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ANOMALIES OF MATTER COUPLING IN THREE-DIMENSIONAL TOPOLOGICAL GRAVITY

T.T. Burwick [1] and A.H. Chamseddine [1]

Institut für Theoretische Physik
Universität Zürich, Schönbergasse 9
CH-8001 Zürich

and

K.A. Meissner [2]

Theoretical Physics Division, CERN
1211 Geneva 23, Switzerland

ABSTRACT

We study the matter coupling of three-dimensional topological gravity. The form of the matter coupling is arrived at by breaking the gauge groups $SO(1,3)$, $SO(2,2)$ or $ISO(1,2)$ to $SO(1,2)$. The perturbative analysis is performed using a background metric on the manifold for gauge fixing. It is shown that the system has divergences which can only be cancelled by counterterms proportional to the background metric. This shows that the quantum theory is metric dependent, a phenomenon which will be referred to as metric anomaly. It is also shown that supersymmetrization of this construction does not cure this problem.

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[1]Supported by the Swiss National Foundation.

[2]Permanent address: Institute of Theoretical Physics, Warsaw University, ul. Hoza 69, 00-681 Warsaw, Poland.

I. Introduction

Many attempts have been made to formulate a quantum theory of four-dimensional gravity [1]. In this approach it is hoped that if gravity can be formulated as a renormalizable theory, then this would improve the prospects of unifying gravity with the other known interactions. The recent developments in three-dimensional gravity provide an excellent testing ground for this program. There it was shown that by formulating three-dimensional gravity as a topological gauge theory of the groups $SO(1,3)$, $SO(2,2)$ or $ISO(1,2)$, the theory becomes finite [2]. The main difficulty in advancing this program lies in introducing non-trivial matter coupling. By "non-trivial" we mean couplings which, in the non-topological phase, reduce to the familiar scalar, spinor and vector interactions. The difficulty arises because a basic ingredient in this construction is not to use the metric, but rather the dreibein field which is also a gauge field. The metric would arise in the non-topological phase as a product of the dreibein fields: $g_{\mu\nu} = e^a_\mu e_{a\nu}$. Any matter interaction must then be written using the dreibein. This, being a part of a gauge multiplet, would necessarily break gauge invariance. It is then inevitable that the gauge symmetries $SO(1,3)$, $SO(2,2)$ or $ISO(1,2)$ must be broken to $SO(1,2)$, the Lorentz group in three dimensions. It is preferable to break the symmetry spontaneously, since our main interest is to investigate in a perturbative setting the renormalization analysis of matter interactions.

The advantage of dealing with three-dimensional gravity is that the graviton propagator can be obtained by expanding e^a_μ around zero, which is a great simplification over expanding around a flat background. The disadvantage is that in order to gauge fix, a background metric on the three-dimensional manifold must be specified. (This metric will be taken to be flat so as to simplify the perturbative treatment.) In the pure gravity case it has been shown that the quantum theory is independent of the background metric showing the topological nature of the theory [2]. This independence on the background metric is, however, not guaranteed [3]. As will be seen, matter interactions will give rise to divergences that could not be cancelled without using the background metric to construct counterterms. There shows that at the quantum level the energy-momentum tensor is non-zero. We deduce that matter interactions destroy the topological nature of the theory. With hindsight, it is possible to understand this phenomenon by examining the partition function. The integration of the matter fields produces determinants which are independent of the background metric but depend only on e^a_μ. For the gravity part one gets a ratio of determinants. In the absence of matter this ratio is topological, the Ray-Singer torsion [2],[4]. In the presence of matter and the associated breakdown of symmetry to $SO(1,2)$, one loses a piece and the ratio of determinants becomes metric dependent. This phenomenon is similar to what happened in two dimensions where the metric tensor appears at the classical level as a background metric. At the quantum level, the metric becomes dynamical and governed by the Liouville action. This was referred to as a gravitational anomaly, but in our case it is more appropriate to call it a metric anomaly.

In the perturbative analysis we shall try to cure the problem of divergences by introducing supersymmetry. This, unfortunately, will turn out to be insufficient. It appears to us that the best hope for such a program to work is to have both gravity and matter as part of a large symmetry.

The plan of this paper is as follows. In Section 2 we give the coupling of three-dimensional topological gravity to matter without using a background metric. In Section 3 the construction is generalized to the supersymmetric case. The perturbative analysis and the arisal of the metric anomalies are given in Section 4. Some comments and the conclusion are in Section 5.

II. Matter coupling to three-dimensional topological gravity

From the work of Witten it is now established that three-dimensional quantum gravity becomes a finite theory when formulated as a gauge theory of $SO(1,3)$, $SO(2,2)$ or $ISO(1,2)$ depending on the cosmological constant [2]. The gauge invariant action is of the Chern-Simons type

$$S_g = 2k_g \int < AdA + \frac{2}{3}A^3 > \tag{2.1}$$

where A is an $SO(2,2)$ gauge field (the other two cases can be recovered by Wick rotation or an Inönu-Wigner group contraction).

$$A = \frac{1}{4}\, A^{AB} J_{AB}, \quad A = a,3; \quad a = 0,1,2$$

and the quadratic form is defined by

$$< J_{AB} J_{CD} >= \epsilon_{ABCD} \tag{2.2}$$

The connection with gravity is made through the identification

$$A^{a3} \equiv e^a, \quad A^{ab} \equiv \omega^{ab} \tag{2.3}$$

In terms of e and the spin connection ω the action (2.1) takes the form

$$S_g = k_g \int \epsilon_{abc} e^a (d\omega^{bc} + \omega^{bd}\omega_d^c - \frac{1}{3} e^b e^c) \tag{2.4}$$

