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Supersymmetry and Supergravity

Jonathan Bagger*

Department of Physics and Astronomy
The Johns Hopkins University
Baltimore, Maryland 21218

and

Julius Wess

Institut für Theoretische Physik
Universität Karlsruhe
D–7500 Karlsruhe 1, Germany

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XXI. THE MINIMAL CHIRAL SUPERGRAVITY MODEL

We now have what we need to construct the supergravity matter couplings. The general case is rather involved, so we shall start here with a simpler example. We take the Lagrangian to be given by

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar\theta\, \Phi_i{}^+\Phi_i \;+\; \left[\int d^2\theta \left(a_i\Phi_i \;+\; \frac{1}{2} m_{ij}\Phi_i\Phi_j \;+\; \frac{1}{3} g_{ijk}\Phi_i\Phi_j\Phi_k \right) \;+\; \text{h.c.} \right] . \tag{21.1}$$

This Lagrangian was first introduced in Chapter V; it is the most general renormalizable supersymmetric Lagrangian involving only scalar superfields. In curved space, the renormalizability of (21.1) is less important. However, the techniques we introduce in analyzing this model will prove useful in discussing the general case in later chapters. Since the result we derive reduces to (21.1) in the limit of flat space, we call it the minimal chiral supergravity model.

We start our construction by writing down an invariant action for the supergravity multiplet,

$$\mathcal{L}_{S.G.} \;=\; -\frac{6}{\kappa^2} \int d^2\Theta\, \mathcal{E} R \;+\; \text{h.c.} \tag{21.2}$$

Here $\kappa^2 = 8\pi G_N$ is the gravitational coupling, which we set equal to one. The chiral density $\mathcal{E}$ and superspace curvature R were computed in Chapter XX. Their Θ expansions are listed later in this chapter. Inserting these expressions into (21.2) gives $\mathcal{L}_{S.G.}$ in terms of the supergravity multiplet:

$$\mathcal{L}_{S.G.} \;=\; -\frac{1}{2} e\mathcal{R} \;-\; \frac{1}{3} eM^*M \;+\; \frac{1}{3} eb^a b_a \;+\; \frac{1}{2} e\epsilon^{k\ell mn}(\bar\psi_k\bar\sigma_\ell\tilde{\mathcal{D}}_m\psi_n \;-\; \psi_k\sigma_\ell\tilde{\mathcal{D}}_m\bar\psi_n) \,. \tag{21.3}$$

The curvature $\mathcal{R}$ was defined in (17.15),

$$\mathcal{R} \;=\; e_a{}^n e_b{}^m(\partial_n\omega_m{}^{ab} \;-\; \partial_m\omega_n{}^{ab} \;+\; \omega_m{}^{ac}\omega_{nc}{}^b \;-\; \omega_n{}^{ac}\omega_{mc}{}^b) \,, \tag{21.4}$$

and the connection $\omega_m{}^{ab}$ was computed in (17.12). From (21.3) we see that $\mathcal{L}_{S.G.}$ contains the Einstein action for the gravitational field. It also contains the Rarita-Schwinger action for the spin-$\frac{3}{2}$ gravitino; the covariant derivative $\tilde{\mathcal{D}}_m\psi_n$ was defined in (17.8). The fields M and b_a do not propagate. They are the auxiliary fields of the supergravity multiplet. Note that they enter (21.3) with opposite signs.

The Lagrangian (21.1) is easily extended to curved superspace. We first write it in chiral form,

$$\mathcal{L} = \int d^2\theta \left[-\frac{1}{8}\bar{D}\bar{D}\Phi_i{}^+\Phi_i + a_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k \right] + \text{h.c.}\,, \tag{21.5}$$

as outlined in Exercise 6 of Chapter IX. We then add the supergravity action (21.2), and replace $\theta \to \Theta$, $d^2\theta \to d^2\Theta\, 2\mathcal{E}$, and $-\frac{1}{4}\bar{D}\bar{D} \to -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)$. This gives the action (21.1) in curved superspace:

$$\begin{aligned}\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \Big[&-3R - \frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\Phi_i{}^+\Phi_i \\ &- \frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)[c_i\Phi_i + \bar{c}_i\Phi_i{}^+] + d + a_i\Phi_i \\ &+ \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k \Big] + \text{h.c.}\end{aligned} \tag{21.6}$$

This Lagrangian describes the minimal chiral model. It reduces to (21.1) in flat space. The c and d terms are included because they arise from shifts in the superfields Φ_i. They vanish in flat space. Note that gauge invariance restricts $c_i = 0$ unless Φ_i is neutral.

Equation (21.6) contains two types of terms: those with the chiral projector $(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)$, and those without. The terms with projector are curved-space generalizations of the chiral kinetic energy. Those without are curved-space extensions of the usual superspace potential. We will emphasize this distinction by writing (21.6) in the following form,

$$\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\Omega(\Phi, \Phi^+) + P(\Phi) \right] + \text{h.c.}\,, \tag{21.7}$$

where $\Omega(\Phi, \Phi^+) = \Phi_i{}^+\Phi_i + c_i\Phi_i + \bar{c}_i\Phi_i{}^+ - 3$ is the superspace kinetic energy, and $P(\Phi) = d + a_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k$ is the superspace potential. In Chapter XXIII we shall see that this distinction is preserved in the general case, where Ω and P are arbitrary functions of their respective superfields.

