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Analvtic Criteria for Stability of Beam Loaded R.F. Systems

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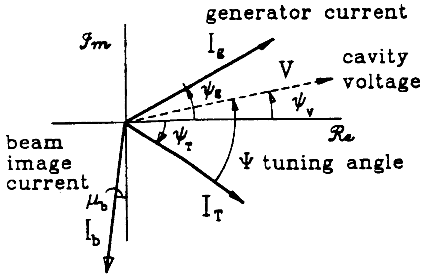
This paper presents the instability analysis of a beam-loaded radio-frequency system with beam phase-loop and cavity tuning-loop for both accelerating and non-accelerating beams. The case of voltage-proportional feedback around the cavity is also included. The symbolic manipulation program SMP [1] was used to expand and simplify the Routh determinantal conditions for a fifth order characteristic polynomial. The paper is a much abridged version of an internal design note [2].

I. Beam Loading Equations



The disposition of steady state phasors is as shown above. We adopt the notation of Reference [2]. The cavity voltage is $\mathbf{V}(t)e^{j\omega t}$ and the total current driving the cavity is $\mathbf{I}_T(t)e^{j\omega t}$, where t indicates time and ω is the drive angular frequency. Bold face indicates complex quantities, and ordinary type denotes scalars. We employ dot notation for time derivatives. The cavity fundamental resonance is modelled as a parallel resonance LCR circuit. Let $\Omega_{res} = 1/\sqrt{LC}$ be the resonance frequency and $\alpha = \Omega_{res}/(2Q) = 1/(2RC)$ be the half-bandwidth. We write the voltage and current as the sum of steady state parts $\mathbf{V}^0 = V^0 e^{j\psi_V}$ and $\mathbf{I}_T^0 = I_T^0 e^{j\psi_T}$, and small time dependent perturbations. We use ψ to denote steady state phases and ϕ perturbation phases. Let $\Psi = \psi_V - \psi_T$.

A. *Steady state*

We must specify the steady state generator current $\mathbf{I}_g^0 = I_g^0 e^{j\psi_g}$ and beam image current $\mathbf{I}_b^0 = I_b^0 e^{j\psi_b}$ which sum to form the total current $\mathbf{I}_T^0$. The beam current is $\approx 90°$ out of phase with the cavity voltage; depending on the synchronous phase angle μ_b. We set $\mu_b = 0$ for a non-accelerating beam. Hence $\psi_b = \pm(\pi/2 + \mu_b)$ and the $-$ sign applies below transition energy and the $+$ above. We adopt the dimensionless current ratios $Y_g = I_g^0/I_V^0$ and $Y_b = I_b^0/I_V^0$, where $I_V^0 = V^0/R$. The components obey:

$$1 = Y_g \cos\psi_g - Y_b \sin\mu_b$$
$$\tan\Psi = Y_b \cos\mu_b - Y_g \sin\psi_g \ .$$

From this follows the detuning $\tan\Psi = (\Omega^2 - \omega^2)/2\alpha\omega$. Until we choose a definite value for ψ_g, there is no direct relation between $\tan\Psi$ and $(Y_b\ , \mu_b)$.

B. *Non steady state*

Let us assume the "slow approximation" $\ddot{\mathbf{V}} \ll \omega\dot{\mathbf{V}}$ and $\dot{\mathbf{I}}_T \ll \omega\mathbf{I}_T$. We allow for a varying resonance frequency $\Omega(t) = \Omega_0(t) + \Delta\Omega(t)$. We introduce the perturbation vectors $\mathbf{e}$ as follows: $\mathbf{V} = \mathbf{V}^0(1 + \mathbf{e}_V)$ and $\mathbf{I}_T^0(1 + \mathbf{e}_T) = = \mathbf{I}_g^0(1 + \mathbf{e}_g) + \mathbf{I}_b^0(1 + \mathbf{e}_b)$. The dimensionless components z_r and ϕ_r of the vector $\mathbf{e}_r = (z_r + j\phi_r)$ are amplitude and phase modulations, respectively. The cavity response is modelled by:

$$z_V(1 + s\tau_c) + \phi_V \tan\Psi + Y_g(\phi_g \sin\psi_g - z_g \cos\psi_g) + + Y_b(z_b \sin\mu_b - \phi_b \cos\mu_b) = 0 \ ,$$

$$\phi_V(1 + s\tau_c) - z_V \tan\Psi - Y_g(\phi_g \cos\psi_g + z_g \sin\psi_g) + + Y_b(z_b \cos\mu_b + \phi_b \sin\mu_b) = \tau_c \Delta\Omega \ .$$

Here $\tau_c = \alpha^{-1}$ is the cavity time constant, and time derivatives are replaced by the Laplace operator s .

