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Average beta-beating from random errors

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Summary

The impact of random errors on average β-beating is studied via analytical derivations and simulations. A systematic positive β-beating is expected from random errors quadratic with the sources or, equivalently, with the rms β-beating. However, random errors do not have a systematic effect on the tune.

1 Introduction

Optics aberrations are a major concern for modern accelerators [1]. Simulations presented in [2–5] show that the ring-average β-function tends to increase with the rms β-beating. Analytical estimates in [6–11] show the appearance of constant terms in the β-beating due to second order contributions from quadrupolar errors.

Section 2 follows the derivation presented in [11] to compute the average β-beating in a transfer line from a single error. Section 3 uses the resonance driving term theory presented in [6,7] to correlate the average and the rms β-beatings in presence of random errors. Section 4 computes the expected tune shift from random errors. A possible experimental application is briefly discussed in Section 5.

2 Average β-beating in a transfer line

Deviations from the design β-function occur due to focusing errors. The perturbed transfer matrix $\boldsymbol{M}_p$ due to a quadrupole error Δk at position s_0 can be derived by multiplying the unperturbed transfer matrix $\boldsymbol{M}$ with a matrix that describes the quadrupole gradient error

$$\boldsymbol{M}_p(s, s_0) = \begin{pmatrix} C_p & S_p \\ C'_p & S'_p \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\Delta k & 1 \end{pmatrix} = \begin{pmatrix} C - \Delta k S & S \\ C' - \Delta k S' & S' \end{pmatrix}. \tag{1}$$

The perturbed β-function can be derived as

$$\beta_p(s) = \beta(s_0)C_p^2 - \alpha(s_0)2S_pC_p + \gamma(s_0)S_p^2. \tag{2}$$

Expressing the transfer matrix elements as function of the unperturbed optics parameters, the perturbed β-function can be written as

$$\begin{aligned}
\beta_p(s) =& \beta(s_0)\left\{\sqrt{\frac{\beta(s)}{\beta(s_0)}}\left[\cos\phi+\alpha(s_0)\sin\phi\right]-\Delta k\sqrt{\beta(s)\beta(s_0)}\sin\phi\right\}^2 \\
&-\alpha(s_0)2\sqrt{\beta(s)\beta(s_0)}\sin\phi \\
&\cdot\left\{\sqrt{\frac{\beta(s)}{\beta(s_0)}}\left[\cos\phi+\alpha(s_0)\sin\phi\right]-\Delta k\sqrt{\beta(s)\beta(s_0)}\sin\phi\right\} \\
&+\gamma(s_0)\left[\sqrt{\beta(s)\beta(s_0)}\sin\phi\right]^2 \qquad (3)\\
=& \beta(s)\left[\cos\phi^2+2\alpha(s_0)\cos\phi\sin\phi+\alpha(s_0)^2\sin\phi^2\right] \\
&-2\Delta k\beta(s)\beta(s_0)\cos\phi\sin\phi-2\Delta k\beta(s)\beta(s_0)\alpha(s_0)\sin\phi^2 \\
&+\Delta k^2\beta(s)\beta(s_0)^2\sin\phi^2-2\alpha(s_0)\beta(s)\sin\phi\cos\phi \\
&-2\alpha(s_0)^2\beta(s)\sin\phi^2+2\Delta k\beta(s)\beta(s_0)\alpha(s_0)\sin\phi^2 \\
&+\beta(s)\sin\phi^2+\alpha(s_0)^2\beta(s)\sin\phi^2 \qquad (4)\\
=& \beta(s)\underbrace{\left[\cos\phi^2+\sin\phi^2\right]}_{=1}-\beta(s)\beta(s_0)\Delta k\sin(2\phi) \\
&+\Delta k^2\beta(s)\beta(s_0)^2\sin\phi^2. \qquad (5)
\end{aligned}$$

With $\Delta\beta(s)=\beta_p(s)-\beta(s)$ follows

$$\begin{aligned}
\frac{\Delta\beta(s)}{\beta(s)} &= -\beta(s_0)\Delta k\sin(2\phi)+\beta(s_0)^2\Delta k^2(\sin\phi)^2 \\
&= -\beta(s_0)\Delta k\sin(2\phi)+\frac{1}{2}\beta(s_0)^2\Delta k^2(1-\cos(2\phi))\ . \qquad (6)
\end{aligned}$$

The average β-beating along the tranfer line after the focusing error is approximately given by

$$\left\langle\frac{\Delta\beta}{\beta}\right\rangle\approx\frac{1}{2}\beta(s_0)^2\Delta k^2\ , \qquad (7)$$

which represents a net positive contribution to the average β-beating, quadratic with the focusing error.

