CERN-TH-2018-013

ON TOPOLOGICAL INVARIANTS OF ALGEBRAIC THREEFOLDS WITH ($\mathbb{Q}$-FACTORIAL) SINGULARITIES

ANTONELLA GRASSI AND TIMO WEIGAND

Abstract. We study local, global and local-to-global properties of threefolds with certain singularities. We prove criteria for these threefolds to be rational homology manifolds and conditions for threefolds to satisfy rational Poincaré duality. We relate the topological Euler characteristic of elliptic Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities to dimensions of Lie algebras and certain representations, Milnor and Tyurina numbers and other birational invariants of an elliptic fibration. We give an interpretation in terms of complex deformations. We state a conjecture on the extension of Kodaira's classification of singular fibers on relatively minimal elliptic surfaces to the class of birationally equivalent relatively minimal genus one fibered varieties and we give results in this direction.

1. Introduction

We study local, global and local-to-global properties of threefolds with singularities, in particular terminal and klt, $\mathbb{Q}$-factorial singularities. The computation of the topological Euler characteristic of a genus one fibered variety is a known illustration of such properties: the non trivial contributions are localized in the stratified singular loci of the fibration and are combined via the Mayer-Vietoris Theorem. Poincaré duality is another. For complex varieties local and global deformations are of interest, and so are local-to-global principles relating local and global deformations. A less known occurrence is the expected correspondence, predicted by physics, between elliptic fibrations of smooth Calabi-Yau varieties and Lie algebras together with their representations.

We state and formalize the correspondence between fibrations and algebras in terms of local and global properties of the stratified singular locus of elliptic fibrations (not necessarily smooth). We then prove related local-to-global properties for Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities. In particular we prove a formula which relates the dimensions of Lie algebras and certain representations to the topological Euler characteristic of elliptic Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities, Milnor and Tyurina numbers and other birational invariants of the (relative minimal) elliptic fibration (Definition 9.1 and Theorem 9.4). We state precise conjectures for this correspondence for more general varieties and fibrations. Our results constitute a step towards what we call a "Grothendieck-Brieskorn"

Date: 10th April 2018.

program regarding the intriguing connection between singular varieties and Lie algebras, together with certain distinguished representations. In fact, Brieskorn, Grothendieck, and later Slodowy [14, 72], discovered beautiful connections between surface singularities and Lie algebras, but a mathematical explanation of the parallelisms of the classification between singularities and Lie algebras remains elusive (Arnol'd, [15]). A well known illustration of the parallelism is the correspondence between surface rational double points and Lie algebras: rational double points are classified by the Dynkin diagrams of the simply laced Lie algebras of type $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$ [17, 21]. These are also the singularities of Weierstrass elliptic surfaces.

Although Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities and elliptic fibrations are the focus of the applications in Section 9, we state our program in more generality, in particular for higher dimensions and different singularities. In Section 8 we review and extend the correspondence between fibrations and Lie algebras and their representations, and state local-to-global principles which relate them. The correspondence is expressed in terms of the components of codimension one and two of the stratified singular loci of the fibrations. (A different mathematical approach using deformation is taken in ongoing projects [32].) Key ingredients for the proofs in Sections 8 and 9 are several other local, global and local-to-global results which we prove in the preceding sections.

In Section 5 we start with a review of different notions of cohomologies which agree in the case of rational homology manifolds; rational homology manifolds satisfy Poincaré duality. We then establish necessary and sufficient conditions for threefolds with certain rational and klt singularities to be rational homology manifolds (Theorem 5.7 with J. Shaneson, and Theorem 5.8) and to satisfy rational Poincaré duality (Theorems 5.13 and 5.16). These conditions are expressed in terms of the properties of local analytic and (global) algebraic $\mathbb{Q}$-factoriality as well as local-to-global principles. To this end, we prove a local analytic $\mathbb{Q}$-factorialization result for isolated klt singularities (Theorem 4.6). The above singularities occur naturally in the minimal model program as well as in various physics models. The minimal varieties in the sense of the Mori Minimal Model Programs have generally terminal $\mathbb{Q}$-factorial singularities. In string theory it is known that some Calabi-Yau fourfolds with terminal singularities are the "correct" models, even when a smooth birationally equivalent minimal Calabi-Yau exists [19]. Even for Calabi-Yau threefolds the appearance of $\mathbb{Q}$-factorial terminal singularities seems oftentimes unavoidable [10, 24, 25, 59]. While klt singularities of varieties (not of pairs (B, Δ)) have not yet appeared in the physics literature that we know of, we speculate that they might occur naturally as boundary components, in the study of the heterotic/F-theory duality. Generalized "Calabi-Yau" threefolds with isolated klt but not canonical singularities are of interest in mathematics, for example regarding their structure and rationality properties (Remark 5.12).

In Section 2 we recall some known properties of smooth Calabi-Yau threefolds, in particular elliptically fibered ones, and state the original motivation of our work. We review the geometric and algebraic definitions of global and local analytic and algebraic $\mathbb{Q}$-factoriality, the local Picard group, as well as of the algebraic and analytic Class groups in Section 3. In Section 4 we also summarize some properties of factoriality and $\mathbb{Q}$-factoriality which are hard to find in the literature. In Section 4.4 we discuss the "Calabi-Yau" condition, in any dimension, and $\mathbb{Q}$-factoriality.

In Section 6 we turn to Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities. We find that the dimension of the complex deformation space is computed by $b_3(X)$ with a modification coming from the dimension of the versal deformations of each singularity, the Tyurina/Milnor number in this case. The decomposition (8) suggests the existence of a "local-to-global principle" (Conjecture 6.8). Section 7 concerns Kähler deformations. The results about complex and Kähler deformations play a role in the proof of the results in Section 9. Some of the results in Section 9 generalize previous results of [33] for smooth elliptic Calabi-Yau threefolds.

In Section 9.2 we review the basics of the predicted correspondence by the physics of F-theory and interpret our results. Applications to physics are studied in companion papers [2] and [35].

We state a precise conjecture (Conjecture 9.8) on the extension of Kodaira's classification of singular fibers on relatively minimal genus one surfaces to the class of birationally equivalent relatively minimal genus one fibered varieties. We support the conjecture with a local-to-global principle, by associating to the stratified discriminant locus of the fibration Σ the non abelian and abelian gauge algebras and their representations. We give results in this direction.

Because this paper spans from algebraic geometry, algebra and topology, with applications to string theory, we have also included some general definitions and properties.

Acknowledgements

We thank P. Aluffi, M. Banagl, J. Kollár, L. Maxim, R. Pardini, M. Rossi, R. Svaldi, S. Verra and especially J. Shaneson, for comments on the draft and useful discussions. We also thank the Aspen Center for Physics, the Fields Institute in Toronto, and the Banff International Research Station for hospitality during various stages of this project. The work of TW is partially supported by DFG under TR33 "The Dark Universe".

2. Motivation: smooth Calabi-Yau varieties

Definition 2.1. Let X be a complex normal algebraic threefold. X is Calabi-Yau if $h^i(X, \mathcal{O}_X) = 0$, $i = 1, 2$, and $K_X \sim \mathcal{O}_X$.

If X is also smooth, Poincaré duality and the Hodge decomposition imply that the Betti numbers (of the singular cohomology) are

$$b_1 = 0, \quad b_2 = h^{1,1}(X), \quad b_3 = 2(1 + h^{2,1}). \tag{1}$$

The topological Euler characteristic can be written as

$$\chi_{top}(X) = 2\{h^{1,1}(X) - h^{2,1}(X)\}. \tag{2}$$

The above expression is particularly relevant in physics applications because the Picard group $\mathrm{Pic}(X)$ is isomorphic to $H^2(X,\mathbb{Z})$; its rank, which is also called the rank of the Néron-Severi group $\mathrm{NS}(X)$, counts the Kähler deformations $h^{1,1}(X)$ of X. Similarly $h^{2,1}(X)$ is the dimension of the space of complex deformations of X, the Kuranishi space of X. When X has singularities, these identifications do not necessarily hold, so we denote by KaDef the dimension of the space of Kähler deformations and by CxDef the dimension of the space of complex deformations. In the smooth case (2) becomes

$$\chi_{top}(X) = 2\{\mathrm{KaDef}(X) - \mathrm{CxDef}(X)\}. \tag{3}$$

The celebrated mirror symmetry implies that mirror pairs of smooth Calabi-Yau threefolds have topological Euler characteristics of opposite sign.

Equation (3) is used in [33] to express the topological Euler characteristic of general elliptically fibered smooth Calabi-Yau threefolds (with a section) in terms of the Lie algebras and their representations which are naturally associated to the singular fibers of the fibration. The results of [33, 34] are motivated by the "anomaly cancellation" mechanism in physics. In fact, the cancellation of anomalies interpreted in the geometry of smooth elliptically fibered Calabi-Yau threefolds [34] involves $\mathrm{KaDef}(X)$ and $\mathrm{CxDef}(X)$ as defined above. In Section 9 we generalize these results to singular Calabi-Yau threefolds. We focus on the singularities which occur naturally in the minimal model program, i.e. terminal, and klt, $\mathbb{Q}$-factorial singularities. We recall the basics in the following Section 3.

3. Terminal, canonical, klt, ($\mathbb{Q}$-)factoriality

Let X be a complex normal reduced algebraic variety.

3.1. Terminal, canonical, klt. A resolution of X is a birational (bimeromorphic) morphism $\rho : Y \to X$, with Y smooth. X has rational singularities if and only if $R^i\rho_*\mathcal{O}_Y = 0,\ i > 0$. X has ℓ-rational singularities if and only if $R^i\rho_*\mathcal{O}_Y = 0,\ 0 < i \leq \ell$. Let $j : X_0 \hookrightarrow X$ be the natural inclusion of the smooth locus, with canonical bundle $\omega_0(X_0)$ and let $\omega_X = j_*(\omega_0(X_0))$. Since X is normal, ω_X is a reflexive sheaf; let K_X be the associated Weil divisor.

Definition 3.1. X is $\mathbb{Q}$-*Gorenstein* if there exists some integer r such that rK_X is a Cartier divisor (that is K_X is $\mathbb{Q}$-Cartier). The minimum such integer r is the index of X.

X is Gorenstein if it is Cohen-Macaulay and of index 1.

Let X be a $\mathbb{Q}$-Gorenstein variety and $\rho : Y \to X$ a smooth resolution with exceptional divisors E_i. If a resolution ρ is an isomorphism in codimension one, the resolution is called small. In general $rK_Y = \rho^* rK_X + \sum_i a_i r E_i$, where $a_i \in \mathbb{Q}$ are the *discrepancies*.

Definition 3.2. (1) If $a_i > 0$ for all i, X is said to have *at worst terminal* singularities.
(2) If $a_i \geq 0$, for all i, X is said to have *at worst canonical* singularities.
(3) If $a_i > -1$ for all i, X is said to have *at worst klt* singularities.
(4) If $a_i \geq -1$ for all i, X is said to have *at worst log canonical* singularities.

A smooth variety has at worst terminal singularities. If X is a surface it can be shown that X has at worst terminal singularities if and only if it is smooth. The canonical surface singularities are the $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$ singularities (rational double points). A singularity is klt if and only if it is a cyclic quotient of an index 1 canonical singularity by an action which is fixed point free in codimension 2 [52, Corollary 5.21].

Theorem 3.3 (See for example 6.2.12, [44])**.** *Klt singularities, hence canonical and terminal singularities, are rational and in particular Cohen-Macaulay.*

Remark 3.4. Recall that threefold terminal singularities are isolated; if they are Gorenstein, they are analytic hypersurface singularities, see for example [52]. X has at worst canonical singularities and the index of K_X is 1 if and only if it is Gorenstein and rational. In particular a Calabi-Yau variety with canonical singularities is Gorenstein.

Log canonical singularities are however not necessarily rational; thus we will only consider here varieties with at worst klt singularities.

3.2. **Factorial, $\mathbb{Q}$-factorial; the geometric definitions.**
The different notions of $\mathbb{Q}$-factoriality - algebraic, analytic, global and local - are quite delicate. We can consider X as a complex algebraic variety, and also its support X^h as a complex analytic variety. Sometimes we will consider complex analytic varieties X. In the following we do not always distinguish between X and X^h, rather we specify if the relevant objects are algebraic and analytic when necessary. Note that X has rational singularities if and only if X^h has rational singularities.

Definition 3.5. Let $\mathrm{WDiv}(X)$ be the group of algebraic (analytic) Weil divisors and $\mathrm{CDiv}(X)$ the group of algebraic (analytic) Cartier divisors.

If X is smooth, then $\mathrm{WDiv}(X) = \mathrm{CDiv}(X)$. This is true more generally if X is factorial, that is if and only if, $\forall\, x \in X$, the local rings $\mathcal{O}_{X,x}$ are unique factorization domains.

Definition 3.6. Let $\mathrm{WDiv}(X)_{\mathbb{Q}} = \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{CDiv}_{\mathbb{Q}}(X) = \mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
If $\mathrm{WDiv}(X)_{\mathbb{Q}} = \mathrm{CDiv}(X)_{\mathbb{Q}}$, X is algebraic $\mathbb{Q}$-*factorial* (globally analytically $\mathbb{Q}$-*factorial*), or it has (analytic) $\mathbb{Q}$-*factorial* singularities.

Often one says $\mathbb{Q}$-factorial instead of algebraic $\mathbb{Q}$-factorial. In the case of hypersurfaces and complete intersection varieties, various results, starting from Grothendieck's work, relate factoriality and $\mathbb{Q}$-factoriality, for example [66] and [53]. Analytic factoriality implies factoriality, but the converse is not true [70, Corollary 2, p. 41]. In particular:

Proposition 3.7 (Reid-Ue; Corollary 5.1 [46]; [53])**.** *Let $(\mathcal{U}, p)$ be a threefold with terminal singularities of index 1. Then any $\mathbb{Q}$-Cartier divisor is Cartier, that is $\mathbb{Q}$-factoriality is equivalent to factoriality.*

Thus a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities is factorial (Remark 3.4). We also consider properties of the germ $(\mathcal{U}, p)$:

Definition 3.8. $(\mathcal{U}, p)$ is algebraic (analytically) $\mathbb{Q}$-factorial if every algebraic (analytic) Weil divisor in a neighborhood of p (Euclidean in the analytic case) is $\mathbb{Q}$-Cartier.

Definition 3.9. X is locally (analytically) $\mathbb{Q}$-factorial if $(\mathcal{U}, p)$ is algebraic (analytically) $\mathbb{Q}$-factorial, for every $p \in X$ and any open set $\mathcal{U}$.

In the analytic case global does not imply local necessarily, since there can be Weil divisors in $(\mathcal{U}, p)$ which do not extend to Weil divisors in X. In general, algebraic $\mathbb{Q}$-factoriality does not imply analytic $\mathbb{Q}$-factoriality; it does so if X is projective algebraic, by Chow's Theorem.