C. *Beam rigid bunch dipole motion*

Suppose the ideal drive frequency is synchronous with a particle travelling with the equilibrium. However, as a result of modulations the cavity phase may advance or lag by an amount ϕ_V. Likewise, the beam centroid may differ from the ideal phase by an amount ϕ_b. Suppose the cavity has relative amplitude modulation z_V.

To first order in perturbation amplitudes, the Laplace transform, of the beam energy deviation δE is:

$$s\,\delta E = K_1[z_V \sin\mu_b + (\phi_V - \phi_b)\cos\mu_b] \ .$$

Because of the energy deviation, the phase error ϕ_b will advance at the rate: $s\,\phi_b = K_2 \delta E$. The product $\sqrt{K_1 \times K_2} = \Omega_s$, the synchrotron frequency *sans* the usual $\cos\mu_b$ term.

D. *Stability conditions*

The system response contains only self-damped oscillations, when all zeros of the characteristic polynomial lie in the left half of the complex plane. Necessary conditions are for the coefficients of s^n and the Routh-Hurwitz criteria [RH(i) for $i = 1, 2, \ldots n{+}1$] for combinations of the coefficients to be greater than zero. We shall omit trivial conditions such as $\tau_c > 0$.

II. Cavity and beam dipole mode

This is the case originally treated by Robinson [3]. The model assumes that the generator current is maintained by an ideal feed-forward.

Characteristic polynomial

$$\Omega_s^2[\cos\mu_b \sec^2\Psi - Y_b \tan\Psi] + 2\Omega_s^2 \cos(\mu_b)\tau_c\, s + + [\sec^2\Psi + (\Omega_s\tau_c)^2 \cos\mu_b]s^2 + 2\tau_c\, s^3 + \tau_c^2\, s^4 \ .$$

Routh determinants

RH(4): $\tan\Psi \geq 0$, hence $\Psi \geq 0$. If RH(4) < 0, then the cavity is detuned in the wrong sense.

RH(5): $\cos\mu_b \sec^2\Psi - Y_b\tan\Psi > 0$ implies the Robinson limit: $Y_b < 2\cos\mu_b/\sin 2\Psi$. If RH(5) < 0, the bunch simply wanders. Substituting the matched generator condition ($\psi_g = 0$) gives the special case $Y_b < 1/\sin\mu_b$.

III. Cavity, beam dipole mode, phase-loop

The model of section II is supplemented with a beam phase-loop intended to damp bunch dipole oscillations. We assume that the feedback has the response of a pure integrator, and modifies the generator phase ϕ_g, that is $\phi_g = (K_p/s)\times(\phi_b - \phi_V)$. If there is an r.f. feedback around the cavity, this loop modifies the demand phase ϕ_d.

Characteristic polynomial

$$\Omega_s^2[\cos\mu_b\sec^2\Psi + K_p\tau_c\sin\mu_b Y_g\sin\psi_g - Y_b\tan\Psi]+$$
$$+[K_p\sec^2\Psi + K_pY_b(\sin\mu_b - \cos\mu_b\tan\Psi)+2\Omega_s^2\cos(\mu_b)\tau_c]s+$$
$$+[\sec^2\Psi + K_p\tau_c(1+Y_b\sin\mu_b)+(\Omega_s\tau_c)^2\cos\mu_b]s^2+2\tau_c s^3+\tau_c^2 s^4$$

A necessary condition for stability is that the coefficient of s^1 be > 0. Unless $\tan\Psi \leq \tan\mu_b$ and $K_p > 0$, we find a condition for Y_b which resembles the Robinson limit;

$$Y_b < \frac{2}{\sin 2\Psi} + \frac{2\Omega_s^2\tau_c}{K_p\tan\Psi} \quad \text{if } \mu_b = 0\,.$$

In most cases this limit is subordinate to RH(5) below.

Routh determinants

RH(3): $2 + K_p\tau_c[\cos 2\Psi + Y_b\cos\Psi\sin(\Psi+\mu_b)] > 0$.
This condition allows a domain of stability with $\Psi+\mu_b < 0$. The damping provided by the phase-loop can overcome (partially) the instability caused by incorrect detuning.

