ASITP

INSTITUTE OF THEORETICAL PHYSICS ACADEMIA SINICA

AS-ITP-93-67
December 1993

THE σ-MODEL AND NON-COMMUTATIVE GEOMETRY

Han-ying GUO

Jian-ming LI

Ke WU

P.O.Box 2735, Beijing 100080, The People's Republic of China

Telefax : (86)-1-2562587 Telephone : 2568348

Telex : 22040 BAOAS CN Cable : 6158

ASITP-93-67

Nov. 5, 1993.

The σ–Model and Non–commutative Geometry [1]

Han-Ying Guo, Jian-Ming Li and Ke Wu

CCAST (World Laboratory), P. O. Box 8730, Beijing 100080, China;

Institute of Theoretical Physics, Academia Sinica, P. O. Box 2735, Beijing 100080, China.[2]

Abstract

Using the knowledge of non-commutative geometry, we show that the original σ–model can be built up by the gauge theory on discrete group.

1 Introduction

It has become widely accepted that strong interactions exhibit a set of approximate symmetries corresponding the chiral groups $SU_L(2) \times SU_R(2)$ and $SU_L(3) \times SU_R(3)$. The best known Lagrangian model based on $SU_L(2) \times SU_R(2)$ is the so called σ-model[1]. Whether there is more profound meaning from the ordinary differential geometrical point of view in σ-model is an open question.

Since Alain Connes[2] applied his non–commutative geometry to the particle physics model building in which the Higgs fields were introduced as gauge fields, many efforts have been done in this direction[3-6]. Especially, Sitarz[7] developed discrete points idea and built a gauge theory on discrete group, which is simple to be understood. Soon after, the physical model building was completed by the authors[8].

In the previous works, the main idea is consider Higgs field as gauge field. In [8], we developed the free fermion Lagrangian on discrete group which contain not only

[1]Work supported in part by The National Natural Science Foundation Of China.

[2]Mailing address

the derivative on space time but also the derivative on discrete group, if we require the Lagrangian is invariant under local transformation of a gauge group, besides the Yang–Mills gauge field was introduced, another gauge field must be introduced also, which is just the Higgs field, corresponding to the standard model, the gauge group is $SU_L(2) \times U_Y(1)$, in the following we will repeat the main idea in detail.

In this letter, we found that σ-model can be set up by the gauge theory in non-commutative geometry. If the system only have global chiral invariant, the gauge group elements are not the function on space time, but they might be the function on discrete group. Using this idea we can build the σ–model by the gauge theory on discrete group.

2 Gauge Theory on Discrete Group

Let us first assign the free fermion field with respect to Z_2 elements as follow

$$\psi(x,e) = \psi_L(x), \quad \psi(x,r) = \psi_R(x), \quad e,r \in \{Z_2\} \tag{1}$$

we write down a Lagrangian on discrete group Z_2 as follow

$$\mathcal{L}(x,r) = \overline{\psi}(x,r)(i\gamma^\mu\partial_\mu + \mu\partial_Z)\psi(x,r) \tag{2}$$

where ∂_Z is the derivative on discrete group, acting on the function of discrete group as follow

$$\partial_Z f(h) = f(h) - R_Z f(h) = f(h) - f(h \cdot Z).$$

By noticing the fact

$$\mathcal{L}(x) = \int_{Z_2} \mathcal{L}(x,Z) == \overline{\psi}(x,r)(i\gamma^\mu\partial_\mu - \mu)\psi(x,Z)$$

just is the Lagrangian of free fermion in space time M^4, then we call that the Lagrangian (2) is the free fermion Lagrangian on $M^4 \times Z_2$

Similar to the reason that leads to the introduction of Yang-Mills fields, it is reasonable to require that the Lagrangian (2) be invariant under gauge transformations

$H(x,h),\ h \in Z_2$, where H not only are functions on M, but also on Z_2 as well

$$\psi(x,h) \to \psi(x,h)' = H(x,h)\psi(x,h). \tag{3}$$

Namely, we require Z_2 symmetry be also gauged. Then the first term in (2) should be replaced by

$$\overline{\psi}(x,h) i\gamma^\mu D_\mu \psi(x,h), \quad D_\mu = \partial_\mu + igA(x,h) \tag{4}$$

where $A(x,e) = L_\mu$ and $A(x,r) = R_\mu$ are gauge potentials valued on the Lie algebras of G_L and G_R, which are supposed to be semi-simple in this sub-section, respectively.

