Recursion Relations for Clebsch-Gordan Coefficients of $U_q(su_2)$ and $U_q(su_{1,1})$

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Abstract

We report in this article three- and four-term recursion relations for Clebsch-Gordan coefficients of the quantum algebras $U_q(su_2)$ and $U_q(su_{1,1})$. These relations were obtained by exploiting the complementarity of three quantum algebras in a q-deformation of $sp(8,\mathbb{R})$.

1. Introduction

The theory of quantum algebras has been the object of numerous investigations both in physics and mathematics. In particular, the one-parameter quantum algebras $U_q(su_2)$ and $U_q(su_{1,1})$ have been investigated by many authors (see, for instance, Refs. [1,2]). In addition, two-parameter deformations of su_2 and u_2 have been worked out in various papers [3-8].

The application to physics of the quantum algebras $U_q(su_2)$ and $U_q(su_{1,1})$ requires the knowledge of the corresponding coupling and recoupling coefficients. Clebsch-Gordan coefficients for $U_q(su_2)$ (and $U_q(su_{1,1})$), in one- and two-parameter formulations, have been calculated by several people (e.g., see Refs. [5,7]). In addition, recursion relations for Clebsch-

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Gordan coefficients of $U_q(su_2)$ and $U_q(su_{1,1})$ have also been derived, as an extension of the relations corresponding to the $q = 1$ case [9-13], in Refs. [14-16].

It is the aim of this contribution to list three- and four-term recursion relations for the Clebsch-Gordan coefficients of $U_q(su_2)$ derived from an algorithm recently described in Ref. [17]. We shall also give a few three-term recursion relations for the Clebsch-Gordan coefficients of $U_q(su_{1,1})$ derived by means of this algorithm.

This work is dedicated to the memory of the late Professor Ya.A. Smorodinskiĭ who contributed, among many other fields, to the Wigner-Racah algebra of the group SU_2.

2. The algorithm

The algorithm applied in this article follows from the work of Schwinger [9] and was implicitly used by Rasmussen [11] and Kibler and Grenet [12] in order to derive recursion relations in the $q = 1$ case. In the $q \neq 1$ case, this algorithm has been described in detail by Smirnov and Kibler [17]. We shall not repeat the description of the algorithm in the present paper. However, we shall briefly mention here its main features.

We start from a q-deformation of the dynamical invariance algebra $sp(8, \mathbb{R})$ of the four-dimensional harmonic oscillator. Following the work of Moshinsky and Quesne [18] on the complementarity of Lie groups, we can extract three complementary algebras from two chains of quantum algebras having the q-deformation of $sp(8, \mathbb{R})$ as the head algebra. These algebras are denoted in Ref. [17] as $U_q(su_2^{\mathcal{J}})$, $U_q(su_2^{\Lambda})$ and $U_q(su_{1,1}^{\mathcal{K}})$ and their generators are collectively indicated by $\mathcal{J}$, Λ and $\mathcal{K}$, respectively. The latter generators are built from the four pairs of q-boson operators and the four number operators corresponding to the four-dimensional q-deformed harmonic oscillator. They are defined in such a way to satisfy the co-product rules for the Hopf algebras $U_q(su_2^{\mathcal{J}})$, $U_q(su_2^{\Lambda})$ and $U_q(su_{1,1}^{\mathcal{K}})$ (see Ref. [17]).

The algorithm amounts to calculating, in two different ways, matrix elements of the type $\langle n_1 n_2 n_3 n_4 | X | j : \mu m \kappa \rangle$, where X is either a linear form or a bilinear form of the generators $\{\mathcal{J}\}$, $\{\Lambda\}$ and $\{\mathcal{K}\}$. Furthermore, the vectors $|n_1 n_2 n_3 n_4\rangle$ are state vectors for the four-dimensional harmonic oscillator and the vectors $|j : \mu m \kappa\rangle$ are common eigenstates of $\mathcal{J}^2$, $\mathcal{J}_3$, Λ_3 and $\mathcal{K}_3$, where $\mathcal{J}^2$ stands for the common Casimir operator of $U_q(su_2^{\mathcal{J}})$, $U_q(su_2^{\Lambda})$ and $U_q(su_{1,1}^{\mathcal{K}})$.