RH(5): $\cos\mu_b\sec^2\Psi + K_p\tau_c\sin\mu_b Y_g\sin\psi_g - Y_b\tan\Psi > 0$.
Unless $\psi_g\times\mu_b > 0$ there is no change to the Robinson limit.

RH(4): $$0 \leq 2K_p\sec^2\Psi[\sec^2\Psi + Y_b(\sin\mu_b - \cos\mu_b\tan\Psi)+$$
$$+(\Omega_s\tau_c)^2\cos\mu_b(\cos 2\Psi + \tan\mu_b\sin 2\Psi)]+$$
$$+2K_p(\Omega_s\tau_c)^2\cos\mu_b Y_b(\cos\mu_b\tan\Psi - \sin\mu_b)+4\Omega_s^2\tau_c Y_b\tan\Psi+$$
$$+\tau_c K_p^2[(1+Y_b\sin\mu_b)^2 - (Y_g\sin\psi_g\tan\Psi)^2]\,.$$

A sufficient condition for $RH(4) > 0$ is $\tan\Psi = \tan\mu_b$. Alternatively, we may substitute $\psi_g = 0$ and so find $RH(4) > 0$ at all points on the matched generator curve. Finally, we note that $\mu_b = 0$, $\tan\Psi < 1/\tan\psi_g$ and RH(5) > 0 are sufficient conditions for RH(4) > 0.

IV. Cavity, beam dipole mode, and tuning loop

A feedforward (or program) accomplishes the bulk of the cavity tuning. The tuning loop endeavours to bring the generator current and gap voltage vectors in-phase by modifying the cavity resonance frequency. The feedback, for small oscillations about the program set-point, is modelled by a pure integrator: $\tau_c\,\Delta\Omega_{res} = (K_t/s)\times(\phi_g - \phi_V)$. Since there are no other loops present, $\phi_g = 0$ for all time. The loop will tend to reduce the phase error to zero (i.e. $\phi_g = \phi_V$) provided K_t is positive.

Characteristic polynomial

$$\Omega_s^2\cos\mu_b K_t(1 - Y_b\sin\mu_b) + 2\tau_c s^4 + \tau_c^2 s^5+$$
$$+\Omega_s^2[\cos\mu_b(\sec^2\Psi + \tau_c K_t) - Y_b\tan\Psi]s+$$
$$+[K_t + 2\Omega_s^2\cos(\mu_b)\tau_c]s^2 + [\sec^2\Psi + \tau_c K_t + (\Omega_s\tau_c)^2\cos\mu_b]s^3.$$

A necessary condition for stability is that the coefficients of s^1 be greater than zero, and this implies

$$Y_b < \cos\mu_b\left[\frac{2}{\sin 2\Psi} + \frac{K_t\tau_c}{\tan\Psi}\right] \quad \text{if } \Psi > 0\,.$$

However, this condition is subordinate to RH(5).

Routh determinants

RH(3): $2\sec^2\Psi + K_t\tau_c \geq 0$.

RH(4):
$K_t(2\sec^2\Psi + K_t\tau_c) + Y_b\Omega_s^2\tau_c[4\tan\Psi - K_t\tau_c\sin 2\mu_b] \geq 0$.
This condition is usually unimportant for positive detuning ($\Psi > 0$), and is subordinate to RH(5) for negative detuning.

RH(5): This expression can be solved for the beam current Y_b, and is found to factor:

$$Y_b < [0.5K_t\sin 2\mu_b(\sec^2\Psi + \tau_c K_t) - K_t\tan\Psi+$$
$$+\Omega_s^2\cos(\mu_b)\tau_c(2\tan\Psi - 0.5K_t\tau_c\sin 2\mu_b)]\times$$
$$(2\sec^2\Psi + K_t\tau_c)/\Omega_s^2\tau_c(2\tan\Psi - 0.5\tau_c K_t\sin 2\mu_b)^2\,.$$

Since the beam current (Y_b) is positive, this leads to a quadratic constraint on the tuning loop gain.