3 Average β-beating in a ring

In [6] the perturbed β function in a ring is expressed as a function of the amplitude and phase of the generating driving term f_{2000} and the unperturbed β_{model} function, as

$$\beta_x=\beta_{x,model}\left(1+32|f_{2000}|^2+8|f_{2000}|\sin q_{2000}\right)\ +O\left(|f_{2000}|^4\right), \qquad (8)$$

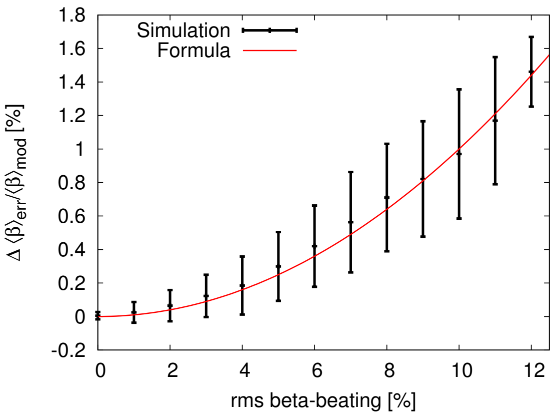

Figure 1: Relative average β deviation versus rms β-beating for LHC injection optics with random errors together with prediction from Eq. (12).

where $|f_{2000}|$ and q_{2000} are the amplitude and phase of the generating function term. With many small random errors $|f_{2000}|$ and $\sin q_{2000}$ would tend to be uncorrelated giving a ring-average β-beating of

$$\left\langle \frac{\Delta\beta}{\beta} \right\rangle = 32 \left\langle |f_{2000}|^2 \right\rangle \ . \tag{9}$$

Note that we neglect all terms of order above $|f_{2000}|^2$. The standard deviation, σ, of the β-beating around the ring is given by

$$\sigma\left(\frac{\Delta\beta}{\beta}\right) = \sqrt{\frac{1}{C}\int_0^C 64|f_{2000}|^2 \sin^2 q_{2000}\, \mathrm{d}s} \ . \tag{10}$$

Using that $\sin^2 x = (1-\cos 2x)/2$ and, again, the assumption that $|f_{2000}|$ and q_{2000} are uncorrelated, the standard deviation takes the form

$$\sigma\left(\frac{\Delta\beta}{\beta}\right) = \sqrt{32\langle |f_{2000}|^2\rangle} \ . \tag{11}$$

From Eqs. (9) and (11) the following identity is obtained

$$\left\langle \frac{\Delta\beta}{\beta} \right\rangle = \sigma^2\left(\frac{\Delta\beta}{\beta}\right) \ , \tag{12}$$

which implies that the ring-average β function increases with the square of the standard deviation of the β-beating, also known as rms β-beating. This seems to be a universal property of all lattices that have a sufficient number of elements for the assumption on error randomness to hold. Figure 1 shows the LHC injection simulations presented in [3] including the prediction from Eq. (12). Figures 2, 3, 4 and 5 illustrate this correlation for the ALBA and ESRF synchrotron light sources and for LHC ballistic [12, 13] and standard collision optics (β^*=40cm), respectively.

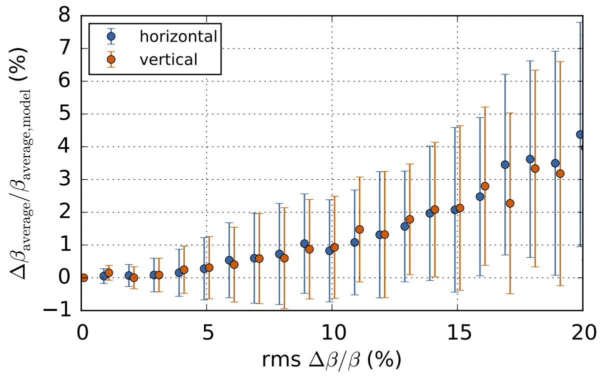

Figure 2: Relative ring-average β deviation versus rms β-beating for the ALBA lattice with random errors.