As for the second term $\mu\overline{\psi(x,h)}\partial_r\psi(x,h)$ in (2), it is also needed to introduce another gauge covariant derivative D_r to replace ∂_r and

$$D_r\psi(x,h) \to [D_r\psi(x,h)]' = H(x,h)D_r\psi(x,h) \tag{5}$$

in order that $\overline{\psi}(x,h)D_r\psi(x,h)$ is Z_2-gauge invariant. This can be realized if we introduce a field $\phi(x,h)$, the Higgs field, as a connection with respect to the Z_2-gauge symmetry and form the covariant derivative as follows

$$D_r\psi = (\partial_r + \frac{\lambda}{\mu}\phi R_r)\psi. \tag{6}$$

Then the transformation law (5) is satisfied if the generalized gauge field $\phi(x,h)$ has the transformation property

$$\frac{\mu}{\lambda} - \phi' = H(\frac{\mu}{\lambda} - \phi)(R_r H^{-1}). \tag{7}$$

We may introduce a new field $\Phi = \frac{\mu}{\lambda} - \phi$ such that the transformation rule (7) becomes

$$\Phi \to \Phi' = H\Phi(R_r H^{-1}). \tag{8}$$

Similar to the usual gauge theory where the covariant derivative is equivalent to the covariant exterior derivative $D_M = d_M + igA_\mu dx^\mu$ and $D_M f = D_\mu f dx^\mu$, for the case at hand, the covariant exterior derivative takes form

$$D_{Z_2} = d_{Z_2} + \frac{\lambda}{\mu}\phi\chi^r. \tag{9}$$

The reason is that

$$(d_{Z_2} + \frac{\lambda}{\mu}\phi\chi^r)f = (\partial_r + \frac{\lambda}{\mu}\phi R_r)f\chi^r = D_r f\chi^r.$$

Thus from (2), it follows the generalized gauge invariant Lagrangian for fermions in each sector characterized by Z_2

$$\begin{aligned}\mathcal{L}_F(x,h) &= \overline{\psi(x,h)}\{i\gamma^\mu(\partial_\mu + ig(h)A_\mu(x,h)) + (\mu\partial_r + \lambda\phi(x,h)R_r)\}\psi(x,h) \\ &= \overline{\psi}(i\gamma^\mu D_\mu + \mu D_r)\psi.\end{aligned} \tag{10}$$

The Hermitian property of operator ϕR_r requires that

$$\phi^\dagger(x,e) = R_r\phi(x,e) = \phi(x,r). \tag{11}$$

After integrating over Z_2 the last terms in $\mathcal{L}(x,h)$ we get

$$\int_{Z_2} \lambda\psi(x,h)\Phi(x,h)R_r\psi(x,h) = -\lambda\overline{\psi}_L(x)\Phi(x)\psi_R(x) - \lambda\overline{\psi}_R(x)\Phi^\dagger(x)\psi_L(x) \tag{12}$$

which are nothing but the Yukawa couplings between the Higgs and chiral fermions.