We now simplify the expressions to a non-accelerating beam, to make a correspondence with Reference [4]. In the limit $\mu_b \to 0$ the stability criterion can be written:

$$Y_b < \left[1 - \frac{K_t}{2\Omega_s^2\tau_c}\right]\left[\frac{2}{\sin 2\Psi} + \frac{K_t\tau_c}{\tan\Psi}\right]\,.$$

The tuner gain condition, for +ve and -ve tuning angles, can be summarized $(K_t - 2\Omega_s^2\tau_c)\times\Psi < 0$. The instability regime where $Y_b \ll 1$, $\Psi > 0$ and $K_t > 2\Omega_s^2\tau_c$ has been experimentally observed in the PSB [4].

V. Tuning loop and beam phase-loop

We supplement the previous model with the ideal phase-loop; $s\phi_g = K_p(\phi_b - \phi_V)$. Because $s\tau_c\Delta\Omega_{res} = K_t(\phi_g - \phi_V)$ there is the possibility for cross-coupling to the tuning loop through the cavity-voltage phase-perturbation ϕ_g.

Characteristic polynomial

$$\Omega_s^2\cos\mu_b K_t(1 - Y_b\sin\mu_b) + 2\tau_c s^4 + \tau_c^2 s^5+$$
$$+\{\Omega_s^2[\cos\mu_b(\sec^2\Psi + \tau_c K_t) - Y_b\tan\Psi]+$$
$$K_p[K_t + \Omega_s^2\tau_c\sin\mu_b(Y_b\cos\mu_b - \tan\Psi)]\}s + \{K_t + 2\Omega_s^2\cos\mu_b\tau_c+$$
$$+K_p[\sec^2\Psi + \tau_c K_t + Y_b(\sin\mu_b - \cos\mu_b\tan\Psi)]\}s^2+$$
$$+[\sec^2\Psi + \tau_c K_t + (\Omega_s\tau_c)^2\cos\mu_b + \tau_c K_p(1 + Y_b\sin\mu_b)]s^3\,.$$

The coefficients of s^1 and s^2 have the possibility to change sign when $\Psi > 0$. For brevity we give the limit $\mu_b = 0$.

$$Y_b < \frac{2}{\sin 2\Psi} + \frac{K_t(\tau_c + K_p/\Omega_s^2)}{\tan\Psi} \quad \text{when } \mu_b = 0\,,\ \Psi > 0\,.$$

The coefficient of s^2 is automatically positive if $\tan\Psi \leq \tan\mu_b$; alternatively,

$$Y_b < \frac{2}{\sin 2\Psi} + \frac{2\Omega_s^2\tau_c + K_t(1+\tau_c K_p)}{K_p \tan\Psi} \quad \text{if } \mu_b = 0\,,\ \Psi > 0\,.$$

Routh determinants

RH(3) factors and simplifies to: $2 + \tau_c K_p \cos 2\Psi$ $+ Y_b \tau_c K_p \cos\Psi \sin(\Psi + \mu_b) + \tau_c K_t(1 - K_p\tau_c)\cos^2\Psi \geq 0$.

This condition is reminiscent of RH(3) in section III and has the effect of allowing some negative detuning. We should also like RH(3) to be satisfied in the limit $Y_b \to 0$; and for $K_p\tau_c \gg 1$ this implies the approximate condition: $K_t\tau_c < 2 - \sec^2\Psi \leq 1$.

RH(4): The Routh determinant has many terms, but simplifies under the substitution $\tan\Psi \Rightarrow Y_b\cos\mu_b$, as occurs when the generator is matched ($\psi_g = 0$); one finds a cubic condition in Y_b. A sufficient stability condition is that the coefficients of Y_b^0, Y_b^1, Y_b^2, Y_b^3 be greater than zero. Only the coefficients of Y_b^0 and Y_b^1 have the possibility to change sign; and so, by inspection, sufficient conditions for RH(4)> 0 are $\tau_c K_t \leq 1$ and $K_p \geq K_t$.

RH(5): The Routh determinant has many decades of monomial terms. Under the condition $\psi_g = 0$, there results a quintic polynomial in Y_b. The condition $\mu_b = 0$ reduces the system to a quadratic in Y_b^2 ; the coefficient of Y_b^4 is unavoidably negative, and so limits the maximum beam current. The allowed domain of Y_b will be maximized when the coefficients of Y_b^0 and Y_b^2 are positive. By inspection, $K_t\tau_c \leq 1$ and $K_p \geq K_t$ is a sufficient condition for both coefficients to be positive.

RH(6): $1 - Y_b \sin\mu_b > 0$ imposes a further constraint on the beam current, which is the same as the no-loop case for a matched generator.