It should be emphasized that the algorithm just described also furnishes an elegant way for deriving the q-analogue of Regge symmetries for $U_q(su_2)$ as well as some connecting formulas between Clebsch-Gordan coefficients of $U_q(su_2)$ and $U_q(su_{1,1})$ [17].

3. Recursion relations

We give below recursion relations for the Clebsch-Gordan coefficients of $U_q(su_2)$ and

$U_q(su_{1,1})$ obtained according to the algorithm described in Section 2.

In what follows, we employ the abbreviations

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$$
$$\mathrm{h}_\pm = \tfrac{1}{2} \pm \tfrac{1}{2} \quad \mathrm{h}_\mp = \tfrac{1}{2} \mp \tfrac{1}{2}$$
$$\mathrm{u}_\pm = 1 \pm 1 \quad \mathrm{u}_\mp = 1 \mp 1$$

In addition, the Clebsch-Gordan coefficients for $U_q(su_{1,1})$ (for the positive discrete series) and $U_q(su_2)$ are written as $(k_1 k_2 \kappa_1 \kappa_2 | j\kappa)_q$ and $(j_1 j_2 m_1 m_2 | jm)_q$, respectively.

3.1. Recursion relations for $U_q(su_2)$

1. The action of $\mathcal{J}_\pm$ on $|j_1 \mp \frac{1}{2}, j_2 \pm \frac{1}{2}, j, m\rangle_q$ leads to

$$\sqrt{[j \mp j_1 \pm j_2 + 1]\,[j \pm j_1 \mp j_2]}\,(j_1 j_2 m_1 m_2 | jm)_q$$
$$= q^{+\frac{1}{2}(j_1 - m_1 - j_2 + m_2)} \sqrt{[j_1 + m_1 + \mathrm{h}_\mp]\,[j_2 + m_2 + \mathrm{h}_\pm]}\,(j_1 \mp \tfrac{1}{2}, j_2 \pm \tfrac{1}{2}, m_1 \mp \tfrac{1}{2}, m_2 \pm \tfrac{1}{2} | jm)_q$$
$$+ q^{-\frac{1}{2}(j_1 + m_1 - j_2 - m_2)} \sqrt{[j_1 - m_1 + \mathrm{h}_\mp]\,[j_2 - m_2 + \mathrm{h}_\pm]}\,(j_1 \mp \tfrac{1}{2}, j_2 \pm \tfrac{1}{2}, m_1 \pm \tfrac{1}{2}, m_2 \mp \tfrac{1}{2} | jm)_q$$

2. The action of $\mathcal{K}_\pm$ on $|j_1 \mp \frac{1}{2}, j_2 \mp \frac{1}{2}, j, m\rangle_q$ leads to

$$\sqrt{[j_1 + j_2 - j + \mathrm{h}_\mp]\,[j_1 + j_2 + j + 1 + \mathrm{h}_\mp]}\,(j_1 j_2 m_1 m_2 | jm)_q$$
$$= q^{+\frac{1}{2}(j_1 - m_1 + j_2 + m_2 + 1)} \sqrt{[j_1 + m_1 + \mathrm{h}_\mp]\,[j_2 - m_2 + \mathrm{h}_\mp]}\,(j_1 \mp \tfrac{1}{2}, j_2 \mp \tfrac{1}{2}, m_1 \mp \tfrac{1}{2}, m_2 \pm \tfrac{1}{2} | jm)_q$$
$$- q^{-\frac{1}{2}(j_1 + m_1 + j_2 - m_2 + 1)} \sqrt{[j_1 - m_1 + \mathrm{h}_\mp]\,[j_2 + m_2 + \mathrm{h}_\mp]}\,(j_1 \mp \tfrac{1}{2}, j_2 \mp \tfrac{1}{2}, m_1 \pm \tfrac{1}{2}, m_2 \mp \tfrac{1}{2} | jm)_q$$