The Lagrangian (21.7) has a long expansion in terms of component fields. To find it, we need the Θ expansions of Φ_i, $\mathcal{E}$, R:

$$\Phi_i = A_i + \sqrt{2}\Theta\chi_i + \Theta\Theta F_i$$

$$2\mathcal{E} = e\{1 + i\Theta\sigma^a\bar{\psi}_a - \Theta\Theta[M^* + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b]\}$$

$$R = -\frac{1}{6}\Big\{M + \Theta[\sigma^a\bar{\sigma}^b\psi_{ab} - i\sigma^a\bar{\psi}_a M + i\psi_a b^a]$$

$$+\ \Theta\Theta\Big[-\frac{1}{2}\mathcal{R} + i\bar\psi^a\bar\sigma^b\psi_{ab} + \frac{2}{3}MM^*$$

$$+\ \frac{1}{3}b^a b_a - ie_a{}^m\mathcal{D}_m b^a + \frac{1}{2}\bar\psi\bar\psi M - \frac{1}{2}\psi_a\sigma^a\bar\psi_c b^c$$

$$+\ \frac{1}{8}\epsilon^{abcd}[\bar\psi_a\bar\sigma_b\psi_{cd} + \psi_a\sigma_b\bar\psi_{cd}]\Big]\Big\}\ . \qquad \mathbf{(21.8)}$$

The components of $\Xi_i = (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\Phi_i{}^+$ can be computed with the help of (19.7):

$$\Xi_i = (\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\Phi_i{}^+$$

$$= (\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\Phi_i{}^+\Big| + \Theta^\alpha\mathcal{D}_\alpha(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\Phi_i{}^+\Big|$$

$$-\ \frac{1}{4}\Theta\Theta\mathcal{D}^\alpha\mathcal{D}_\alpha(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\Phi_i{}^+\Big|\ . \qquad (21.9)$$

The necessary ingredients are given in Exercise 1. We find:

$$\Xi_i = -4F_i^* + \frac{4}{3}MA_i^*$$

$$+\ \Theta\Big\{-4i\sqrt{2}\sigma^c\hat{D}_c\bar\chi_i - \frac{2}{3}\sqrt{2}\sigma^a b_a\bar\chi_i$$

$$+\ \frac{4}{3}A_i^*(2\sigma^{ab}\psi_{ab} - i\sigma^a\bar\psi_a M + i\psi_a b^a)\Big\}\ .$$

$$+\ \Theta\Theta\Big\{-4e_a{}^m\mathcal{D}_m\hat{D}^a A_i^* - \frac{8}{3}ib_a\hat{D}^a A_i^*$$

$$-\ \frac{2}{3}\sqrt{2}\bar\psi_{ab}\bar\sigma^{ab}\bar\chi_i + 2\sqrt{2}\bar\psi_a\hat{D}^a\bar\chi_i - \frac{8}{3}M^*F_i^*$$

$$-\ \frac{2}{3}i\sqrt{2}\bar\psi_a\bar\chi_i b^a + \frac{1}{3}i\sqrt{2}\bar\psi_a\bar\sigma^a\sigma^c\bar\chi_i b_c$$

$$+\ \frac{4}{3}A_i^*\Big[-\frac{1}{2}\mathcal{R} + i\bar\psi^a\bar\sigma^b\psi_{ab} - ie_a{}^m\mathcal{D}_m b^a$$

$$+\ \frac{2}{3}M^*M + \frac{1}{3}b_a b^a + \frac{1}{2}\bar\psi\bar\psi M - \frac{1}{2}\psi_a\sigma^a\bar\psi_c b^c$$

$$+\ \frac{1}{8}\epsilon^{abcd}(\bar\psi_a\bar\sigma_b\psi_{cd} + \psi_a\sigma_b\bar\psi_{cd})\Big]\Big\}\ . \qquad \mathbf{(21.10)}$$

Here we have used the following supercovariant derivatives:

$$\hat{D}_a A_i^* = e_a{}^m\partial_m A_i^* - \frac{1}{2}\sqrt{2}\bar\psi_{a\dot\kappa}\bar\chi_i^{\dot\kappa}$$

$$\hat{D}_a\bar\chi_{i\dot\alpha} = e_a{}^m\mathcal{D}_m\bar\chi_{i\dot\alpha} + \frac{i}{2}\sqrt{2}\psi_a{}^\beta\sigma_{\beta\dot\alpha}{}^b\hat{D}_b A_i^*$$

$$-\ \frac{1}{2}\sqrt{2}\bar\psi_{a\dot\alpha}F_i^*\ , \qquad \mathbf{(21.11)}$$

where $\mathcal{D}_m\bar\chi_{i\dot\alpha} = \partial_m\bar\chi_{i\dot\alpha} + \bar\chi_{i\dot\beta}\omega_m{}^{\dot\beta}{}_{\dot\alpha}$. The superfields $\mathcal{E}$, R, Φ_i and Ξ_i allow us to compute any supergravity Lagrangian involving only chiral fields.

The expansion of the Lagrangian (21.7) contains kinetic terms for the physical fields A_i, χ_i, $e_m{}^a$ and $\psi_m{}^\alpha$, as well as terms involving the auxiliary fields M, b_a and F_i. The

Lagrangian also has higher-order interaction terms, such as nonrenormalizable four-fermion couplings, which are suppressed by powers of Newton's constant. For ease of exposition, we will write the full Lagrangian as follows,

$$\mathcal{L} \;=\; \mathcal{L}_{kin} \;+\; \mathcal{L}_{aux} \;+\; \mathcal{L}_{quartic} \tag{21.12}$$