A. *R.F. feedback around the cavity*

Including a voltage proportional feedback around the cavity modifies the equations. This type of feedback, as discussed in Reference [5], requires a high power summing junction since it is the entire r.f. signal which is fed back. The current $\mathbf{I}_g$ becomes the sum of the demand current $\mathbf{I}_d^0$ and the feedback current $\mathbf{I}_f = -h\mathbf{I}_V$. It is found that the characteristic polynomials are identical with those of sections II, III, IV, V except with the substitutions: $\tau_c \Rightarrow \tau_c/(1+h)$, $\tan\Psi \Rightarrow \tan\Psi/(1+h)$, $Y_b \Rightarrow Y_b/(1+h)$ made throughout. This being so, we can take over all previous results regarding the polynomial coefficients and Routh-Hurwitz determinants. Generally, the stability limit is enhanced by a factor $(1+h)$.

VI. Cavity, beam dipole and quadrupole modes

Robinson type stability for dipole-quadrupole mode coupling has been investigated in Reference [6], for the case $\mu_b = 0$. We generalize to the case of an accelerating beam.

A. *Rigid bunch quadrupole motion*

Let bunch half-length be $\Theta = \Theta_0 + \theta$, the sum of a steady state part Θ_0 and a small perturbation $\theta(t)$. The Laplace transform of the envelope oscillation can be derived from:

$s\,\theta = \Omega_s^2\,\delta W$ and $s\,\delta W = -4\cos\mu_b \times \theta - z_V \Theta_0 \cos\mu_b$.

where the variable δW is conjugate to θ. To complete our description of the beam coupling to the cavity, we give the relation between θ and amplitude modulation of the beam current z_b. To first order $z_b + F_0 \times \theta = 0$. The form factor F_0 depends on the bunch shape, λ. Let J_n be Bessel functions. For the functions $\lambda = (\Theta_0^2 - x^2)^\alpha$ with $\alpha > 0$,

$$F_0(\Theta_0) = (2\alpha+1)/\Theta_0 \; - \; J_{\alpha-1/2}(\Theta_0)/J_{\alpha+1/2}(\Theta_0)\;.$$

For example, if $\alpha = 1$ then $F_0 \approx \Theta_0/5$ when $\Theta_0 < 1$.

Characteristic polynomial

The polynomial is too lengthy to reproduce here. We consider $\mu_b > 0$, in which case only the coefficient of s^2 has the possibility to change sign when $\Psi > 0$; this implies a beam current limit, but the condition is subordinate to those below.

Routh determinants

RH(3): $2\sec^2\Psi - Y_b F_0 \Theta_0 (\Omega_s\tau_c)^2 \cos\mu_b \sin\mu_b > 0$.

This constraint is quite severe for small tuning angles and long bunches, but is subordinate to RH(6).

If RH(3)> 0 then a sufficient condition for RH(4)> 0 is: $\tan\Psi \geq \sin 2\mu_b F_0\Theta_0(1+2\Omega_s^2\tau_c^2\cos\mu_b)/2(1+F_0\Theta_0\cos^2\mu_b)$.

RH(5) simplifies very slightly to a condition with 24 monomial terms, and there is no simple interpretation. In the limit of large tuning angle, short bunch length, and $\Omega_s\tau_c$ order of or less than unity, we find the approximation:

$$2\tan\Psi[2\cos\mu_b - Y_b\sin 2\Psi] + F_0\Theta_0\cos\mu_b[16\cos^2\mu_b\tan\Psi + 4\sin 2\mu_b\sec^2\Psi + 2Y_b(2\cos\mu_b - \sin\mu_b\tan\Psi - 4\sin^2\Psi)] > 0.$$

The leading term in $\tan\Psi$ contains the Robinson limit.

RH(6) factors; if RH(3) > 0 and RH(4) > 0 this leaves the new condition $Y_b < 3\tan\Psi/[F_0\Theta_0\cos\mu_b]$ which poses a severe constraint at small tuning angles unless μ_b is large or the bunches are short.

RH(7): $4(\cos\mu_b\sec^2\Psi - Y_b\tan\Psi) + Y_b F_0\Theta_0\cos\mu_b(\sin\mu_b - \cos\mu_b\tan\Psi + Y_b) > 0$. The term in Y_b^2 in this quadratic will favourably modify the stability compared with the Robinson limit. However, for small tuning angles condition RH(6) supersedes RH(7).

VII. References

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