Example 3.10 ([46], page 104, and [44])**.** $(\mathcal{U}, 0) = (xy + zw + z^3 + w^3 = 0, 0)$ is $\mathbb{Q}$-factorial, actually factorial, but not locally analytically $\mathbb{Q}$-factorial, since the local analytic equation in $(\mathcal{U}, 0)$ is $z_1z_2 + z_3z_4 = 0$.

When the isolated singularity is toric, the singularity is $\mathbb{Q}$-factorial if and only if the maximal cone corresponding to the toric singular point is simplicial. More generally a variety with orbifold singularities is $\mathbb{Q}$-factorial.

Following Kawamata we make

Definition 3.11.
(1) $\sigma(X) = \dim(\mathrm{WDiv}(X)_{\mathbb{Q}} / \mathrm{CDiv}(X)_{\mathbb{Q}})$.
(2) $\sigma((\mathcal{U}, p)) = \dim(\mathrm{WDiv}((\mathcal{U}, p))_{\mathbb{Q}} / \mathrm{CDiv}((\mathcal{U}, p))_{\mathbb{Q}})$.

Remark 3.12. X (respectively, $(\mathcal{U}, p)$) is $\mathbb{Q}$-factorial if and only if $\sigma(X) = 0$ (respectively, $\sigma(\mathcal{U}, p) = 0$), that is if and only if $\mathrm{WDiv}(X)/\mathrm{CDiv}(X)$ (respectively, $\mathrm{WDiv}(\mathcal{U}, p))/\mathrm{CDiv}(\mathcal{U}, p))$ is torsion.

In Example 3.10 $\sigma((\mathcal{U}^h, 0)) = 1$.

Proposition 3.13 (Kawamata [46], Lemma 1.2)**.** *Let X be an algebraic variety. If X has rational singularities, then $\sigma(X)$ is finite.*

Proposition 3.14 (Kawamata [46], Lemma 1.12)**.** *Let $(\mathcal{U}, p)$ analytic. If $(\mathcal{U}, p)$ has rational singularities, then $\sigma((\mathcal{U}, p))$ is finite.*

In particular we use Property 3.14 in Theorem 4.6. In Section 4 we study $\mathbb{Q}$-factoriality and local analytic $\mathbb{Q}$-factoriality by analyzing properties of the local class group and the local Picard group.

Definition 3.15 (The divisor class group, the local divisor class group)**.**
The Divisor Class group $\mathcal{C}l(X)$ is the group of Weil divisors modulo the principal divisors.
The Picard group $\operatorname{Pic}(X)$ is the group of Cartier divisors modulo the principal divisors.
The local (analytic) divisor class group, $\mathcal{C}l(\mathcal{O}_{X,x})$ and $\mathcal{C}l(\mathcal{O}^h_{X,x})$, is geometrically the group of local Weil divisors modulo the principal divisors.

The following results 3.16, 3.17 hold both in the algebraic and analytic settings [40]:

Proposition 3.16. *$\mathcal{C}l(\mathcal{O}_{X,x}) = \varinjlim \operatorname{Pic}(\mathcal{V})$, where the limit is taken over the set of $\mathcal{V} \subset \operatorname{Spec}(\mathcal{O}_{X,x})$ such that $\mathcal{O}_z$ is factorial $\forall\ z \in \mathcal{V}$.*

$\mathcal{C}l(\mathcal{O}^h_{X,x}) = \varinjlim \operatorname{Pic}(\mathcal{V})$, where the limit is taken over the set of $\mathcal{V} \subset \operatorname{Spec}(\mathcal{O}^h_{X,x})$ such that $\mathcal{O}_z$ is factorial $\forall\ z \in \mathcal{V}$.

Corollary 3.17. *Let $\mathcal{U}$ be an open set and $Z \subset \mathcal{U}$ proper, closed, with $\operatorname{codim} Z \geq 2$. Then*

$$\mathcal{C}l(\mathcal{U}) \simeq \mathcal{C}l(\mathcal{U} \setminus Z). \tag{4}$$

If Z is the singular locus of X, then $Pic(\mathcal{U} \setminus Z) \simeq \mathcal{C}l(\mathcal{U})$. If $\dim \mathcal{U} \geq 3$ and $p \in \mathcal{U}$ an isolated singularity, $\operatorname{Pic}(\mathcal{U} \setminus p) \simeq \mathcal{C}l(\mathcal{O}_{\mathcal{U},p})$.

Definition 3.18. $\operatorname{Pic}(\mathcal{U} \setminus p)$ is usually called the local Picard group, but the notation in the literature is not uniform; we refer to [50]. In the following Section 4 we review equivalent definitions and prove some properties.

4. Almost factoriality and other properties; $\mathbb{Q}$-factorializations

4.1. **The algebraic definition.** Let A be a local noetherian normal domain.

Definition 4.1 (The local divisor class group)**.** $\mathcal{C}l(A)$ is the quotient of the divisorial ideals $\operatorname{WDiv}(A)$ modulo the principal ideals in A; $\operatorname{Pic}(A)$ is the quotient of the invertible ideals $\operatorname{CDiv}(A)$ modulo the principal ideals in A.

The divisorial ideals are the Weil divisors in $\mathcal{U} = \operatorname{Spec}(A)$. We will apply the definitions and results stated below to the local ring $\mathcal{O}_{X,x} = A$ and $\operatorname{Spec}(\mathcal{O}_{X,x}) = \mathcal{U}$. Recall that A is factorial if and only if it is a unique factorization domain. In the algebra literature, see for example [26], $\mathbb{Q}$-factorial is referred to as almost factorial:

Definition 4.2. A is almost factorial (respectively factorial) if $\mathcal{C}l(A)$ is torsion (respectively zero).

Proposition 4.3. *The local class group $\mathcal{C}l(A)$ is torsion (zero) if and only if* $\operatorname{WDiv}(A)/\operatorname{CDiv}(A)$ *is torsion (respectively* $\operatorname{WDiv}(A)/\operatorname{CDiv}(A) = 0$*).*

Proof. It follows from [7, 26, 73]. □

Equivalently, A is almost factorial (respectively factorial) if and only if $\operatorname{WDiv}(A)_{\mathbb{Q}}/\operatorname{CDiv}(A)_{\mathbb{Q}}$ is torsion (respectively $\operatorname{WDiv}(A)/\operatorname{CDiv}(A) = 0$).

4.2. **Geometry.** In the case of isolated singularities one can directly prove

Proposition 4.4. *Let $(\mathcal{U}, p)$ be an analytic contractible germ open set and $p \in \mathcal{U}$ an isolated singularity. Then*

$$\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p}) \simeq \operatorname{Pic}(\mathcal{U} \setminus p) \simeq \operatorname{WDiv}((\mathcal{U}, p))/\operatorname{CDiv}((\mathcal{U}, p)) \,. \tag{5}$$

Therefore, by Remark 3.12 $(\mathcal{U}, p)$ is analytically $\mathbb{Q}$-factorial if and only if $\sigma(\mathcal{U}, p) = 0$, that is the local analytic divisor class group $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$ is torsion.

4.3. **$\mathbb{Q}$-factorializations.** Recall that the nodal quintic threefold $X \subset \mathbb{P}^4$ of equation $x_0 g_0 + x_1 g_1 = 0$, with g_0, g_1 general quartic polynomials in the variables $[x_0, \cdots, x_4]$, is not $\mathbb{Q}$-factorial, as the Weil divisor D defined by $x_0 = g_1 = 0$ is not Cartier. Recall that the birational morphism obtained by blowing up $\mathbb{P}^4$ along D provides a small projective resolution $X_1 \to X$; in particular X_1 is a $\mathbb{Q}$-factorial variety. When the isolated singularity is toric, X is not $\mathbb{Q}$-factorial if and only if the maximal cone corresponding to the toric singular point is not simplicial; a $\mathbb{Q}$-factorial birational model X_1 together with a small morphism $X_1 \to X$ is achieved by a simplicial subdivision of the cone. More generally, we have

Theorem 4.5 (Corollary 1.4.3, [9])**.** *Let X be an algebraic threefold with klt singularities. If X is not $\mathbb{Q}$-factorial, there exists a small projective birational morphism $\phi : X_1 \to X$, where X_1 is $\mathbb{Q}$-factorial with klt singularities.*

The above algebraic $\mathbb{Q}$-factorialization Theorem 4.5 was first proved by Kawamata in Corollary 4.5 [46], for a threefold analytic germ with at most terminal singularities. In the proof Kawamata uses the classification of threefold terminal singularities and he reduces to singularities of index one. Birkar, Cascini, Hacon and M$^{\text{c}}$Kernan prove Corollary 1.4.3 in [9] as a consequence of their celebrated Theorem of the finite generation of the canonical ring.

Kawamata in Corollary 4.5' [46] also proves the analytic $\mathbb{Q}$-factorialization for a threefold analytic germ with terminal singularities. We need an analytic $\mathbb{Q}$-factorialization result for klt singularities.

Theorem 4.6. *Let $(\mathcal{U}^h, p)$ be a threefold analytic germ with at most an isolated klt singularity at p. If $(\mathcal{U}^h, p)$ is not analytically $\mathbb{Q}$-factorial, there exists a small bimeromorphic morphism $\phi : \mathcal{V}^h \to \mathcal{U}^h$, where $\mathcal{V}^h$ is analytically $\mathbb{Q}$-factorial with klt singularities.*

We first need[1]

Theorem 4.7 (Algebraic Approximation, [3], Th. 3.8)**.** *Let $(\mathcal{U}^h, p)$ be a threefold analytic germ with an isolated singularity at p and D^h a Weil divisor. There is a normal quasi-projective variety X, an open neighborhood $\mathcal{U} \subset X$, $p \in \mathcal{U}$, and a Weil divisor D such that, possibly after restricting $\mathcal{U}^h$, there exists a biholomorphic map $m : \mathcal{U}^h \to \mathcal{U}$ with $m(D^h) = D$.*

In Artin's result the singularities are isolated.

Proof of Theorem 4.6. Let D be a generator of $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U}^h,p})$. After possibly further restricting to smaller open neighborhoods of p, we have that D is a generator of $\mathcal{C}l(\mathcal{O}_{\mathcal{U},p})$ and $\mathrm{WDiv}(\mathcal{U})/\mathrm{CDiv}(\mathcal{U})$ (Theorem 4.7). By Theorem 4.5 there exists a small projective birational morphism $\phi : \mathcal{U}_1 \to \mathcal{U}$, where $\mathcal{U}_1$ is $\mathbb{Q}$-factorial with klt singularities. Then $m^{-1} \cdot \phi : \mathcal{U}_1^h \to \mathcal{U}^h$ is a small bimeromorphic morphism, and $\sigma(\mathcal{U}_1^h) < \sigma(\mathcal{U}^h)$. In addition, $\mathcal{U}_1^h$ has klt singularities. Since $\sigma(\mathcal{U}^h)$ is finite (Theorems 3.3 and 3.14) by repeating the process if necessary we obtain the Theorem. □

Also Kollár kindly provided us another proof: Theorem 3.10 in [3] and [42] implies that for any finitely generated subgroup $G \subset \mathcal{C}l(\mathcal{O}^h_{\mathcal{U}^h,p})$ there is an algebraic approximation $\mathcal{U}$ such that G is contained in the image of $\mathcal{C}l(\mathcal{O}_{\mathcal{U},p})$.

In general the relation between the algebraic and analytic class groups is quite delicate. See for example [57] for surface rational double points as well as [11]. There are examples of isolated analytic singularities such that the local (analytic) class group cannot be reconstructed from the local (algebraic) class group of any algebraic approximation [50] (in case of isolated singularities the class group and the local Picard group are identified (Definition 3.18).

4.4. $\mathbb{Q}$-factoriality and the "Calabi-Yau condition".

Theorem 4.8 (Kollár, see also Proposition 3.5, [27])**.** *Let X be a normal $\mathbb{Q}$-factorial scheme, $\tilde{X}$ a normal scheme and $\phi : \tilde{X} \to X$ a birational morphism. Then any irreducible component of the exceptional locus has codimension one.*

Note that the proof also works in the analytic setting. Combining Kollár's Theorem and the previous $\mathbb{Q}$-factorialization results we have:

Corollary 4.9. *Let X be an algebraic threefold with klt singularities. There exists a small birational bimeromorphic morphism $\phi : \tilde{X} \to X$ if and only if X is not $\mathbb{Q}$-factorial.*

Corollary 4.10. *Let $(\mathcal{U}^h, p)$ be a threefold analytic germ with at most an isolated klt singularity at p. There exists a small bimeromorphic morphism $\phi : \tilde{U}^h \to U^h$ if and only if $(\mathcal{U}^h, p)$ is not analytically $\mathbb{Q}$-factorial.*

---

[1]We thank J. Kollár for pointing out Artin's [3].

If X is a threefold with canonical singularities and $K_X \simeq \mathcal{O}_X$, then a smooth resolution $\tilde{X} \to X$ with $K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$ can exist only if either X is canonical but not terminal or X is not $\mathbb{Q}$-factorial. It is often of interest in the physics literature to determine the existence of such resolutions, which are said to preserve the "Calabi-Yau condition", see for example [2].

5. Cohomologies, local and global (analytic) $\mathbb{Q}$-factoriality, Rational Poincaré

Theorem 5.1 ([16], [29], [55]). *Let X be a complex compact analytic threefold. The intersection cohomology of X has Poincaré duality over $\mathbb{Q}$: $IH^i(X,\mathbb{Q})^\vee \simeq IH^{6-i}(X,\mathbb{Q})$.*

Theorem 5.2 ([16], [18], [65]; conjectured in [55]). *Let X be a complex projective algebraic variety. The intersection coholomogy $IH^k(X,\mathbb{Q})$ has a pure Hodge structure of weight k, in particular there is a Hodge decomposition with the property $IH^{i,j}(X,\mathbb{C}) = \overline{IH^{j,i}(X,\mathbb{C})}$, $k = i + j$.*

If X is smooth then the intersection cohomology equals the regular cohomology. This is also true for other types of singular varieties, in particular for rational homology manifolds:

5.1. **Rational homology manifolds.** Recall that all varieties are assumed to be normal and connected.

Definition 5.3. A complex threefold X is a rational homology manifold if and only if for every point $p \in X$, $H_6(X, X \setminus p;\ \mathbb{Q}) = \mathbb{Q}$ and $H_i(X, X \setminus p;\ \mathbb{Q}) = 0,\quad i \leq 5$.

Orbifolds are examples of rational homology manifolds. We are interested in rational homology manifolds which are not orbifolds. In fact, an application and motivation of this paper is the study of singular Calabi Yau threefolds, while three-dimensional Gorenstein orbifold singularities are smooth.