3. The action of $\Lambda_\pm$ on $|j_1, j_2, j, m \mp 1\rangle_q$ leads to

$$\sqrt{[j \pm m]\,[j \mp m + 1]}\,(j_1 j_2 m_1 m_2 | jm)_q$$
$$= q^{+m_2} \sqrt{[j_1 \mp m_1 + 1]\,[j_1 \pm m_1]}\,(j_1, j_2, m_1 \mp 1, m_2 | jm \mp 1)_q$$
$$+ q^{-m_1} \sqrt{[j_2 \mp m_2 + 1]\,[j_2 \pm m_2]}\,(j_1, j_2, m_1, m_2 \mp 1 | jm \mp 1)_q$$

In the limiting case where $q = 1$, the recursion relations 1 to 3 give back well-known relations for the Lie algebra su_2 (see Refs. [12,13]). The six preceding recursion relations are in accordance with the results by Nomura [14], Groza *et al.* [15], Kachurik and Klimyk [15], and Aizawa [16] who derived recursion relations for the Clebsch-Gordan coefficients of $U_q(su_2)$ from q-deformed hypergeometric functions.

We now give twenty four-term recursion relations that can be obtained from twenty operators of type $\mathcal{K}\mathcal{K}$, $\mathcal{J}\mathcal{J}$, $\mathcal{J}\mathcal{K}$, $\Lambda\Lambda$, $\Lambda\mathcal{K}$ and $\Lambda\mathcal{J}$.

4. The action of $\mathcal{K}_\pm \mathcal{K}_\pm$ on $|j_1 \mp 1, j_2 \mp 1, j, m\rangle_q$ leads to

$$\sqrt{[j_1+j_2+j+1+\mathrm{u}_\mp]\,[j_1+j_2+j+\mathrm{u}_\mp]\,[j_1+j_2-j+\mathrm{u}_\mp]\,[j_1+j_2-j\mp 1]}$$
$$(j_1 j_2 m_1 m_2|jm)_q$$
$$= q^{+j_1-m_1+j_2+m_2+1}\sqrt{[j_1+m_1\mp 1]\,[j_1+m_1+\mathrm{u}_\mp]\,[j_2-m_2\mp 1]\,[j_2-m_2+\mathrm{u}_\mp]}$$
$$(j_1\mp 1, j_2\mp 1, m_1\mp 1, m_2\pm 1|jm)_q$$
$$-\,[2]\,q^{m_2-m_1}\sqrt{[j_1+m_1+\mathrm{h}_\mp]\,[j_1-m_1+\mathrm{h}_\mp]\,[j_2+m_2+\mathrm{h}_\mp]\,[j_2-m_2+\mathrm{h}_\mp]}$$
$$(j_1\mp 1, j_2\mp 1, m_1, m_2|jm)_q$$
$$+\,q^{-j_1-m_1-j_2+m_2-1}\sqrt{[j_1-m_1\mp 1]\,[j_1-m_1+\mathrm{u}_\mp]\,[j_2+m_2\mp 1]\,[j_2+m_2+\mathrm{u}_\mp]}$$
$$(j_1\mp 1, j_2\mp 1, m_1\pm 1, m_2\mp 1|jm)_q$$