At the classical level, when e_μ^a is restricted to the subspace of invertible fields, the action (2.4) is equivalent to the Einstein-Hilbert action. However, this equivalence breaks down at the quantum level, where the quantum theory of (2.4) is finite. The main disadvantage in this formulation is the difficulty of introducing matter. This stems from the fact that e_μ^a is a part of the gauge field A and cannot be used by itself without breaking gauge invariance. It is then suggestive to break the gauge symmetry to $SO(1,2)$ so that e_μ^a would correspond to the broken generator J_{a3}. To break the symmetry spontaneously some kind of Higgs mechanism must be employed. However, because of the absence of a metric no potential for the Higgs field can be introduced and a non-zero vacuum expectation value can only be obtained through a constraint. At this point we introduce a Higgs field H_A subject to the constraint [5]

$$H^A H_A - 1 = 0 \tag{2.5}$$

which can be easily imposed through a Lagrange multiplier in the action. The most general Higgs interaction terms are [5]

$$S_h = -\int \epsilon_{ABCD} H^A [k_{h1} D\ H^B F^{CD} + k_{h2} D\ H^B D\ H^C D\ H^D] \tag{2.6}$$

where

$$\begin{aligned} D_\mu H^A &= \partial_\mu H^A + A_\mu^{AB} H_B \\ F^{AB} &= d\, A^{AB} + A^{AC} A_C{}^B \end{aligned}$$

To understand the rôle of the H_A field it is useful to go to a physical gauge where

$$H^3 = 1, \quad H^a = 0 \tag{2.7}$$

which can always be reached by using the gauge parameter ξ^{a3} associated with J_{a3} as well as the constraint (2.5). In the gauge (2.7) the covariant derivative $D_\mu H^A$ takes the simple form

$$D_\mu H^3 = 0, \quad D_\mu H^a = e_\mu^a \tag{2.8}$$

This shows that to formulate a covariant matter interaction the rôle of the dreibein is replaced by $D_\mu H^A$ which in a physical gauge would take the familiar form. The interaction (2.6) also takes a simple form in the gauge (2.7):

$$S_h = \int \epsilon_{abc}(k_{h1}\ e^a F^{bc} + k_{h2}\ e^a e^b e^c) \tag{2.9}$$

where

$$F^{ab} = d\omega^{ab} + \omega^{ac}\omega_c^b - e^a e^b \tag{2.10}$$

Combining (2.4) and (2.9) the sum reduces to

$$S_{g+h} = \int \epsilon_{abc}[(k_g + k_{h1})e^a(d\omega^{bc} + \omega^{be}\omega_e{}^c) + (k_{h2} - k_{h1} - \frac{1}{3}k_g)e^a e^b e^c] \tag{2.11}$$

After rescaling the field e_μ^a we see that the action (2.11) is of the same form as (2.4) and could differ only by the value of the cosmological constant. In other words, the gauge-fixed action (2.11) has a resurrected symmetry of the form $SO(1,3), SO(2,2)$ or $ISO(1,2)$ depending on whether the combination $3k_{h2} - 3k_{h1} - k_g$ is positive, negative or zero. In what follows and for simplicity we are going to assume that the cosmological constant vanishes and set $3k_{h2} - 3k_{h1} - k_g$ to zero.

The simplest matter interaction to construct is that of a scalar multiplet. Let X^A be a scalar multiplet in the fundamental representation of $SO(2,2)$ with the identifications $X^a = \pi^a, X^3 = \varphi$. One possible action that reproduces the familiar form at the classical level is

$$S_m = k_m \int \epsilon_{ABCD} H^A\ DH^B\ DH^C (X^D\ DX^E H_E) \tag{2.12}$$

In the physical gauge (2.7) this takes the form

$$S_m = -k_m \int d^3x \epsilon^{\mu\nu\rho} \epsilon_{abc} e_\mu^a e_\nu^b \pi^c (\partial_\rho \varphi - e_\rho^d \pi_d) \tag{2.13}$$

The action (2.13) is just the first-order formulation of a scalar field action. To see this, assume the non-topological phase where e_μ^a is invertible, and substitute the equation of motion of π_a,

$$\pi_a = \frac{1}{2} e_a^\mu \partial_\mu \varphi \tag{2.14}$$

into the action (2.13) to get

$$S_m = -\frac{k_m}{2}\int d^3x\; e\; e^{\mu}_a e^{\nu a}\partial_\mu\varphi\;\partial_\nu\varphi \tag{2.15}$$

Thus (2.13) reproduces the canonical form at the classical level.

The total action, the sum of (2.11) and (2.15), has only the $SO(1,2)$ gauge symmetry. The perturbative analysis of this action will be performed in Section 4. There we shall find divergences that can be cancelled by counterterms proportional to the background metric introduced in gauge fixing. One potential mechanism to cancel the divergences is to supersymmetrize our construction. This will be the subject of the next section.

III. Topological supergravity and matter coupling

Since $SO(2,2) \cong SO(1,2)\times SO(1,2)$ and $OSP(2|1)$ is the graded version of $SO(1,2)$, the supersymmetric analogue of the construction given in the previous section is achieved by gauging $OSP(2\mid 1)\times OSP(2\mid 1)$ [6], [2].

We shall adopt the notation of [7] for the matrix representation of $OSP(2\mid 1)$. Let Φ_1 and Φ_2 be the gauge fields of the two $OSP(2\mid 1)$ gauge groups transforming as

$$\begin{aligned}\Phi_1 &\to \Omega_1\Phi_1\Omega_1^{-1} + \Omega_1 d\Omega_1^{-1}\\ \Phi_2 &\to \Omega_2\Phi_2\Omega_2^{-1} + \Omega_2 d\Omega_2^{-1}\end{aligned} \tag{3.1}$$

where Ω_1 and Ω_2 are two elements of the two respective groups. These can be represented in the matrix form

$$\Phi = \begin{pmatrix} A_\alpha{}^\beta & \psi_\alpha \\ \bar\psi^\beta & 0\end{pmatrix} \tag{3.2}$$

where

$$A_{\alpha\beta} = A_{\beta\alpha}, \quad \psi_\alpha = \epsilon_{\alpha\beta}\bar\psi^\beta \tag{3.3}$$

It is also convenient to write

$$A_\alpha{}^\beta = A^a(\tau_a)_\alpha{}^\beta$$

where the τ_a are the $SO(2,1)$-generators

$$\tau_0 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \tau_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \tau_2 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Introduce now the Higgs field G transforming as

$$G \to \Omega_1\; G\; \Omega_2^{-1} \tag{3.4}$$

and the covariant derivative of G, transforming as G, is defined by

$$DG = dG + \Phi_1 G - G\Phi_2 \tag{3.5}$$

In order to distinguish the group indices of the second $OSP(2|1)$ let us denote them by $\dot\alpha, \dot\beta, \ldots$. Then the matrix representation of G is

$$G = \begin{pmatrix} H_\alpha^{\dot\beta} & \eta_\alpha \\ \bar\xi^{\dot\beta} & \varphi\end{pmatrix} \tag{3.6}$$

where both η_α and $\xi_{\dot\alpha}$ are Majorana spinors, and $H_\alpha^{\dot\beta}$ and φ are real.

