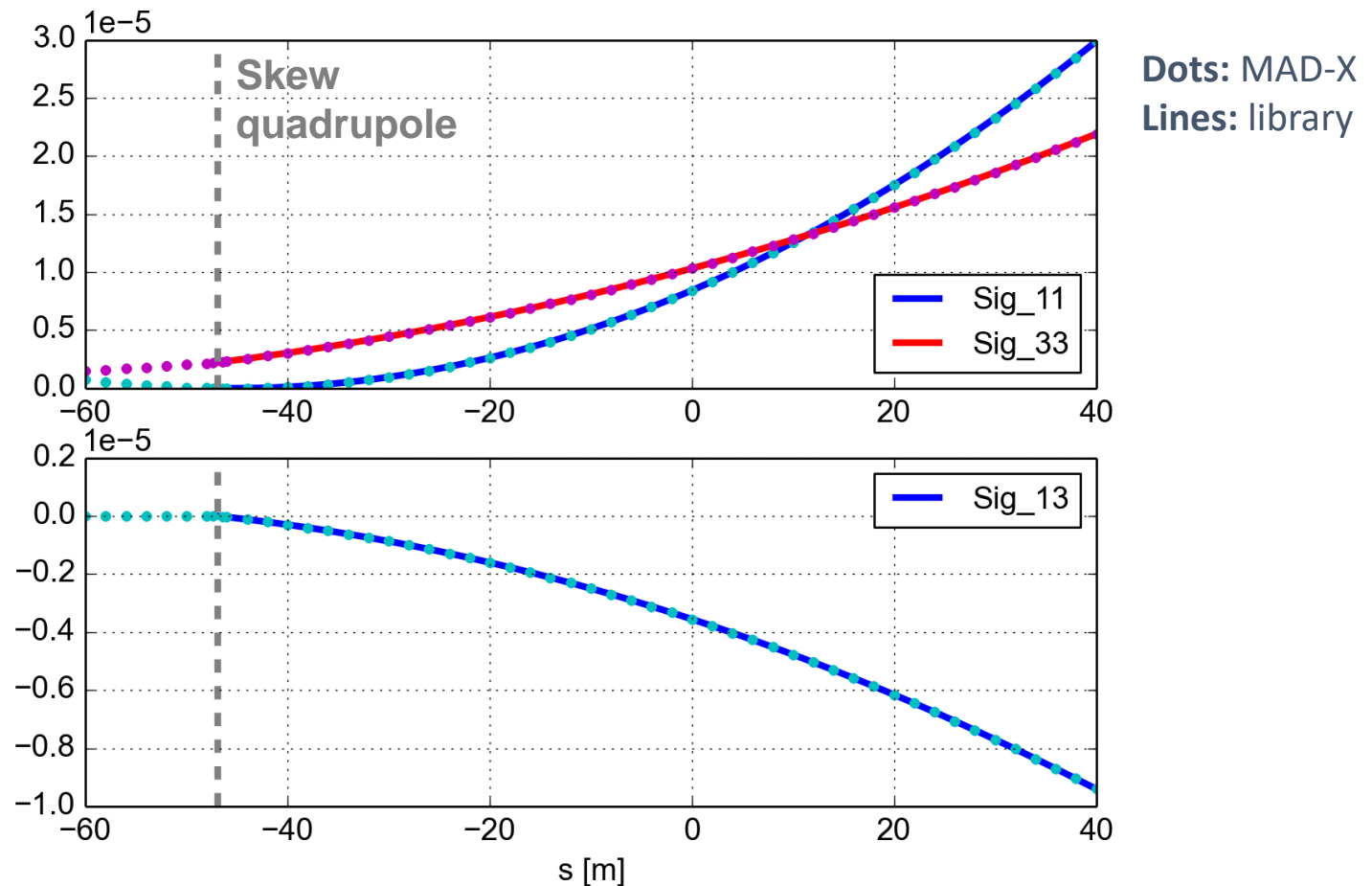


Σ -matrix propagation with linear coupling

Library tested against MAD-X:

- Built a simple line with a strong skew quadrupole
- Entering with a de-coupled beam
- Saves Σ -matrix at regularly spaced markers for comparison against library

Check optics propagation against MAD-X

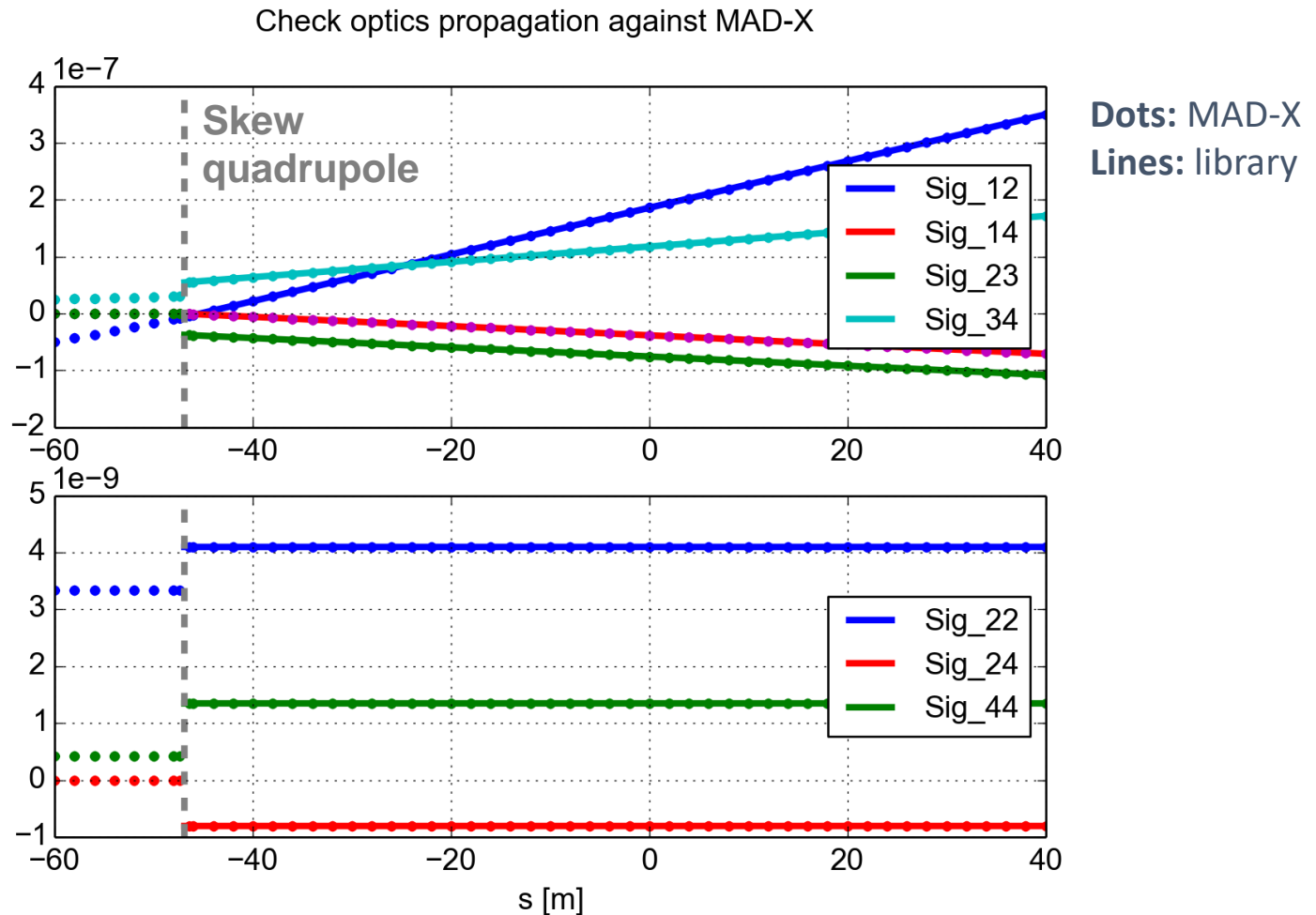




Σ -matrix propagation with linear coupling

Library tested against MAD-X:

- Built a simple line with a strong skew quadrupole
- Entering with a de-coupled beam
- Saves Σ -matrix at regularly spaced markers for comparison against library





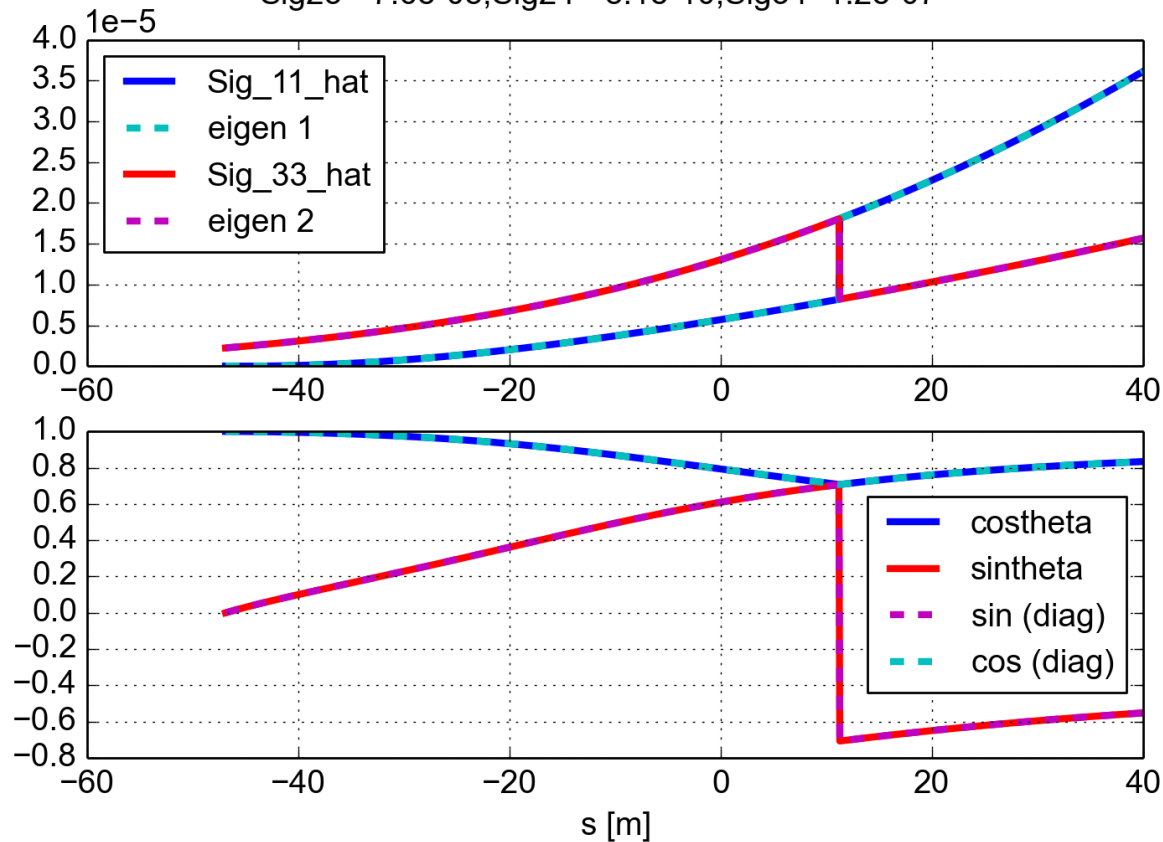
- **Introduction**
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Σ -matrix transformation to un-coupled frame