5. The action of $\mathcal{J}_\pm \mathcal{J}_\pm$ on $|j_1 \mp 1, j_2 \pm 1, j, m\rangle_q$ leads to

$$\sqrt{[j\mp j_1\pm j_2+2]\,[j\pm j_1\mp j_2-1]\,[j\mp j_1\pm j_2+1]\,[j\pm j_1\mp j_2]}(j_1 j_2 m_1 m_2|jm)_q$$
$$= q^{+j_1-m_1-j_2+m_2}\sqrt{[j_1+m_1\mp 1]\,[j_1+m_1+\mathrm{u}_\mp]\,[j_2+m_2\pm 1]\,[j_2+m_2+\mathrm{u}_\pm]}$$
$$(j_1\mp 1, j_2\pm 1, m_1\mp 1, m_2\pm 1|jm)_q$$
$$+\,[2]\,q^{m_2-m_1}\sqrt{[j_1-m_1+\mathrm{h}_\mp]\,[j_1+m_1+\mathrm{h}_\mp]\,[j_2-m_2+\mathrm{h}_\pm]\,[j_2+m_2+\mathrm{h}_\pm]}$$
$$(j_1\mp 1, j_2\pm 1, m_1, m_2|jm)_q$$
$$+\,q^{-j_1-m_1+j_2+m_2}\sqrt{[j_1-m_1\mp 1]\,[j_1-m_1+\mathrm{u}_\mp]\,[j_2-m_2\pm 1]\,[j_2-m_2+\mathrm{u}_\pm]}$$
$$(j_1\mp 1, j_2\pm 1, m_1\pm 1, m_2\mp 1|jm)_q$$

6. The action of $\mathcal{J}_\pm \mathcal{K}_\pm$ on $|j_1 \mp 1, j_2, j, m\rangle_q$ leads to

$$\sqrt{[j_1+j_2\mp j+\mathrm{u}_\mp]\,[j_1+j_2\pm j+1]\,[j\pm j_1\mp j_2]\,[j\mp j_1\pm j_2+1]}(j_1 j_2 m_1 m_2|jm)_q$$
$$= \; q^{+j_1-m_1+m_2+\mathrm{h}_\pm}\sqrt{[j_1+m_1+\mathrm{u}_\mp]\,[j_1+m_1\mp 1]\,[j_2\mp m_2]\,[j_2\pm m_2+1]}$$
$$(j_1\mp 1, j_2, m_1\mp 1, m_2\pm 1|jm)_q$$
$$+\; q^{m_2-m_1}\sqrt{[j_1+m_1+\mathrm{h}_\mp]\,[j_1-m_1+\mathrm{h}_\mp]}$$
$$\left\{[j_2-m_2+\mathrm{h}_\pm]\,q^{+j_2+\mathrm{h}_\mp}-[j_2+m_2+\mathrm{h}_\pm]\,q^{-j_2-\mathrm{h}_\mp}\right\}(j_1\mp 1, j_2, m_1, m_2|jm)_q$$
$$-\; q^{-j_1-m_1+m_2-\mathrm{h}_\pm}\sqrt{[j_1-m_1+\mathrm{u}_\mp]\,[j_1-m_1\mp 1]\,[j_2\pm m_2]\,[j_2\mp m_2+1]}$$
$$(j_1\mp 1, j_2, m_1\pm 1, m_2\mp 1|jm)_q$$