It will also be necessary to define the equivalent representation $\tilde G$ transforming as

$$\tilde G \to \Omega_2\, \tilde G\, \Omega_1^{-1} \tag{3.7}$$

and whose matrix form is

$$\tilde G = \begin{pmatrix} [\epsilon H^T \epsilon^{-1}]_{\dot\alpha}^{\beta} & -\xi_{\dot\alpha} \\ -\bar\eta^{\beta} & \varphi \end{pmatrix} \tag{3.8}$$

To break the symmetry spontaneously we must arrange for G to have a non-zero vacuum. This can be done by employing the gauge invariant constraint

$$G\,\tilde G - \mathbf{1} = 0 \tag{3.9}$$

where $\mathbf{1}$ is the unit matrix. This constraint can also be imposed through a Lagrange multiplier. The unconstrained G has nine components, four fermionic and five bosonic. With the constraint (3.9) the independent components of G are reduced to five, two fermionic and three bosonic. At this point, it is helpful to go to a physical gauge where some of the gauge symmetries are fixed. Using five components of the $OSP(2\mid 1)\times OSP(2\mid 1)$ gauge parameters, two fermionic and three bosonic, we can set

$$G = \mathbf{1} \tag{3.10}$$

This breaks the symmetry to the diagonal part of the product $OSP(2\mid 1)\times OSP(2\mid 1)$.

We first write the pure supergravity action [6]

$$S_{sg} = -\frac{k_{sg}}{2}\int [Str(\Phi_1 d\Phi_1 + \frac{2}{3}\Phi_1^3) - 1 \to 2] \tag{3.11}$$

whose component form is

$$S_{sg} = \frac{k_{sg}}{4}\int [(A_{1a} dA_1^a - \frac{1}{3}\epsilon_{abc}A_1^a A_1^b A_1^c) + 4\bar\psi_1 D_1 \psi_1 - 1 \to 2] \tag{3.12}$$

where $D_i = d + A_i$. The action in (3.12) can be put into a more familiar form by re-expressing it in terms of [6]

$$\begin{aligned} \omega^a &= \frac{1}{2}\,(A_1^a + A_2^a) \\ e^a &= \frac{1}{2}\,(A_1^a - A_2^a) \\ \psi_\pm &= \frac{1}{2}(\psi_1 \pm \psi_2) \end{aligned} \tag{3.13}$$

Then

$$\begin{aligned} S_{sg} = k_{sg} \int \Big[\; & e^a(d\omega_a - \frac{1}{2}\epsilon_{abc}\omega^b\omega^c - \frac{1}{6}\epsilon_{abc}e^b e^c) \\ & + 2\bar\psi_+(d+\omega)\psi_- + 2\bar\psi_-(d+\omega)\psi_+ + 2\bar\psi_+ e\psi_+ + 2\bar\psi_- e\psi_- \Big] \end{aligned} \tag{3.14}$$

Using $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ the bosonic part agrees with (2.4).

The most general expression for the Higgs interactions compatible with (3.11) and the diagonalization in (3.13) is

$$S_{sh} = \int \quad \frac{k_{sh1}}{2}[Str(G\widetilde{DG}(d\Phi_1 + \Phi_1^2)] - Str[\tilde{G}dG(d\Phi_2 + \Phi_2^2)]) \\ +4k_{sh2}Str(G\widetilde{DG}\ DG\ \widetilde{DG}) \tag{3.15}$$

In the physical gauge (3.10) and in terms of the variables (3.13) the Higgs action (3.15) takes the form

$$S_{sh} = k_{sh1}\int \quad [e^a(d\omega_a - \frac{1}{2}\epsilon_{abc}\omega^b\omega^c - \frac{1}{2}\epsilon_{abc}e^be^c) \\ +2\bar{\psi}_+(d+\omega)\psi_- + 2\bar{\psi}_-(d+\omega)\psi_+ + 6\bar{\psi}_-e\psi_- + 2\bar{\psi}_+e\psi_+] \\ +k_{sh2}\int \quad (\epsilon_{abc}e^ae^be^c - 12\bar{\psi}_-e\psi_-) \tag{3.16}$$

One can recombine (3.14) and (3.16) to get

$$S_{sg+sh} = \int \quad \{(k_{sg} + k_{sh1})\ [e^a(d\omega_a - \frac{1}{2}\epsilon_{abc}\omega^b\omega^c) \\ +2\bar{\psi}_+(d+\omega)\psi_- + 2\bar{\psi}_-(d+\omega)\psi_+ + 2\bar{\psi}_+e\psi_+] \\ -(k_{sg} + 3k_{sh1} - 6k_{sh2})[\frac{1}{3}\epsilon_{abc}e^ae^be^c - 2\bar{\psi}_-e\psi_-]\} \tag{3.17}$$

If we tune the coefficients such that

$$k_{sg} + 3k_{sh1} - 6k_{sh2} = 0$$

then the cosmological constant vanishes and the $\bar{\psi}_-e\psi_-$ also drops out. This is an indication that there is a residual supersymmetry between e and ψ_-. In this case the action (3.17) simplifies to

$$k'_{sg}\int\left[e^a(d\omega_a - \frac{1}{2}\epsilon_{abc}\omega^b\omega^c) + 2\bar{\psi}_+(d+\omega)\psi_- + 2\bar{\psi}_-(d+\omega)\psi_+ + 2\bar{\psi}_+e\psi_-\right] \tag{3.18}$$