where

$$\begin{aligned}
\mathcal{L}_{kin} \;=\; & \frac{1}{6} e\,\Omega\mathcal{R} \;-\; e\,\partial_m A_i \partial^m A_i^* \\
& - \frac{i}{2} e\,[\chi_i \sigma^m \mathcal{D}_m \bar{\chi}_i \;+\; \bar{\chi}_i \bar{\sigma}^m \mathcal{D}_m \chi_i] \\
& - \frac{1}{12} e\,\Omega\,\epsilon^{k\ell mn}\,[\bar{\psi}_k \bar{\sigma}_\ell \psi_{mn} \;-\; \psi_k \sigma_\ell \bar{\psi}_{mn}] \\
& + \frac{1}{4} e\,\epsilon^{k\ell mn}\,(\Omega_i\,\partial_k A_i \;-\; \Omega_{i^*}\,\partial_k A_i^*)\,\psi_\ell \sigma_m \bar{\psi}_n \\
& - \frac{1}{2}\sqrt{2}\,e\,[\bar{\psi}_n \bar{\sigma}^m \sigma^n \bar{\chi}_i\,\partial_m A_i \;+\; \psi_n \sigma^m \bar{\sigma}^n \chi_i\,\partial_m A_i^*] \\
& + \frac{1}{3}\sqrt{2}\,e\,[\Omega_i\,\chi_i \sigma^{mn} \psi_{mn} \;+\; \Omega_{i^*}\,\bar{\chi}_i \bar{\sigma}^{mn} \bar{\psi}_{mn}] \\
& - \frac{i}{2}\sqrt{2}\,e\,P_i\,\chi_i \sigma^a \bar{\psi}_a \;-\; \frac{i}{2}\sqrt{2}\,e\,P_{i^*}^*\,\bar{\chi}_i \bar{\sigma}^a \psi_a \\
& - \frac{1}{2} e\,P_{ij}\,\chi_i \chi_j \;-\; \frac{1}{2} e\,P_{i^* j^*}^*\,\bar{\chi}_i \bar{\chi}_j \\
& - e\,P\,\bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \;-\; e\,P^*\,\psi_a \sigma^{ab} \psi_b
\end{aligned} \tag{21.13}$$

is the kinetic part of the Lagrangian,

$$\begin{aligned}
\mathcal{L}_{aux} \;=\; & \frac{1}{9} e\,\Omega\,\Big|M \;-\; 3\,(\log\Omega)_{i^*}\,F_i^*\Big|^2 \\
& + e\,\Omega\,(\log\Omega)_{ij^*}\,F_i F_j^* \;-\; \frac{1}{9} e\,\Omega\,b_a b^a \\
& - \frac{i}{3} e\,(\Omega_i\,\partial_m A_i \;-\; \Omega_{i^*}\,\partial_m A_i^*)\,b^m \;-\; \frac{1}{6} e\,\chi_i \sigma^a \bar{\chi}_i\,b_a \\
& + \frac{i}{6}\sqrt{2}\,e\,(\Omega_i\,\chi_i \psi_m \;-\; \Omega_{i^*}\,\bar{\chi}_i \bar{\psi}_m)\,b^m \\
& - e\,PM^* \;-\; e\,P^* M \;+\; e\,P_i F_i \;+\; e\,P_{i^*}^* F_i^*
\end{aligned} \tag{21.14}$$

is the auxiliary field contribution, and $\mathcal{L}_{quartic}$ contains four-fermi terms that we ignore for the moment. In (21.13) and (21.14), Ω and P are the same as before, except that the superfields Φ_i and Φ_i^+ are replaced by their lowest components A_i and A_i^*. The subscripts on Ω and P denote derivatives with respect to the scalar fields. For example, $P_i = (\partial/\partial A_i)\,P$ and $\Omega_{i^*} = (\partial/\partial A_i^*)\,\Omega$.

To proceed further, we must eliminate the auxiliary fields from $\mathcal{L}_{aux}$. This is most readily done by shifting, $N = M - 3(\log\Omega)_{i^*}\,F_i^*$. The shift decouples N and F_i, and allows the equations of motion to be easily solved,

$$N \;= 9P\Omega^{-1}$$

$$\Omega\,(\log\Omega)_{ij^*}\,F_i \;=\; -\,P^*_{j^*} \;+\; 3P^*\,(\log\Omega)_{j^*}$$

$$\begin{aligned} b_a \;=\; & -\frac{3}{2}\,i\,(\Omega_i\,\partial_a A_i \;-\; \Omega_{i^*}\,\partial_a A_i^*)\,\Omega^{-1} \\ & -\;\frac{3}{4}\,\chi_i\sigma_a\bar{\chi}_i\,\Omega^{-1} \\ & +\;\frac{3}{4}\,\sqrt{2}\,i\,(\Omega_i\,\chi_i\psi_a \;-\; \Omega_{i^*}\,\bar{\chi}_i\bar{\psi}_a)\,\Omega^{-1}\;. \end{aligned} \tag{21.15}$$

Substituting (21.15) into (21.14), we find

$$\begin{aligned} \mathcal{L}_{aux} \;=\; & -9\,e\,PP^*\Omega^{-1} \\ & -\;e\,(\log\Omega)^{-1}_{ij^*}\,[P_i \;-\; 3P(\log\Omega)_i] \\ & \quad\times\;[P^*_{j^*} \;-\; 3P^*(\log\Omega)_{j^*}]\,\Omega^{-1} \\ & -\;\frac{1}{4}\,e\,[\Omega_i\,\partial_a A_i \;-\; \Omega_{i^*}\,\partial_a A_i^*] \\ & \quad\times\;\Big[(\Omega_j\,\partial^a A_j \;-\; \Omega_{j^*}\,\partial^a A_{*j}) \;-\; i\chi_j\sigma^a\bar{\chi}_j \\ & -\;\sqrt{2}\,(\Omega_j\,\chi_j\psi^a \;-\; \Omega_{j^*}\,\bar{\chi}_j\bar{\psi}^a)\Big]\,\Omega^{-1} \;+\;\ldots \end{aligned} \tag{21.16}$$

The dots denote additional four-fermi terms that we absorb in $\mathcal{L}_{quartic}$. Equation (21.16) contains derivative terms,

fermion masses, Yukawa couplings, and the scalar potential, which we shall call $\mathcal{V}(A,A^*)$.