Library tested against **numerical diagonalization** of the Σ -matrix

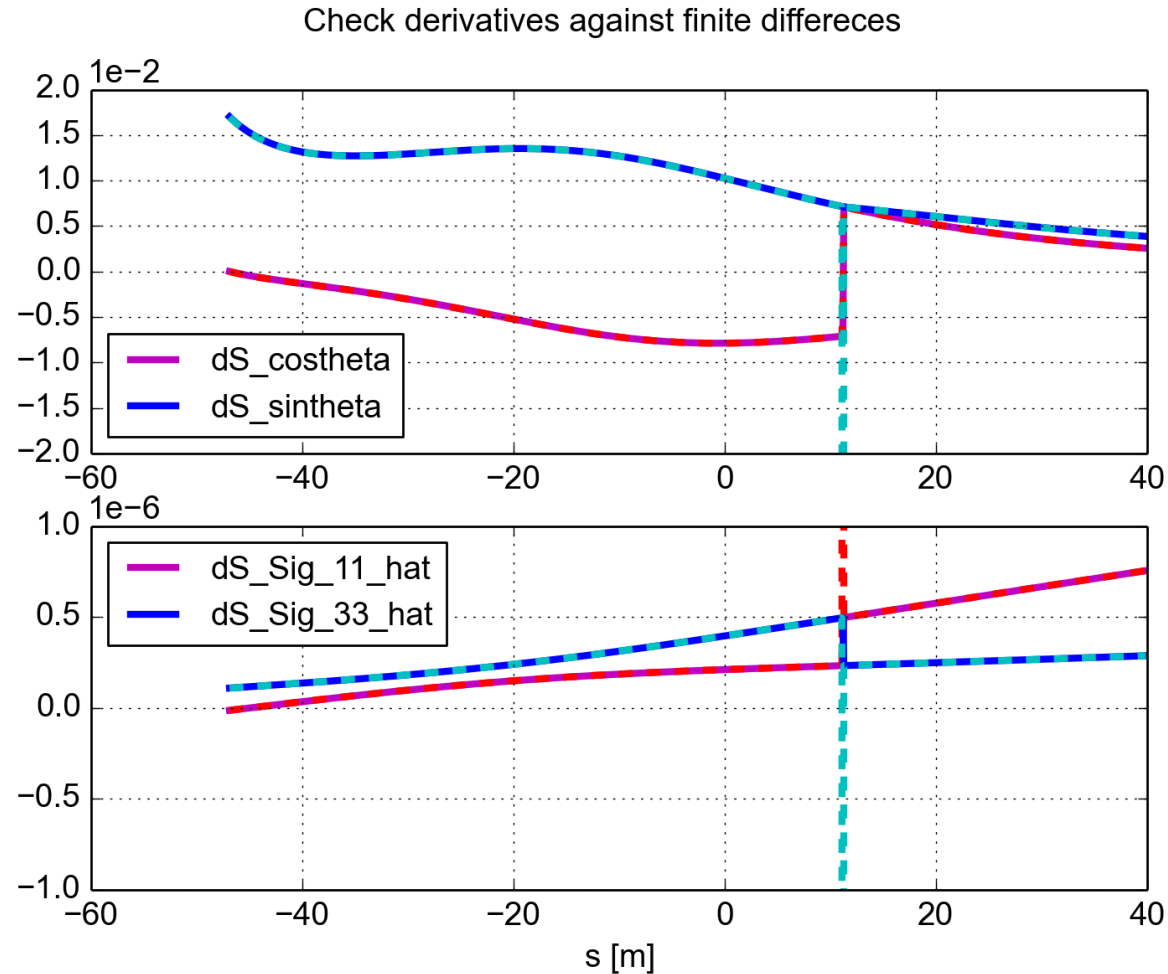
Check rotation against matrix diagonalization
At $s=0$: Sig11=8.4e-06, Sig22=4.1e-09, Sig33=1.0e-05, Sig44=1.3e-09
Sig12=1.9e-07, Sig13=-3.6e-06, Sig14=-3.8e-08,
Sig23=-7.6e-08, Sig24=-8.1e-10, Sig34=1.2e-07





Σ -matrix transformation to un-coupled frame

Library tested against **numerical diagonalization** of the Σ -matrix





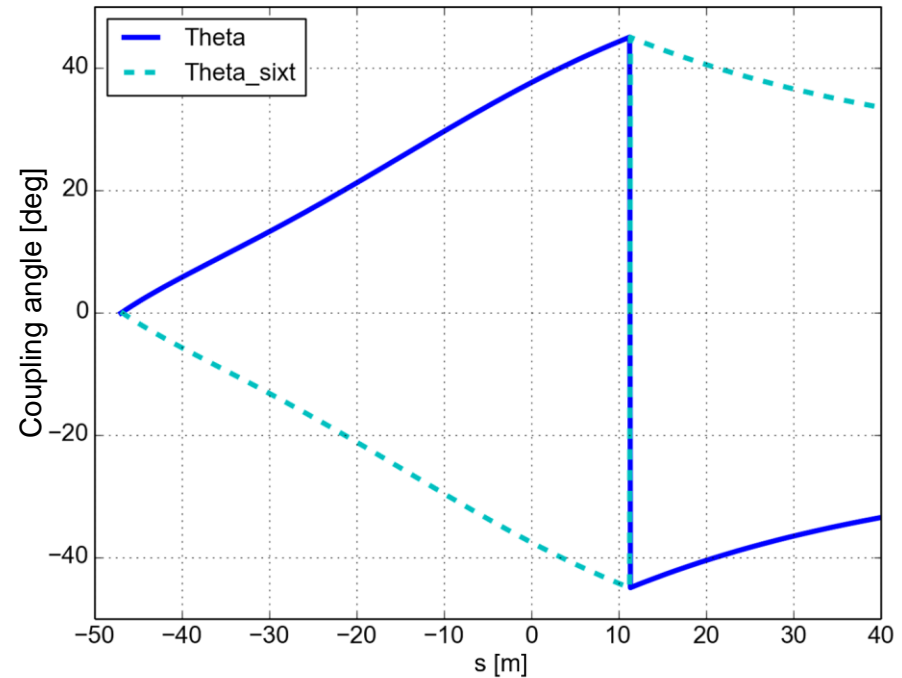
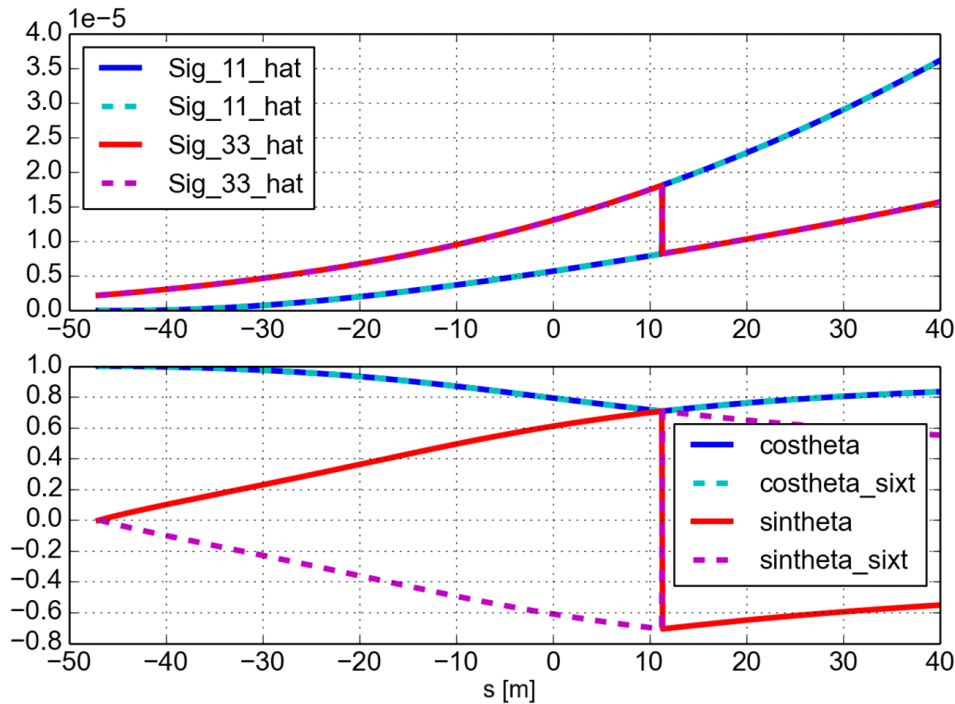
Σ -matrix transformation to un-coupled frame

SixTrack tested against library: **test failed!**

Sign error in the computation of the coupling angle

Original source code:

```
if(abs(sinth).gt.pieni) then
    sinth=(-1d0*sfac)*sqrt(sinth)
else
    sinth=zero
endif
```



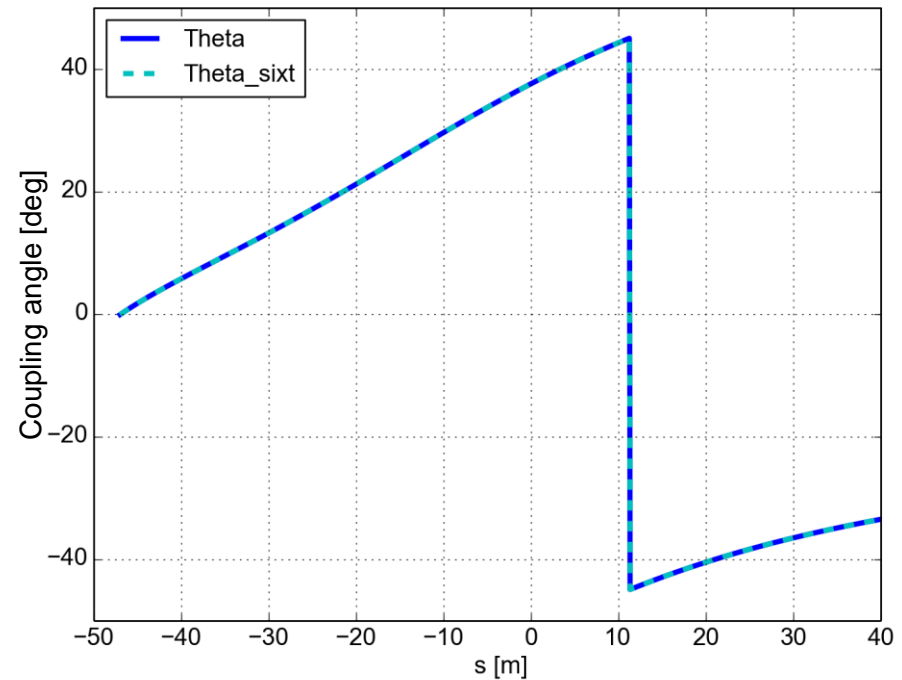
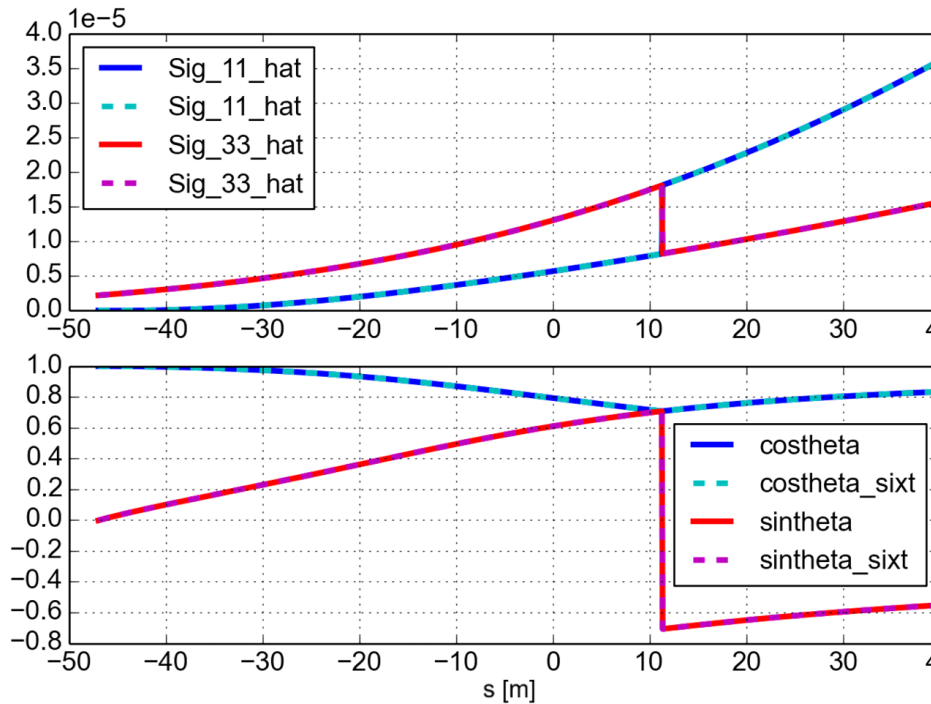