This action has the symmetry $\frac{SxOSP(2|1)}{SO(1,2)}$ where S is the supersymmetric extension of the $ISO(1,2)$ of pure gravity. The transformation properties are

$$\begin{aligned} \delta e^a &= (d+\omega)\xi^a + \bar{\epsilon}_-\sigma^a\psi_+ + \bar{\epsilon}_+\sigma^a\psi_- \\ \delta\psi_- &= -\xi^a\sigma_a\psi_+ + (d+\omega)\epsilon_- + e\epsilon_+ \\ \delta\psi_+ &= (d+\omega)\epsilon_+ \\ \delta\omega^a &= \frac{1}{2}\bar{\epsilon}_+\sigma^a\psi_+ \end{aligned} \tag{3.19}$$

where the ξ^a, ϵ_- extends $SO(1,2)$ with gauge field ω^a to S, while ϵ_+ extends the same $SO(1,2)$ to $OSP(2\mid 1)$. The symmetries with ξ^a and ϵ_- gauge parameters are present only when there are no matter interactions, and they would be broken in the presence of matter. As we are mainly interested in matter interactions these symmetries will be lost and need not be gauge fixed. Only the $OSP(2\mid 1)$ symmetry would survive. This would pose a problem for evaluating the $<\psi_-\psi_->$ propagator since the kinetic operator $\bar{\psi}_-d\psi_-$ is non-invertible and no gauge fixing condition such as $\partial^\mu \psi_\mu = 0$ is available. In this case we are forced to abandon the physical gauge (3.10) and take instead $H^a = 0$, where $H = H^a \tau_a + h$. Then

$$G = \begin{pmatrix} 1 - \frac{1}{4}\eta\bar{\eta} & \eta \\ \bar{\eta} & 1 + \frac{1}{2}\bar{\eta}\eta \end{pmatrix} = \mathbf{1} + 0(\eta) \tag{3.20}$$

which is obtained by solving (3.9). In this gauge, the action in (3.18) would be corrected by $\bar{\eta}\eta$ terms and the gauge invariance associated with the fermionic parts of $OSP(2 \mid 1) \times OSP(2 \mid 1)$ could be used to gauge fix both ψ_- and ψ_+.

The matter interactions which reproduce the bosonic matter interactions (2.13) are now

$$S_{sm} = 4k_{sm} \int str\ (\widetilde{DG}\ DG\ \tilde{G}\ X) Str(\tilde{G}DX) \tag{3.21}$$

where X is a multiplet transforming like G. The matrix representation of X is given by

$$X = \begin{pmatrix} (\varphi + s)\delta_\alpha^{\dot{\alpha}} + \pi^a (\sigma_a)_\alpha^{\dot{\alpha}}, & \lambda_\alpha - \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} & S \end{pmatrix} \tag{3.22}$$

In the physical gauge (3.20) the matter interactions (3.21) simplify to

$$S_{sm} = -k_{sm} \int (\epsilon_{abc} e^a e^b \pi^c + 4\bar{\psi}_- \sigma_a \psi_- \pi^a - 4\bar{\psi}_- e\lambda)\ (d\varphi - e^d \pi_d - 2\bar{\psi}_- \lambda) + 0(\eta) \tag{3.23}$$

Although this action has the correct bosonic interactions for π^a and φ, however, S and χ decouples, and λ does not acquire a propagator.

IV. Perturbative analysis

We now proceed to analyse perturbatively the quantum theory of (2.1), (2.6) and (2.12). By choosing the physical gauge (2.7) the action reduces to (2.11) and (2.13) in addition to the ghost part corresponding to this gauge fixing. It is easily seen that this ghost part decouples and one can equivalently start by analysing the sum of (2.11) and (2.13). First we have to gauge fix the remaining gauge degrees of freedom. These are $SO(1,2)$ gauge invariance and general coordinate transformations (GCT) on the space-time manifold M. It is well known that for quantizing Chern-Simons theories in general and $3D$ pure gravity in particular one has to pick a background metric on M. The partition function for such theories, although expressed in terms of the background metric, is a topological invariant and is independent of the metric. In general, it is not guaranteed that a theory which is metric independent at the classical level would remain so at the quantum level [3]. Our purpose is then to examine whether the matter interactions coupled to gravity as given in (2.11) and (2.13) would remain metric independent at the quantum level.

For convenience we choose the background metric on M to be the flat Minkowski metric $\eta_{\mu\nu}$ [8]. The $SO(1,2)$ and GCT symmetries are fixed by the conditions

$$\eta^{\mu\nu} \partial_\mu \omega_\nu^a = 0, \quad \eta^{\mu\nu} \partial_\mu e_\nu^a = 0 \tag{4.1}$$

where we defined $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$. The gauge fixing and ghost terms obtained through the Fadeev-Popov procedure are:

$$\begin{aligned}\mathcal{L}_{GF} = & - \frac{1}{2\alpha}(\partial^\mu\omega^a_\mu)(\partial^\nu\omega^a_\nu) - \frac{1}{2\beta}(\partial^\mu e^a_\mu)(\partial^\nu e^a_\nu) + \partial^\mu c^*_a \partial_\mu c^a \\ & + c^*_a\left[\epsilon^{abc}\omega_{\mu b}\partial^\mu c_c + \partial^\mu(\partial_\mu d^\nu \omega^a_\nu + d^\nu \partial_\nu \omega^a_\mu)\right] \\ & + d^*_a[\epsilon^{abc}e_{\mu b}\partial^\mu c_c + \partial^\mu(\partial_\mu d^\nu e^a_\nu + d^\nu\partial_\nu e^a_\mu)] \end{aligned} \quad (4.2)$$

where c, d, c^* and d^* are the ghost fields.