The above expressions are not quite ready for model-building. We must still check the normalizations of the physical fields. From (21.13), we see that the gravitational action has an unconventional Brans-Dicke form. This normalization can be fixed by performing a field-dependent Weyl rescaling of the gravitational field:

$$e_n{}^a \;\to\; e_n{}^a\,\exp(\lambda) \tag{21.17}$$

where

$$\exp(2\lambda) \;=\; -\frac{3}{\Omega}\;. \tag{21.18}$$

This transformation restores the canonical normalization (21.3) for the Einstein action. The matter-field normalizations can be restored through a field-dependent redefinition of the spinors,

$$\begin{aligned} \chi_i \;&\to\; \exp(-\lambda/2)\,\chi_i \\ \psi_m \;&\to\; \exp(\lambda/2)\,\psi_m\,, \end{aligned} \tag{21.19}$$

followed by an additional shift of the gravitino,

$$\psi_m \;\to\; \psi_m \;+\; i\sqrt{2}\,\sigma_m\bar{\chi}_i\,\lambda_{i^*}\;. \tag{21.20}$$

Adding the four-fermi terms from $\mathcal{L}_{quartic}$, and performing the transformations (21.17) – (21.20), we find our final result for the supergravity matter coupling:

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2} e\,\mathcal{R} \;-\; e\,K_{ij^*}\,\partial_m A_i \partial^m A_j^* \\
& - \frac{i}{2} e\,K_{ij^*}\,[\chi_i \sigma^m \mathcal{D}_m \bar\chi_j \;+\; \bar\chi_j \bar\sigma^m \mathcal{D}_m \chi_i] \\
& + \frac{1}{2} e\,\epsilon^{k\ell mn}\,[\bar\psi_k \bar\sigma_\ell \tilde{\mathcal{D}}_m \psi_n \;-\; \psi_k \sigma_\ell \tilde{\mathcal{D}}_m \bar\psi_n] \\
& + \frac{1}{4} e\,\epsilon^{k\ell mn}\,(K_i\,\partial_k A_i \;-\; K_{i^*}\,\partial_k A_i^*)\,\psi_\ell \sigma_m \bar\psi_n \\
& - \frac{i}{4} e\,[K_{ij^*}\,(K_k\,\partial_m A_k \;-\; K_{k^*}\,\partial_m A_k^*) \\
& - 2(K_{ij^*k}\,\partial_m A_k \;-\; K_{ij^*k^*}\,\partial_m A_k^*)]\,\chi_i \sigma^m \bar\chi_j \\
& - \frac{1}{2}\sqrt{2}\, e\,K_{ij^*}\,\partial_n A_j^*\,\chi_i \sigma^m \bar\sigma^n \psi_m \\
& - \frac{1}{2}\sqrt{2}\, e\,K_{ij^*}\,\partial_n A_i\,\bar\chi_j \bar\sigma^m \sigma^n \bar\psi_m \\
& + \frac{1}{4} e\,K_{ij^*}\,[i\epsilon^{k\ell mn}\psi_k \sigma_\ell \bar\psi_m \;+\; \psi_m \sigma^n \bar\psi^m]\,\chi_i \sigma_n \bar\chi_j \\
& + \frac{1}{16} e\,[K_{ij^*}K_{k\ell^*} \;-\; 2K_{ij^*k\ell^*} \;+\; 2K^{rs^*}K_{iks^*}K_{j^*\ell^*r}] \\
& \quad \times\; \chi_i \sigma_m \bar\chi_j\,\chi_k \sigma^m \bar\chi_\ell \\
& + e\,\exp(K/2)\Big\{ - P^*\,\psi_a \sigma^{ab} \psi_b \;-\; P\,\bar\psi_a \bar\sigma^{ab} \bar\psi_b \\
& - \frac{i}{2}\sqrt{2}\,D_i P\,\chi_i \sigma^a \bar\psi_a \;-\; \frac{i}{2}\sqrt{2}\,D_{i^*} P^*\,\bar\chi_i \bar\sigma^a \psi_a \\
& - \frac{1}{2}[P_{ij} \;+\; K_{ij}\,P \;+\; K_i\,D_j P \;+\; K_j\,D_i P \\
& - K_i K_j\,P \;-\; K^{k\ell^*} K_{ij\ell^*} D_k P]\,\chi_i \chi_j \;-\; \frac{1}{2}[P_{i^*j^*}^* \\
& + K_{i^*j^*}\,P^* \;+\; K_{i^*}\,D_{j^*}P^* \;+\; K_{j^*}\,D_{i^*}P^* \;-\; K_{i^*}K_{j^*}\,P^* \\
& - K^{k^*\ell} K_{i^*j^*\ell} D_{k^*}P^*]\,\bar\chi_i \bar\chi_j \Big\} \\
& - e\,\exp(K)\,[K^{ij^*}(D_i P)(D_j P)^* \;-\; 3P^* P]\,, \qquad \textbf{(21.21)}
\end{aligned}
$$

where $D_i P = P_i + K_i P$ and $K(A, A^*) = -3\log(-\Omega/3)$. In this expression, $\Omega(A, A^*) = A_i^* A_i + c_i(A_i^* + A_i) - 3$ and $P(A) = d + a_i A_i + \frac{1}{2} m_{ij} A_i A_j + \frac{1}{3} g_{ijk} A_i A_j A_k$. In Chapter XXIII we will derive a similar result for the general chiral coupling.

Equation (21.21) gives the full supergravity coupling of the minimal chiral model. It has properly normalized kinetic energies for all physical fields, and the full set of four-fermi terms is included. Equation (21.21) is automatically invariant under supergravity transformations (up to total derivatives) because it was derived from a superspace formalism. It also has the correct flat-space limit.

From (21.21) we see that the supergravity scalar potential emerges in a form that will turn out to be quite general:

$$\mathcal{V}(A_i, A_j^*) \;=\; \exp(K)\,[K^{ij^*}(D_iP)(D_jP)^* \;-\; 3P^*P]\,. \qquad \mathbf{(21.22)}$$

Note that this expression is not positive definite, so the connection between the potential and supersymmetry breaking is more subtle than before. In Chapter XXIII we shall see that the signal for spontaneous supersymmetry breaking is $\langle D_iP\rangle \neq 0$. Equation (21.22) shows that supersymmetry can be spontaneously broken with *zero* vacuum energy.

The preceeding expressions are all written in terms of the real function $K(A, A^*)$. In the coming chapters, we shall see that this function is called a *Kähler potential*, and that (21.21) and (21.22) have a natural interpretation in the language of complex geometry.