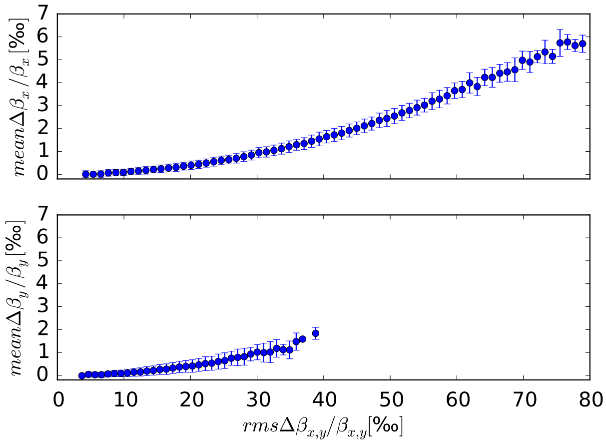

Figure 3: Relative horizontal (top) and vertical (bottom) ring-average β deviation versus rms β-beating for the ESRF lattice with random errors.

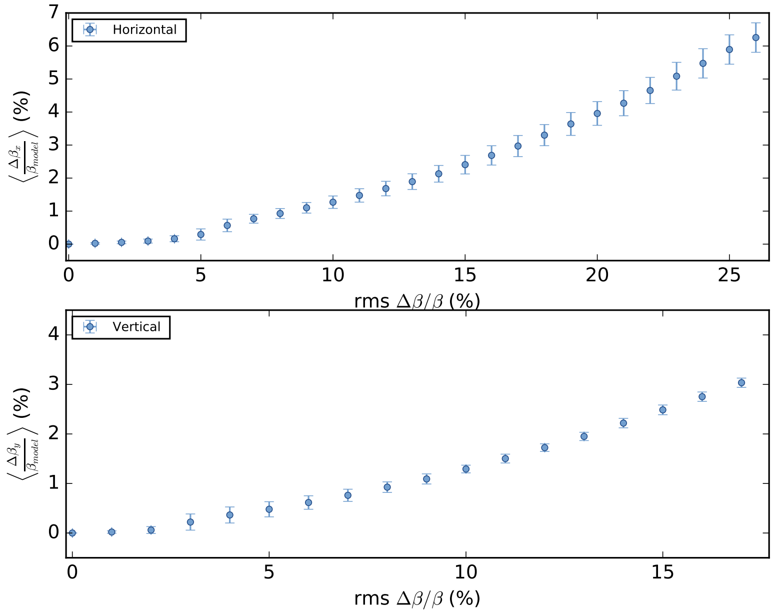

Figure 4: Relative horizontal (top) and vertical (bottom) ring-average β deviation versus rms β-beating for the LHC ballistic optics with random errors.

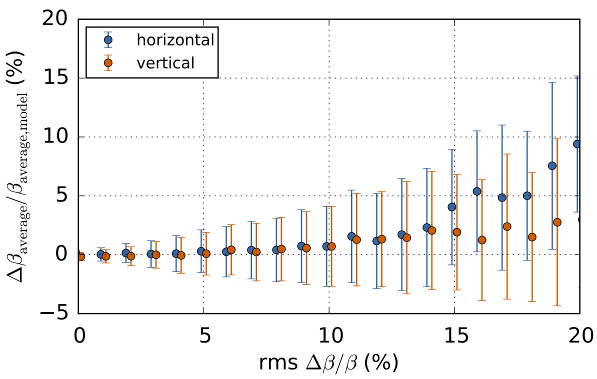

Figure 5: Relative ring-average β deviation versus rms β-beating for the LHC β^*=40cm optics with random errors.

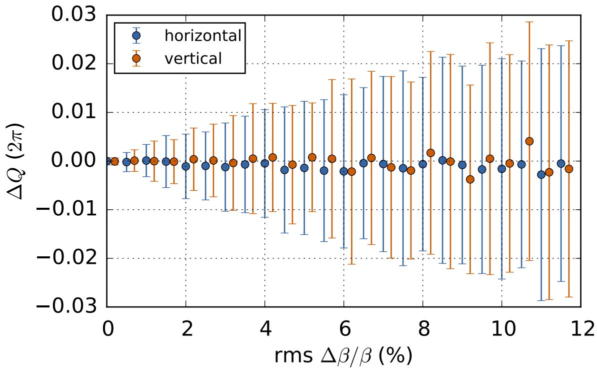

Figure 6: Expected tune shift versus rms β-beating for the LHC injection optics with random errors.