A basic assumption of topological gravity is that any background satisfying the classical equations of motion can be used in the perturbative expansion, including the topological phase $e^a_\mu = 0$. In this unbroken phase of gravity, perturbation theory will take its simplest form. This ability to expand around the $e^a_\mu = 0$ phase is unique to three dimensions. In the higher dimensional generalization of topological gravity [9], perturbation theory requires expansion around a flat background $e^a_\mu = \delta^a_\mu$, because otherwise the graviton cannot be given a propagator.

Working in the topological phase, the only propagators come from the gravity and ghost sectors. Defining $k'_g \equiv k_g + k_h$, the $\mathcal{L}_g + \mathcal{L}_h + \mathcal{L}_{GF}$ parts of the Lagrangian give [8]

$$< e^a_\mu \omega^b_\nu > \; = \; -\frac{i}{k'_g}\eta^{ab}\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2} \quad (4.3)$$

$$< \omega^a_\mu \omega^b_\nu > \; = \; -\alpha\eta^{ab}\frac{p_\mu p_\nu}{p^4} \quad (4.4)$$

$$< e^a_\mu e^b_\nu > \; = \; -\beta\eta^{ab}\frac{p_\mu p_\nu}{p^4} \quad (4.5)$$

$$< c^*_a c_b > \; = \; -\eta_{ab}\frac{1}{p^2} \quad (4.6)$$

It is most convenient to choose the Landau gauge where $\alpha, \beta \to 0$ so that in the gravity sector, we are left only with the off-diagonal propagator $< \omega e >$.

We are now ready to analyse renormalizability of three-dimensional gravity coupled to matter in its topological form as given in (2.11) and (2.13). Let us first look at the one-loop contributions coming from the matter part. We only have $< \omega e >$ and $< c^* c >$ propagators at our disposal and so matter will enter as external lines. These can couple to internal dreibeins e^a_μ. Simple power counting then indicates that the only dangerous diagrams are the two in Fig. 1a,b. These may be expected to be linearly divergent. Including one more propagator will make the integrand odd so that the diagram vanishes. Any additional propagator will give finite diagrams.

The vertices appearing in Fig. 1a,b are

$$(V^{\varphi\pi ee})^{\mu\nu}_{abc}(p_1, \ldots, p_4) \; = \; i(2\pi)^3\delta^{(3)}(p_1 + \ldots + p_4)\cdot(-2k_m)\;\epsilon^{\mu\nu\rho}\epsilon_{abc}(-ip_1)_\rho \quad (4.7)$$

$$(V^{\pi\pi eee})^{\mu\nu\rho}_{abcde}(p_1, \ldots, p_5) \; = \; i(2\pi)^3\delta^{(3)}(p_1 + \ldots + p_5)\cdot 4k_m\;\eta_{ab}\epsilon^{\mu\nu\rho}\epsilon_{cde} \quad (4.8)$$

$$(V^{e\omega\omega})^{\mu\nu\rho}_{abc}(p_1, p_2, p_3) \; = \; i(2\pi)^3\delta^{(3)}(p_1 + p_2 + p_3)\cdot(-k'_g)\epsilon^{\mu\nu\rho}\epsilon_{abc} \quad (4.9)$$

To calculate the diagrams we introduce a Pauli-Villars regulator $\Lambda \to \infty$ in the propagators (4.3)-(4.6) by substituting

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 - \Lambda^2} \quad (4.10)$$

This allows us to calculate the diagrams of Fig.1 and (using here and in the following formulas that can be found in [8]) the results turn out to be

$$\begin{aligned} S_{1a} &= i\frac{k_m}{k_g'}\,\eta_{ab}p_\nu\Lambda^4\int\frac{d^3q}{(2\pi)^3}\,\frac{q^\mu(q-p-k)^\nu+q^\nu(q-p-k)^\mu}{q^2(q^2-\Lambda^2)(q-p-k)^2([q-p-k]^2-\Lambda^2)} \\ &= i\frac{k_m}{k_g'}\frac{\Lambda}{12\pi}\,\eta^{\mu\nu}\eta_{ab}(-ip_\nu) \end{aligned} \tag{4.11}$$

$$\begin{aligned} S_{1b} &= -\frac{2k_m}{k_g'}\,\eta_{bd}\eta_{ac}\Lambda^4\int\frac{d^3q}{(2\pi)^3}\,\frac{q^\mu(q-p-k-\ell)^\nu+q^\nu(q-p-k-\ell)^\mu}{q^2(q^2-\Lambda^2)(q-p-k-\ell)^2([q-p-k-\ell]^2-\Lambda^2)} \\ &= -i\frac{k_m}{k_g'}\frac{\Lambda}{6\pi}\,\eta^{\mu\nu}\eta_{bd}\eta_{ac} \end{aligned} \tag{4.12}$$

as $\Lambda\to\infty$. Notice that there is a second, topologically inequivalent diagram contributing to Fig. 1b. This is obtained by letting the π-lines originate from the other vertex. It is easily seen that this additional diagram doubles the divergence (4.12).

These results show the linear divergences that were expected from power counting. There are no other diagrams to cancel these. Thus coupling bosonic matter in its topological form leads to a system which is neither finite nor renormalizable. The surprising feature is that the form of the divergences (4.11), (4.12) requires counterterms $\mathcal{L}^c$ that include the metric

$$\mathcal{L}^c_{1a} = -\frac{\Lambda}{12\pi}\frac{k_m}{k_g'}\,\eta^{\mu\nu}e^a_\mu\pi_a\partial_\nu\varphi \tag{4.13}$$

$$\mathcal{L}^c_{1b} = +\frac{\Lambda}{12\pi}\frac{k_m}{k_g'}\,\eta^{\mu\nu}\pi^a\pi_a e^b_\mu e_{\nu b} \tag{4.14}$$

Using only fields of the originally topological Lagrangian, no counterterm can be written down to cancel the divergences. Thus, we observe a new kind of gravitational anomaly, resulting in a metric dependence appearing at the quantum level of a classically topological system.