Σ -matrix transformation to un-coupled frame

SixTrack tested against library: **test failed!**
Sign error in the computation of the coupling angle

Corrected source code:

```
if(abs(sinth).gt.pieni) then
    sinth=(sfac)*sqrt(sinth)
else
    sinth=zero
endif
```





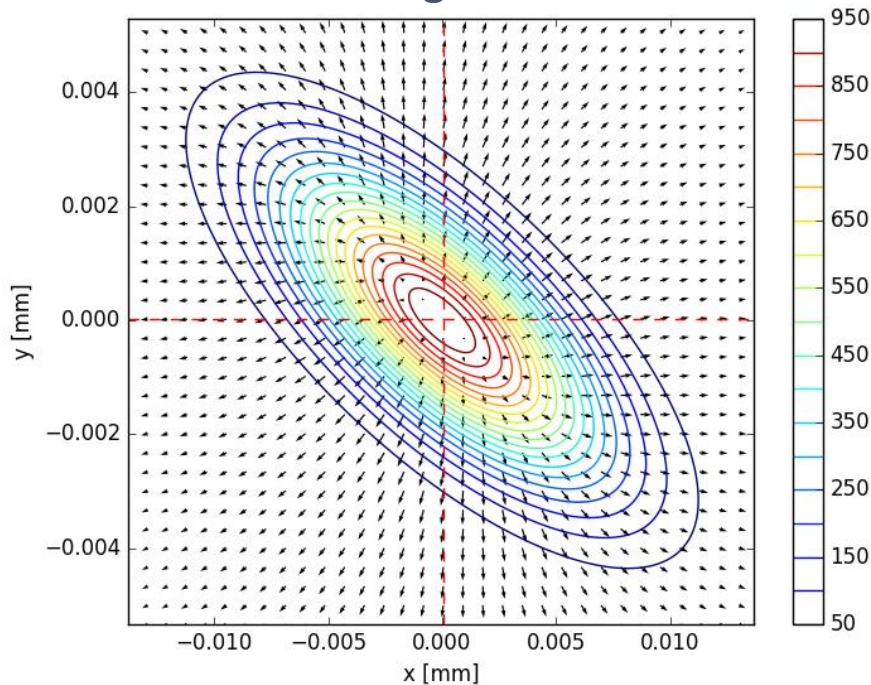
Σ -matrix transformation to un-coupled frame

Input sigma matrix:

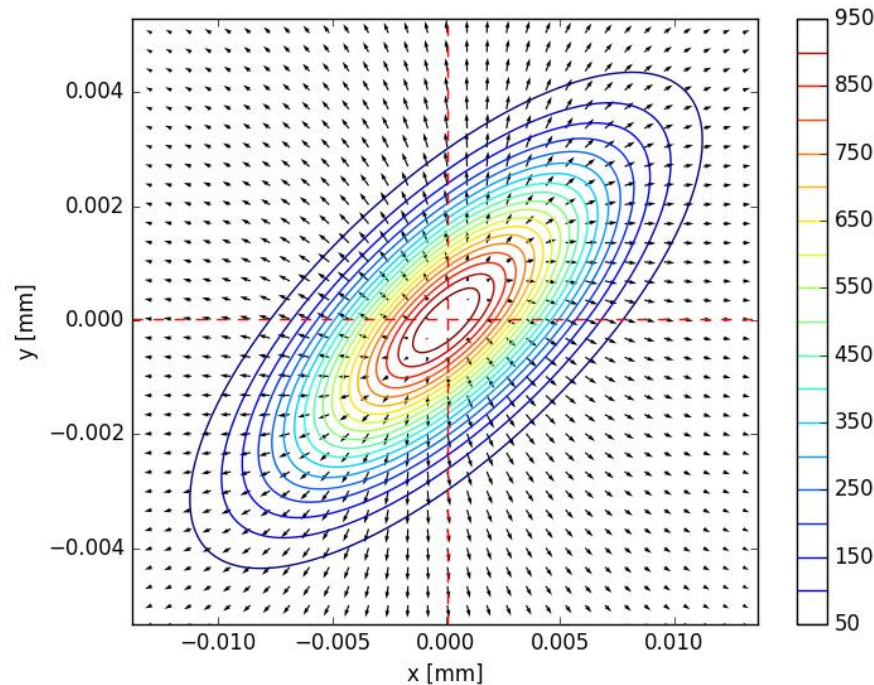
```
{'Sig_11_0': 2.1046670129999999e-05,  
'Sig_12_0': 2.7725426699999999e-07,  
'Sig_13_0': 5.9207071659999999e-06,  
'Sig_14_0': 1.2224001670000001e-07,  
'Sig_22_0': 3.6622825020000002e-09,  
'Sig_23_0': 7.4141336339999994e-08,  
'Sig_24_0': 1.495491124e-09,  
'Sig_33_0': 3.165637487e-06,  
'Sig_34_0': 7.9058234540000002e-08,  
'Sig_44_0': 2.040387648e-09}
```

Checked by Kyrre using full SixTrack simulations (numerical divergence of the computed kicks)

Original



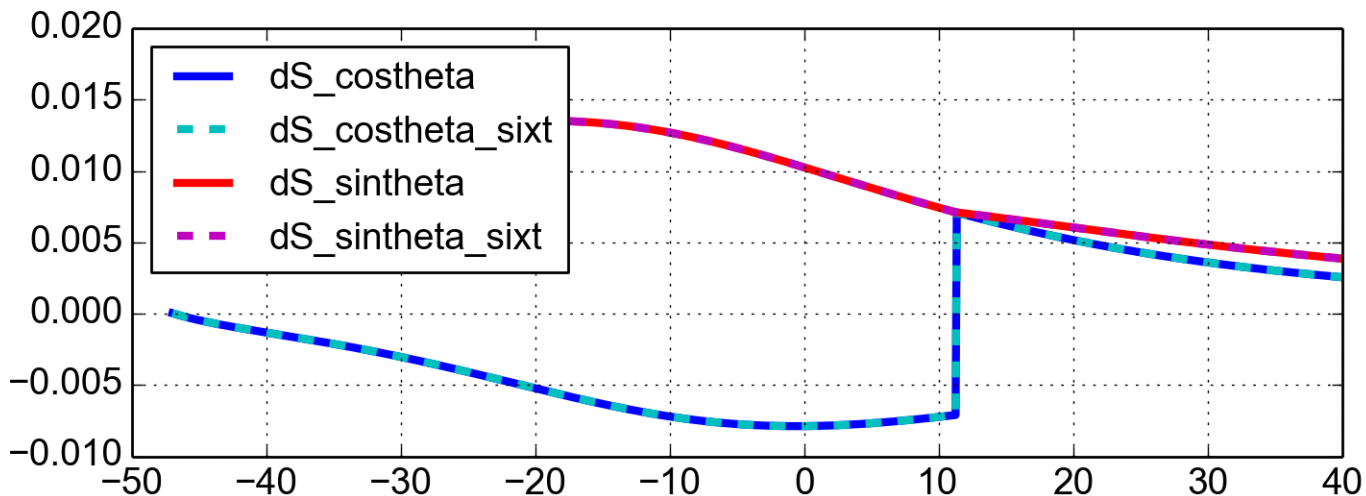
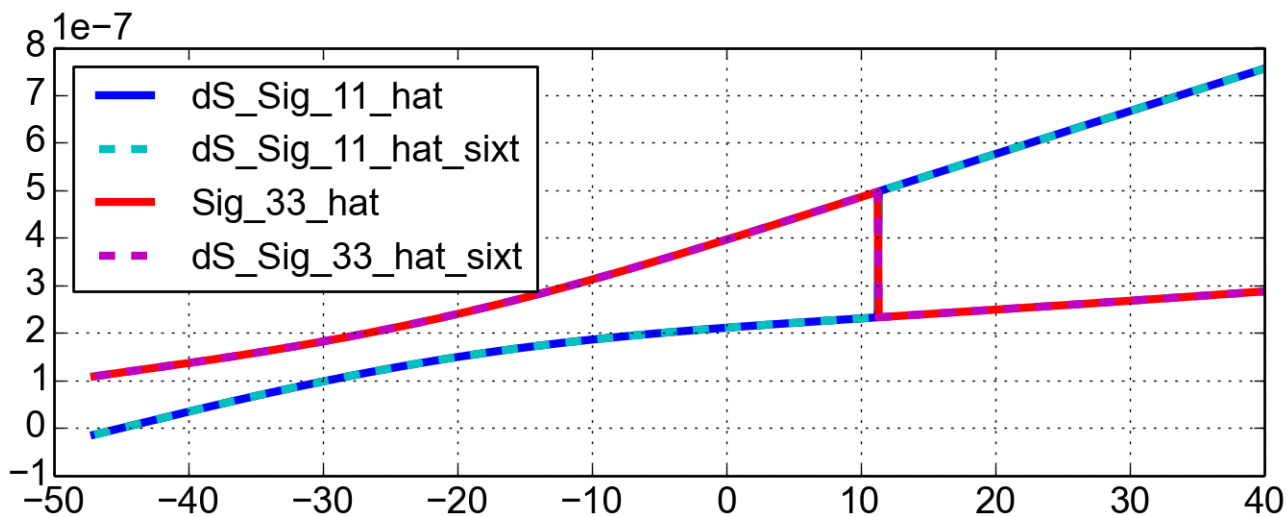
Corrected





Σ -matrix transformation to un-coupled frame

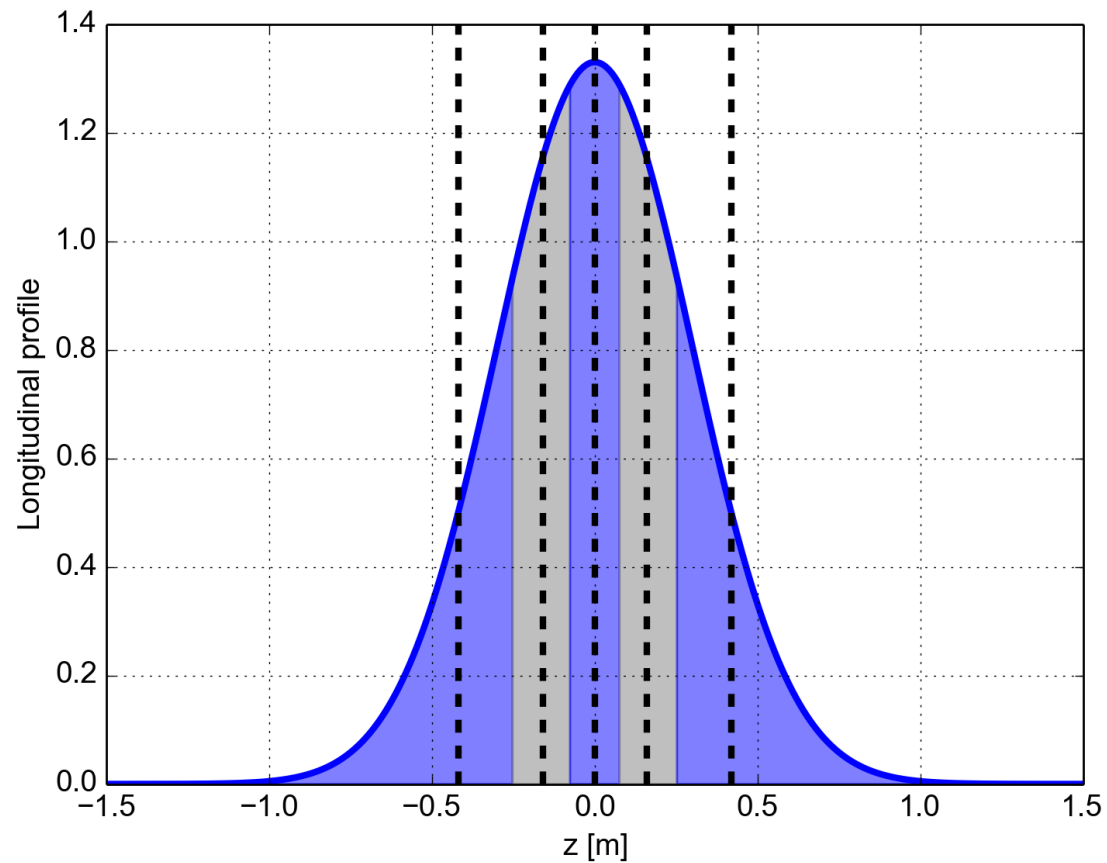
After bug correction **derivatives were also found to be ok**