Theorem 5.4 ([28, 29]). *Let X be a complex analytic variety which is a rational homology manifold. The intersection cohomology and the ordinary (simplicial) cohomology coincide. In particular if X is compact, Poincaré duality holds and $\chi_{top}(X)$ can be computed with any of these theories.*

Definition 5.5. L is a rational homology 5-sphere if and only if $H_0(L,\mathbb{Q}) = H_5(L,\mathbb{Q}) = \mathbb{Q}$ and $H_i(L,\mathbb{Q}) = 0,\quad 0 < i < 5$.

Let $p \in \mathcal{U} \subset \mathbb{C}^n$, $\dim_{\mathbb{C}} \mathcal{U} = 3$, D a suitable small ball around p, $\mathcal{V}_p = \mathcal{U} \cap D$, $S = \partial D$ and $L_p = \mathcal{U} \cap S$. L_p is the link of $p \in \mathcal{U}$. (If p is an isolated singular point, then $\mathcal{V}_p \cap D$ is a cone over L_p.)

Then, see for example [56]:

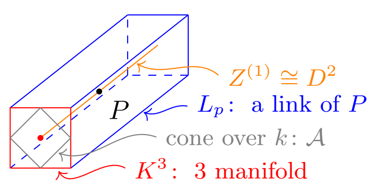
Proposition 5.6. *X is a rational homology manifold if and only if, $\forall\ p \in X$, the link L_p is a rational homology sphere.*

5.2. Rational homology manifolds and $\mathbb{Q}$-factoriality.

Theorem 5.7 (with J. Shaneson)**.** *Let $\mathcal{U} \subset \mathbb{C}^n$, $\dim_{\mathbb{C}} \mathcal{U} = 3$ an algebraic (analytic) variety with rational singularities. Assume that $\mathcal{U} \setminus \Gamma$ is smooth and that for every $p \in \mathcal{U}$, $\pi_1(L_p)$ is finite. Then $\mathcal{U}$ is a rational homology manifold if and only if $\mathcal{U}$ is locally analytically $\mathbb{Q}$-factorial.*

Proof. We will prove that L_p is a rational homology sphere, for all $p \in \mathcal{U}$ if and only if $(\mathcal{U}, p)$ is locally analytically $\mathbb{Q}$-factorial. The statement then follows from Proposition 5.6. By assumption, $H_1(L_p, \mathbb{Q}) = 0$.

$\mathcal{U}$ is complex and normal, thus $\operatorname{codim} \Gamma \geq 2$. Let Z be the support of Γ as a topological variety, $Z^{(2)}$ the singular locus of Z and $Z^{(1)} \stackrel{def}{=} Z \setminus Z^{(2)}$.
Case 1 : $p \in Z$ an isolated singularity. By restricting $\mathcal{U}$ let us assume that $(\mathcal{U}, p)$ is a small neighborhood with an isolated singularity at p. We can then follow the argument of [48, Lemma 4.2] for terminal singularities. Since $H_1(L_p, \mathbb{Q}) = 0$ by Poincaré duality we have $H_4(L_p, \mathbb{Q}) = 0$. Because the singularities are rational, Flenner's result [23, Satz 6.1] implies that $H^2(L_p, \mathbb{Z}) \simeq \operatorname{Pic}^h(\mathcal{U} \setminus p) = \mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$. Then $H_3(L_p, \mathbb{Q}) = H_2(L_p, \mathbb{Q}) = 0$ if and only if the local class group $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$ is torsion, that is, $(\mathcal{U}, p)$ is analytically $\mathbb{Q}$-factorial (Proposition 4.4).
Case 2 [2]: $p \in Z^{(1)}$. By suitably restricting $\mathcal{U}$ we can assume that $\mathcal{U}$ is a small open homotopically equivalent neighborhood of $\mathcal{W} \stackrel{def}{=} D^2 \times c\mathcal{K}$, where $c\mathcal{K}$ is the cone over a real 3-manifold $\mathcal{K}$, with cone point C and $D^2 \subset \mathbb{C}^3$ a ball [29]. Let L_p be the link of p.



Then $\mathcal{U} \supset \mathcal{W}$ and

$$L_p = \partial W = (\partial D^2 \times c\mathcal{K}) \cup (D^2 \times \mathcal{K}), \tag{6}$$

$L_p \setminus Z \cap L_p = [\partial D^2 \times (c\mathcal{K} \setminus C)] \cup (D^2 \times \mathcal{K})$. The Mayer-Vietoris sequence for cohomology implies that $H^2(\mathcal{W} \setminus Z, \mathbb{Z}) \simeq H^2(L_p \setminus Z, \mathbb{Z}) = H^2(\mathcal{K}, \mathbb{Z})$. Because the singularities are rational, Flenner's result [23, Satz 6.1] implies that $H^2(L_p \setminus Z, \mathbb{Z}) \simeq \mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$. Then $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$ is finite if and only $H^2(\mathcal{K}, \mathbb{Q}) = 0$. Since $\mathcal{K}$ is a manifold, Poincaré duality implies that $H^2(\mathcal{K}, \mathbb{Q}) = 0$ if and only if $\mathcal{K}$ is a homology 3 sphere. We already remarked that $H_1(L_p, \mathbb{Z}) = 0$; we now use the Mayer-Vietoris sequence for homology and the decomposition in (6) to compute $H_i(L_p, \mathbb{Z}),\ i = 2, 3, 4$. Then $H_i(L_p, \mathbb{Z}) = 0,\ i = 2, 3, 4$ if and only if $H^2(\mathcal{K}, \mathbb{Q}) = 0$, that is if and only if $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$ is finite. The statement from $\mathcal{W}$ follows from Proposition 4.3.

[2]We are grateful to M. Bies for providing us with the TeX code for the figure.

Case 3 : $p \in Z^{(2)}$. Again, by suitably restricting $\mathcal{U}$, we can assume that $Z^{(2)} \cap \mathcal{U} = \{p\}$, and $L_p = (L_p \setminus Z \cap L_p) \cup_\ell S^1_\ell$, $S^1_\ell \subset Z^{(1)} \cap L_p$. Write:

$$L_p = (L_p \setminus Z \cap L_p) \cup_\ell N_\ell, \tag{7}$$

where N_ℓ is open, it has the same homology of S^1 and $N_\ell \to S^1$, with fiber $c\mathcal{K}$, a cone on 3 manifold $\mathcal{K}$. Take $q \in S^1_\ell$, $\mathcal{W}_q \stackrel{def}{=} D^2 \times c\mathcal{K}$ to be a small neighborhood of q in $\mathcal{U}$ [29].

Claim 1: L_p is a rational homology manifold if an only if $\mathcal{W}_q$ is locally analytically $\mathbb{Q}$-factorial, $\forall q \in Z \cap L_p$.

Proof of Claim 1. Recall that $q \in Z^{(1)}$, as in Case 2. We will prove that $\mathcal{W}_q \cap L_p$ is a rational homology sphere if and only if $\mathcal{W}_q$ is locally analytically $\mathbb{Q}$-factorial, for all q. In fact the previous description implies that $\mathcal{W}_q \cap L_p = I \times c\mathcal{K}$ is a neighborhood of q in L_p, hence $\mathcal{W}_q \cap L_p$ is a rational homology sphere if and only if $\mathcal{K}$ is a rational homology sphere [29]. From Case 2 we know that $\mathcal{K}$ is a rational homology sphere if and only if $\mathcal{W}_q$ is analytically $\mathbb{Q}$-factorial.

Claim 2: If L_p is a rational homology manifold $H_4(L_p, \mathbb{Q}) = 0$.

Proof of Claim 2. Poincaré duality.

Claim 3: $H_2(L_p, \mathbb{Q}) = 0$, if an only if $\mathcal{U}$ is locally analytically $\mathbb{Q}$-factorial.

Proof of Claim 3. To compute $H_2(L_p, \mathbb{Q})$ we use again the Mayer-Vietoris sequence for homology and the decomposition in (7). Note that for each q, $(L_p \setminus Z \cap L_p) \cap N_q$ is homologically a bundle over S^1, with fiber $\mathcal{K}$. Then $H_2(\cup N_q, \mathbb{Q}) = 0$; recall that $H_2(L_p \setminus Z \cap L_p, \mathbb{Q}) = 0$ if and only if $\mathcal{C}l(\mathcal{O}^h_{\mathcal{U},p})$ is finite [23]. We conclude again by Proposition 4.3.

Claim 4: If L_p is a rational homology manifold and $H_2(L_p, \mathbb{Q}) = 0$ then $H_3(L_p, \mathbb{Q}) = 0$.

Proof of Claim 4. By Poincaré duality. □

Theorem 5.8. *Let $\mathcal{U} \subset \mathbb{C}^n$, $\dim_{\mathbb{C}} \mathcal{U} = 3$ an algebraic threefold with klt singularities. Then $\mathcal{U}$ is a rational homology manifold if and only if $\mathcal{U}$ is locally analytically $\mathbb{Q}$-factorial.*

Proof. In fact $\pi_1(L_p)$ is finite [76, Cor 1.5]. The statement then follows from Definition 3.9, Proposition 5.6 and Theorem 5.7. □

Banagl shows that a threefold X with canonical singularities, trivial canonical divisor and $h^1(X, \mathcal{O}_X) > 0$ is a rational homology manifold [6, Remark 6.4, page 29]. The threefolds in the examples 5.10 and 5.11, a rational homology manifold and one which is not, below can be taken to be Calabi-Yau (trivial canonical divisor and $h^1(X, \mathcal{O}_X) = 0$).

Example 5.9. (Rational homology manifolds, non rational homology manifolds, local)

Let $(\mathcal{U}, p)$ be an A_{a-1} Kleinian threefold singularity, that is $(\mathcal{U}, p)$ is the zero-locus of $f(z, x_1, x_2, x_3) = z^a + x_1^2 + x_2^2 + x_3^2$ with $a \geq 2 \in \mathbb{N}$. These are terminal (and non-canonical) singularities [67, Th. 1.1]. A local, small resolution is possible if and only if a is even, [67, Cor. 1.6], [5,13]. Then Corollaries 4.9 and 4.10 imply that $(\mathcal{U}, p)$ is $\mathbb{Q}$-factorial (locally analytic

$\mathbb{Q}$-factorial) if and only if a is odd. On the other hand one can also find directly that $(\mathcal{U}, p)$ is a rational homology manifold if and only if a is odd [12], [20, Theorem 4.10]. Flenner [23] proves the statement directly using the local divisor class group (see Section 4).

Both these types of examples occur in examples of (elliptically fibered) threefolds with $\mathbb{Q}$-factorial terminal singularities:

Example 5.10. (A threefold with $\mathbb{Q}$-factorial terminal singularities and a rational homology manifold.) Let $\pi : X \to B$ be an elliptic fibration in Weierstrass form with general singular Kodaira fibers of type II (cusps) and I_1 (nodes) over the smooth points of the discriminant of the fibration. Then the singularities of X are $\mathbb{Q}$-factorial, terminal, but not smooth, with local equation $z_0^a + \sum_{i=1,3} z_i^2 = 0$, with a odd. These singularities are analytically $\mathbb{Q}$-factorial and X is a rational homology manifold. This is Case 2 in the proof of Theorem 9.4, see also [2].

Example 5.11. (A threefold with $\mathbb{Q}$-factorial terminal singularities but not a rational homology manifold.) Let $\pi : X \to B$ be a general elliptic fibration in Weierstrass form with singular Kodaira fibers of type I_1 over the smooth points of the discriminant of the fibration such that the singular points of the discriminant are cusps and simple normal crossing divisors. Then the singularities $p \in X$ occur over the simple normal crossing points of the discriminant; these singularities are ordinary double points ("conifold") terminal $\mathbb{Q}$-factorial, but not smooth, with local equation $z_0^2 + \sum_{i=1,3} z_i^2 = 0$. These are not analytically $\mathbb{Q}$-factorial, and X is not a rational homology manifold. The Jacobian elliptic fibration of a general genus one fibration has exactly this type of singularities see for example [10, 59]. This is in fact Case 1 in the proof of Theorem 9.4, see also [2].

In the next Section we show that nevertheless threefolds with isolated klt singularities satisfy some Poincaré duality over the rationals.

5.3. **Rational Poincaré Duality.**

Remark 5.12. $\bar{R}_3 = E^3 / < -\omega I_3 >$ is an example of a "Calabi-Yau" threefold with isolated klt but not canonical singularities [63]. Here $E = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ and $\omega = e^{\frac{2}{3}\pi i}$. The canonical divisor is numerically trivial, $h^1(\bar{R}_3, \mathcal{O}_X) = h^2(\bar{R}_3, \mathcal{O}_X) = 0$, it is elliptically fibered and it is a rational homology manifold. $\bar{R}_3$ is also rational [64]. Klt varieties with numerically trivial canonical divisors have Bochner's type properties [37].

More generally, we can prove

Theorem 5.13. *Let X be a projective algebraic threefold with isolated klt singularities.*

(1) *The cup product with the fundamental class gives an isomorphism $H^5(X, \mathbb{Q}) \xrightarrow{\sim} H_1(X, \mathbb{Q})$, that is $H_5(X, \mathbb{Q})$ and $H_1(X, \mathbb{Q})$ are Poincaré duals over the rationals.*

(2) *If in addition $b_2(X) = b_4(X)$, X satisfies rational Poincaré duality, i.e. the cup product with the fundamental class gives Poincaré duality with rational coefficients.*

Proof. If X is a rational homology manifold, both statements follow from Theorem 5.4 (no assumption of $b_2(X) = b_4(X)$ is needed). If X is not a rational homology manifold, then it is not locally analytically $\mathbb{Q}$-factorial by Theorem 5.8. Let $\{(\mathcal{U}, P)\}$ be a collection of contractible neighborhoods of the isolated singular points $\{P\}$ which are not analyticaly $\mathbb{Q}$-factorial; without loss of generality we can assume that any two neighborhoods $(\mathcal{U}, P)$ do not intersect. Let $\phi_P : (\mathcal{U}'_P, \Gamma_P) \to (\mathcal{U}_P, P)$ be the $\mathbb{Q}$-factorialization, that is a small bimeromorphic morphism, an isomorphism in codimension 1, with $(\mathcal{U}'_P, \Gamma_P)$ analytically $\mathbb{Q}$-factorial and klt (Theorem 4.6). $\mathcal{U}'_P$ is simply connected [74][3] and $\mathcal{U}'_P$ is contratctible to the exceptional locus Γ_P. Hence $H_1(\mathcal{U}'_P) = H_3(\mathcal{U}'_P) = H_4(\mathcal{U}'_P) = H_5(\mathcal{U}'_P) = 0$. Let X' be the complex analytic threefold obtained by patching in $X \backslash \cup \{\mathcal{U}_P\}$ the collection of the singular neighbhoroods $\{\mathcal{U}'_P\}$. Let $f : X' \to X$ be the induced morphism. The commutative diagrams obtained by combining the Mayer-Vietoris sequences in homology and cohomology for $X = (X \setminus \cup_P P) \coprod_P \mathcal{U}_P$ and $X' = (X \setminus \cup_P \Gamma_P) \coprod_P \mathcal{U}'_P$ imply that:
$H_1(X) \simeq H_1(X')$, $H_5(X) \simeq H_5(X')$, $H^4(X) \simeq H^4(X')$, and $H^5(X) \simeq H^5(X')$.