References

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Equations

$$\mathcal{L}_{S.G.} \;=\; -\frac{6}{\kappa^2}\int d^2\Theta\,\mathcal{E}R \;+\; \text{h.c.} \qquad \mathbf{(21.2)}$$

$$\begin{aligned}\mathcal{L}_{S.G.} \;=\; & -\frac{1}{2}e\mathcal{R} \;-\; \frac{1}{3}eM^*M \;+\; \frac{1}{3}eb^ab_a \\ & +\; \frac{1}{2}e\epsilon^{k\ell mn}(\bar\psi_k\bar\sigma_\ell\tilde{\mathcal{D}}_m\psi_n \;-\; \psi_k\sigma_\ell\tilde{\mathcal{D}}_m\bar\psi_n)\,.\end{aligned} \qquad \mathbf{(21.3)}$$

$$\begin{aligned}\mathcal{L} \;=\; \int d^2\Theta\,2\mathcal{E}\,\Big[& -3\,R \;-\; \frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\Phi_i{}^+\Phi_i \\ & -\; \frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\,[\,c_i\Phi_i + \bar c_i\Phi_i{}^+\,] \;+\; d \;+\; a_i\Phi_i \\ & +\; \frac{1}{2}m_{ij}\Phi_i\Phi_j \;+\; \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\Big] \;+\; \text{h.c.}\end{aligned} \qquad \mathbf{(21.6)}$$

$$\Phi_i \;=\; A_i \;+\; \sqrt{2}\Theta\chi_i \;+\; \Theta\Theta F_i$$

$$2\mathcal{E} \;=\; e\{1 \;+\; i\Theta\sigma^a\bar\psi_a \;-\; \Theta\Theta[M^* \;+\; \bar\psi_a\bar\sigma^{ab}\bar\psi_b]\}$$

$$R \;=\; -\frac{1}{6}\Big\{M \;+\; \Theta[\sigma^a\bar\sigma^b\psi_{ab} \;-\; i\sigma^a\bar\psi_aM \;+\; i\psi_ab^a]$$

$$
\begin{aligned}
&+ \ \Theta\Theta\Bigg[-\frac{1}{2}\mathcal{R} \ + \ i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} \ + \ \frac{2}{3}MM^* \\
&+ \ \frac{1}{3}b^a b_a \ - \ ie_a{}^m\mathcal{D}_m b^a \ + \ \frac{1}{2}\bar{\psi}\bar{\psi}M \ - \ \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c \\
&+ \ \frac{1}{8}\epsilon^{abcd}[\bar{\psi}_a\bar{\sigma}_b\psi_{cd} \ + \ \psi_a\sigma_b\bar{\psi}_{cd}]\Bigg]\Bigg\} \, .
\end{aligned}
\tag{21.8}
$$

$$
\begin{aligned}
\Xi_i \ = \ & -4F_i^* \ + \ \frac{4}{3}MA_i^* \\
&+ \ \Theta\Bigg\{ -4i\sqrt{2}\sigma^c\hat{D}_c\bar{\chi}_i \ - \ \frac{2}{3}\sqrt{2}\sigma^a b_a\bar{\chi}_i \\
&+ \ \frac{4}{3}A_i^*(2\sigma^{ab}\psi_{ab} \ - \ i\sigma^a\bar{\psi}_a M \ + \ i\psi_a b^a)\Bigg\} \\
&+ \ \Theta\Theta\Bigg\{ -4e_a{}^m\mathcal{D}_m\hat{D}^a A_i^* \ - \ \frac{8}{3}ib_a\hat{D}^a A_i^* \\
&- \ \frac{2}{3}\sqrt{2}\bar{\psi}_{ab}\bar{\sigma}^{ab}\bar{\chi}_i \ + \ 2\sqrt{2}\bar{\psi}_a\hat{D}^a\bar{\chi}_i \ - \ \frac{8}{3}M^*F_i^* \\
&- \ \frac{2}{3}i\sqrt{2}\bar{\psi}_a\bar{\chi}_i b^a \ + \ \frac{1}{3}i\sqrt{2}\bar{\psi}_a\bar{\sigma}^a\sigma^c\bar{\chi}_i b_c \\
&+ \ \frac{4}{3}A_i^*\Bigg[-\frac{1}{2}\mathcal{R} \ + \ i\bar{\psi}^a\bar{\sigma}^b\psi_{ab} \ - \ ie_a{}^m\mathcal{D}_m b^a \\
&+ \ \frac{2}{3}M^*M \ + \ \frac{1}{3}b_a b^a \ + \ \frac{1}{2}\bar{\psi}\bar{\psi}M \ - \ \frac{1}{2}\psi_a\sigma^a\bar{\psi}_c b^c \\
&+ \ \frac{1}{8}\epsilon^{abcd}(\bar{\psi}_a\bar{\sigma}_b\psi_{cd} \ + \ \psi_a\sigma_b\bar{\psi}_{cd})\Bigg]\Bigg\} \, .
\end{aligned}
\tag{21.10}
$$

$$
\begin{aligned}
\hat{D}_a A_i^* \ &= \ e_a{}^m\partial_m A_i^* \ - \ \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\kappa}}\bar{\chi}_i^{\dot{\kappa}} \\
\hat{D}_a\bar{\chi}_{i\dot{\alpha}} \ &= \ e_a{}^m\mathcal{D}_m\bar{\chi}_{i\dot{\alpha}} \ + \ \frac{i}{2}\sqrt{2}\psi_a{}^\beta\sigma_{\beta\dot{\alpha}}{}^b\hat{D}_b A_i^* \\
&\quad - \ \frac{1}{2}\sqrt{2}\bar{\psi}_{a\dot{\alpha}}F_i^* \, .
\end{aligned}
\tag{21.11}
$$