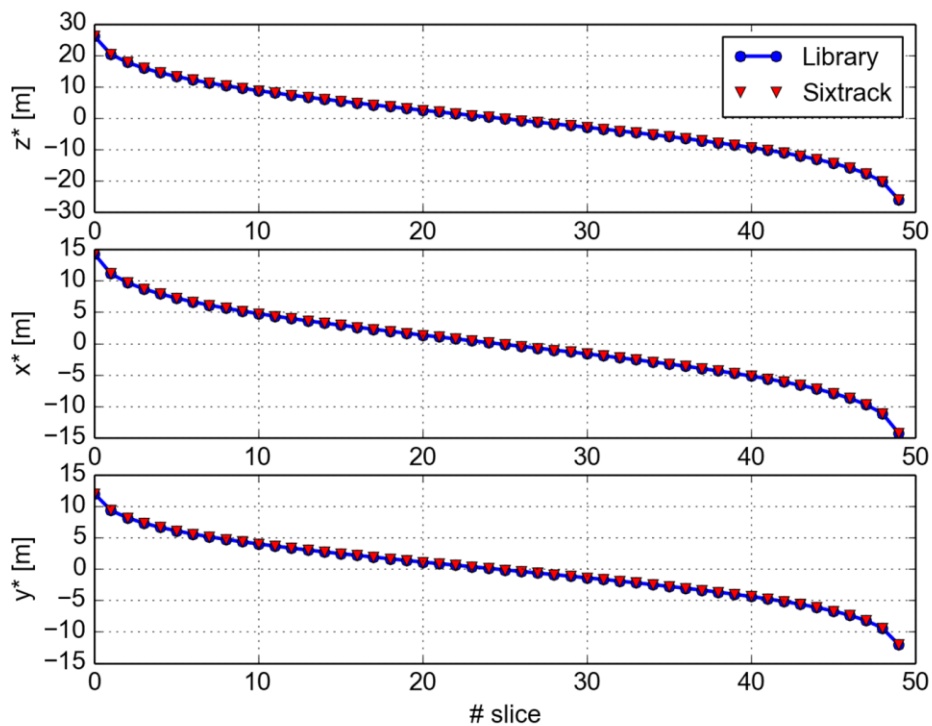
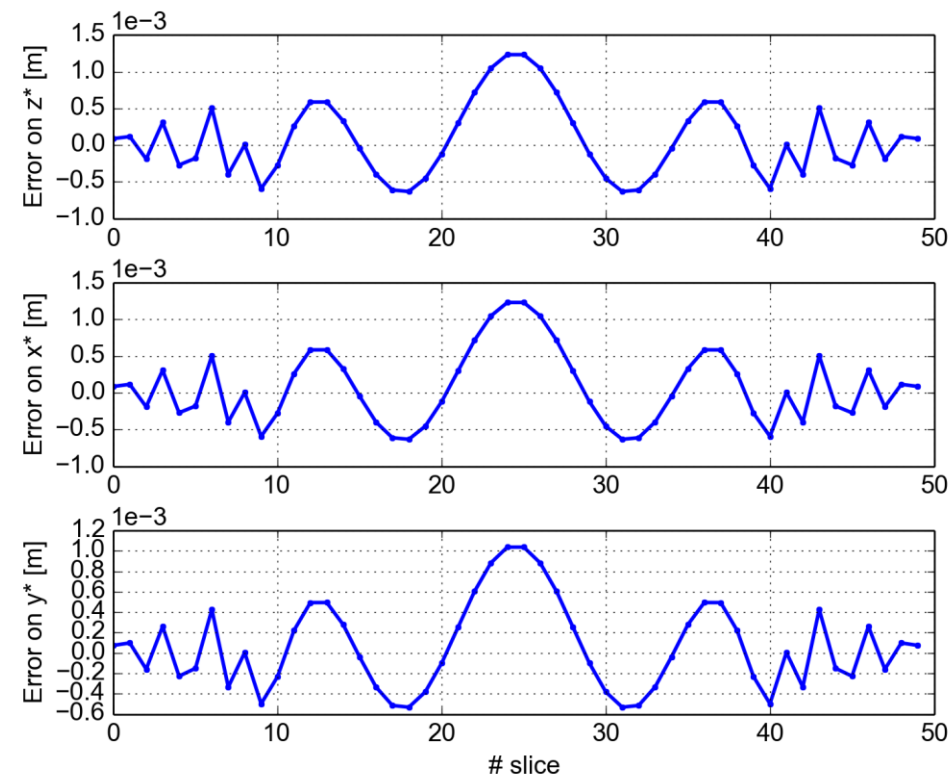


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Library: slicing could be easily re-implemented using python inverse error function



Sixtrack: implementation is correct but not very accurate





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Sixtrack (corrected) vs library: agreement to the 6th digit!

```
Compare kicks against sixtrack:  
D_x -2.32123980148e-07 -2.32123980355e-07 err=2.08e-16  
D_px 4.62575633839e-08 4.62575633839e-08 err=0.00e+00  
D_y -1.95977011284e-07 -1.9597701092e-07 err=-3.64e-16  
D_py 3.88258677153e-08 3.88258677153e-08 err=0.00e+00  
D_sigma -5.29477794942e-10 -5.29477350852e-10 err=-4.44e-16  
D_delta 6.18915584942e-08 6.18915584951e-08 err=-8.67e-19
```



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Case $T > 0, |\Sigma_{13}^*| > 0$

We use the expression that we have derived before:

$$\begin{aligned} R(S) &= \Sigma_{11}^* - \Sigma_{33}^* \\ W(S) &= \Sigma_{11}^* + \Sigma_{33}^* \\ T(S) &= R^2 + 4\Sigma_{13}^{*2} \end{aligned}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\begin{aligned} \hat{\Sigma}_{11}^* &= \frac{1}{2} (W + \operatorname{sgn}(R)\sqrt{T}) \\ \hat{\Sigma}_{33}^* &= \frac{1}{2} (W - \operatorname{sgn}(R)\sqrt{T}) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} + \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \\ \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} - \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \end{aligned}$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

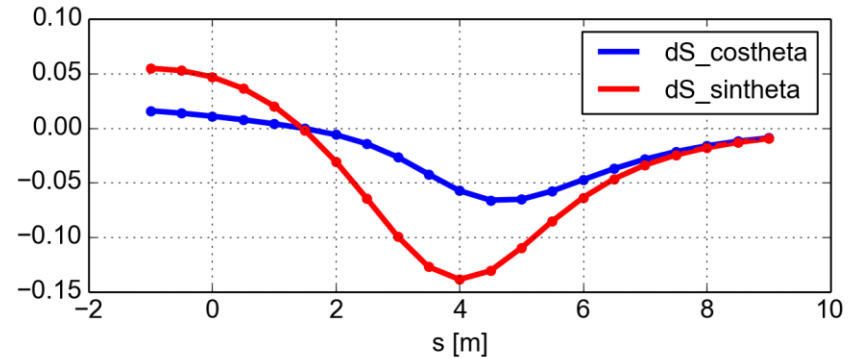
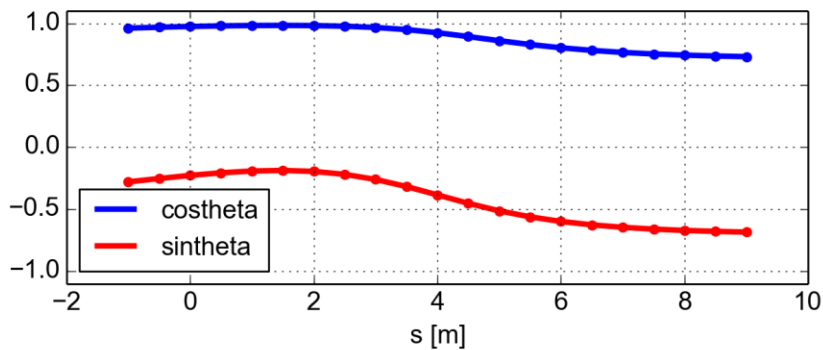
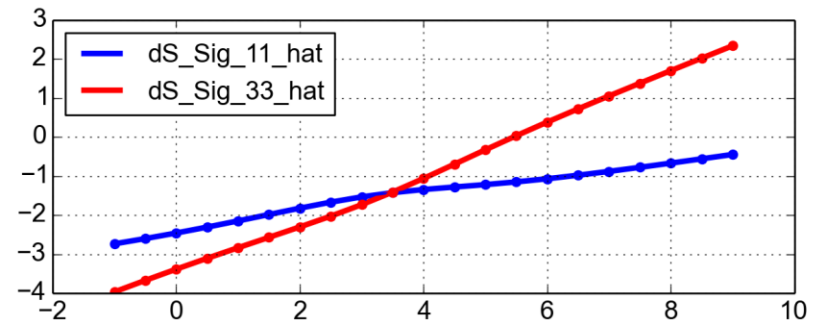
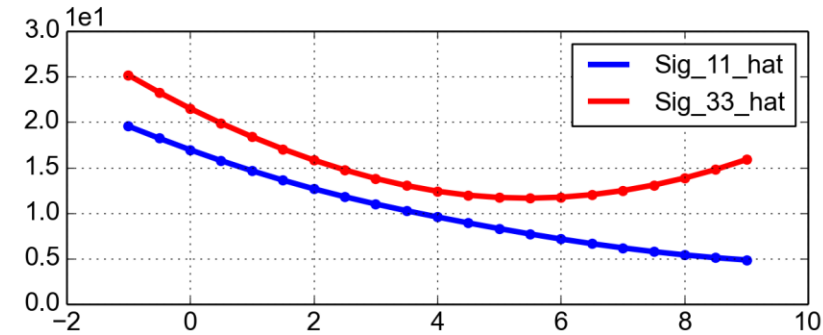
$$\frac{\partial}{\partial S} \sin \theta = -\frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$



Case $T > 0, |\Sigma_{13}^*| > 0$

Tests:

Mode: check_singularities At s=4.0:
SIG13=1.0 T=8.0, a=2.0e-01, b=-3.0e-02, c=4.0e-01, d=1.0e-01