Theorems 5.7 and 5.8 show that X' is a rational homology manifold, then Poincaré duality on X' holds and the top arrows in the diagrams below are isomorphisms:

$$\begin{array}{ccc} H^5(X') & \xrightarrow{\cap[X']} & H_1(X') \\ \uparrow \simeq & & \simeq \downarrow \\ H^5(X) & \xrightarrow{\cap[X]} & H_1(X) \end{array} \qquad \begin{array}{ccc} H^4(X') & \xrightarrow{\cap[X']} & H_2(X') \\ \uparrow \simeq & & f_* \downarrow \\ H^4(X) & \xrightarrow{\cap[X]} & H_2(X) \end{array}$$

The first statement follows immediately. The Mayer-Vietoris sequence implies also that $f_* : H_2(X') \to H_2(X)$ is surjective, and thus, if $b_2(X) = b_4(X)$, $H^4(X) \overset{\cap[X]}{\to} H_2(X)$ is an isomorphism, as a surjective morphism between spaces of the same dimension. □

Proposition 5.14. *Let X be a projective $\mathbb{Q}$-factorial threefold with isolated rational hypersurface singularities and $h^2(X, \mathcal{O}_X) = 0$, then $b_2(X) = b_4(X)$.*

Proof. It follows from [62, Theorem 3.2]. □

Corollary 5.15. *Let X be a projective $\mathbb{Q}$-factorial threefold with klt isolated hypersurface singularities and $h^2(X, \mathcal{O}_X) = 0$. Then X satisfies rational Poincaré duality.*

Proof. It follows from Theorem 5.13 and Proposition 5.14. □

[3]Shepperd-Barron proved in an unpublished note in 1989 that $\mathcal{U}'_P$ is simply connected when P is a canonical singularity.

Theorem 5.16. *Let X be a projective Gorenstein $\mathbb{Q}$-factorial threefold with terminal singularities and $h^2(X, \mathcal{O}_X) = 0$. Then X satisfies rational Poincaré duality.*

Proof. In fact, threefold Gorenstein terminal singularities are isolated rational hypersurface singularities (see Theorem 3.3 and Remark 3.4). The statement follows from the previous Corollary 5.15. □

Corollary 5.17. *Let X be a projective minimal threefold of Kodaira dimension 0. Then X satisfies rational Poincaré duality.*

Proof. If $h^1(X, \mathcal{O}_X) \neq 0$ then X is a rational homology manifold [6, Remark 6.4, page 29]. If $h^1(X, \mathcal{O}_X) = 0$ then X is either a Calabi-Yau or a quotient of a Calabi-Yau by a finite group and the statement follows from Theorem 5.16. □

In particular:

Corollary 5.18. *Let X be a projective $\mathbb{Q}$-factorial Calabi-Yau or Gorenstein Fano threefold with terminal singularities. Then*

$$\chi_{top}(X) = 2\{1 + b_2(X)\} - b_3\,.$$

Proof. In fact X satisfies rational Poincaré duality.

□

6. The third Betti number (and complex deformations of Calabi-Yau threefolds)

6.1. **Milnor and Tyurina numbers.** Let $(\mathcal{U}, 0) \subset \mathbb{C}^{n+1}$ be a neighborhood of an isolated hypersurface singularity $P = 0$, defined by $f = 0$.

Definition 6.1. The Milnor number of P can be defined as

$$m(P) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/ < \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} >)\,.$$

Definition 6.2. The Tyurina number $\tau(P)$ is the dimension of the space of versal deformations of the hypersurface singularity at P in $\mathcal{U}$ and it is computed algebraically as

$$\tau(P) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/ < f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} >).$$

Remark 6.3. $m(P) \geq \tau(P)$ and Saito proved that $\tau(P) = m(P)$ if and only if P is a weighted hypersurface singularity [54, 69]. Saito's Theorem has been generalized to complete intersections by Greuel [38].

$m(P)$ and $\tau(P)$ are also computable by SINGULAR [39] and Maple [68].

6.2. **Complex deformations and $b_3(X)$.**

Proposition 6.4. *Let X be a $\mathbb{Q}$-factorial Calabi-Yau threefold with terminal singularities and let* $\mathrm{CxDef}(X)$ *denote the dimension of the Kuranishi space of X, the space of complex deformations of X. Then*

$$\mathrm{CxDef}(X) = \mathrm{CxDef}(X_t) = \frac{1}{2}b_3(X_t) - 1 = \frac{1}{2}\{b_3(X) + \sum_P m(P)\} - 1,$$

where X_t is the smoothing of X and $m(P)$ the Milnor number of the singular point P.

Proposition 6.5. *Let X be a $\mathbb{Q}$-factorial Calabi-Yau threefold with terminal singularities which are weighted hypersurface singularities and let* $\mathrm{CxDef}(X)$ *denote the dimension of the Kuranishi space of X, the space of complex deformations of X. Then*

$$\mathrm{CxDef}(X) = \frac{1}{2}\{b_3(X) + \sum_P \tau(P)\} - 1,$$

where $\tau(P)$ is the Tyurina number of the singular points P.

Proof of Propositions 6.4 and 6.5. The singularities are hypersurface singularities since they are terminal of index 1. Proposition 6.4 and Proposition 6.5 follow from Theorems 1.3 and 3.2 in [62] and from Remark 6.3 above [69]. Theorem 1.3 proves that a $\mathbb{Q}$-factorial Calabi-Yau threefold with terminal singularities admits a smoothing to a Calabi-Yau X_t. Theorem 3.2 also proves that if a threefold X with isolated rational hypersurface singularities has a smoothing X_t and $h^2(X, \mathcal{O}_X) = 0$, then $b_3(X) = b_3(X_t) - \sum_{\mathrm{sing}\ P} m(P)$, where $m(P)$ are the Milnor numbers of the singularities. □

Remark 6.6. Proposition 6.4 implies that $\frac{1}{2}b_3(X) + \frac{1}{2}\sum_P m(P) \in \mathbb{Z}$, but in general $\frac{1}{2}b_3(X) \notin \mathbb{Z}$ because the Hodge decomposition and Hodge duality may not hold. The Kleinian hypersurface singularities of type A_{a-1} of Example 5.9 are weighted hypersurface singularities, and thus $m(P) = \tau(P) = a - 1$. They are rational homology manifolds if and only if a is odd, in which case both $b_3 = 2h^{1,2}$ and $a - 1$ are even.

Remark 6.7. If X is a Calabi-Yau variety with $\mathbb{Q}$-factorial terminal singularities, the dimension of the space of complex deformations splits into a "localized" and “non-localized" contribution, given by the dimension of the versal deformation space of the singularities and, respectively, the remaining deformations:

$$\mathrm{CxDef}(X) = \underbrace{-1 + \frac{1}{2}\{b_3(X) + \sum_P (m(P) - 2\tau(P))\}}_{non-localized} + \underbrace{\sum_P \tau(P)}_{localized} . \tag{8}$$

In Section 9.2 we present a natural interpretation of this decomposition for physics.

The decomposition (8) suggests the existence of a “local-to-global principle" for deformations of Calabi-Yau threefolds with $\mathbb{Q}$-factorial terminal singularities:

Conjecture 6.8. *There exists a natural decomposition of the Kuranishi space of X into the space of complex structure deformations of X which deform the isolated singularities, whose dimension is the sum of the dimensions of the versal deformations (Tyurina numbers), and the remaining space of deformations of X which do not change the location or form of the isolated singularities.*

Note also that in the general hypothesis considered in Section 9 and [2] $m_P = \tau_P$.

7. The second Betti number (and Kähler deformations); topological Euler characteristic

Let X be a (normal) complex threefold with $h^2(X, \mathcal{O}_X) = 0$. The exponential sequence, see for example [36, pg. 142], implies that $b_2(X)$ is the rank of the Néron-Severi group, namely $b_2(X) = \mathrm{KaDef}(X)$ (Section 2).

More generally the following holds:

Theorem 7.1 (Srinivas, see Appendix)**.** *Let X be a normal projective variety over the field $\mathbb{C}$ of complex numbers. Let $\pi : Y \to X$ be a resolution of singularities. Assume $R^1\pi_*(\mathcal{O}_Y) = 0$ (this condition is independent of the choice of resolution). Then*

1. *the singular cohomology $H^2(X, \mathbb{Z})$ supports a pure Hodge structure;*
2. *the Néron-Severi group, $\mathrm{NS}(X) \stackrel{def}{=} c_1(Pic(X)) \subset H^2(X, \mathbb{Z})$ coincides with the subgroup of (1, 1) classes, i.e. with the subgroup* $\{\alpha \in H^2(X, \mathbb{Z}) | \alpha_{\mathbb{C}} \in H^2(X, \mathbb{C}) \text{ is of type } (1, 1)\}$.

The above Theorem is used in the Section 8.

Corollary 7.2. *Let X be a projective $\mathbb{Q}$-factorial threefold with isolated klt hypersurface singularities and $h^2(X, \mathcal{O}_X) = 0$. Then*

$$\chi_{top}(X) = 2\{1 - b_1(X) + \mathrm{KaDef}(X)\} - b_3(X)\,.$$

Proof. In fact $b_2(X) = \mathrm{KaDef}(X)$, as we observed at the beginning of this Section. The statement then follows from Corollary 5.18. □

Corollary 7.3. *Let X be a projective threefold with $h^2(X, \mathcal{O}_X) = 0$ and $\mathbb{Q}$-factorial rational singularities which are analytically $\mathbb{Q}$-factorial, then*

$$\chi_{top}(X) = 2 - 4h^{1,0}(X) + 2\,\mathrm{KaDef}(X) - b_3(X) \qquad (9)$$

$$= 2 - 2h^{0,3}(X) - 4h^{1,0}(X) + 2\{\mathrm{KaDef}(X) - h^{1,2}(X)\}. \qquad (10)$$

Theorem 7.4. *Let X be a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities. Then*

$$\chi_{top}(X) = 2\{\mathrm{KaDef}(X) - \mathrm{CxDef}(X)\} + \sum_P m(P),$$

where $m(P)$ is the Milnor number of the singular point P.

Proof. The statement follows from Proposition 6.4 and Corollary 7.2. □

Corollary 7.5. *All the terms in the equation of Theorem 7.4 do not depend on the choice of minimal model X.*

Proof. In fact, let X and X' be birationally equivalent minimal threefolds with $\mathbb{Q}$-factorial terminal singularities. Then $b_j(X) = b_j(X')$ $\forall j$ [49, Theorem 3.2.2]. Since birationally equivalent minimal models are related by flop transitions, $\mathrm{KaDef}(X) = \mathrm{KaDef}(X')$. Also, X and X' have the same analytic type of singularities [48, Theorem 2.4]; hence they also have the same Milnor numbers. Furthermore the dimensions of the miniversal deformation spaces are the same [51, Theorem 12.6.2]. □

8. Gauge algebras and representations (a Brieskorn-Grothendieck Program)

It was noted by Du Val and Coxeter [17,21] that rational double points are classified by the Dynkin diagrams of the simply laced Lie algebras of type $\mathfrak{a}_n, \mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. In fact, if we resolve the singularity by blowing up, the dual diagram of the exceptional divisors is one of the above Dynkin diagrams. Further connections between the Lie algebras and surface singularities were discovered in works by Brieskorn, Grothendieck, Tyurina, and Slodowy. A mathematical explanation of the parallelisms of the classification remains elusive. String theory provides a framework in which a Lie algebra $\mathfrak{g}$, the *"gauge algebra"*, is naturally associated to an elliptic fibration between Calabi-Yau manifolds. All the Dynkin diagrams, including the non-simply laced ones, occur. Deep relations between $\mathfrak{g}$, its representations and the geometry of the fibration have emerged, however the correspondence is often case by case and the assumption of smoothness imposes restrictions. In this section we review and state the expected correspondence in mathematical terms, and extend it to singular varieties, in particular to $\mathbb{Q}$-factorial terminal singularities.

In Sections 8.1.2 and 8.1.3 we construct the algebras, in Sections 8.2.1 and 8.2.2 we define a map between the codimension one and two strata of the discriminant locus of the fibration and the representations of the algebra. Theorem 9.4 in the following Section 9 provides evidence for a "Brieskorn-Grothendieck Program", associating Lie algebras and their representations to singularities of varieties.

Although Calabi-Yau threefolds with terminal singularities and elliptic fibrations are the focus of the applications in Section 9, in this Section we state definitions and results in more generality.

Definition 8.1. A genus one fibration is a morphism $\pi : X \to B$ whose fibers over a dense set are genus one curves. The complement of this dense locus is the discriminant of the fibration and it is denoted by Σ. X is relatively minimal if $K_X \cdot \Gamma \geq 0$, for all the curves Γ contracted by π (or equivalently K_X is π-nef).

If a genus one fibration π has a section $\sigma : B \to X$, it is called an elliptic fibration.

The support of the discriminant locus of a $X \to B$ a genus one fibration has a stratified structure given by its singularities; we analyze this in Section 8.2. If X and B are smooth and π is an elliptic fibration, X is the resolution of the Weierstrass model W of the fibration, which has Gorenstein singularities [61]:

Definition 8.2. A Weierstrass model W is defined by $y^2z-(x^3+\alpha xz^2+\beta z^3)=0$ where α, β are sections of $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 6}$ with $\mathcal{L}$ a line bundle on B.

If $\dim W = 2$, the singularities of W are the rational double points. If $\mathcal{L} = \mathcal{O}(-K_B)$, $K_W \simeq \mathcal{O}_W$. We are mostly interested in Weierstrass models W which are birationally Calabi-Yau varieties. By a rescaling of the Weierstrass equation, possibly together with a suitable blowup of the base, we assume that α and β nowhere vanish simultaneously with multiplicity equal to or higher than 4 and 6, that is there there are no “non-minimal" points. The assumption is necessary for the existence of an equidimensional birationally equivalent elliptic fibration $X \to B$; X is a relative minimal model of $W \to B$. The condition is also sufficient when $\dim W = 3$ [31].

Assuming X to be smooth imposes restrictions. However, without loss of generality we can still assume that B is smooth if $\dim(X) = 3$ [30, 61].

8.1. (Gauge) algebras and the codimension one strata of the discriminant.

8.1.1. *The abelian components of the gauge algebra.* If the Mordell-Weil group of the elliptic fibration has rank $r > 0$, the gauge algebra includes an abelian part $\mathfrak{u}(1)^{\oplus r}$. We briefly discuss the Mordell-Weil group and the abelian part of the gauge algebra after Theorem 9.4; in the present work we focus on the non-abelian part of the gauge algebra.