$$
\begin{aligned}
\mathcal{L} \ = \ & -\frac{1}{2}e\mathcal{R} \ - \ eK_{ij^*}\partial_m A_i\partial^m A_j^* \\
&- \ \frac{i}{2}eK_{ij^*}[\chi_i\sigma^m\mathcal{D}_m\bar{\chi}_j \ + \ \bar{\chi}_j\bar{\sigma}^m\mathcal{D}_m\chi_i] \\
&+ \ \frac{1}{2}e\epsilon^{k\ell mn}[\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathcal{D}}_m\psi_n \ - \ \psi_k\sigma_\ell\tilde{\mathcal{D}}_m\bar{\psi}_n] \\
&+ \ \frac{1}{4}e\epsilon^{k\ell mn}(K_i\partial_k A_i \ - \ K_{i^*}\partial_k A_i^*)\psi_\ell\sigma_m\bar{\psi}_n \\
&- \ \frac{i}{4}e[K_{ij^*}(K_k\partial_m A_k \ - \ K_{k^*}\partial_m A_k^*) \\
&- \ 2(K_{ij^*k}\partial_m A_k \ - \ K_{ij^*k^*}\partial_m A_k^*)]\chi_i\sigma^m\bar{\chi}_j \\
&- \ \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_j^*\chi_i\sigma^m\bar{\sigma}^n\psi_m \\
&- \ \frac{1}{2}\sqrt{2}eK_{ij^*}\partial_n A_i\bar{\chi}_j\bar{\sigma}^m\sigma^n\bar{\psi}_m
\end{aligned}
$$

$$+\ \frac{1}{4}\, e\, K_{ij^*}\, [i\epsilon^{k\ell mn}\psi_k\sigma_\ell\bar{\psi}_m \ +\ \psi_m\sigma^n\bar{\psi}^m]\,\chi_i\sigma_n\bar{\chi}_j$$

$$+\ \frac{1}{16}\, e\, [K_{ij^*}K_{k\ell^*} \ -\ 2K_{ij^*k\ell^*} \ +\ 2K^{rs^*}K_{iks^*}K_{j^*\ell^*r}]$$

$$\times\ \chi_i\sigma_m\bar{\chi}_j\,\chi_k\sigma^m\bar{\chi}_\ell$$

$$+\ e\,\exp(K/2)\Big\{ -\ P^*\,\psi_a\sigma^{ab}\psi_b \ -\ P\,\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b$$

$$-\ \frac{i}{2}\sqrt{2}\,D_iP\,\chi_i\sigma^a\bar{\psi}_a - \frac{i}{2}\sqrt{2}\,D_{i^*}P^*\,\bar{\chi}_i\bar{\sigma}^a\psi_a$$

$$-\ \frac{1}{2}[P_{ij} \ +\ K_{ij}\,P \ +\ K_i\,D_jP \ +\ K_j\,D_iP$$

$$-\ K_iK_j\,P \ -\ K^{k\ell^*}K_{ij\ell^*}D_kP]\,\chi_i\chi_j \ -\ \frac{1}{2}[P^*_{i^*j^*}$$

$$+\ K_{i^*j^*}\,P^* \ +\ K_{i^*}\,D_{j^*}P^* \ +\ K_{j^*}\,D_{i^*}P^* \ -\ K_{i^*}K_{j^*}\,P^*$$

$$-\ K^{k^*\ell}K_{i^*j^*\ell}D_{k^*}P^*]\,\bar{\chi}_i\bar{\chi}_j\Big\}$$

$$-\ e\,\exp(K)\,[K^{ij^*}(D_iP)(D_jP)^* \ -\ 3P^*P]\,. \qquad \textbf{(21.21)}$$

$$\mathcal{V}(A_i, A_j^*) \ =\ \exp(K)\,[K^{ij^*}(D_iP)(D_jP)^* \ -\ 3P^*P]\,. \qquad \textbf{(21.22)}$$

Exercises

(1) Verify as many of the following relations as you wish:

(a) $\Phi^+| \ =\ A^*$

(b) $\mathcal{D}_\beta\Phi^+| \ =\ 0$

(c) $\bar{\mathcal{D}}_{\dot\alpha}\Phi^+| \ =\ \sqrt{2}\,\bar{\chi}_{\dot\alpha}$

(d) $\mathcal{D}_\alpha\mathcal{D}_\beta\Phi^+| \ =\ 0$

(e) $\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}_{\dot\beta}\Phi^+| \ =\ 2\epsilon_{\dot\alpha\dot\beta}F^*$

(f) $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\beta\Phi^+| \ =\ 0$

(g) $\mathcal{D}_\beta\bar{\mathcal{D}}_{\dot\alpha}\Phi^+| \ =\ -2i\sigma_{\beta\dot\alpha}{}^a\hat{D}_aA^*$

(h) $\mathcal{D}_\alpha\mathcal{D}_\beta\mathcal{D}_\gamma\Phi^+| \ =\ 0$

(i) $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\mathcal{D}_\beta\Phi^+| \ =\ 0$

(j) $\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\beta\Phi^+| \ =\ 0$

(k) $\mathcal{D}_\alpha\mathcal{D}_\beta\bar{\mathcal{D}}_{\dot\alpha}\Phi^+| \ =\ -\frac{2}{3}\sqrt{2}\,\epsilon_{\alpha\beta}\bar{\chi}_{\dot\alpha}\,M^*$

(l) $\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha\Phi^+| = 0$

(m)
$$\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\beta}\Phi^+| = -i2\sqrt{2}\,\sigma_{\alpha\dot\beta}{}^c\hat{D}_c\bar\chi_{\dot\alpha} + \frac{1}{6}\sqrt{2}\,(\bar\chi_{\dot\beta}b_{\alpha\dot\alpha} - 3\bar\chi_{\dot\alpha}b_{\alpha\dot\beta} - 3\epsilon_{\dot\alpha\dot\beta}b_{\alpha\dot\gamma}\bar\chi^{\dot\gamma})$$