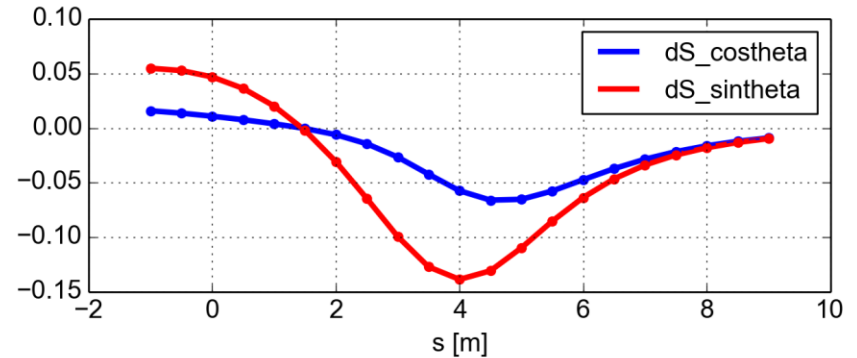
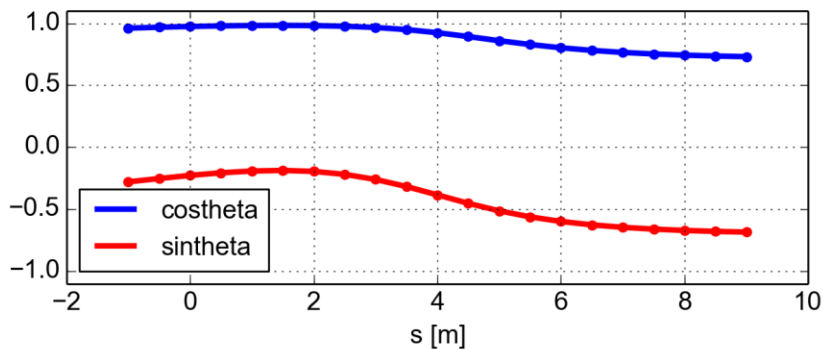
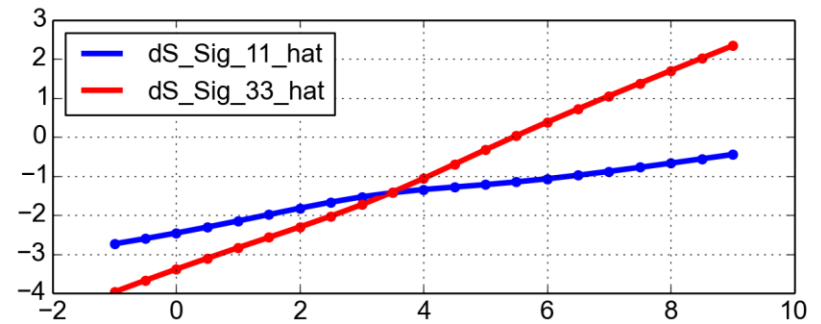
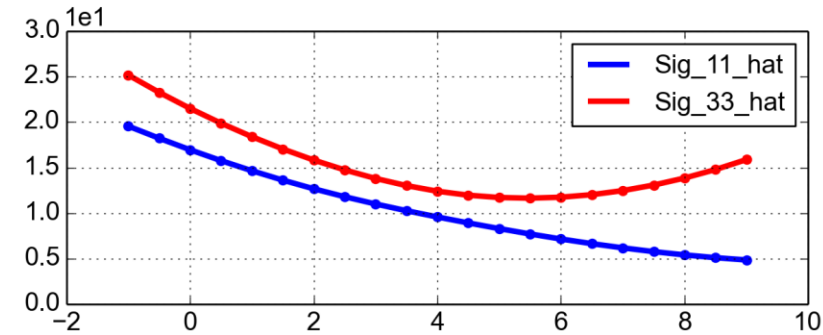
- Expression with denominator (apparently singular)
- - ● - - Expression with correction



Case $T > 0$, $|\Sigma_{13}^*| > 0$

Tests against Sixtrack:

Mode: vs_sixtrack At s=4.0:
SIG13=1.0 T=8.0, a=2.0e-01, b=-3.0e-02, c=4.0e-01, d=1.0e-01



— Library (with correction)
- - • - - Sixtrack



Case $T > 0, |\Sigma_{13}^*| = 0$:

The highlighted formulas break and **alternative expressions** need to be found:

$$\begin{aligned}R(S) &= \Sigma_{11}^* - \Sigma_{33}^* \\W(S) &= \Sigma_{11}^* + \Sigma_{33}^* \\T(S) &= R^2 + 4\Sigma_{13}^{*2}\end{aligned}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\begin{aligned}\hat{\Sigma}_{11}^* &= \frac{1}{2} (W + \operatorname{sgn}(R)\sqrt{T}) \\ \hat{\Sigma}_{33}^* &= \frac{1}{2} (W - \operatorname{sgn}(R)\sqrt{T})\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} + \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \\ \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} - \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right)\end{aligned}$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

$$\frac{\partial}{\partial S} \sin \theta = - \frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$



Case $T > 0, |\Sigma_{13}^*| = 0$:

$$\cos 2\theta = \operatorname{sgn}(\Sigma_{11}^* - \Sigma_{33}^*) \frac{\Sigma_{11}^* - \Sigma_{33}^*}{\sqrt{(\Sigma_{11}^* - \Sigma_{33}^*)^2 + 4\Sigma_{13}^{*2}}} \quad \longrightarrow \quad \sin \theta = 0$$

$$\begin{aligned} \frac{\partial}{\partial S} \cos \theta &= \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta \\ \frac{\partial}{\partial S} \sin \theta &= \frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta \end{aligned}$$



Handling the denominators: case #1

Case $T > 0$, $|\Sigma_{13}^*| = 0$:

Around the singular point we can write:

$$\Sigma_{13}^* = c\Delta S + d\Delta S^2 \quad \text{with}$$

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$\cos 2\theta = \frac{|R|}{\sqrt{R^2 + 4\Sigma_{13}^{*2}}} = \frac{1}{\sqrt{1 + 4\frac{\Sigma_{13}^{*2}}{R^2}}} \simeq \frac{1}{1 + 2\frac{\Sigma_{13}^{*2}}{R^2}} \simeq 1 - 2\frac{\Sigma_{13}^{*2}}{R^2}$$

$$\sin \theta = \text{sgn}(R)\text{sgn}(\Sigma_{13}^*) \frac{|\Sigma_{13}^*|}{|R|} = \frac{\Sigma_{13}^*}{R}$$

$$\frac{\partial}{\partial S} \sin \theta = \frac{1}{R^2} \left[(c + 2d\Delta S) R - \frac{\partial R}{\partial S} (c\Delta S + d\Delta S^2) \right] \quad \longrightarrow \quad \text{At the singular point} \quad \frac{\partial}{\partial S} \sin \theta = \frac{c}{R}$$

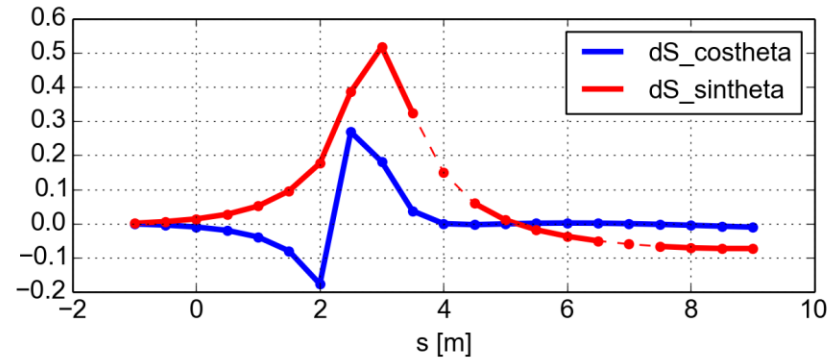
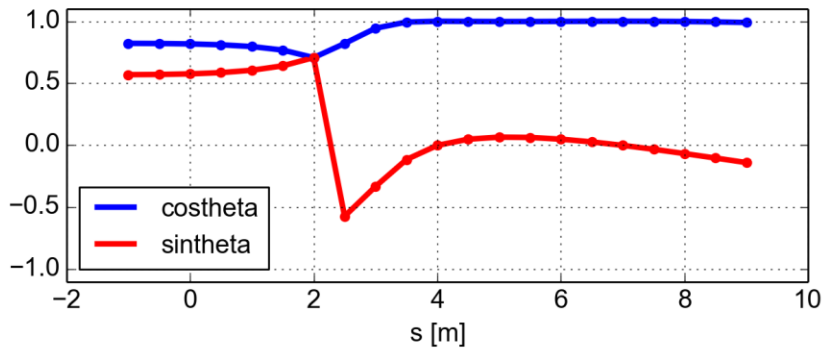
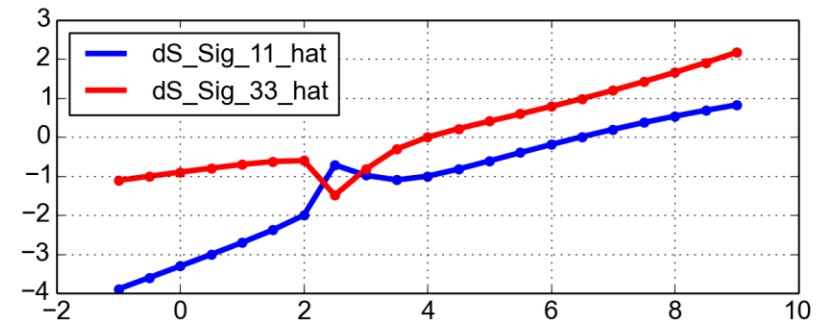
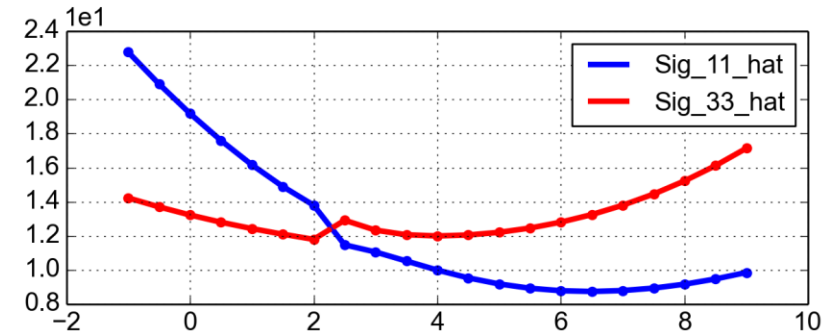
Which is always regular once we assume $T > 0$ and therefore $R^2 > 0$



Case $T > 0$, $|\Sigma_{13}^*| = 0$:

Tests:

Mode: check_singularities At s=4.0:
SIG13=0.0 T=4.0, a=-5.0e-01, b=0.0, c=-3.0e-01, d=1.0e-01