7. The action of $\mathcal{J}_\pm \mathcal{K}_\mp$ on $|j_1, j_2 \pm 1, j, m\rangle_q$ leads to

$$\sqrt{[j_1+j_2\pm j+\mathrm{u}_\pm]\ [j_1+j_2\mp j+1]\ [j\mp j_1\pm j_2+1]\ [j\pm j_1\mp j_2]}(j_1j_2m_1m_2|jm)_q$$
$$=-\,q^{-j_2+m_2-m_1-\mathrm{h}_\mp}\sqrt{[j_1\pm m_1]\ [j_1\mp m_1+1]\ [j_2+m_2+\mathrm{u}_\pm]\ [j_2+m_2\pm 1]}$$
$$(j_1,j_2\pm 1,m_1\mp 1,m_2\pm 1|jm)_q$$
$$+\,q^{m_2-m_1}\sqrt{[j_2+m_2+\mathrm{h}_\pm]\ [j_2-m_2+\mathrm{h}_\pm]}$$
$$\left\{[j_1+m_1+\mathrm{h}_\mp]\ q^{+j_1+\mathrm{h}_\pm}-[j_1-m_1+\mathrm{h}_\mp]\ q^{-j_1-\mathrm{h}_\pm}\right\}$$
$$(j_1,j_2\pm 1,m_1,m_2|jm)_q$$
$$+\,q^{+j_2+m_2-m_1+\mathrm{h}_\mp}\sqrt{[j_1\mp m_1]\ [j_1\pm m_1+1]\ [j_2-m_2+\mathrm{u}_\pm]\ [j_2-m_2\pm 1]}$$
$$(j_1,j_2\pm 1,m_1\pm 1,m_2\mp 1|jm)_q$$

8. The action of $\Lambda_\pm\Lambda_\pm$ on $|j_1,j_2,j,m\mp 2\rangle_q$ leads to

$$\sqrt{[j\pm m-1]\ [j\pm m]\ [j\mp m+1]\ [j\mp m+2]}(j_1j_2m_1m_2|jm)_q$$
$$=q^{+2m_2}\sqrt{[j_1\pm m_1-1]\ [j_1\pm m_1]\ [j_1\mp m_1+1]\ [j_1\mp m_1+2]}$$
$$(j_1,j_2,m_1\mp 2,m_2|j,m\mp 2)_q$$
$$+\,[2]\ q^{m_2-m_1}\sqrt{[j_1\pm m_1]\ [j_1\mp m_1+1]\ [j_2\pm m_2]\ [j_2\mp m_2+1]}$$
$$(j_1,j_2,m_1\mp 1,m_2\mp 1|j,m\mp 2)_q$$
$$+\,q^{-2m_1}\sqrt{[j_2\pm m_2-1]\ [j_2\pm m_2]\ [j_2\mp m_2+1]\ [j_2\mp m_2+2]}$$
$$(j_1,j_2,m_1,m_2\mp 2|j,m\mp 2)_q$$

9. The action of $\Lambda_+\mathcal{J}_\pm$ on $|j_1\mp\frac{1}{2},j_2\pm\frac{1}{2},j,m-1\rangle_q$ leads to

$$\sqrt{[j-m+1]\ [j+m]\ [j\mp j_1\pm j_2+1]\ [j\pm j_1\mp j_2]}(j_1j_2m_1m_2|jm)_q$$
$$=q^{\pm\frac{1}{2}(j_1\mp m_1-j_2\pm 3m_2\pm 1)}\sqrt{[j_1\pm m_1\mp 1]\ [j_1-m_1+1+\mathrm{h}_\mp]\ [j_1+m_1]\ [j_2\pm m_2+\mathrm{h}_\pm]}$$
$$(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1-\tfrac{3}{2},m_2+\tfrac{1}{2}|j,m-1)_q$$
$$+\left\{[j_1\mp m_1+\mathrm{h}_\pm]\ q^{\mp\frac{1}{2}(j_1\pm m_1-j_2\mp 3m_2\mp 1)}+[j_2\pm m_2+\mathrm{h}_\mp]\ q^{\pm\frac{1}{2}(j_1\mp 3m_1-j_2\pm m_2\mp 1)}\right\}$$
$$\sqrt{[j_1\pm m_1+\mathrm{h}_\mp]\ [j_2\mp m_2+\mathrm{h}_\pm]}(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1-\tfrac{1}{2},m_2-\tfrac{1}{2}|j,m-1)_q$$
$$+q^{\mp\frac{1}{2}(j_1\pm 3m_1-j_2\mp m_2\pm 1)}\sqrt{[j_2\mp m_2\pm 1]\ [j_2-m_2+1+\mathrm{h}_\pm]\ [j_2+m_2]\ [j_1\mp m_1+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1+\tfrac{1}{2},m_2-\tfrac{3}{2}|j,m-1)_q$$