Next we present two methods to describe the non-abelian gauge algebras associated to the fibration, the first one uses the existence of a section. One novelty in our analysis is also the presence of singularities on X.

8.1.2. *The non-abelian components of the gauge algebra, through the “Tate algorithm".*

The proofs of the following Lemmas are along the general arguments of [34], but there they are not always stated explicitly.

Proposition 8.3. *Let B be smooth. To a Weierstrass model $W \to B$ there is a naturally associated Lie algebra $\mathfrak{g} = \bigoplus_{\Sigma_j} \mathfrak{g}(\Sigma_j)$, where the sum is taken over the irreducible components of the discriminant locus. For any irreducible component Σ_j, the Kodaira fiber over the general point of Σ_j and the possible $\mathfrak{g}(\Sigma_j)$ are listed in the second and third column of Table A.*

Proof. The singular fibers for smooth (relatively) minimal elliptic surfaces were classified by Kodaira [47], and the associated algebra is the one associated to the rational double point of the singular Weierstrass model. For Weierstrass threefolds which are equisingular along all components, this association gives the simply-laced algebras of Table A. By a careful analysis of Tate's algorithm [75] we extend the association to all dimensions. The correspondence does not use the existence of a smooth (relatively) mimimal model of the Weierstrass model $W \to B$. The Kodaira classification and Tate's algorithm only depend on the generic structure of the elliptic fibration along each irreducible component. The Kodaira-Tate algorithm as elaborated in Appendix B of [34] can still be applied, since B is smooth and each irreducible component of the discriminant Σ is a Cartier divisor, which is locally principal. The modified algorithm starts by constructing resolutions of the general singularities of the Weierstrass model W, which are the singularities over the general points of the discriminant locus. Then the analysis along the irreducible components Σ_j of the discriminant locus determines the algebra, together with a possible associated “monodromy" which leads to the non-simply laced algebras. □

The modified Tate algorithm also describes the structure of the partially resolved fibration near each component:

Lemma 8.4. *Let B be smooth, $W \to B$ be a Weierstrass model and $W_{\Sigma_j} \to B$ the partial general resolution in the Proof of Proposition 8.3. Let D^l_j be an irreducible Weil divisor which maps surjectively onto Σ_j. Then the elliptic fibration induces on D^l_j the structure of a surface generically ruled either over Σ_j or over Σ'_j, a finite branched cover of Σ_j. Let $\ell_{j,l}$ be the general fiber of the ruling. In the non-simply laced case, the cover is of degree* 2 *unless $\mathfrak{g}(\Sigma_j) = \mathfrak{g}_2$; in this case, the degree of the cover is* 3.

Note that D^l_j is not always normal and also that $\ell_{j,l}$ is not always reduced or irreducible.

8.1.3. *The non-abelian components of the gauge algebra, through the intersection matrix.*

From now on we consider genus one fibrations of threefolds, although most of what we write can be generalized to higher dimensions and for more general fibrations:

Definition 8.5. Let Y be a threefold and assume that $H_2(Y, \mathbb{Q})$ and $H_4(Y, \mathbb{Q})$ are Poincaré dual. Let $<,>$: $H_2(Y, \mathbb{Q}) \times H_4(Y, \mathbb{Q}) \to \mathbb{Q}$ be the induced non-degenerate pairing. For $E \in H_2(Y, \mathbb{Q})$, let $[E]^\perp = \{D \in H_4(Y, \mathbb{Q}) \text{ s. t. } < E, D >= 0\}$. For $B_1 \in H_4(Y, \mathbb{Q})$, let $[B_1]^\perp = \{C \in H_2(Y, \mathbb{Q}) \text{ s. t. } < C, B_1 >= 0\}$.

We also denote by $[E]^\perp$ its dual in $H^2(Y, \mathbb{Q})$. When Y is $\mathbb{Q}$-factorial the pairing is the intersection pairing between curves and $\mathbb{Q}$-Cartier divisors.

Definition-Proposition 8.6. Let $\pi : Y \to B$ be a fibration with general fiber E and assume that $H_2(Y, \mathbb{Q})$ and $H_4(Y, \mathbb{Q})$ are Poincaré dual. Assume also that B and Y are 1-rational.
Then $\pi^*(\mathrm{NS}(B)) \subseteq H^{1,1}(Y, \mathbb{Q}) \cap H^2(Y, \mathbb{Z}) \cap [E]^\perp$.

Set $\overline{\Lambda} \overset{def}{=} H^{1,1}(Y, \mathbb{Q}) \cap H^2(Y, \mathbb{Z}) \cap [E]^\perp$ and $\Lambda \overset{def}{=} \overline{\Lambda}/\pi^*(\mathrm{NS}(B))$.

Proof. Note that $\pi^*(\mathrm{NS}(B)) \subseteq \mathrm{NS}(Y) \cap [E]^\perp$. Since B and Y are 1-rational (Section 3) $\mathrm{NS}(Y) \subseteq H^{1,1}(Y,\mathbb{C}) \cap H^2(Y,\mathbb{Z})$, by Theorem 7.1. □

If $\pi : Y \to B$ is a factorial relatively minimal threefold (with terminal singularities) of a Weierstrass fibration $W \to B$, with B smooth as in Section 8.1.2, Λ is generated by the exceptional divisors D_j^l described in Lemma 8.4, because the fibration is equidimensional [31]. The identification depends on the choice of the Weierstrass model or equivalently on the choice of the section of the fibration π.

Definition 8.7. Let $\pi : Y \to B$ be a fibration, with general fiber E and Y $\mathbb{Q}$-factorial. Let $H_2^{(\pi)}(Y,\mathbb{Q})$ be the span in $H_2(Y,\mathbb{Q})$ of $\mathrm{NE}(Y/B)$, the classes of effective curves contracted by π, that is the set of ℓ such that $\pi_*(\ell) = 0$. Let L_1 be $H_2^{(\pi)}(Y,\mathbb{Q})$ modulo the numerical equivalence class of E.

Proposition 8.8. *Let Y, B and $\pi : Y \to B$ as in Definitions 8.6 and 8.7. Then the Poincaré pairing induces an integral pairing when restricted to the classes of algebraic curves in L_1 and Λ.*

L_1 and Λ can be defined also for genus one fibrations. When there is a section we have:

Definition 8.9. Let $\pi : Y \to B$ be an elliptic fibration with section $B_1 \simeq B$. Assume that $H_2(Y,\mathbb{Q})$ and $H_4(Y,\mathbb{Q})$ are Poincaré dual. Let $[B_1]^\perp$ be the orthogonal complement within $H_2(Y,\mathbb{Q})$. Let $H_2^{(\pi)}(Y,\mathbb{Q})$ be the span in $H_2(Y,\mathbb{Q})$ of the effective curves contracted by π, $\bar{L} \subset H_2(Y,\mathbb{Q})$ the subspace spanned by the $\ell_{j,l}$ in $H_2^{(\pi)}(Y,\mathbb{Q})$ and $L_2 = H_2(Y,\mathbb{Z}) \cap \bar{L} \cap [B_1]^\perp$.

If $\pi : Y \to B$ is a $\mathbb{Q}$-factorial relatively minimal model (with terminal singularities) of a Weierstrass fibration $W \to B$, as in Section 8.1.2, $L_1 \simeq L_2$ and the isomorphism between the two definitions depends on the choice of a section; in this case we write $L \stackrel{def}{=} L_1 \simeq L_2$.

Corollary 8.10. *Let $X \to B$ be a genus one threefold with Gorenstein $\mathbb{Q}$-factorial terminal singularities and $h^2(X,\mathcal{O}_X) = 0$. Then L_1 and Λ are well-defined and the pairing is integral when restricted to L_1 and to Λ.*

L_1 and in particular Λ in the statement are well defined if X has isolated klt singularities and $b_2(X) = b_4(X)$, by Theorem 5.13.

Proof. Theorem 5.16 implies that $H_2(X,\mathbb{Q})$ and $H_4(X,\mathbb{Q})$ are Poincaré dual. We noted in Section 3.2 that a Gorenstein threefold with $\mathbb{Q}$-factorial terminal singularities is actually factorial. □

In particular a Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities satisfies the hypothesis of the Corollary. Note also that if $h^2(W,\mathcal{O}_W) = 0$ then $\mathrm{NS}(Y) \subseteq H^2(Y,\mathbb{Z})$.

Proposition 8.11. *Let B be smooth, $W \to B$ be a Weierstrass model and $X \to B$ the birationally equivalent minimal model with $\mathbb{Q}$-factorial terminal Gorenstein singularities. Equivalently, let $X \to B$ be an elliptic threefold with Gorenstein $\mathbb{Q}$-factorial terminal singularities and $W \to B$ the associated minimal model. The pairing restricted to L and to Λ gives the negative of the Cartan matrices of the algebras $\mathfrak{g}(\Sigma_j)$ as in Proposition 8.3.*

Λ serves as the coroot lattice of the Lie group G associated with $\mathfrak{g}$, and $\Lambda \otimes U(1)$ serves as the Cartan subgroup, as in [33, Lemma 1.2].

Proof. The partial resolution constructed in Tate's algorithm is isomorphic to X over the general points of Σ. Recall that the fibration $\pi : X \to B$ is equidimensional [31]. Let D_j^l and $\ell_{j,l}$ be defined as in Corollary 8.4. The curves $\{\ell_{j,l}\}$ generate L over $\mathbb{Q}$, $\{D_j^l\}$ generate Λ; $< \ell_{j,k}, D_j^l >= \ell_{j,k} \cdot D_j^l$ gives the negative of the entries of a block of the Cartan matrix. □

Remark 8.12. The Poincaré pairing between Λ and L gives the transpose Cartan matrix; recall that the Cartan matrix is not symmetric if $\mathfrak{g}$ is not simply laced. Note also that the rows of (a block in) the Cartan matrix are the Dynkin coefficients of the roots, the weights of the adjoint representation. In fact, we will see in the following Section 8.2 that associated to Σ_j is an “unlocalized" representation, which is precisely given by the adjoint representation for the simply laced algebras.

8.2. **Representations of (gauge) algebras and the codimension two strata.**

The support of the discriminant locus Σ of a genus one fibration has a stratified structure given by its singularities. In this paper we focus on the codimension one strata, given by the irreducible components of codimension one in B, and the codimension two strata, given by the singular locus of Σ and more generally by the codimension two components of Σ in B. To simplify the statements we assume that $\dim B = 2$, however the statements also hold for higher dimensions with appropriate modifications. If $\dim B = 2$, we denote by Q a singular point of Σ.

The “unlocalized "and “localized" representations in the physics language are associated to the different codimension of the strata of Σ and both occur with a certain multiplicity which depends on the dimension of the base B. We will present methods for computing the multiplicities if B is a surface in the following sections. The methods can be extended to the presence of singularities, as we prove under certain general assumptions in Section 9.

To simplify the statements we assume that B is smooth, $W \to B$ be a Weierstrass model and $X \to B$ the birationally equivalent minimal model with $\mathbb{Q}$-factorial terminal Gorenstein singularities. Equivalently, let $X \to B$ be an elliptic variety with Gorenstein $\mathbb{Q}$-factorial terminal singularities and $W \to B$ the associated minimal model. We can assume $X \to B$ to be equidimensional and B smooth. Let B_1 be a section of the fibration.

8.2.1. *The unlocalized representations.* [Codimension one strata]

Lemma 8.13.

(i) *The intersection product between* $\ell_{j,k}$ *and* D^l_j *described in Proposition 8.11 gives the positive simple root vectors* $\alpha^l_k = -\ell_{j,k} \cdot D^l_j$*, which are weight vectors associated with the representation* $\mathrm{adj}_{\mathfrak{g}(\Sigma_j)}$*.*

(ii) *If* $\mathfrak{g}(\Sigma_j)$ *is not simply laced, there is another naturally associated representation* $\rho_0^{d-1}(\Sigma_j)$*, described in Table A.*

Proof. X is smooth over the generic point of the codimension one strata of Σ; see also Remark 8.12. This proves (i). If $\mathfrak{g}(\Sigma_j)$ is not simply laced, let $\tilde{\mathfrak{g}}(\Sigma_j)$ be the Lie algebra associated with the generic fiber of Σ_j; it is a cover of the Lie algebra $\mathfrak{g}(\Sigma_j)$. $\rho_0(\Sigma_j)$ is the representation determined through the "branching rules" which decompose $\mathrm{adj}_{\tilde{\mathfrak{g}}(\Sigma_j)}$ as $\mathrm{adj}_{\mathfrak{g}(\Sigma_j)} \oplus \rho_0^{d-1}(\Sigma_j)$, where d is the degree of the cover introduced in Lemma 8.4 [33]. □

The Lemma motivates the following

Assignment 8.14 (Unlocalized representations)**.** *To each irreducible component of the codimension one strata* Σ_j *one associates an unlocalized representation of* $\mathfrak{g}(\Sigma_j)$ *as follows:*
If $\mathfrak{g}(\Sigma_j)$ *is simply laced, the unlocalized representation is* $\mathrm{adj}_{\mathfrak{g}(\Sigma_j)}$*.*
If $\mathfrak{g}(\Sigma_j)$ *is not simply laced, the unlocalized representation associated with* Σ_j *is* $\mathrm{adj}_{\mathfrak{g}(\Sigma_j)} \oplus \rho_0(\Sigma_j)$*, where* $\rho_0(\Sigma_j)$ *is summarized in Table A.*

If $\mathfrak{g}(\Sigma_j)$ *is simply laced, the multiplicity of the representation is the genus of* Σ_j*,* $g(\Sigma_j)$*; if* $\mathfrak{g}(\Sigma_j)$ *is not simply laced, the multiplicity is* $g(\Sigma'_j) - g(\Sigma_j)$*, with* Σ'_j *as in Lemma 8.4.*

8.2.2. *The localized representations.* [Codimension two strata]

Let Q be a singular point of the discriminant Σ. The physics predicts that certain representations are associated to Q. We present two ways, Method 8.17 and Method 8.21, to compute the unlocalized representations, building on [77], [4, 43] and then [45] as elaborated further in [34]. Our general results imply that the methods can be extended to the case of $\mathbb{Q}$-factorial terminal singularities, as we will verify explicitly under certain genericity assumptions in Section 9.

The underlying principles and some first computations were outlined in [77], [4,43] and [45]. Various refinements and verifications, on smooth fibrations, have been made in the physics literature since. In the case of Calabi-Yau varieties, we will also show that the representations are independent of the choice of the minimal model.

The constructions described below will give the trivial representation if Q is replaced either by a general point of B, or a general point in Σ (the codimension one strata).