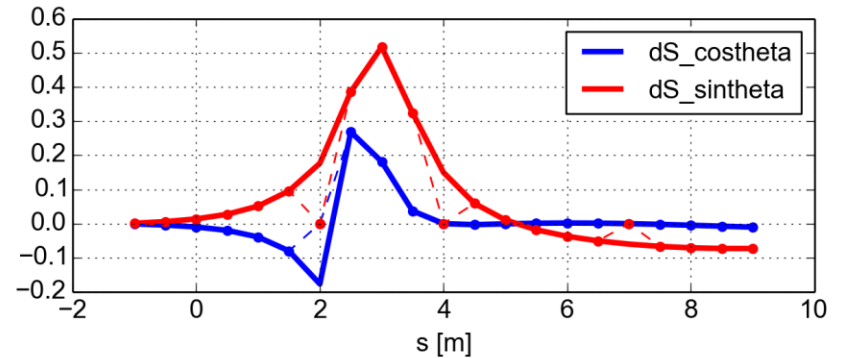
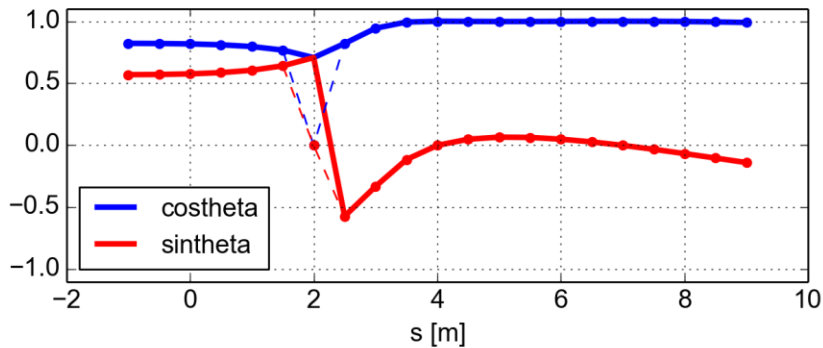
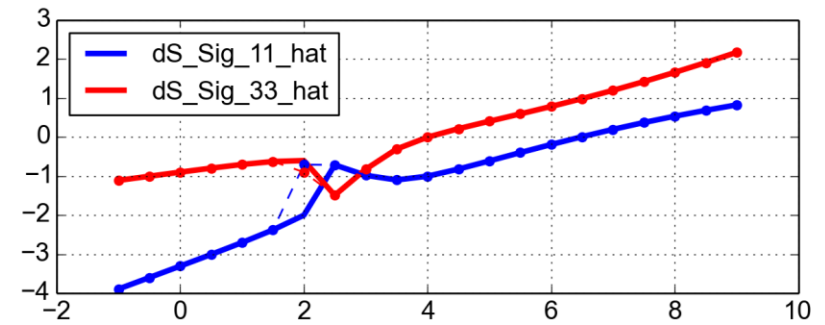
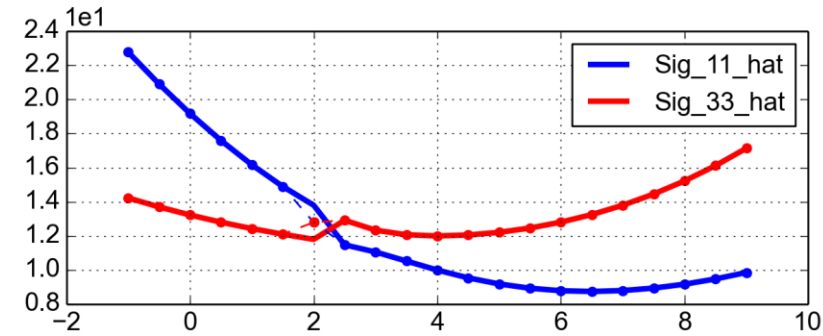
- Expression with denominator (apparently singular)
- - • - - Expression with correction



Case $T > 0, |\Sigma_{13}^*| = 0$:

Tests against Sixtrack:

Mode: vs_sixtrack At s=4.0:
SIG13=0.0 T=4.0, a=-5.0e-01, b=0.0, c=-3.0e-01, d=1.0e-01



— Library (with correction)
- - • - - Sixtrack



Case $T=0, |c|>0$

The highlighted formulas break and **alternative expressions** need to be found:

$$\begin{aligned}
 R(S) &= \Sigma_{11}^* - \Sigma_{33}^* \\
 W(S) &= \Sigma_{11}^* + \Sigma_{33}^* \\
 T(S) &= R^2 + 4\Sigma_{13}^{*2}
 \end{aligned}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\begin{aligned}
 \hat{\Sigma}_{11}^* &= \frac{1}{2} (W + \operatorname{sgn}(R)\sqrt{T}) \\
 \hat{\Sigma}_{33}^* &= \frac{1}{2} (W - \operatorname{sgn}(R)\sqrt{T})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} + \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right) \\
 \frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] &= \frac{1}{2} \left(\frac{\partial W}{\partial S} - \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right)
 \end{aligned}$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

$$\frac{\partial}{\partial S} \sin \theta = -\frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$



Case $T=0, |c|>0$

Around the singular point we can write:

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4(c + d\Delta S)^2 \right]$$

$$\cos 2\theta = \frac{|2a + b\Delta S|}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}}$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(2a + b\Delta S) \left[\frac{b}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}} - \frac{(2a + b\Delta S)(2ab + b^2\Delta S + 4cd + 4d^2\Delta S)}{\left(\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}\right)^3} \right]$$

$$\Delta S = 0 \quad \Downarrow$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(2a) \left[\frac{b}{2\sqrt{a^2 + c^2}} - \frac{a(ab + 2cd)}{2(\sqrt{a^2 + c^2})^3} \right]$$



Case $T=0, |c|>0$

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$R = 2a\Delta S + b\Delta S^2$$

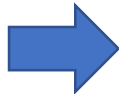
$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4(c + d\Delta S)^2 \right]$$

$$\hat{\Sigma}_{11}^* = \frac{W}{2} + \frac{1}{2} \operatorname{sgn} \left(2a\Delta S + b\Delta S^2 \right) |\Delta S| \sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}$$

$$\hat{\Sigma}_{33}^* = \frac{W}{2} - \frac{1}{2} \operatorname{sgn} \left(2a\Delta S + b\Delta S^2 \right) |\Delta S| \sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}$$

$$\frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] = \frac{1}{2} \frac{\partial W}{\partial S} + \frac{1}{2} \operatorname{sgn} \left(2a\Delta S + b\Delta S^2 \right) \operatorname{sgn}(\Delta S) \left[\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2} + \frac{\Delta S (2ab + b^2\Delta S + 4cd + 4d^2\Delta S)}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}} \right]$$
$$\frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] = \frac{1}{2} \frac{\partial W}{\partial S} - \frac{1}{2} \operatorname{sgn} \left(2a\Delta S + b\Delta S^2 \right) \operatorname{sgn}(\Delta S) \left[\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2} + \frac{\Delta S (2ab + b^2\Delta S + 4cd + 4d^2\Delta S)}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}} \right]$$

$$\Delta S = 0$$



$$\hat{\Sigma}_{11}^* = \frac{W}{2}$$

$$\hat{\Sigma}_{33}^* = \frac{W}{2}$$

$$\frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] = \frac{1}{2} \frac{\partial W}{\partial S} + \operatorname{sgn}(2a) \sqrt{a^2 + c^2}$$

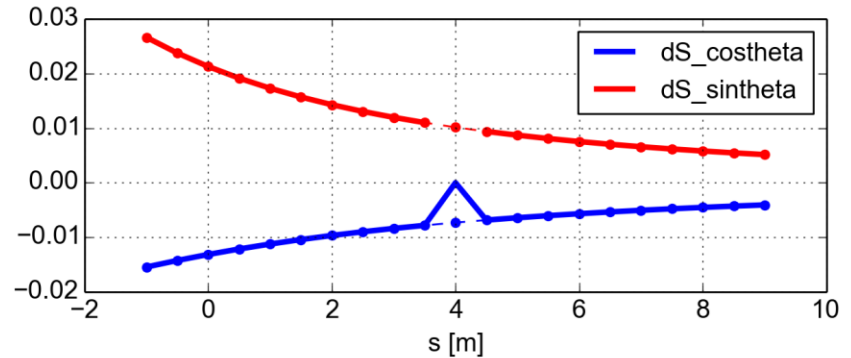
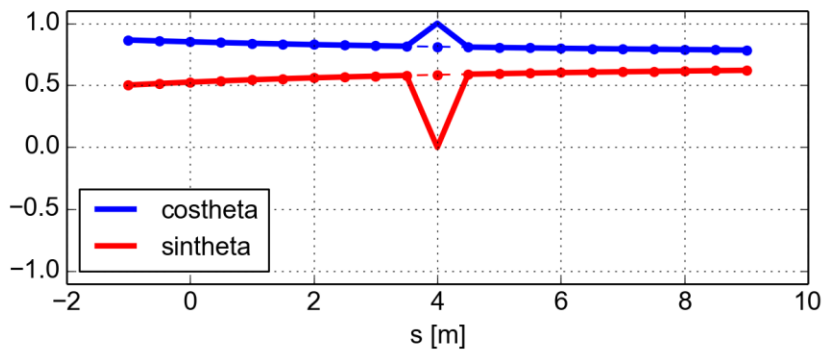
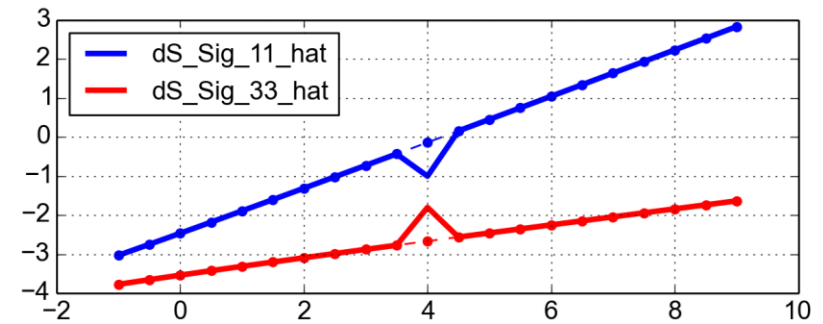
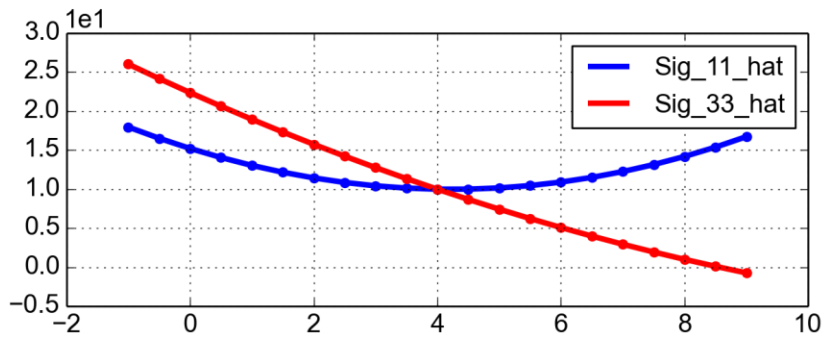
$$\frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] = \frac{1}{2} \frac{\partial W}{\partial S} - \operatorname{sgn}(2a) \sqrt{a^2 + c^2}$$



Case $T=0, |c|>0$

Tests:

Mode: check_singularities At s=4.0:
SIG13=0.0 T=0.0, a=4.0e-01, b=0.0, c=1.2, d=1.0e-01