Actually, adding matter terms (2.13) even destroys finiteness in the gravity sector. It is highly instructive to understand the reason for these anomalies, since it may point to a fundamental reason behind the non-renormalizability of Einstein-Hilbert gravity in dimensions $D>3$ and ways to cure it. The key rôle is played by the ghosts coming from the gauge fixing (4.1). These gauge fixing conditions are the same as for pure gravity [2],[8]. There, however, they served to fix $ISO(1,2)$-invariance. Introducing the ω-equation of motion as a constraint so that the torsion vanishes, the $ISO(1,2)$ transformations coincided with GCT. Therefore it was not required for pure gravity to fix the GCT separately because that would correspond to over gauge-fixing. Having included the matter terms, the $ISO(1,2)$-symmetry is broken down to an $SO(1,2)$-symmetry and GCT symmetry has to be fixed separately. The condition $\partial\omega=0$ fixes the $SO(1,2)$-invariance, while $\partial e=0$ fixes GCT invariance. Fixing GCT by $\partial e=0$ leads to the ghosts d^*,d in (4.2). Unfortunately, these are not propagating in the $e^a=0$ phase, unlike the ghosts that arose when the gauge fixing condition $\partial e=0$ had to fix the translation part of $ISO(1,2)$.

The linear divergent one-loop diagrams in the gravity sector are shown in Fig. 2a,b. Using also the vertex

$$(V^{\omega c^* c})^{\mu bc}_a(p_1,p_2,p_3) = i(2\pi)^3\delta^{(3)}(p_1+p_2+p_3)i\;p_3^\mu\epsilon_a^{\;bc} \tag{4.15}$$

the amplitudes turn out to be

$$
\begin{aligned}
S_{2a} &= 2\eta_{ab}\Lambda^4 \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu(q-p)^\nu + q^\nu(q-p)^\mu}{q^2(q^2-\Lambda^2)(q-p)^2([q-p]^2-\Lambda^2)} \\
&= i\frac{-\Lambda}{6\pi}\,\eta^{\mu\nu}\eta_{ab} \qquad (4.16) \\
S_{2b} &= 2\eta_{ab}\Lambda^4 \int \frac{d^3q}{(2\pi)^3} \frac{(q-p)^\mu q^\nu}{q^2(q^2-\Lambda^2)(q-p)^2([q-p]^2-\Lambda^2)} \\
&= i\frac{\Lambda}{12\pi}\,\eta^{\mu\nu}\eta_{ab} \qquad (4.17)
\end{aligned}
$$

as $\Lambda \to \infty$. For pure gravity there would be a third diagram with internal d^*, d ghosts giving the same amplitude as (4.17) and therefore cancelling the total one-loop divergences [8].

Here, however, a counterterm

$$\mathcal{L}_2^c = \frac{\Lambda}{24\pi}\,\eta^{\mu\nu}\omega_\mu^a\omega_{\nu a} \qquad (4.18)$$

has to be introduced. This is again of the geometrical structure that was observed in the matter sector.

Notice that the decisive point in discussing the gravity sector was the loss of invariance under the translationary part of $ISO(1,2)$, leaving only the Lorentz-group $SO(2,1)$ and GCT as symmetry. These are also the symmetries of Einstein-Hilbert gravity in dimension $D > 3$. The vielbeins e_μ^a no longer come as gauge fields. However, in $D > 3$ it is not possible to see the problem of non-renormalizability in terms of non-propagating ghosts since the perturbation analysis has to be performed by expanding e_μ^a around a flat background. The problem must be seen to arise because the Einstein-Hilbert action does not have enough gauge symmetry.

Having shown the occurrence of anomalous divergences, it is natural to look for an anomaly cancelling mechanism. Examining Fig. 1a,b, one might think that by replacing the internal $e-\omega$ lines by fermionic ones then a cancellation between the anomalous diagrams might be possible. Indeed the new diagrams with internal fermionic lines could be generated by supersymmetrizing the bosonic construction. This was carried out in Section 3 where the supergravity and matter actions have $OSP(2\mid 1)\times OSP(2\mid 1)$ gauge invariance. For the supersymmetric case we want to fix the gauge by imposing

$$\partial^\mu\omega_\mu^a = 0, \quad \partial^\mu\psi_\mu^+ = 0 \qquad (4.19)$$

$$H^a = 0, \quad \partial^\mu\psi_\mu^- = 0 \qquad (4.20)$$

$$\partial^\mu e_\mu^a = 0 \qquad (4.21)$$

The conditions (4.19), (4.20) fix the $OSP(1,2)\times OSP(1,2)$, while (4.21) has to fix GCT. As was mentioned already in Section 3 for the perturbative analysis (4.20) has to be used instead of (3.10). This is due to the fact that otherwise the $\psi^\pm$-kinetic terms cannot be inverted to give a propagator. Using (4.20) instead of (3.10) leads to (3.20) and so the actions (3.18) and (3.23) will be corrected by $\eta\bar{\eta}$-terms. These corrections, however, would not change the conclusions of this paper and will therefore be ignored in what follows. The gauge fixing and ghost terms corresponding to (4.19)-(4.21) are

$$\mathcal{L}_{SGF} = -\frac{1}{2\alpha}(\partial^\mu\omega_\mu^a)(\partial^\nu\omega_{\nu a}) - \frac{1}{2\alpha'}(\partial^\mu\overline{\psi_\mu^+})(\partial^\nu\psi_\nu^+)$$

$$
\begin{aligned}
&- \frac{1}{2\beta}\,(\partial^\mu e^a_\mu)\,(\partial^\nu e_{\nu a}) - \frac{1}{2\beta'}(\partial^\mu \overline{\psi^-_\mu})\,(\partial^\nu \overline{\psi^-_\nu}) - \frac{1}{\gamma'}(H^a)^2 \\
&+ \partial^\mu c^{*a}\partial_\mu c_a + \partial^\mu \overline{\gamma^*}\partial_\mu \gamma \\
&+ \epsilon_{abc}\Big[c^{*a}\omega^b_\mu \partial^\mu c^c + d^{*a}e^b_\mu\partial^\mu c^c\Big] + 2\overline{\psi^+_\mu}\Big[c^*\partial^\mu\gamma + \partial^\mu c\gamma^*\Big] \\
&+ 2\overline{\psi^-_\mu}\Big[d^*\partial^\mu\gamma + \partial^\mu c\delta^*\Big] - \overline{\gamma^*}\omega_\mu\partial^\mu\gamma - \overline{\delta^*}e_\mu\partial^\mu\gamma \qquad (4.22)
\end{aligned}
$$

where $c^* = c^{*a}\tau_a, c = c^a\tau_a$ are scalar and γ^*, γ are spinor ghosts. The ghost terms coming from GCT have not been written down, since they decouple as for the non-supersymmetric case.