10. The action of $\Lambda_+\mathcal{K}_\pm$ on $|j_1\mp\frac{1}{2},j_2\mp\frac{1}{2},j,m-1\rangle_q$ leads to

$$\sqrt{[j-m+1]\ [j+m]\ [j_1+j_2\pm j+1]\ [j_1+j_2\mp j+\mathrm{u}_\mp]}(j_1j_2m_1m_2|jm)_q$$
$$=\pm\, q^{\pm\frac{1}{2}(j_1\mp m_1+j_2\pm 3m_2+\mathrm{u}_\pm)}\sqrt{[j_1+m_1]\ [j_1\pm m_1\mp 1]\ [j_1-m_1+1+\mathrm{h}_\mp]\ [j_2\mp m_2+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1-\tfrac{3}{2},m_2+\tfrac{1}{2}|j,m-1)_q$$
$$\pm\left\{[j_2\mp m_2+\mathrm{h}_\pm]\ q^{\pm\frac{1}{2}(j_1\mp 3m_1+j_2\pm m_2+\mathrm{u}_\mp)}-[j_1\mp m_1+\mathrm{h}_\pm]\ q^{\mp\frac{1}{2}(j_1\pm m_1+j_2\mp 3m_2+\mathrm{u}_\mp)}\right\}$$
$$\sqrt{[j_1\pm m_1+\mathrm{h}_\mp]\ [j_2\pm m_2+\mathrm{h}_\mp]}(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1-\tfrac{1}{2},m_2-\tfrac{1}{2}|j,m-1)_q$$
$$\mp\, q^{\mp\frac{1}{2}(j_1\pm 3m_1+j_2\mp m_2+\mathrm{u}_\pm)}\sqrt{[j_2+m_2]\ [j_2\pm m_2\mp 1]\ [j_2-m_2+1+\mathrm{h}_\mp]\ [j_1\mp m_1+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1+\tfrac{1}{2},m_2-\tfrac{3}{2}|j,m-1)_q$$

11. The action of $\Lambda_-\mathcal{J}_\pm$ on $|j_1\mp\frac{1}{2},j_2\pm\frac{1}{2},j,m+1\rangle_q$ leads to

$$\sqrt{[j-m]\ [j+m+1]\ [j\mp j_1\pm j_2+1]\ [j\pm j_1\mp j_2]}(j_1j_2m_1m_2|jm)_q$$
$$=q^{\pm\frac{1}{2}(j_1\mp 3m_1-j_2\pm m_2\pm 1)}\sqrt{[j_2-m_2]\ [j_2\pm m_2\pm 1]\ [j_2+m_2+1+\mathrm{h}_\pm]\ [j_1\pm m_1+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1-\tfrac{1}{2},m_2+\tfrac{3}{2}|j,m+1)_q$$
$$+\left\{[j_1\pm m_1+\mathrm{h}_\pm]\ q^{\pm\frac{1}{2}(j_1\mp m_1-j_2\pm 3m_2\mp 1)}+[j_2\mp m_2+\mathrm{h}_\mp]\ q^{\mp\frac{1}{2}(j_1\pm 3m_1-j_2\mp m_2\mp 1)}\right\}$$
$$\sqrt{[j_1\mp m_1+\mathrm{h}_\mp]\ [j_2\pm m_2+\mathrm{h}_\pm]}(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1+\tfrac{1}{2},m_2+\tfrac{1}{2}|j,m+1)_q$$
$$+\, q^{\mp\frac{1}{2}(j_1\pm m_1-j_2\mp 3m_2\pm 1)}\sqrt{[j_1-m_1]\ [j_1\mp m_1\mp 1]\ [j_1+m_1+1+\mathrm{h}_\mp]\ [j_2\mp m_2+\mathrm{h}_\pm]}$$
$$(j_1\mp\tfrac{1}{2},j_2\pm\tfrac{1}{2},m_1+\tfrac{3}{2},m_2-\tfrac{1}{2}|j,m+1)_q$$