From direct calculation, similarly the antisymmetric tensor $F_{\mu\nu}$ is related to the covariant derivative as

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = igF_{\mu\nu}\psi, \tag{13}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].$$

We have $F_{r\mu}$ and F_{rr} are related to the covariant derivatives respectively

$$[D_r, D_\mu]\psi = \frac{\lambda}{\mu}F_{r\mu}R_r\psi = -\frac{\lambda}{\mu}F_{\mu r}R_r\psi, \tag{14}$$

$$(D_r D_r - 2D_r) = \frac{\lambda^2}{\mu^2}F_{rr}\psi.$$

where

$$F_{r\mu} = \partial_\mu\Phi + igA_\mu\Phi - \Phi R_r(igA_\mu), \tag{15}$$

$$F_{rr} = \Phi\Phi^\dagger - \frac{\mu^2}{\lambda^2}.$$

In [7], Sitarz show that if we consider the linear term of the field strength, all the constrains on the parameter should be relaxed. Then we write down the most general Lagrangian for the gauge bosons and Higgs as following:

$$\begin{aligned}\mathcal{L}_{YM-H}(x,h) &= -\tfrac{1}{4}TrL_{\mu\nu}(x)L^{\mu\nu}(x) - \tfrac{1}{4}TrR_{\mu\nu}(x)R^{\mu\nu}(x)\\ &+TrF_{\mu r}F_r^{\dagger\mu} - \eta TrF_{rr}F_{rr}^{\dagger}.\end{aligned} \tag{16}$$

where η is a positive real constant. Then we get the entire Lagrangian of the system

$$\mathcal{L}(x) = \mathcal{L}_F + \mathcal{L}_{YM-H}, \tag{17}$$

where $\mathcal{L}_F$ is the gauge invariant one for fermions with Yukawa couplings to Higgs and $\mathcal{L}_{YM-H}$ is the one for Yang-Mills fields and Higgs

$$\begin{aligned}\mathcal{L}_F = &\ \overline{\psi(x)}i\gamma^\mu\partial_\mu\psi(x)\\ &-g_1\overline{\psi}_L(x)\gamma^\mu L_\mu\psi_L(x) - g_2\overline{\psi}_R(x)\gamma^\mu R_\mu(x)\psi_R(x)\\ &-\lambda(\overline{\psi}_L(x)\Phi(x)\psi_R(x) + \overline{\psi}_R(x)\Phi(x)^\dagger\psi_L(x)),\\ \mathcal{L}_{YM-H} = &-\tfrac{1}{4}TrL_{\mu\nu}(x)L^{\mu\nu}(x) - \tfrac{1}{4}TrR_{\mu\nu}(x)R^{\mu\nu}(x)\\ &+Tr(D_\mu\Phi(x))(D^\mu\Phi(x))^\dagger\\ &-\eta Tr(\Phi(x)\Phi(x)^\dagger - \tfrac{\mu^2}{\lambda^2})^2.\end{aligned} \tag{18}$$

In this work, the most important idea is that the Yukawa coupling is introduce as gauge coupling, especially, even though Yang-Mills fields were zero, the scale field might exist also. As an example, we will study σ model in next section.

3 $SU_L(2) \times SU_R(2)$ σ model

Now we are ready to build the σ model by the knowledge of non-commutative geometry, for the nucleon doublet field $N = \begin{pmatrix} p \\ n \end{pmatrix}$, we set

$$\psi(x,e) = N_L, \quad \psi(x,r) = N_R \tag{19}$$

the gauge transformation $H(e) \in SU_L(2)$ and $H(r) \in SU_R(2)$ are global to the space time, but are local to the discrete group. For the elements of gauge group are not

depend on the space time, then the Yang-Mills gauge field $A(x,e) = A(x,r) = 0$; but they depend on the discrete group, therefore, the scale field Φ is nonzero. More general we can write Φ as 2×2 Hermitian matrices

$$\Phi(x) = \sigma I + i\tau^i\pi_i \tag{20}$$

where I is 2×2 identity and τ_i are three pauli matrices, σ and π are real scale fields. From the (17), we have the lagragian for the σ model

$$\begin{aligned}\mathcal{L} = &\ i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \lambda\overline{\psi}_L(x)\Phi(x)\psi_R(x) - \lambda\overline{\psi}_R(x)\Phi^\dagger(x)\psi_R(x)\\ &+\partial_\mu\Phi(\partial^\mu\Phi)^\dagger - \eta(\Phi\Phi^\dagger - \tfrac{\mu^2}{\lambda^2})^2\end{aligned} \tag{21}$$