(n)
$$\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}_{\dot\beta}\Phi^+| = i2\sqrt{2}\,\epsilon_{\dot\alpha\dot\beta}\sigma_{\alpha\dot\gamma}{}^c\hat{D}_c\bar\chi^{\dot\gamma} + \frac{1}{3}\sqrt{2}\,\epsilon_{\dot\alpha\dot\beta}b_{\alpha\dot\gamma}\bar\chi^{\dot\gamma}$$

(o) $\mathcal{D}_a\Phi^+| = \hat{D}_aA^* = e_a{}^m\mathcal{D}_mA^* - \frac{1}{\sqrt{2}}\bar\psi_{a\dot\alpha}\bar\chi^{\dot\alpha}$

(p) $\mathcal{D}_a\mathcal{D}_\beta\Phi^+| = 0$

(q) $\mathcal{D}_\alpha\mathcal{D}_c\Phi^+| = -\frac{i}{6}\sqrt{2}\,\sigma_{c\alpha\dot\beta}\bar\chi^{\dot\beta}M^*$

(r)
$$\mathcal{D}_c\bar{\mathcal{D}}_{\dot\alpha}\Phi^+| = \sqrt{2}\hat{D}_c\bar\chi_{\dot\alpha} = \sqrt{2}\left(e_c{}^m\mathcal{D}_m\bar\chi_{\dot\alpha} + \frac{i}{\sqrt{2}}\psi_c{}^\beta\sigma_{\beta\dot\alpha}{}^a\hat{D}_aA^* - \frac{1}{\sqrt{2}}\bar\psi_{c\dot\alpha}F^*\right)$$

(s)
$$\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_c\Phi^+| = \sqrt{2}\left\{\hat{D}_c\bar\chi_{\dot\alpha} - \frac{i}{24}\bar\sigma_c{}^{\dot\beta\beta}(\bar\chi_{\dot\beta}b_{\beta\dot\alpha} - 3\bar\chi_{\dot\alpha}b_{\beta\dot\beta} - 3\epsilon_{\dot\alpha\dot\beta}b_{\beta\dot\kappa}\bar\chi^{\dot\kappa})\right\}$$

(t)
$$\mathcal{D}_a\mathcal{D}_b\Phi^+| = e_a{}^m\mathcal{D}_m\hat{D}_bA^* + \frac{i}{12}\sqrt{2}\,\psi_a{}^\alpha\sigma_{b\alpha\dot\beta}\bar\chi^{\dot\beta}M^* + \frac{1}{2}\sqrt{2}\,\bar\psi_a{}^{\dot\alpha}\left\{\hat{D}_b\bar\chi_{\dot\alpha} - \frac{i}{24}\bar\sigma_b{}^{\dot\beta\beta}(\bar\chi_{\dot\beta}b_{\beta\dot\alpha} - 3\bar\chi_{\dot\alpha}b_{\beta\dot\beta} - 3\epsilon_{\dot\alpha\dot\beta}b_{\beta\dot\kappa}\bar\chi^{\dot\kappa})\right\}$$

(u) $\mathcal{D}_c\mathcal{D}_\alpha\mathcal{D}_\beta\Phi^+| = 0$

(v) $\mathcal{D}_\alpha\mathcal{D}_c\mathcal{D}_\beta\Phi^+| = 0$

(w)
$$\mathcal{D}_\alpha\mathcal{D}_\beta\mathcal{D}_c\Phi^+| = i\sqrt{2}\,\sigma_{c\beta\dot\beta}\bar\chi^{\dot\beta}\mathcal{D}_\alpha R^+| - \frac{1}{3}(\sigma^a\bar\sigma^c\epsilon)_{\alpha\beta}M^*\hat{D}_aA^*$$

(x)
$$\mathcal{D}^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\Phi^+| = 16e_a{}^m\mathcal{D}_m\hat{D}_aA^* + \frac{32}{3}ib^a\hat{D}_aA^* - 8\sqrt{2}\,\bar\psi_a\hat{D}^a\bar\chi + \frac{32}{3}M^*F^* + \frac{8}{3}\sqrt{2}\,\bar\psi_{mn}\bar\sigma^{mn}\bar\chi + \frac{8}{3}i\sqrt{2}\,\bar\psi_a\bar\chi\,b^a - \frac{4}{3}i\sqrt{2}\,\bar\psi_a\bar\sigma^a\sigma^c\bar\chi b_c\,.$$

(2) Use the above relations to derive (21.10).

(3) Verify that $\mathcal{L}_{kin}$ and $\mathcal{L}_{aux}$ are given by (21.13) and (21.14), respectively.

(4) Show that the Weyl rescaling (21.17) takes

$$\frac{1}{6}\,e\,\Omega\mathcal{R} \;\rightarrow\; -\frac{1}{2}\,e\,\mathcal{R} \;-\; \frac{3}{4}\,e\,\Omega^{-2}\,\partial_m\Omega\,\partial^m\Omega \;+\; \frac{3}{2}\,\partial_m\left[\,e g^{mn}\,\Omega^{-1}\partial_n\Omega\,\right].$$

(5) Use the result of Exercise 4 to show that (21.17) - (21.20) restore the proper kinetic energies in (21.21).

(6) Use (21.14) and (21.17) - (21.20) to check that the potential (21.22) is indeed correct.

(7) Show that for $\Omega(A, A^*) = A_i{}^* A_i - 3$, the field redefinition

$$A_1' \;=\; \frac{A_1 + a}{1 + \frac{1}{3}a^* A_1}\,, \quad A_i' \;=\; \frac{A_i\sqrt{1 - \frac{1}{3}|a|^2}}{1 + \frac{1}{3}a^* A_1} \quad (i \neq 1)$$

induces the Kähler transformation

$$K(A', A'^*) \;=\; K(A, A^*) \;-\; 3\log\frac{1 - \frac{1}{3}|a|^2}{|1 + \frac{1}{3}a^* A_1|^2}\,.$$

As discussed in Appendix C, this is an isometry transformation – it leaves invariant the Kähler geometry.