Using $k'_{sg} \equiv k_{sg} + k_{sh1}$ and the gauge fixing terms, the propagators for the gravitinos turn out to be

$$
\begin{aligned}
< \overline{\psi^-_\mu}^\alpha \psi^+_{\nu\beta} > &= \frac{-i}{k'_{sg}}\delta^\alpha_\beta\ \epsilon_{\mu\nu\rho}\ \frac{p^\rho}{p^2} \\
< \overline{\psi^+_\mu}^\alpha \psi^+_{\nu\beta} > &= -2\alpha'\delta^\alpha_\beta\ \frac{p_\mu p_\nu}{p^2} \\
< \overline{\psi^-_\mu}\psi^-_\nu > &= -2\beta'\delta^\alpha_\beta\ \frac{p_\mu p_\nu}{p^2} \qquad (4.23)
\end{aligned}
$$

The graviton propagators are of the same form as in (4.3)-(4.5). Again, it is most convenient to use the Landau gauge $\alpha_1, \ldots, \beta' \to 0$, leaving only off-diagonal propagators.

Working in the topological phase, there are no matter propagators. Matter can only enter as external lines, like for the non-supersymmetric case. The vertices can be read off from (3.18) and (3.23). For example,

$$
\begin{aligned}
(V^{\varphi\pi\psi^-\psi^-})^{\mu\nu,\beta}_{a,\alpha}(p_1,p_2,p_3,p_4) &= i\ (2\pi)^3\delta^{(3)}(p_1+p_2+p_3+p_4)(-4k_{sm})\epsilon^{\mu\nu\rho}(\tau_a)^\beta_\alpha(-ip_1)_\rho \\
(V^{\varphi\lambda e\psi^-})^{\mu\nu,\beta}_{a,\alpha}(p_1,p_2,p_3,p_4) &= i\ (2\pi)^3\delta^{(3)}(p_1+p_2+p_3+p_4)4k_{sm}\epsilon^{\mu\nu\rho}(\tau_a)^\beta_\alpha(-ip_1)_\rho \\
(V^{\psi^+\omega\psi^-})^{\mu\nu\rho,\beta}_{a,\alpha}(p_1,p_2,p_3) &= i\ (2\pi)^3\delta^{(3)}(p_1+p_2+p_3)2k'_{sg}\epsilon^{\mu\nu\rho}(\tau_a)^\beta_\alpha \\
(V^{\psi^+e\psi^+})^{\mu\nu\rho,\beta}_{a,\alpha}(p_1,p_2,p_3) &= i\ (2\pi)^3\delta^{(3)}(p_1+p_2+p_3)2k'_{sg}\epsilon^{\mu\nu\rho}(\tau_a)^\beta_\alpha \qquad (4.24)
\end{aligned}
$$

It turns out that for example the diagram in Fig. 3a realizes our hope to cancel the diagram in Fig. 1a (4.11):

$$
\begin{aligned}
S_{3a} &= 4i\ \frac{k_{sm}}{k'_{sg}}(\tau_a)^\gamma_\alpha\ (\tau_b)^\alpha_\gamma p_\nu\Lambda^4\int\frac{d^3q}{(2\pi)^3}\ \frac{q^\nu(q-p-k)^\mu + q^\mu(q-p-k)^\nu}{q^2(q^2-\Lambda^2)(q-p-k)^2([q-p-k]^2-\Lambda^2)} \\
&= -i\ \frac{k_{sm}}{k'_{sg}}\frac{\Lambda}{12\pi}\ \eta_{ab}\eta^{\mu\nu}(-ip_\nu) \qquad (4.25)
\end{aligned}
$$

However, now there are new diagrams like the one shown in Fig. 3c with an amplitude

$$
\begin{aligned}
S_{3c} &= -i\ 4\frac{k_{sm}}{k'_{sg}}\eta^{ab}(\tau_a\tau_b)^\beta_\alpha\ p_\nu\int\frac{d^3q}{(2\pi)^3}\ \frac{q^\nu(q-p-k)^\mu + q^\mu(q-p-k)^\nu}{q^2(q^2-\Lambda^2)(q-p-k)^2([q-p-k]^2-\Lambda^2)} \\
&= -i\ \frac{k_{sm}}{k'_{sg}}\frac{\Lambda}{3\pi}\ \eta_{ab}(\tau_a\tau_b)^\beta_\alpha\eta^{\mu\nu}(-ip_\nu)
\end{aligned}
$$

This is not cancelled by any other diagram and the divergency is again of the geometric type. Thus, we conclude that not even supersymmetrization is able to cure the anomalous behaviour resulting from including matter to three-dimensional topological gravity.

V. Conclusions

In this paper we have constructed matter interactions coupled to gravity and supergravity by spontaneously breaking gauge symmetries. The actions used are metric independent, and the metric itself arises in the non-topological phase as a product of the dreibein fields e_μ^a which are gauge fields. Changing the matter action of a scalar boson to its second-order formulation and projecting into the space of invertible e_μ^a we arrive at the canonical action of a scalar field. The main advantage of working with topological gravity in three dimensions is that the graviton propagator can be obtained without expanding e_μ^a around a non-zero classical background. This makes it easy to analyse the quantum theory. A background metric is, however, needed to fix the gauge symmetries present. We have shown that the topological interactions considered here are divergent. The divergences cannot be cancelled without using the background metric. Thus, although the energy-momentum tensor $T_{\mu\nu}$ was zero classically, it is non-zero at the quantum level. This indicates that the theory loses its topological nature at the quantum level. We examined supersymmetrization as a possible mechanism to cancel the metric anomaly. Unfortunately, this was not enough to cure the problem.