12. The action of $\Lambda_-\mathcal{K}_\pm$ on $|j_1\mp\frac{1}{2},j_2\mp\frac{1}{2},j,m+1\rangle_q$ leads to

$$\sqrt{[j-m]\ [j+m+1]\ [j_1+j_2\mp j+\mathrm{u}_\mp]\ [j_1+j_2\pm j+1]}(j_1j_2m_1m_2|jm)_q$$
$$=\pm\, q^{\pm\frac{1}{2}(j_1\mp 3m_1+j_2\pm m_2+\mathrm{u}_\pm)}\sqrt{[j_1\pm m_1+\mathrm{h}_\mp]\ [j_2-m_2]\ [j_2\mp m_2\mp 1]\ [j_2+m_2+1+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1-\tfrac{1}{2},m_2+\tfrac{3}{2}|j,m+1)_q$$
$$\pm\left\{[j_1\pm m_1+\mathrm{h}_\pm]\ q^{\pm\frac{1}{2}(j_1\mp m_1+j_2\pm 3m_2+\mathrm{u}_\mp)}-[j_2\pm m_2+\mathrm{h}_\pm]\ q^{\mp\frac{1}{2}(j_1\pm 3m_1+j_2\mp m_2+\mathrm{u}_\mp)}\right\}$$
$$\sqrt{[j_1\mp m_1+\mathrm{h}_\mp]\ [j_2\mp m_2+\mathrm{h}_\mp]}(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1+\tfrac{1}{2},m_2+\tfrac{1}{2}|j,m+1)_q$$
$$\mp\, q^{\mp\frac{1}{2}(j_1\pm m_1+j_2\mp 3m_2+\mathrm{u}_\pm)}\sqrt{[j_2\pm m_2+\mathrm{h}_\mp]\ [j_1-m_1]\ [j_1\mp m_1\mp 1]\ [j_1+m_1+1+\mathrm{h}_\mp]}$$
$$(j_1\mp\tfrac{1}{2},j_2\mp\tfrac{1}{2},m_1+\tfrac{3}{2},m_2-\tfrac{1}{2}|j,m+1)_q$$

13. The action of $\Lambda_\pm\Lambda_\mp$ on $|j_1,j_2,j,m\rangle_q$ leads to

$$[j\pm m]\ [j\mp m+1](j_1j_2m_1m_2|jm)_q$$
$$=q^{m_2-m_1+1}\sqrt{[j_1-m_1+1]\ [j_1+m_1]\ [j_2+m_2+1]\ [j_2-m_2]}\,(j_1,j_2,m_1-1,m_2+1|jm)_q$$
$$+\left\{q^{+2m_2}\ [j_1\mp m_1+1]\ [j_1\pm m_1]+q^{-2m_1}\ [j_2\mp m_2+1]\ [j_2\pm m_2]\right\}(j_1j_2m_1m_2|jm)_q$$
$$+\, q^{m_2-m_1-1}\sqrt{[j_1+m_1+1]\ [j_1-m_1]\ [j_2-m_2+1]\ [j_2+m_2]}\,(j_1,j_2,m_1+1,m_2-1|jm)_q$$

In the limiting case where $q = 1$, the recursion relations 4 to 12 are in agreement with the results of Kibler and Grenet [12].