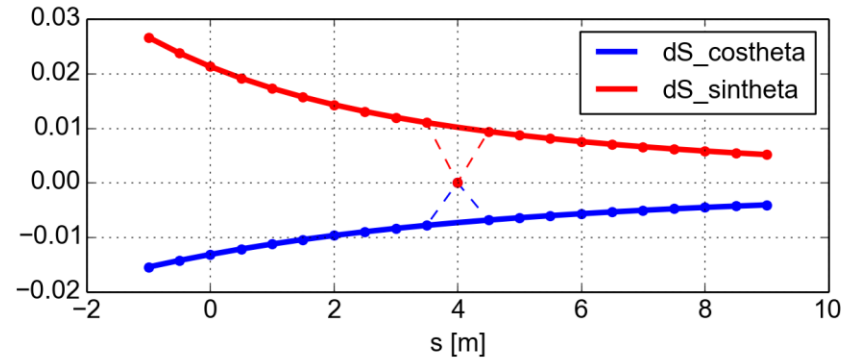
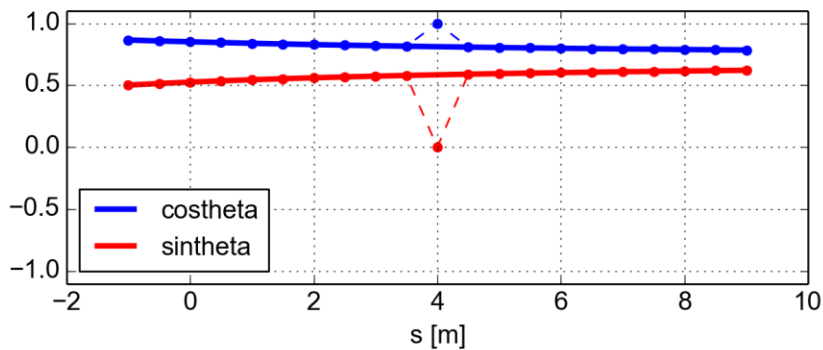
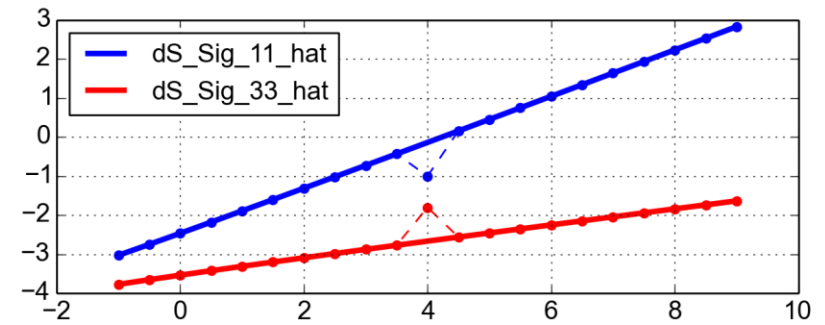
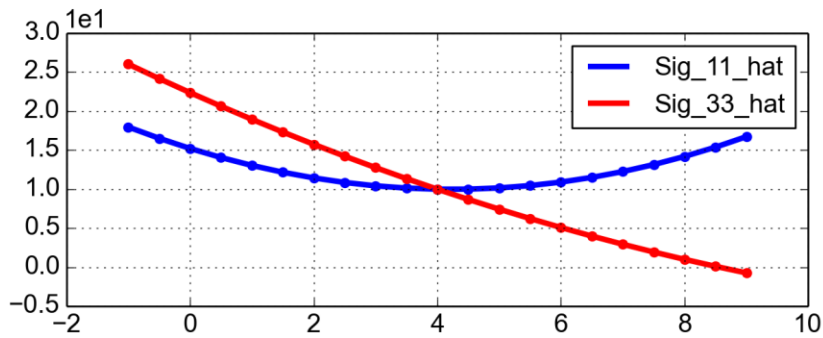


- Expression with denominator (apparently singular)
- - • - - Expression with correction

Case $T=0, |c|>0$

Tests against Sixtrack:

Mode: vs_sixtrack At s=4.0:
 SIG13=0.0 T=0.0, a=4.0e-01, b=0.0, c=1.2, d=1.0e-01



— Library (with correction)
 - - • - - Sixtrack



Case $T=0, c=0, |a|>0$

The highlighted formulas break and **alternative expressions** need to be found:

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$

$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}_{11}^* = \frac{1}{2} (W + \operatorname{sgn}(R)\sqrt{T})$$

$$\hat{\Sigma}_{33}^* = \frac{1}{2} (W - \operatorname{sgn}(R)\sqrt{T})$$

$$\frac{\partial}{\partial S} [\hat{\Sigma}_{11}^*] = \frac{1}{2} \left(\frac{\partial W}{\partial S} + \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right)$$

$$\frac{\partial}{\partial S} [\hat{\Sigma}_{33}^*] = \frac{1}{2} \left(\frac{\partial W}{\partial S} - \operatorname{sgn}(R) \frac{1}{2\sqrt{T}} \frac{\partial T}{\partial S} \right)$$

$$\frac{\partial}{\partial S} [\cos 2\theta] = \operatorname{sgn}(R) \left(\frac{\partial R}{\partial S} \frac{1}{\sqrt{T}} - \frac{R}{2(\sqrt{T})^3} \frac{\partial T}{\partial S} \right)$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

$$\frac{\partial}{\partial S} \sin \theta = -\frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$



Case $T=0, c=0, |a|>0$

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4(c + d\Delta S)^2 \right]$$

We proceed as before:

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}} \quad \longrightarrow \quad \cos 2\theta = \frac{|2a + b\Delta S|}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}} \quad \longrightarrow \quad \cos 2\theta = \frac{|2a|}{2\sqrt{a^2 + c^2}}$$

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)}$$

$$\sin \theta = \operatorname{sgn}(R) \operatorname{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S} \cos \theta = \frac{1}{4 \cos \theta} \frac{\partial}{\partial S} \cos 2\theta$$

$$\frac{\partial}{\partial S} \sin \theta = - \frac{1}{4 \sin \theta} \frac{\partial}{\partial S} \cos 2\theta$$

Same as before but this
denominator becomes zero



Case $T=0, c=0, |a|>0$

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4(c + d\Delta S)^2 \right]$$

We need to **expand to higher order**:

$$\cos 2\theta = \frac{1}{\sqrt{1 + \frac{4d^2\Delta S^2}{(2a+b\Delta S)^2}}} \simeq 1 - \frac{2d^2\Delta S^2}{(2a+b\Delta S)^2}$$

$$\sin \theta = \text{sgn}(R)\text{sgn}(\Sigma_{13}^*) \sqrt{\frac{1}{2} (1 - \cos 2\theta)}$$

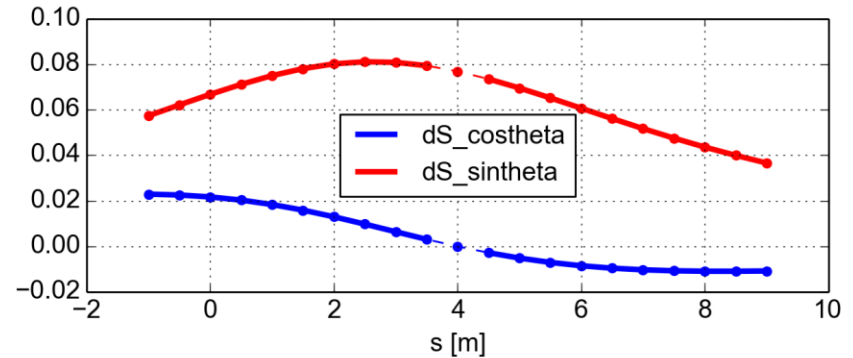
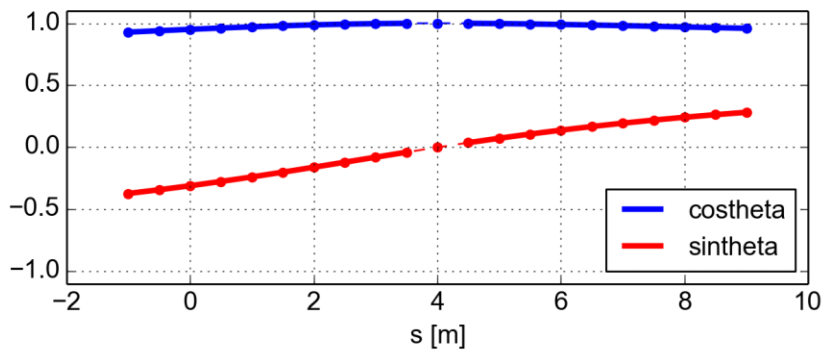
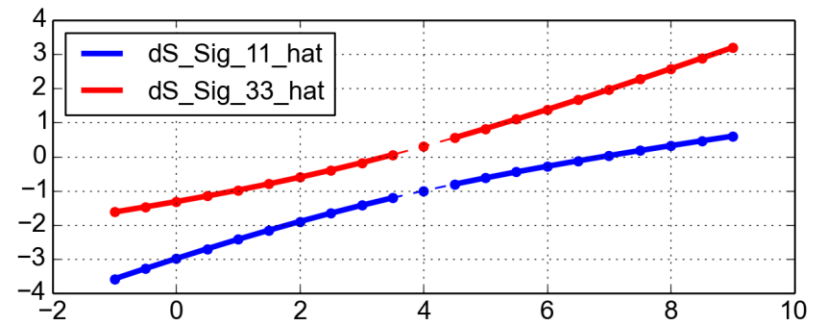
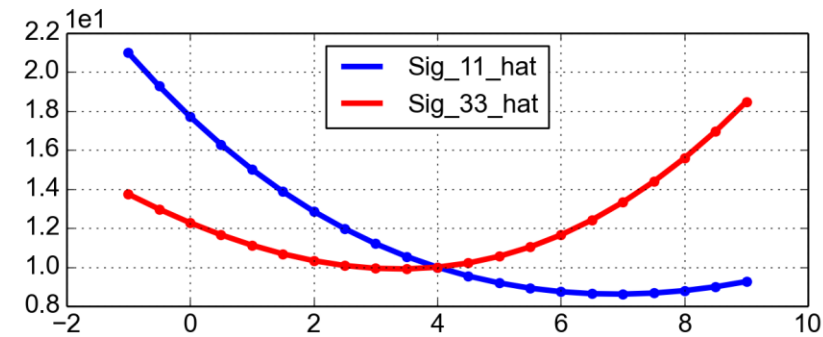
$$\sin \theta = \frac{d\Delta S}{2a} \left| 1 - \frac{b\Delta S}{2a} \right|$$

$$\frac{\partial}{\partial S} \sin \theta = \frac{d}{2a}$$

Case $T=0, c=0, |a|>0$

Tests:

Mode: check_singularities At s=4.0:
 SIG13=0.0 T=0.0, a=-6.5e-01, b=-5.0e-02, c=0.0, d=-1.0e-01



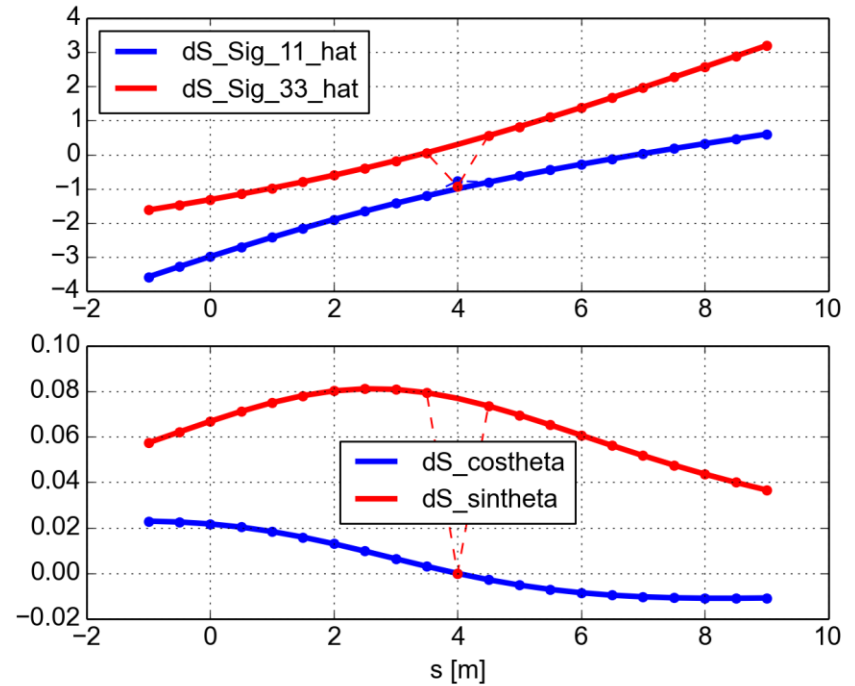
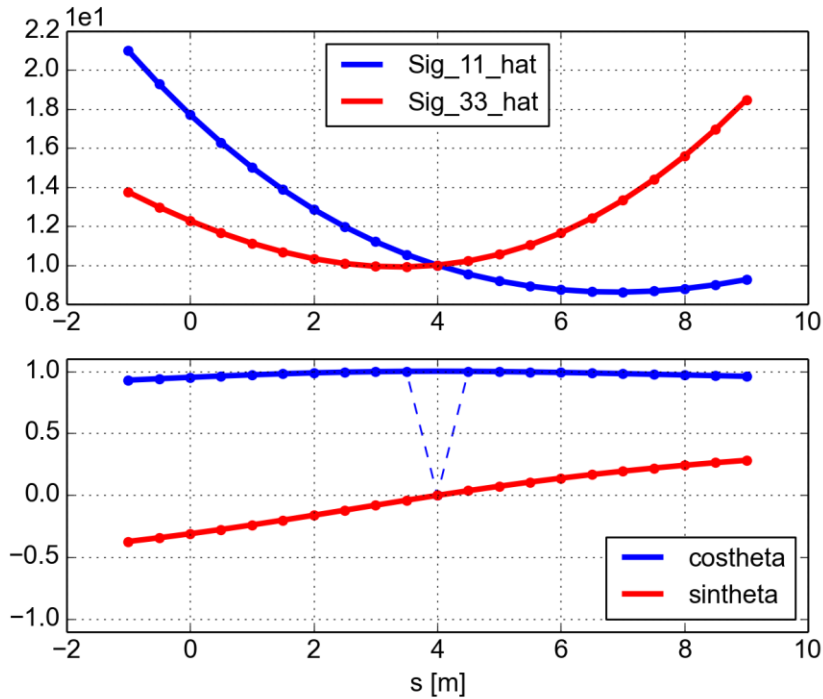
- Expression with denominator (apparently singular)
- - ● - - Expression with correction