We make the following Conjecture, which has been verified under general conditions even in the presence of $\mathbb{Q}$-factorial terminal singularities, as we will prove in this paper (see Table A), and in various other examples [2]:

Conjecture 8.15. *For Q a singular point of the discriminant Σ as above, denote by X_Q the corresponding fiber in X. Let ℓ^a_Q be the class of an irreducible component of X_Q in $H_2(X,\mathbb{Z}) \cap [B_1]^{\perp}$.*

1. *The intersection numbers with the ruled divisors D^l_j,*
$$\beta^l(\ell^a_Q) = -\ell^a_Q \cdot D^l_j, \quad l = 1, \ldots, \mathrm{rk}(\mathfrak{g}(\Sigma_j)),$$
form the entries of a weight vector of an irreducible representation $\rho_{Q,a}$ of $\mathfrak{g}(\Sigma_j)$.
2. *All weight vectors $\beta^l_p(\rho_{Q,a})$, labeled by $p \in \{1, \ldots, \dim(\rho_{Q,a})\}$, are obtained by $\beta^l_p(\rho_{Q,a}) = -C_p(\rho_{Q,a}) \cdot D^l_j$ with*
$$C_p(\rho_{Q,a}) = \ell^a_Q + \sum_{k=1}^{\mathrm{rk}(\mathfrak{g}(\Sigma_j))} n^k_p\, \ell_{j,k}, \quad n^k_p \in \mathbb{Z}.$$
3. *Some of the curve classes $[C_p(\rho_{Q,a})]$ are represented by effective curves, and the remaining ones by anti-effective ones.*

In the algebra-geometry dictionary, (2) states that all the weights are obtained by adding to a weight vector $\beta^l(\ell^a_Q)$ the linear combinations of the positive simple roots α^l_k with suitable coefficients $n^k_p \in \mathbb{Z}$, $\beta^l_p(\rho_{Q,a}) = \beta^l(\ell^a_Q) + \sum_{k=1}^{\mathrm{rk}(\mathfrak{g}(\Sigma_j))} n^k_p\, \alpha^l_k$. If $\ell^a_Q = \ell_{j,l}$ is the class of a ruling, then $\rho_{Q,a} = \mathrm{adj}(\mathfrak{g}(\Sigma_j))$, in agreement with the first observation in the proof of Lemma 8.13.

Definition 8.16. Let $C_p(\rho_{Q,a}) = \ell^a_Q + \sum_{k=1}^{\mathrm{rk}(\mathfrak{g}(\Sigma_j))} n^k_p\, \ell_{j,k}, \quad n^k_p \in \mathbb{Z}$, as in Conjecture 8.15. Let $M(\ell^a_Q) \overset{def}{=} \{C_p(\rho_{Q,a})\}$ the collection of such curves and $-M(\ell^a_Q) \overset{def}{=} \{-C_p(\rho_{Q,a})\}$.

Assignment 8.17 (Localized representations, via weight lattices from intersection theory)**.** *With the notation above, we make the following assignments:*

1. *To each irreducible fiber components ℓ^b_Q, assign a representation $\rho_{Q,b}$ as in Conjecture 8.15.*
2. *$\ell^a_Q \neq \ell^b_Q$ give independent representations if and only if $M(\ell^b_Q) \neq \pm M(\ell^a_Q)$.*
3. *If $M(\ell^a_Q) = -M(\ell^a_Q)$ as a set, then the assigned multiplicity to $\rho_{Q,a}$ is $\delta_a = \frac{1}{2}$, otherwise $\delta_a = 1$.*
4. *The full representation associated with Q, with respect to $\mathfrak{g}(\Sigma_j)$, is then*
$$\rho_Q = \sum_{\rho_{Q,a} \neq \mathrm{adj}_{\mathfrak{g}(\Sigma_j)}} \delta_a\, \rho_{Q,a},$$
where the sum is over the independent representations different from $\mathrm{adj}(\mathfrak{g}(\Sigma_j))$.

Remark 8.18. If Q is at the intersection of two different components, for example $\Sigma_i \cap \Sigma_j$, and $\mathfrak{g}(\Sigma_i) \neq \{e\}$ and $\mathfrak{g}(\Sigma_j) \neq \{e\}$, then Q gives rise to representations of $\mathfrak{g}(\Sigma_i) \oplus \mathfrak{g}(\Sigma_j)$. This must be taken into consideration in determining the final multiplicity of the representations at Q.

Remark 8.19. In this sense L as in Definitions 8.7 and 8.9 together with the intersection pairing with Λ defines the weight lattice of the total algebra $\mathfrak{g}$. When X is Calabi-Yau, two birationally equivalent resolutions X and X' of the same Weierstrass model W give rise to the same representations ρ_Q defined above, as studied in the physics literature e.g. in [22, 41, 43]. We prove this in general in Theorem 9.7.

The novel aspect of this Section is also that the procedure outlined in Method 8.17 continues to be applicable if X has $\mathbb{Q}$-factorial terminal singularities. In this case, X is the relative minimal model of W. However, special care must be taken in evaluating the intersection numbers determining the weights due to the presence of the singularities. As we show in Case 3 in the proof of Theorem 9.4 below, the singularity is associated to a (non)-trivial localized representation of the algebra.

Before describing the second method to determine the localized representations, we need the following

Definition 8.20. Let $W \to Z$ be a minimal Weierstrass model over a smooth surface Z. Let $z \in Z$ be a point and $C \subset Z$ be a general curve through z in a (Euclidean) neighborhood of z. Consider the Weierstrass surface $W_{|C}$ restricted to C. Without loss of generality we assume also that $W_{|C}$ defines a minimal Weierstrass surface. Then by $\mathfrak{g}(z)$ we denote the "gauge" algebra associated to $W_{|C}$ at the point z.

Note that the singularity of $W_{|C}$ in the fiber over z is a rational double point, and $\mathfrak{g}(z)$ is the simply laced Lie algebra with Dynkin diagram the dual graph of the curve of resolution.

Assignment 8.21 (Localized representations, via "Katz-Vafa's method")**.** *Consider W, B and X as stated at the beginning of Section 8.2, and let Q be a singular point of the discriminant Σ.*

(1) *Up to a change of parameter $t = z^d$, there is a family of disks C_t intersecting Σ at P_t with $P_0 = Q$ such that the singularities of $\pi^{-1}(C_t)$ admit a simultaneous resolution.*
(2) *Furthermore, there exists a space of versal deformations of $\pi^{-1}(C_0)$ which is simultaneously resolvable, and the parameter curve $\{z\}$ of C_{z^d} is a ramified cover of the parameter curve of the versal deformations with ramification c at $z = 0$. Let $b = d/c$ and locally $z^d = s^b$.*
(3) *Then one can decompose*

$$\mathrm{adj}_{\mathfrak{g}(Q)} = \mathrm{adj}_{\mathfrak{g}(Q_{s^b})} \oplus \hat{\rho}_Q \oplus \bar{\hat{\rho}}_Q \oplus 1^{\oplus(\mathrm{rk}(\mathfrak{g}_Q) - \mathrm{rk}(\mathfrak{g}(Q_{s^b})))},$$

with $\mathfrak{g}(Q)$ and $\mathfrak{g}(Q_{s^b})$ as in Definition 8.20.

(4) *The representation ρ_Q associated to Q in X is*

$$\rho_Q = \frac{1}{b}\rho'_Q,$$

where ρ'_Q follows from decomposing $\hat{\rho}_Q$ into representations of $\mathfrak{g}$ as

$$\hat{\rho}_Q = \rho'_Q \oplus \rho_{\text{sing}} \oplus \oplus_i \frac{1}{2}\rho_0(\Sigma_i). \tag{11}$$

Here ρ_{sing} (if non-zero) is a singlet with respect to $\mathfrak{g}$ and the factors $\frac{1}{2}\rho_0(\Sigma_i)$ may appear in the decomposition for those Σ_i with $Q \in \Sigma_i$ only if $\mathfrak{g}(\Sigma_i)$ is non-simply laced and Q is a ramification point for the associated monodromy. In this case $\rho_0(\Sigma_i)$ is as defined in Lemma 8.13.

Note that (1) and (2) are guaranteed by the Tyurina and Brieskorn-Grothendieck theorems [14, 72].

Remark 8.22. We show in the following section that Method 8.21 continues to be applicable in the presence of $\mathbb{Q}$-factorial terminal singularities.

Conjecture 8.23. *Refinements of Method 8.17 and Method 8.21 determine the same localized representations.*

The conjecture has been confirmed for various classes of examples, but no general proof has been obtained.

9. (Gauge) algebras, representations, singularities and the topological Euler characteristic of Calabi-Yau threefolds

Let $X \to B$ be an elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities and Weierstrass model W, as in the previous sections.

Singular varieties are in fact unavoidable also in the physics interpretation, even in the case of $\mathbb{Q}$-factorial Calabi-Yau threefolds with terminal singularities, when there is a smoothing [62], but the smooth Calabi-Yau lies outside the loci of interest. This is the case of the Weierstrass models associated with the Jacobian of general genus one fibrations without a section, which has $\mathbb{Q}$-factorial terminal singularities [10]. For Calabi-Yau fourfolds it is known that even simple examples of isolated terminal singularities of the type $\mathbb{C}^4/\Gamma$ cannot be smoothed [60, 71].

We find that while "the gauge algebra" can be associated as in the smooth case, the dictionary described in [8, 33, 34] between the "anomaly constraints" in physics and the geometry of the Calabi-Yau must be modified when the Calabi-Yau is singular.

If $X = W$ is smooth, that is the gauge algebra is trivial, $30K_B^2 + \frac{1}{2}\chi_{top}(X) = 0$ [33, Theorem 2.2]. More generally, we define the following invariant $\mathcal{R}$ and prove that, under general conditions, it contains information about (the dimensions of) certain representations of the associated gauge algebra.

Definition 9.1. Let

$$\mathcal{R} = 30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) + \sum_P m(P)\right), \tag{12}$$

where the sum is over the singular points P of X with Milnor number $m(P)$.

By Corollary 7.5, $\mathcal{R}$ is independent of the choice of the particular minimal model X, the $\mathbb{Q}$-factorial terminal resolution of W. $\mathcal{R}$ is a topological invariant of X.

9.1. **Gauge algebra, general.** As in [33], we assume that the discriminant is of the form $\Sigma = \Sigma_1 \cup \Sigma_0$, where Σ_1 is a smooth curve and Σ_0 denotes the locus where the general fiber is nodal (I_1 fiber), $\mathfrak{g}(\Sigma_0) = \emptyset$ and $\mathfrak{g}(\Sigma_1)$ is the associated Lie algebra as in Proposition 8.3. We also assume that the Weierstrass model is otherwise general (“genericity assumption"). Let λ be the multiplicity of Σ along Σ_1. Our assumptions have the following implications, as summarized in

Proposition 9.2 (Proposition 4.4 in [33])**.**

1. $\Sigma_0 \cap \Sigma_1 = \{Q_1^1, \cdots, Q_1^{B_1}, Q_2^1, \cdots, Q_2^{B_2}\}$. *The numbers B_i are determined by the algebra $\mathfrak{g}$.*
2. *(Equivalently:) $\Sigma_0 \cdot \Sigma_1 = (-12K_B - \lambda\Sigma_1) \cdot \Sigma_1 = r_1B_1 + r_2B_2$, where the numbers r_i and λ are determined by the algebra $\mathfrak{g}$.*
3. *The local equation around each point Q_i^ℓ does not depend on ℓ, but only on $i = 1, 2$; then without loss of generality we write $X_{Q_i} = \pi^{-1}(Q_i^\ell)$.*

As in [33] we make the following

Definition 9.3. Let ρ be a representation of a Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$. The *charged dimension of* ρ is $(\dim\rho)_{ch} = \dim(\rho) - \dim(ker\rho|_{\mathfrak{h}})$.

For example, if ρ is the adjoint representation then

$$(\dim \mathrm{adj})_{ch} = \dim\mathfrak{g} - \dim\mathfrak{h} = \dim G - \mathrm{rk}\, G.$$

Theorem 9.4. *Let X, W and Σ be as above and $\mathcal{R}$ as in Definition 9.1; let $Q \in \Sigma_1$ denote the singular points of Σ. Let ρ_Q be the associated localized representation obtained as in Section 8.2.2. ρ_Q is given in Table A, a modified version of Table A in [34]. Let P denote the singular points of X with Tyurina number $\tau(P)$. Then*

$$\mathcal{R} \;=\; (g-1)(\dim \mathrm{adj})_{ch} + (g'-g)(\dim\rho_0)_{ch} + \sum_Q (\dim\rho_Q)_{ch} + \sum_P \tau(P)\,. \tag{13}$$

Here $g \overset{def}{=} g(\Sigma_1)$ and $g' \overset{def}{=} g(\Sigma_1')$ denote the genus of the discriminant component Σ_1 and, respectively, of its finite branched cover Σ_1' occurring in Lemma 8.4 and $\mathrm{adj} = \mathrm{adj}_{\mathfrak{g}(\Sigma_1)}$, $\rho_0 =$ $\rho_0(\Sigma_1)$ *are the unlocalized representations according to Lemma 8.13.*

Number	Type	$\mathfrak{g}$	ρ_0	$\rho_{Q_1^\ell}$	$\rho_{Q_2^\ell}$	$(\dim \mathrm{adj})_{ch}$	$(\dim \rho_0)_{ch}$	$\dim (\rho_{Q_1^\ell})_{ch}$	$\dim (\rho_{Q_2^\ell})_{ch}$
1	I_1	$\{e\}$		–	–	0	0	0	0
2	I_2	$\mathfrak{su}(2)$		–	fund	2	0	0	2
3	I_3	$\mathfrak{su}(3)$		–	fund	6	0	0	3
4	I_{2k}, $k \geq 2$	$\mathfrak{sp}(k)$	Λ^2_0	–	fund	$2k^2$	$2k^2 - 2k$	0	$2k$
5	I_{2k+1}, $k \geq 1$	$\mathfrak{sp}(k)$	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund	fund	$2k^2$	$2k^2 + 2k$	k	$2k$
6	I_n, $n \geq 4$	$\mathfrak{su}(n)$		Λ^2	fund	$n^2 - n$	0	$\frac{1}{2}(n^2 - n)$	n
7	II	$\{e\}$		–		0	0	0	
8	III	$\mathfrak{su}(2)$		$2 \times$ fund		2	0	4	
9	IV	$\mathfrak{sp}(1)$	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund		2	4	1	
10	IV	$\mathfrak{su}(3)$		$3 \times$ fund		6	0	9	
11	I_0^*	$\mathfrak{g}_2$	**7**	–		12	6	0	
12	I_0^*	$\mathfrak{so}(7)$	vect	–	spin	18	6	0	8
13	I_0^*	$\mathfrak{so}(8)$		vect	spin$_\pm$	24	0	8	8
14	I_1^*	$\mathfrak{so}(9)$	vect	–	spin	32	8	0	16
15	I_1^*	$\mathfrak{so}(10)$		vect	spin$_\pm$	40	0	10	16
16	I_2^*	$\mathfrak{so}(11)$	vect	–	$\frac{1}{2}$ spin	50	10	0	16
17	I_2^*	$\mathfrak{so}(12)$		vect	$\frac{1}{2}$ spin$_\pm$	60	0	12	16
18	I_n^*, $n \geq 3$	$\mathfrak{so}(2n+7)$	vect	–	NM	$2(n+3)^2$	$2n+6$	0	NM
19	I_n^*, $n \geq 3$	$\mathfrak{so}(2n+8)$		vect	NM	$2(n+3)(n+4)$	0	$2n+8$	NM
20	IV^*	$\mathfrak{f}_4$	**26**	–		48	24	0	
21	IV^*	$\mathfrak{e}_6$		**27**		72	0	27	
22	III^*	$\mathfrak{e}_7$		$\frac{1}{2}$**56**		126	0	28	
23	II^*	$\mathfrak{e}_8$		NM		240	0	NM	

Table A. The representations which occur under our "generic" hypotheses. The associated representation is independent of the particular resolution. Cases with non-minimal Weierstrass model are denoted "NM". $(\dim \rho_i)_{ch} = \mathcal{R}_i$.