3.2. Recursion relations for $U_q(su_{1,1})$

14. The action of $\mathcal{J}_\pm$ on $|k_1 \mp \frac{1}{2}, k_2 \pm \frac{1}{2}, j, \kappa\rangle_q$ leads to

$$\begin{aligned}
&\sqrt{[j \mp k_1 \pm k_2 + 1]\ [j \pm k_1 \mp k_2]}\ (k_1 k_2 \kappa_1 \kappa_2 | j\kappa)_q \\
&= q^{\frac{1}{2}(\pm\kappa_1 \mp \kappa_2 - k_1 + k_2)} \sqrt{[\kappa_1 \pm k_1]\ [\kappa_2 \pm k_2 \pm 1]}\ (k_1 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}, \kappa_1 \mp \tfrac{1}{2}, \kappa_2 \pm \tfrac{1}{2} | j\kappa)_q \\
&+ q^{\frac{1}{2}(\mp\kappa_1 \pm \kappa_2 - k_1 + k_2)} \sqrt{[\kappa_2 \mp k_2]\ [\kappa_1 \mp k_1 \mp 1]}\ (k_1 + \tfrac{1}{2}, k_2 - \tfrac{1}{2}, \kappa_1 \mp \tfrac{1}{2}, \kappa_2 \pm \tfrac{1}{2} | j\kappa)_q
\end{aligned}$$

15. The action of $\mathcal{K}_\pm$ on $|k_1, k_2, j, \kappa \mp 1\rangle_q$ leads to

$$\begin{aligned}
&\sqrt{[\kappa \mp j \mp 1]\ [\kappa \pm j]}\ (k_1 k_2 \kappa_1 \kappa_2 | j\kappa)_q \\
&= \pm q^{\frac{1}{2}(\pm\kappa_1 \pm \kappa_2 - k_1 + k_2)} \sqrt{[\kappa_1 \pm k_1]\ [\kappa_2 \mp k_2 \mp 1]}\ (k_1 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}, \kappa_1 \mp \tfrac{1}{2}, \kappa_2 \mp \tfrac{1}{2} | j, \kappa \mp 1)_q \\
&\mp q^{\frac{1}{2}(\mp\kappa_1 \mp \kappa_2 - k_1 + k_2)} \sqrt{[\kappa_2 \pm k_2]\ [\kappa_1 \mp k_1 \mp 1]}\ (k_1 + \tfrac{1}{2}, k_2 - \tfrac{1}{2}, \kappa_1 \mp \tfrac{1}{2}, \kappa_2 \mp \tfrac{1}{2} | j, \kappa \mp 1)_q
\end{aligned}$$

16. The action of $\Lambda_\pm$ on $|k_1 \mp \frac{1}{2}, k_2 \mp \frac{1}{2}, j, \kappa\rangle_q$ leads to

$$\begin{aligned}
&\sqrt{[j \pm k_1 \pm k_2 \pm 1]\ [j \mp k_1 \mp k_2 + \mathrm{u}_\mp]}\ (k_1 k_2 \kappa_1 \kappa_2 | j\kappa)_q \\
&= q^{+\frac{1}{2}(k_1 + k_2 - \kappa_1 + \kappa_2 + \mathrm{u}_\mp)} \sqrt{[\kappa_1 - k_1 - \mathrm{h}_\mp]\ [\kappa_1 + k_1 + \mathrm{h}_\mp]}\ (k_1 \mp 1, k_2, \kappa_1, \kappa_2 | j\kappa)_q \\
&+ q^{-\frac{1}{2}(k_1 + k_2 + \kappa_1 - \kappa_2 + \mathrm{u}_\mp)} \sqrt{[\kappa_2 - k_2 - \mathrm{h}_\mp]\ [\kappa_2 + k_2 + \mathrm{h}_\mp]}\ (k_1, k_2 \mp 1, \kappa_1, \kappa_2 | j\kappa)_q
\end{aligned}$$

The recursion relations 14 to 16 are in accordance with the results by Nomura [14], Groza *et al.* [15], Kachurik and Klimyk [15], and Aizawa [16] who derived recursion relations for the Clebsch-Gordan coefficients of $U_q(su_{1,1})$ from q-deformed hypergeometric functions.

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