Case $T=0, c=0, |a|>0$

Tests against Sixtrack:

Mode: vs_sixtrack At s=4.0:
SIG13=0.0 T=0.0, a=-6.5e-01, b=-5.0e-02, c=0.0, d=-1.0e-01



- Library (with correction)
- - • - - Sixtrack



Case $T=0, c=0, a=0$

$$a = \Sigma_{12}^* - \Sigma_{34}^*$$

$$b = \Sigma_{22}^* - \Sigma_{44}^*$$

$$c = \Sigma_{14}^* + \Sigma_{23}^*$$

$$d = \Sigma_{24}^*$$

$$R = b\Delta S^2$$

$$\Sigma_{13}^* = d\Delta S^2$$

$$T(S) = R^2 + 4\Sigma_{13}^{*2}$$

$$\cos 2\theta = \operatorname{sgn}(R) \frac{R}{\sqrt{T}}$$

$$\cos 2\theta = \frac{|b|}{\sqrt{b^2 + 4d^2}}$$

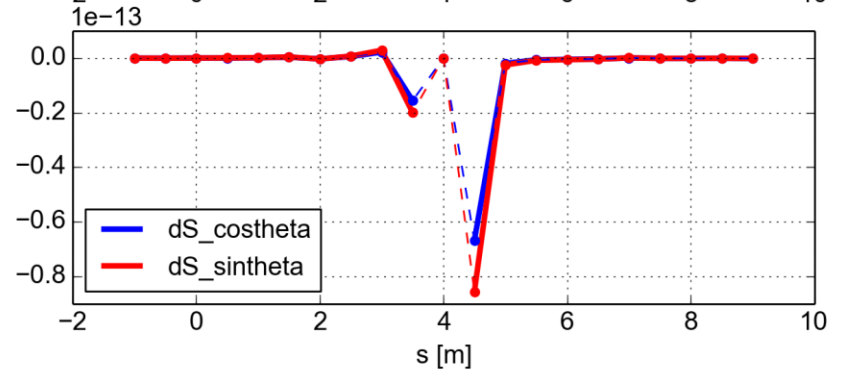
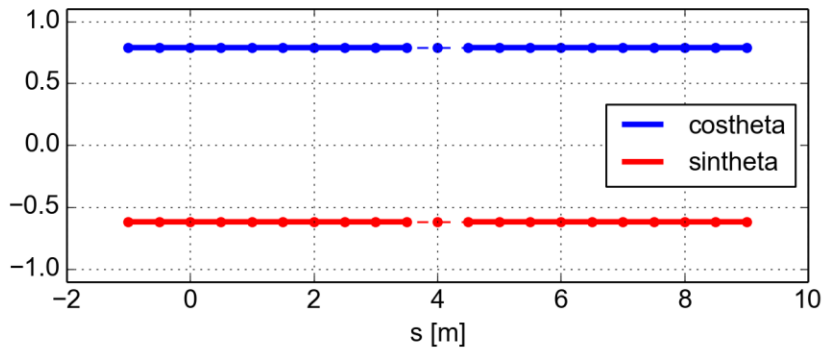
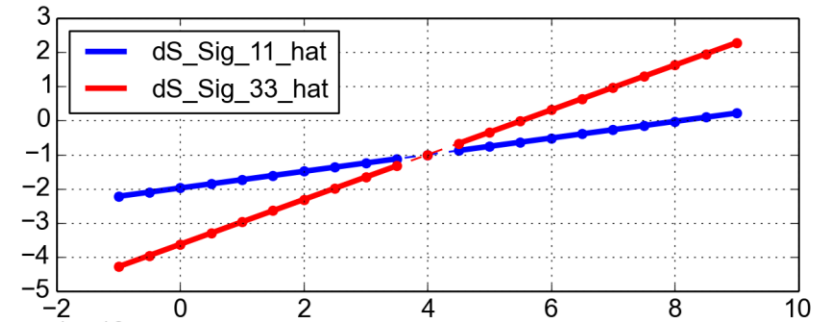
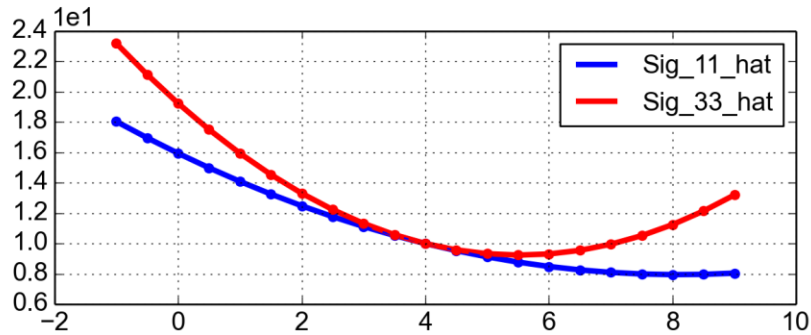
which is a constant...



Case $T=0, c=0, a=0$

Tests:

Mode: check_singularities At s=4.0:
SIG13=0.0 T=0.0, a=0.0, b=-5.0e-02, c=0.0, d=1.0e-01



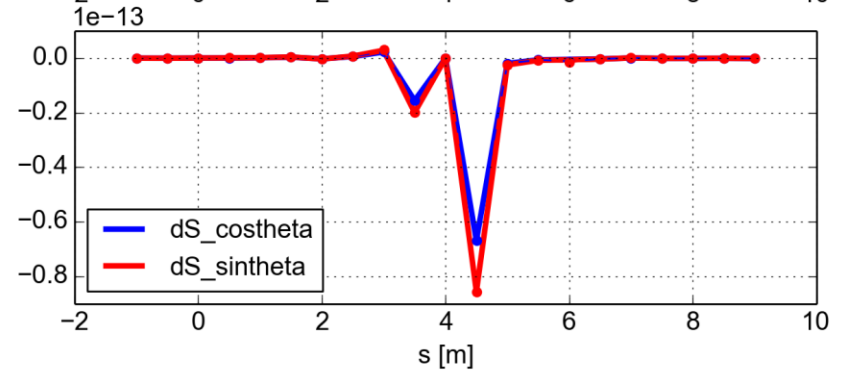
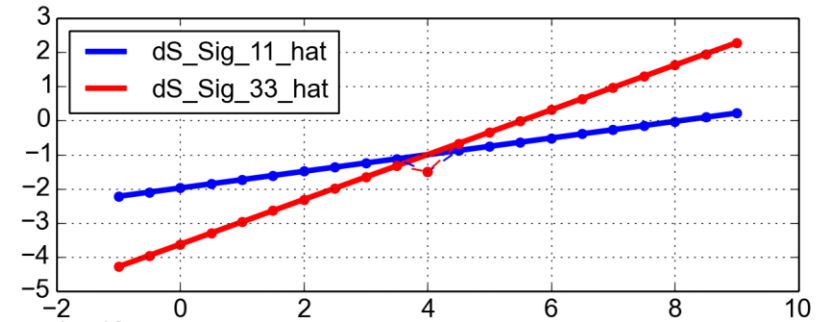
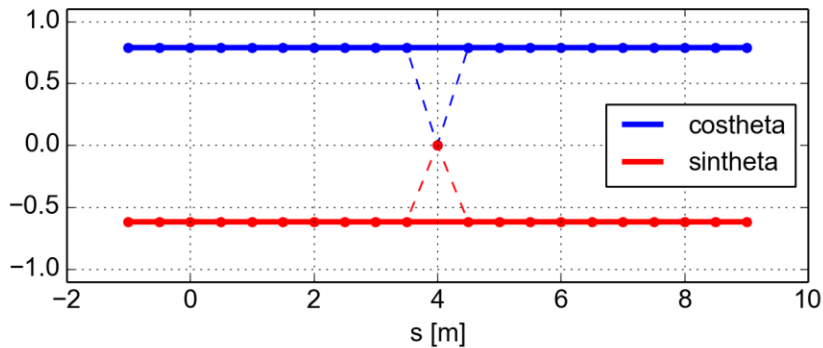
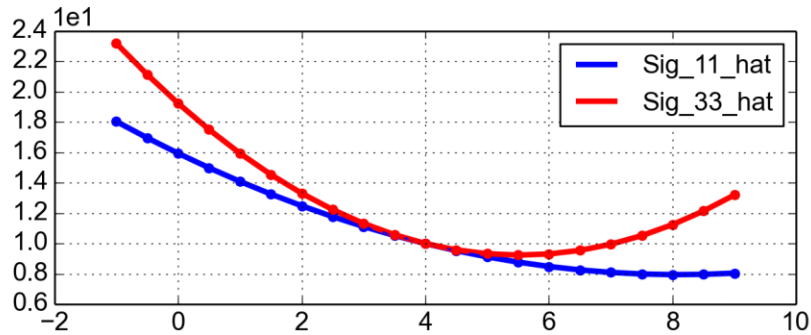
- Expression with denominator (apparently singular)
- - • - - Expression with correction



Case $T=0, c=0, a=0$

Tests against Sixtrack:

Mode: vs_sixtrack At s=4.0:
SIG13=0.0 T=0.0, a=0.0, b=-5.0e-02, c=0.0, d=1.0e-01



— Library (with correction)
- - • - - Sixtrack



- Complete **mathematical derivation** needed for implementation available in the prepared note (CERN-ACC-NOTE-2018-0023)
- Implemented in a **Python/C library** for usage in other simulation codes (SixtrackLib, PyHEADTAIL) and compatible with **GPU**
 - **“Stress tests”** performed on the different functional blocks of the library → **Passed**
- **Source code** including all tests available [on github](#)
- **SixTrack implementation tested** against library. Outcome:
 - **Uncoupled case:**
 - **Bug identified** in “inverse boost” → **corrected** (now in the production version)
 - **Other tests passed**
 - **Coupled case:**
 - Suffering from a **serious bug** (wrong sign) → **corrected** (now in the production version)
 - **Apparently singular cases** (denominators) not correctly handled → **strategy to be defined** (requires serious re-structuring, should we just replace everything with the library code?)
- **Next steps:**
 - Tests on GPU
 - Performance profiling and, if needed, optimization
 - Real life usage (fancy GPUs in Bologna should be coming soon 😊)