Number	Type	$\mathfrak{g}$	Q_1^ℓ	Q_2^ℓ	$\chi(X_{Q_1^\ell})$	$\chi(X_{Q_2^\ell})$	$\tau(P_1)$	$\tau(P_2)$
1	I_1	$\{e\}$	II	I_1 (NSR)	2 (II)	1	0	1
5	$I_{2k+1}, k \geq 1$	$\mathfrak{sp}(k)$	I^*_{2k-2}	I_{2k+1} (NSR)	$k+2$ (br.)	2k+1	1	0
7	II	$\{e\}$	III (NSR)		2		2	

Table B. The fiber types listed in column 4 and in column 5 correspond to the vanishing orders of the Weierstrass model and not to the topology of the fiber $X_{Q_i^\ell}$ of the minimal terminal $\mathbb{Q}$-factorial resolution. In the last two columns we list the Milnor-Tyurina numbers at the points with $\mathbb{Q}$-factorial terminal singularities (with no small resolution NSR).

Proof of Theorem 9.4. If X is smooth, $m(P) = \tau(P) = 0$, the Theorem has been proved in [33] by deconstructing $\chi_{top}(X)$ with the help of the Mayer-Vietoris sequence and explicitly comparing both sides of (13) for all possible 20 types of Weierstrass models subject to the stated assumptions. In particular, the dimensions $\mathcal{R}_i = (\dim \rho_i)_{ch}$ include the multiplicities of the representations as given in Table A . If X is singular, in our general hypothesis the Milnor and Tyurina number are equal [69]. There are three more cases to consider. These are listed as Models number 1, 5 and 7 in Tables A, B taken from [33] with the information on the singular models completed. In all these cases, $\chi_{top}(X)$ is computed via deconstruction [33] as summarized in equ. (A.11) of [2].

Case 1 (Model 1 in Table A): The fiber over generic points of Σ_1 is of Kodaira Type I_1 and the gauge algebra is trivial. In the fibers over the points Q_2^ℓ there are $\mathbb{Q}$-factorial terminal singularities with Milnor and Tyurina numbers $m(P_2) = \tau(P_2) = 1$ (Kleinian A_2 or conifold); note that these are not locally analytically $\mathbb{Q}$-factorial. Since the gauge group is trivial, no charged representations are present. The claim then follows.

Case 2 (Model 7 in Table A): The fiber over generic points of Σ_1 is of Kodaira Type II and the gauge algebra is trivial. In the fibers over the points Q_1^ℓ there are $\mathbb{Q}$-factorial terminal singularities (Kleinian A_3) with $m(P_1) = \tau(P_1) = 2$. There are again no charged representations, and the Theorem follows.

Case 3 (Model 5 in Table A): The fiber over generic points of Σ_1 is locally of Kodaira Type I_{2k+1} and associated gauge algebra $\mathfrak{g} = \mathfrak{sp}(k)$. In the fibers over the points Q_2^ℓ there are $\mathbb{Q}$-factorial terminal singularities with $m(P_2) = \tau(P_2) = 1$; topologically the fiber at Q_2^ℓ is the same as the general fiber over Σ_1. We claim that the singularity induces a representation associated to Q_2^ℓ, and that it is $\rho_2 = \mathrm{fund}_{\mathfrak{sp}(k)}$. This representation can be understood by considering the double cover $\hat{X}$ of the fibration with I_{2k+1} fibers over generic points of Σ_1, with associated algebra $\mathfrak{su}(2k+2)$. In the language of Method 8.21, this double cover admits a simultaneous resolution. Hence $t = z^2$, i.e. $d = 2$. In the double cover, the the fiber at Q_2^ℓ becomes I_{2k+2}. The versal deformations of this singularity in $\hat{X}$ are parametrized by a deformation parameter $s = t = z^2$, leading to the parameter $b = 1$. Following Method 8.21 (and the "branching rules") we decompose the adjoint of $\mathfrak{g}(Q_2^\ell) = \mathfrak{su}(2k+2)$ into a representation of $\mathfrak{g}((Q_2^\ell)_s) = \mathfrak{su}(2k+1)$,

$$(14) \qquad \mathfrak{su}(2k+2) \quad \to \quad \mathfrak{su}(2k+1) \oplus \mathfrak{u}(1)$$

$$(15) \qquad \mathrm{adj}_{\mathfrak{su}(2k+2)} \quad \to \quad (\mathrm{adj}_{\mathfrak{su}(2k+1)})_0 + (\mathrm{adj}_{\mathfrak{u}(1)})_0 + (\mathrm{fund}_{\mathfrak{su}(2k+1)})_1 + \overline{(\mathrm{fund}_{\mathfrak{su}(2k+1)})_1}$$

and further decompose $\mathrm{fund}_{\mathfrak{su}(2k+1)}$ into a representation of $\mathfrak{sp}(k)$,

$$(16) \qquad \mathrm{fund}_{\mathfrak{su}(2k+1)}\,|_{\mathfrak{sp}(k)} = 1 + \mathrm{fund}_{\mathfrak{sp}(k)}\,.$$

Note that this decomposition is consistent with the form of formula (11) because Q_2^ℓ is not a branch point for monodromy. The $\mathfrak{sp}(k)$ charged part of the decomposition is the representation associated with Q_2^ℓ, i.e. $\rho_2 = \mathrm{fund}_{\mathfrak{sp}(k)}$. In addition $\rho_0 = \mathbf{\Lambda}^2 + 2\,\mathrm{fund}$ [33]. The RHS of (13) evaluates to $(g-1)(\dim(\mathfrak{g}) - \mathrm{rk}(\mathfrak{g})) + (g'-g)\mathcal{R}_0 + B_1\mathcal{R}_1 + B_2\mathcal{R}_2 + B_2$, with $\mathcal{R}_i = (\dim \rho_i)_{ch}$

as given in Table A and with $g' - g = \frac{1}{2}\Sigma_1 \cdot (\Sigma_1 - K_B)$, $g - 1 = \frac{1}{2}\Sigma_1 \cdot (\Sigma_1 + K_B)$ [33]. The claim follows. □

9.2. **F-theory interpretation.** In the physics literature mostly smooth models have been considered; in particular the "F-theory" interpretation of the correpondence between singularities and algebras is on manifolds. However, $\mathbb{Q}$-factorial terminal singularities occur naturally, for example in the Jacobian variety of a genus one fibration [10], in certain fiber products of rational elliptic surfaces [59], as well as F-theory duals of generic non-geometric compactifications of the heterotic string as studied in [24, 25].

Theorem 9.4 is consistent with the cancellation of gravitational anomalies in the six-dimensional effective theory obtained by compactification of "F-theory" on X,

$$H - V + 29T = 273, \tag{17}$$

even when X has singularities. Here

$$\dim(\mathfrak{g}) = V, \quad h^{1,1}(B) - 1 = T$$

are the "number of vector multiplets" and the "number of tensor multiplets", respectively. The "number of hypermultiplets" H splits into the number of "charged" and "uncharged" hypermultiplets,

$$H = H_{unch} + H_{ch},$$

where H_{unch} is the dimension of the complex deformations +1 and H_{ch} is related to the dimension of the algebra and its representations, with multiplicities. Theorem 9.4 indicates that H_{ch} and H_{unch} decompose in localized and unlocalized summands. We make the following

Definition 9.5.

(i) $H_{ch} = H_{ch}^{unloc} + H_{ch}^{loc}$, with
$H_{ch}^{unloc} = g\,(\dim \mathrm{adj}_{\mathfrak{g}})_{ch} + (g' - g)(\dim \rho_0)_{ch}$ and $H_{ch}^{loc} = \sum_Q (\dim \rho_Q)_{ch}$.

(ii) $H_{unch} = H_{unch}^{unloc} + H_{unch}^{loc}$, with
$H_{unch} = 1 + \mathrm{CxDef}(X)$ and $H_{unch}^{loc} = \sum_P \tau(P)$.

The "uncharged localized" hypermultiplets counted by $H_{unch}^{loc} = \sum_P \tau(P)$ are the number of versal deformations of the singularities at P in Remark 6.7. The splitting motivates Conjecture 6.8.

With this definition, and in presence of singularities, Theorem 9.4 is equivalent to the gravitational anomaly cancellation condition (17), where one has to use that $K_B^2 = 10 - h^{1,1}(B)$ and, for the models satisfying the "genericity assumption" of Theorem 9.4, $h^{1,1}(X) = 1 + h^{1,1}(B) + \mathrm{rk}(\mathfrak{g})$ as well as $m(P) = \tau(P)$ [34], [2].

9.3. **Mordell-Weil group and several components.**

When the Mordell-Weil group $\mathrm{MW}(X)$ of the elliptic fibration has rank r, the gauge algebra includes also an abelian part $\mathfrak{u}(1)^{\oplus r}$, see Section 8.1.1. The codimension two strata of Σ_0 will include additional points C_r (over which the fiber is Kodaira type I_2, X locally smooth). Then according to our assignment, we expect an associated representation associated to the abelian part of the algebra $\mathfrak{u}(1)^{\oplus r}$. In the physics framework, this is the contribution of the extra "charged singlets" localized at C_r to H_{ch} [58]. If X is otherwise general then (13) in Theorem 9.4 continues to hold, but on the RHS we add the additional contribution of $\sum_{C_r}$. The explicit counting of the points Q furthermore changes accordingly.

Conjecture 9.6. *Let $X \to B$ be an elliptic Calabi-Yau threefold with $\mathbb{Q}$-factorial terminal singularities $\{P\}$, the relative minimal model of a Weirstrass model $W \to B$.*
Assume $\mathrm{rk}(\mathrm{MW}(X)) = r$, *that the discriminant is of the form $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_N$ with the simple algebra $\mathfrak{g}_i$ associated to Σ_i and that the Weierstrass model is otherwise general. Let*

$$\mathcal{R}' \overset{def}{=} 30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) - \sum_P m(P) + 2\sum_P \tau(P)\right).$$

Then

$$\mathcal{R}' = \sum_i (g_i - 1)(\dim \mathrm{adj}_i)_{ch} + (g_i' - g_i)(\dim \rho_{0,i})_{ch} + \sum_Q (\dim \rho_Q)_{ch} + \sum_{C_r} 1 + \sum_P \tau(P),$$

where $g_i = g(\Sigma_i)$, $g_i' = g(\Sigma_i')$, and Q, C_r are the codimension two strata of Σ.

If $Q \in \Sigma_i \cap \Sigma_j$, the associated representation ρ_Q is a tensor product representation with respect to $\mathfrak{g}_i \oplus \mathfrak{g}_j$.

If $m(P) = \tau(P)$ for all P, $\mathcal{R} = \mathcal{R}'$.

9.4. **Birational Kodaira Classification and elliptic fibration of higher dimensional varieties.**

Theorem 9.7. *The algebra and the representations are birational invariants of the $\mathbb{Q}$-factorial terminal minimal model $X \to B$.*

Proof. $\mathcal{R}$ is a birational invariant of the minimal model of the fibration $X \to B$ (Corollary 7.5). The gauge algebra and the unlocalized representations are also invariant by construction. The proof of Theorem 9.4 shows that the possible flops in the fibers over the points Q are isomorphisms in the neighborhood of the fibers over Q' for $\rho_Q \neq \rho_{Q'}$. □

The dimensions of the representations, listed in last four columns of Table A uniquely, determine the Kodaira type of the general fibers over the codimension one strata of the discriminant, as well as the fibers over the codimension two strata, up to birational transformations of the relative minimal model of the fibration. Based on Theorem 9.7 we make the following

Conjecture 9.8. *The Kodaira classification of singular fibers on relatively minimal elliptic surfaces, with section, extends to the class of birationally equivalent relatively minimal elliptic threefolds. The classification is obtained by associating to the stratified discriminant locus of the fibration Σ the non abelian gauge algebras and their representations.*

We speculate that the classifications can be suitably extended in higher dimension, for example, in the case of fourfolds with the addition of information on the Yukawa coupling. Recent work [1] suggests that multiple fibers in a Calabi-Yau of type ${}_mI_0$ are associated to discrete torsion.

Conjecture 9.9. *The Kodaira classification of singular fibers on relatively minimal genus one surfaces extends to the class of birationally equivalent relatively minimal genus one fibered varieties as in 9.8. The multiple fibers of multiplicity m in a Calabi-Yau are associated to discrete torsion $\mathbb{Z}/m\mathbb{Z}$.*

References

[1] L. B. Anderson, A. Grassi, J. Gray, and P.-K. Oehlmann, *F-theory on Quotient Threefolds with (2,0) Discrete Superconformal Matter*. arXiv:1801.08658, 2018.

[2] P. Arras, A. Grassi, and T. Weigand, *Terminal singularities, Milnor numbers, and matter in F-theory*, J. Geom. Phys., 123 (2018), pp. 71–97.

[3] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math., (1969), pp. 23–58.

[4] P. S. Aspinwall, S. H. Katz, and D. R. Morrison, *Lie groups, Calabi-Yau threefolds, and F theory*, Adv. Theor. Math. Phys., 4 (2000), pp. 95–126.

[5] M. F. Atiyah, *On analytic surfaces with double points*, Proc. Roy. Soc. London. Ser. A, 247 (1958), pp. 237–244.

[6] M. Banagl, *Topological and Hodge L-classes of singular covering spaces and varieties with trivial canonical class.* arXiv:1706.03578, 2017.

[7] M. Beltrametti and F. Odetti, *Semifattorialità e aperti affini*, Ann. Univ. Ferrara Sez. VII (N.S.), 23 (1977), pp. 11–16 (1978).

[8] M. Bies, C. Mayrhofer, and T. Weigand, *Algebraic Cycles and Local Anomalies in F-Theory*, JHEP, 11 (2017), p. 100.

[9] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc., 23 (2010), pp. 405–468.

[10] V. Braun and D. R. Morrison, *F-theory on genus-one fibrations*, J. High Energy Phys., (2014), p. 132.