Rewrite the Lagrangian in terms of the fields σ,π, we get another form expression

$$\mathcal{L} = i\overline{N}\gamma^\mu\partial_\mu N - \lambda\overline{N}(\sigma + i\gamma^5\tau\cdot\pi)N + (\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2 - \eta(\sigma^2+\pi^2-\frac{\mu^2}{\lambda^2})^2. \tag{22}$$

Now we discuss the transformation property of the fields. For the gauge transformation

$$H(r) = 1 + ih(r)\cdot\tau, \quad h\in\mathcal{A}, \quad r\in Z_2. \tag{23}$$

If we set $h(e) = \alpha$, $h(Z) = \beta$, gauge transformation (23) is

$$H(e) = SU_L(2) = 1 + i\alpha\cdot\tau, \quad H(Z) = SU_R(2) = 1 + i\beta\cdot\tau.$$

From (8), we have that under the gauge transformation

$$\begin{aligned}&N \to N + i\tfrac{\alpha+\beta}{2}\cdot\tau N + i\tfrac{\alpha-\beta}{2}\cdot\tau\gamma^5 N\\ &\sigma + i\pi\cdot\tau \to \sigma - (\alpha+\beta)\cdot\pi + i[\pi - \sigma(\alpha+\beta) - (\alpha-\beta)\wedge\pi]\cdot\tau\end{aligned} \tag{24}$$

If we redefine the parameter as $\alpha' = \alpha+\beta, \beta' = \beta-\alpha$, we get

$$\begin{aligned}&N \to N + i\tfrac{\alpha'}{2}\cdot\tau N - i\tfrac{\beta'}{2}\cdot\tau\gamma^5 N\\ &\sigma \to \sigma - \alpha'\cdot\pi\\ &\pi \to \pi - \sigma\alpha' + \beta'\wedge\pi\end{aligned} \tag{25}$$

which is just the one for the σ model.

As the Higgs fields, the σ model has its geometry origin. Similarly, the other effective Lagrangian models can be studied in non-commutative geometry also. We will discuss these topics in the following papers.

References

1. J.Schwinger, Ann.Phys.**2**,(1958)407; M.Gell-Mann and M.Levy, Nuovo Cimento-**16**(1960)705.
2. A. Connes, in: The Interface of Mathematics and Particle Physics, eds. D. Quillen, G. Segal and S. Tsou (Oxford U. P, Oxford 1990); A. Connes and J. Lott, Nucl. Phys. (Proc. Suppl.) **B18**, 44 (1990); A. Connes and Lott, Proceedings of 1991 Cargese Summer Conference; See also A. Connes, *Non-Commutative Geometry*. **IHES/M/93/12.**
3. D. Kastler, Marseille, CPT preprint **CPT-91/P.2610**, **CPT-91/P.2611**.
4. R. Coquereaux, G. Esposito-Farése and G Vaillant, Nucl Phys **B353** 689 (1991).
5. M. Dubois-Violette, R. Kerner, J. Madore, J. Math. Phy. **31**. (1990) 316; B. S. Balakrishna, F Gürsey and K. C. Wali, Phys Lett **B254**, 430 (1991).
6. A. H. Chamseddine, G Felder and J. Fröhlich, Phys. Lett. **296B** (1993) 109.
7. A. Sitarz, Non-commutative Geometry and Gauge Theory on Discrete Groups, preprint **TPJU**-7/1992; A.Sitarz, Phys. Lett **308B**(1993) 311.
8. Hao-Gang Ding, Han-Ying Guo, Jian-Ming Li and Ke Wu, Higgs as Gauge Fields On Discrete Groups and Standard Model For Electroweak and Electroweak-Strong Interactions, preprint **ASITP**-93-23,**CCAST**-93-5.