From our study we can now extract the following lessons. Firstly, to construct non-trivial matter interactions in a topological theory we must adopt the mechanism of spontaneously breaking the symmetry to liberate e_μ^a from the other gauge fields. Secondly, the external matter although topological at the classical level, loses this property at the quantum level. Finally, to cure the problem of metric anomalies it seems that the system of gravity and matter must be protected by a larger symmetry. The difficulty is now that the gauge symmetries cannot be used to unify scalars and spinors with the gravitational gauge sectors. Such a unification, however, does occur in the Kaluza-Klein approach where the scalars and spinors are produced as the components of gravitons and gravitinos in the extra dimensions. We cannot judge without actually analysing such a mechanism whether the problems encountered here would be avoided. The disadvantage in this case is that the perturbative analysis cannot be performed in the topological phase and one must expand e_μ^a around a non-zero classical background. Another possibility of unifying scalars and spinors with the gravitational sector may be an unconventional one, like finding the analogue of the construction of Connes [10].

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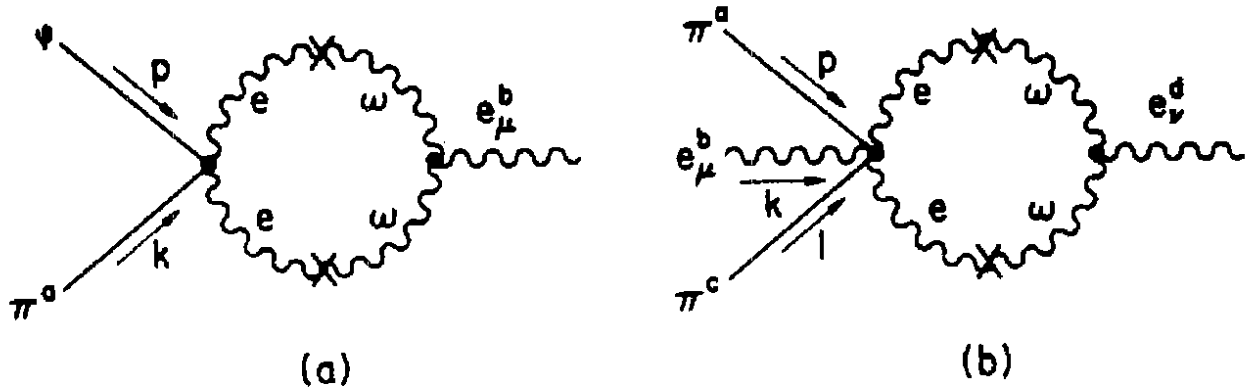

Fig. 1

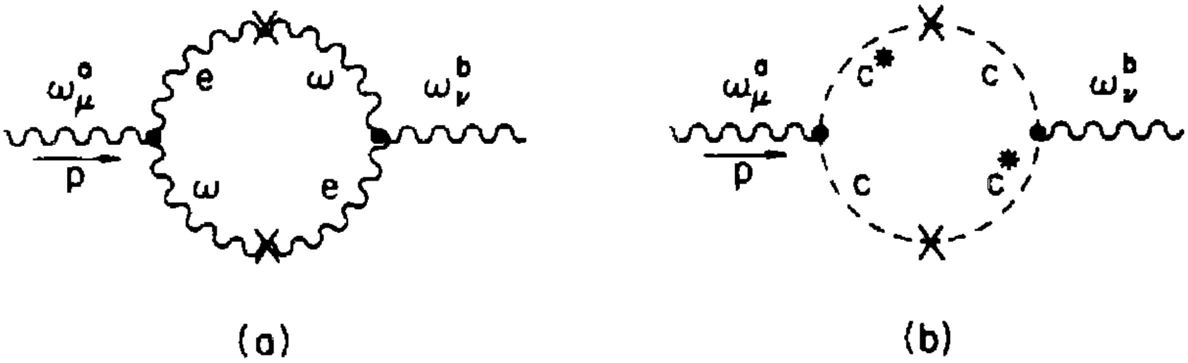

Fig. 2

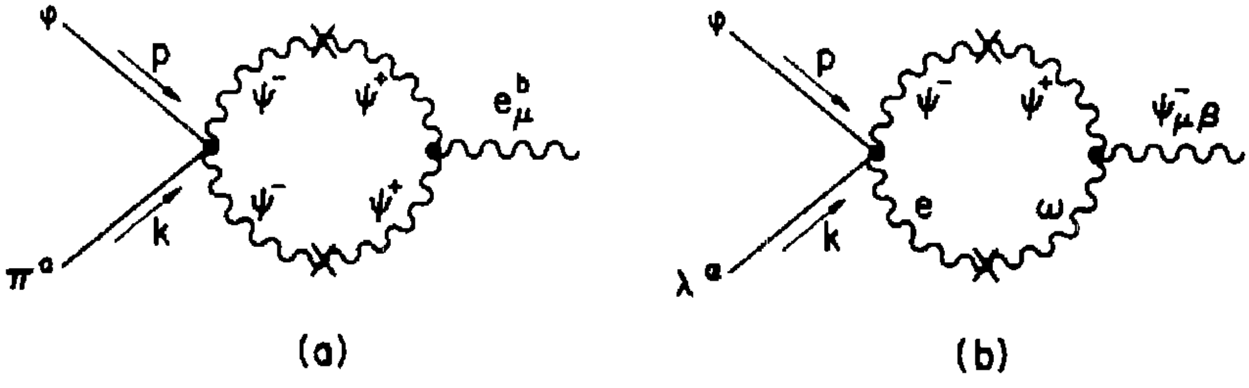

Fig. 3