[11] J. Brevik and S. Nollet, *Srinivas' problem for rational double points*, Michigan Math. J., 64 (2015), pp. 155–168.

[12] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math., 2 (1966), pp. 1–14.

[13] ———, *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen*, Math. Ann., 166 (1966), pp. 76–102.

[14] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, in Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 279–284.

[15] F. E. Browder, ed., *Mathematical developments arising from Hilbert problems*, Proceedings of Symposia in Pure Mathematics, Vol. XXVIII, American Mathematical Society, Providence, R. I., 1976.

[16] J. Cheeger, M. Goresky, and R. MacPherson, *L^2-cohomology and intersection homology of singular algebraic varieties*, in Seminar on Differential Geometry, vol. 102 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, N.J., 1982, pp. 303–340.

[17] H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. (2), 35 (1934), pp. 588–621.

[18] M. A. A. de Cataldo and L. Migliorini, *The Hodge theory of algebraic maps*, Ann. Sci. École Norm. Sup. (4), 38 (2005), pp. 693–750.

[19] F. Denef, M. R. Douglas, B. Florea, A. Grassi, and S. Kachru, *Fixing all moduli in a simple F-theory compactification*, Adv. Theor. Math. Phys., 9 (2005), pp. 861–929.

[20] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.

[21] P. Du Val, *Homographies, quaternions and rotations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[22] M. Esole, S.-H. Shao, and S.-T. Yau, *Singularities and Gauge Theory Phases*, Adv. Theor. Math. Phys., 19 (2015), pp. 1183–1247.

[23] H. Flenner, *Divisorenklassengruppen quasihomogener Singularitäten*, J. Reine Angew. Math., 328 (1981), pp. 128–160.

[24] A. Font, I. García-Etxebarria, D. Lust, S. Massai, and C. Mayrhofer, *Heterotic T-fects, 6D SCFTs, and F-Theory*, JHEP, 08 (2016), p. 175.

[25] A. Font and C. Mayrhofer, *Non-geometric vacua of the Spin$(\mathbf{32})/\mathbb{Z}_\mathbf{2}$ heterotic string and little string theories*, JHEP, 11 (2017), p. 064.

[26] R. M. Fossum, *The divisor class group of a Krull domain*, Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74.

[27] O. Fujino, *What is log terminal?*, in Flips for 3-folds and 4-folds, vol. 35 of Oxford Lecture Ser. Math. Appl., Oxford Univ. Press, Oxford, 2007, pp. 49–62.

[28] M. Goresky and R. MacPherson, *Intersection homology. II*, Invent. Math., 72 (1983), pp. 77–129.

[29] ———, *Stratified Morse theory*, vol. 14 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1988.

[30] A. Grassi, *On minimal models of elliptic threefolds*, Math. Ann., 290 (1991), pp. 287–301.

[31] A. Grassi, *Log contractions and equidimensional models of elliptic threefolds*, J. Algebraic Geom., 4 (1995), pp. 255–276.

[32] A. Grassi, J. Halverson, and J. L. Shaneson, *Geometry and topology of string junctions*, J. Sing., 15 (2016), pp. 36–52.

[33] A. Grassi and D. R. Morrison, *Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds*, J. Algebraic Geom., 12 (2003), pp. 321–356.

[34] A. Grassi and D. R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, Commun. Number Theory Phys., 6 (2012), pp. 51–127.

[35] A. Grassi and T. Weigand, *Work in progress.* To appear, 2018.

[36] H. Grauert and R. Remmert, *Theory of Stein spaces*, Classics in Mathematics, Springer-Verlag, Berlin, 2004. Translated from the German by Alan Huckleberry, Reprint of the 1979 translation.

[37] S. Greb, G. Henri, and K. Daniel, *Klt varieties with trivial canonical class, holonomy, differential forms, and fundamental groups.* arXiv:1704.01408, 2017.

[38] G.-M. Greuel, *Dualität in der lokalen Kohomologie isolierter Singularitäten*, Math. Ann., 250 (1980), pp. 157–173.

[39] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[40] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[41] H. Hayashi, C. Lawrie, D. R. Morrison, and S. Schafer-Nameki, *Box Graphs and Singular Fibers*, JHEP, 05 (2014), p. 048.

[42] H. Hironaka, *Formal line bundles along exceptional loci*, in Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 201–218.

[43] K. A. Intriligator, D. R. Morrison, and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, Nucl. Phys., B497 (1997), pp. 56–100.

[44] S. Ishii, *Introduction to singularities*, Springer, Tokyo, 2014.

[45] S. Katz and C. Vafa, *Matter from geometry*, Nuclear Phys. B, 497 (1997), pp. 146–154.

[46] Y. Kawamata, *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2), 127 (1988), pp. 93–163.

[47] K. Kodaira, *On compact analytic surfaces. II, III*, Ann. of Math. (2) 77 (1963), 563–626; ibid., 78 (1963), pp. 1–40.

[48] J. Kollár, *Flops*, Nagoya Math. J., 113 (1989), pp. 15–36.

[49] J. Kollár, *Flips, flops, minimal models, etc*, in Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 113–199.

[50] ———, *Maps between local Picard groups*, Algebr. Geom., 3 (2016), pp. 461–495.

[51] J. Kollár and S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc., 5 (1992), pp. 533–703.

[52] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, vol. 134 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[53] H.-W. Lin, *On crepant resolution of some hypersurface singularities and a criterion for UFD*, Trans. Amer. Math. Soc., 354 (2002), pp. 1861–1868 (electronic).

[54] E. J. N. Looijenga, *Isolated singular points on complete intersections*, vol. 5 of Surveys of Modern Mathematics, International Press, Somerville, MA; Higher Education Press, Beijing, second ed., 2013.

[55] R. MacPherson, *Global questions in the topology of singular spaces*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), PWN, Warsaw, 1984, pp. 213–235.

[56] L. Maxim, *Intersection homology & perverse sheaves with applications to singularities.*

[57] N. Mohan Kumar, *Rational double points on a rational surface*, Invent. Math., 65 (1981/82), pp. 251–268.

[58] D. R. Morrison and D. S. Park, *F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, JHEP, 10 (2012), p. 128.

[59] D. R. Morrison, D. S. Park, and W. Taylor, *Non-Higgsable abelian gauge symmetry and F-theory on fiber products of rational elliptic surfaces*, (2016).

[60] D. R. Morrison and G. Stevens, *Terminal quotient singularities in dimensions three and four*, Proc. Amer. Math. Soc., 90 (1984), pp. 15–20.

[61] N. Nakayama, *On Weierstrass models*, in Algebraic geometry and commutative algebra, Vol. II, Kinokuniya, Tokyo, 1988, pp. 405–431.

[62] Y. Namikawa and J. H. M. Steenbrink, *Global smoothing of Calabi-Yau threefolds*, Invent. Math., 122 (1995), pp. 403–419.

[63] K. Oguiso, *Pisot units, Salem numbers and higher dimensional projective manifolds with primitive automorphisms of positive entropy*. arXiv:1608.03122, 2018.

[64] K. Oguiso and T. T. Truong, *Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy*, J. Math. Sci. Univ. Tokyo, 22 (2015), pp. 361–385.

[65] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, vol. 52 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Springer-Verlag, Berlin, 2008.

[66] F. Polizzi, A. Rapagnetta, and P. Sabatino, *On factoriality of threefolds with isolated singularities*, Michigan Math. J., 63 (2014), pp. 781–801.

[67] M. Reid, *Minimal models of canonical* 3*-folds*, in Algebraic varieties and analytic varieties (Tokyo, 1981), vol. 1 of Adv. Stud. Pure Math., North-Holland, Amsterdam, 1983, pp. 131–180.

[68] M. Rossi and L. Terracini, *MAPLE subroutines for computing Milnor and Tyurina numbers of hypersurface singularities with application to arnol'd adjacencies*. To appear in Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino.

[69] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math., 14 (1971), pp. 123–142.

[70] P. Samuel, *Lectures on unique factorization domains*, Notes by M. Pavman Murthy. Tata Institute of Fundamental Research Lectures on Mathematics, No. 30, Tata Institute of Fundamental Research, Bombay, 1964.

[71] M. Schlessinger, *Rigidity of quotient singularities*, Invent. Math., 14 (1971), pp. 17–26.

[72] P. Slodowy, *Groups and special singularities*, in Singularity theory (Trieste, 1991), World Sci. Publ., River Edge, NJ, 1995, pp. 731–799.

[73] U. Storch, *Fastfaktorielle Ringe*, Schr. Math. Inst. Univ. Münster No., 36 (1967), pp. iv+42.

[74] S. Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math., 14 (2003), pp. 825–836.

[75] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, (1975), pp. 33–52. Lecture Notes in Math., Vol. 476.

[76] Z. Tian and C. Xu, *Finiteness of fundamental groups*, Compos. Math., 153 (2017), pp. 257–273.

[77] E. Witten, *Phase transitions in M theory and F theory*, Nucl. Phys., B471 (1996), pp. 195–216.

Antonella Grassi, Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104

E-mail address: grassi@math.upenn.edu

Timo Weigand, Theory Department, CERN, CH-1211 Geneva, Switzerland and Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany

ON THE COHOMOLOGY CLASSES OF DIVISORS ON SOME NORMAL PROJECTIVE COMPLEX VARIETIES

V. SRINIVAS

Our goal here is to prove a result (presumably well-known to experts), giving a context where the "Lefschetz $(1,1)$-theorem" holds for certain singular complex projective varieties.

Theorem 1. *Let X be a normal projective variety over the field $\mathbb{C}$ of complex numbers. Let $\pi : Y \to X$ be a resolution of singularities. Assume: $R^1\pi_*\mathcal{O}_Y = 0$ (this condition is independent of the choice of resolution). Then*

(1) *the singular cohomology $H^2(X,\mathbb{Z})$ supports a* pure *Hodge structure*
(2) *the image of the Chern class map $c_1 : Pic\,(X) \to H^2(X,\mathbb{Z})$ concides with the subgroup of $(1,1)$ classes, i.e. with the subgroup*

$$\{\alpha \in H^2(X,\mathbb{Z}) \,|\, \alpha_{\mathbb{C}} \in H^2(X,\mathbb{C}) \text{ is of type } (1,1)\}.$$

Example 1. Let X be a normal projective variety over $\mathbb{C}$ with $\dim X \geq 3$, and only isolated Cohen-Macaulay singularities (eg., isolated complete intersection singularities). Then X satisfies the hypotheses of Theorem 1. (The argument below is presumably also well known to experts.)

Indeed, let $S = \{x_1, \dots, x_r\}$ be the singular locus. If $\pi : Y \to X$ is a resolution, then $R^1\pi_*\mathcal{O}_Y$ is supported within the singular locus, and so is a direct sum of skyscraper sheaves supported at the points x_i. So it suffices to show that the stalk at each x_i vanishes.

Let $Z_i = Y \times_X \operatorname{Spec} \mathcal{O}_{X,x_i}$, so that the stalk of $R^1\pi_*\mathcal{O}_Y$ at x_i is $H^1(Z_i, \mathcal{O}_{Z_i})$. Let $E_i = \pi^{-1}(x_i)$.

Then we have that

$$H^1_{E_i}(Z_i, \mathcal{O}_{Z_i}) = 0$$

from the Grauert-Riemenschneider theorem.

If $U_i = Z_i \setminus E_i$, which is also isomorphic to the punctured specrum of $\mathcal{O}_{X,x_i}$, then we have an exact sequence in local cohomology

$$\cdots \to H^1_{E_i}(Z_i, \mathcal{O}_{Z_i}) \to H^1(Z_i, \mathcal{O}_{Z_i}) \to H^1(U_i, \mathcal{O}_{U_i}) \to \cdots$$

On the other hand, we also have a sequence in local cohomology for the structure sehaf of $\operatorname{Spec} \mathcal{O}_{X,x_i}$ and its punctured spectrum; this yields an isomorphism

$$H^1(U_i, \mathcal{O}_{U_i}) \cong H^2_{\mathfrak{M}_{x_i}}(\mathcal{O}_{X,x_i}).$$

Finally, since I assumed X is Cohen-Macaulay, *of dimension* ≥ 3, the local cohomologies satisfy

$$H^j_{\mathfrak{M}_{x_i}}(\mathcal{O}_{X,x_i}) = 0 \;\; \forall\, j < \dim X,$$

and in particular for $j = 2$.

Thus $H^1(Z_i, \mathcal{O}_{Z_i})$ vanishes as claimed. (In fact, one has that $R^j\pi_*\mathcal{O}_Y = 0$ for $j < \dim X - 1$, by a similar argument.)

Now we sketch the proof of Theorem 1.

Proof. We now work with the analytic topology. By GAGA, the condition $R^1\pi_*\mathcal{O}_Y = 0$, for the analytic topology, is the same as the similar condition in the Zariski topology. Note also that $\mathrm{Pic}\,(X)$ is identified with the isomorphism classes of analytic line bundles.

From the exponential sheaf sequence on Y, we have a long exact sequence of sheaves on X

$$0 \to \pi_*\mathbb{Z}_Y \to \pi_*\mathcal{O}_Y \to \mathcal{O}_Y^* \stackrel{\partial}{\to} R^1\pi_*\mathbb{Z}_Y \to R^1\pi_*\mathcal{O}_Y \cdots$$

Since X is normal, the first three terms are just the exponential sequence on X, and in particular, $\partial = 0$. Since we assumed $R^1\pi_*\mathcal{O}_Y = 0$, we get also that $R^1\pi_*\mathbb{Z}_Y = 0$.

Now the Leray spectral sequence implies that

(i) $H^1(X,\mathbb{Z}) \to H^1(Y,\mathbb{Z})$ and $H^1(X,\mathcal{O}_X) \to H^1(Y,\mathcal{O}_Y)$ are isomorphisms (this implies that $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ is an isomorphism)

(ii) $H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ and $H^2(X,\mathcal{O}_X) \to H^2(Y,\mathcal{O}_Y)$ are injective.

From (ii), it follows that $H^2(X,\mathbb{Z})$ supports a *pure* Hodge structure, which necessarily satisfies $F^1H^2(X,\mathbb{C}) = H^2(X,\mathbb{C}) \cap F^1H^2(Y,\mathbb{C})$, and further, using the exponential sheaf sequence, that

$$NS(X) = \ker\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)\right) = \ker\left(H^2(X,\mathbb{Z}) \to H^2(Y,\mathcal{O}_Y)\right) = \ker\left(H^2(X,\mathbb{Z}) \to \frac{H^2(Y,\mathbb{C})}{F^1H^2(Y,\mathbb{C})}\right) = \ker\left(H^2(X,\mathbb{Z}) \to \frac{H^2(X,\mathbb{C})}{F^1H^2(X,\mathbb{C})}\right).$$

Thus, X satisfies the "Lefschetz $(1,1)$ theorem".

□