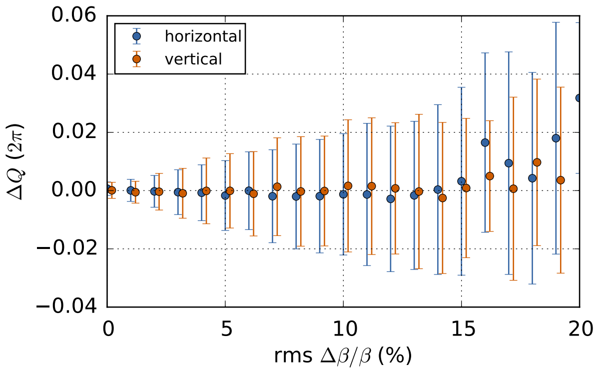

Figure 7: Expected tune shift versus rms β-beating for the LHC β^*=40cm optics with random errors.

4 Expected tune shift

Expanding the tune formula, $2\pi Q = \int \mathrm{d}s/\beta$, up to second order in the generating term $|f_{2000}|$ and assuming similar absence of correlations as in Section 3 the tune is expected to remain unchanged up to order $O(|f_{2000}|^4)$,

$$\begin{aligned} Q_x &= \frac{1}{2\pi}\int_0^C \frac{\mathrm{d}s}{\beta_{x,model}}\left[1-32|f_{2000}|^2-8|f_{2000}|\sin q_{2000}+32|f_{2000}|^2(1-\cos 2q_{2000})\right] \\ &= Q_{x,model}-\frac{4}{\pi}\int_0^C \frac{\mathrm{d}s}{\beta_{x,model}}\left[|f_{2000}|\sin q_{2000}+4|f_{2000}|^2\cos 2q_{2000}\right] \\ &= Q_{x,model}\ . \end{aligned} \tag{13}$$

Figures 6, 7 and 8 illustrate the lack of correlation between expected tune shift and rms beta-beating for different machines. In general for all the plot the available statistics is poorer for the larger values of rms $\Delta\beta/\beta$.

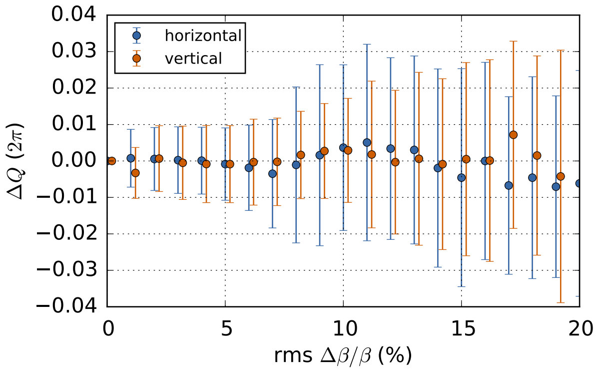

Figure 8: Expected tune shift versus rms β-beating for the ALBA lattice with random errors.

5 Summary and outlook

A universal relation for all lattices has been found between the expected increase in the ring average β-function and the rms β-beating generated by random errors,

$$\left\langle \frac{\Delta\beta}{\beta} \right\rangle = \sigma^2 \left(\frac{\Delta\beta}{\beta} \right) \ .$$

The assumption on the randomness of the errors might not be applicable for lattices with very few elements. Contrary to intuition this increase in average β-function is not translated into a decrease in the expected tune. The expected tune has no correlation with the rms β-beating for random errors. Simulations with LHC, ALBA and ESRF lattices have been used to confirm the findings.

When measuring β functions from the amplitude of betatron oscillations a BPM wrong global calibration factor of α affects β by a factor α^2. Measured β functions in this way would feature a modified behaviour as

$$\left\langle \frac{\Delta\beta}{\beta} \right\rangle = \alpha^2\sigma^2 \left(\frac{\Delta\beta}{\beta} \right) \ ,$$

which would allow to actually measure α provided enough statistics can be acquired.

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