Note that after such a transformation, the supergravity potential $\mathcal{V}$ can again be related to a superpotential P of third order. This feature characterizes the minimal chiral supergravity model.

(8) Let $\Omega = A^*A - 3$ and $P = \mu(1 + \frac{1}{3}\sqrt{3}A)^3$. Show that the potential $\mathcal{V}$ vanishes. Since the potential does not determine the expectation value $\langle A\rangle$, the field A is known as a "sliding singlet." Show that the gravitino mass slides as well,

$$m_{3/2} \;=\; \mu\left(\frac{1 + a}{\sqrt{1 - |a|^2}}\right)^3$$

where $\langle A\rangle = \sqrt{3}a$.

XXII. CHIRAL MODELS AND KAHLER GEOMETRY

In the previous chapter, we constructed the minimal coupling of chiral superfields to supergravity. We found that the resulting Lagrangian could be written in terms of a Kähler potential and its derivatives. In what follows, we will begin to explore the relation between matter couplings and Kähler geometry. We will work in flat space, where the connection first appears, leaving the curved-space generalization until Chapter XXIII. A brief introduction to Kähler geometry is given in Appendix C.

We start by studying the most general Lagrangian that can be built from chiral superfields Φ^i, for $i = 1, ..., n$. This Lagrangian takes a very simple form,

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar\theta\, K(\Phi^i, \Phi^{+j}) \;+\; \left[\int d^2\theta\, P(\Phi^i) + \text{ h.c.}\right] . \tag{22.1}$$

Here K and P are superfields, with power series expansions in terms of the scalar superfields Φ^i,

$$\begin{aligned} K(\Phi, \Phi^+) \;&=\; \sum c_{i_1 \ldots i_N, j_1 \ldots j_M}\, \Phi^{i_1} \ldots \Phi^{i_N} \Phi^{+j_1} \ldots \Phi^{+j_M} \\ P(\Phi) \;&=\; \sum g_{i_1 \ldots i_N}\, \Phi^{i_1} \ldots \Phi^{i_N} . \end{aligned} \tag{22.2}$$

To find the component Lagrangian, we must expand K and P in terms of the θ variables. The expansions of the individual fields were given in equations (5.3) and (5.5). For the superpotential P, equations (5.7) and (5.8) immediately extend to

$$\begin{aligned} P(\Phi) \;&=\; P(A) \;+\; \sqrt{2}\,\theta\chi^i\, \frac{\partial P(A)}{\partial A^i} \\ &\quad +\; \theta\theta \left\{ F^i\, \frac{\partial P(A)}{\partial A^i} - \frac{1}{2}\chi^i\chi^j\, \frac{\partial^2 P(A)}{\partial A^i \partial A^j} \right\} . \end{aligned} \tag{22.3}$$

Here all component fields are functions of $y^m = x^m + i\theta\sigma^m\bar\theta$. The conjugate superpotential P^+ has an analogous expansion,

$$\begin{aligned} P^+(\Phi^+) \;&=\; P^*(A^*) \;+\; \sqrt{2}\,\bar\theta\bar\chi^i\, \frac{\partial P^*(A^*)}{\partial A^{*i}} \\ &\quad +\; \bar\theta\bar\theta \left\{ F^{*i}\, \frac{\partial P^*(A^*)}{\partial A^{*i}} \;-\; \frac{1}{2}\bar\chi^i\bar\chi^j\, \frac{\partial^2 P^*(A^*)}{\partial A^{*i} \partial A^{*j}} \right\} , \end{aligned} \tag{22.4}$$

where the fields now depend on y^+.

The θ expansion of $K(\Phi, \Phi^+)$ can be computed starting from the monomial

$$K_{NM} \;=\; \Phi^{i_1} \ldots \Phi^{i_N} \Phi^{+j_1} \ldots \Phi^{+j_M} \;. \qquad (22.5)$$

Its $\theta\theta\bar\theta\bar\theta$-component may be found with the help of (5.9) and an appropriate interpretation of (22.3) and (22.4),

$$\begin{aligned} K_{NM} \;=&\; \ldots \\ &+ \theta\theta\bar\theta\bar\theta \left\{ \left[F^k \frac{\partial}{\partial A^k} (A^{i_1} \ldots A^{i_N}) \;-\; \frac{1}{2} \chi^k \chi^\ell \frac{\partial^2 (A^{i_1} \ldots A^{i_N})}{\partial A^k \partial A^\ell} \right] \right. \\ &\times \left[F^{*k} \frac{\partial}{\partial A^{*k}} (A^{*j_1} \ldots A^{*j_M}) \;-\; \frac{1}{2} \bar\chi^k \bar\chi^\ell \frac{\partial^2 (A^{*j_1} \ldots A^{*j_M})}{\partial A^{*k} \partial A^{*\ell}} \right] \\ &- \partial_m (A^{i_1} \ldots A^{i_N}) \, \partial^m (A^{*j_1} \ldots A^{*j_M}) \\ &\left. - \, i \frac{\partial}{\partial A^{*k}} \left(A^{*j_1} \ldots A^{*j_M} \right) \bar\chi^k \bar\sigma^m \partial_m \left(\chi^\ell \frac{\partial (A^{i_1} \ldots A^{i_N})}{\partial A^\ell} \right) \right\} , \end{aligned} \qquad (22.6)$$

where we have used partial integration. This result can be rewritten more elegantly in terms of derivatives of $K_{NM}|$, the lowest component of the superfield (22.5),

$$\begin{aligned} K_{NM} \;=&\; \ldots \;+\; \theta\theta\bar\theta\bar\theta \left\{ \frac{\partial^2 K_{NM}|}{\partial A^i \partial A^{*j}} F^i F^{*j} \right. \\ &- \frac{1}{2} \frac{\partial^3 K_{NM}|}{\partial A^i \partial A^{*j} \partial A^{*k}} F^i \bar\chi^j \bar\chi^k \;-\; \frac{1}{2} \frac{\partial^3 K_{NM}|}{\partial A^{*i} \partial A^j \partial A^k} F^{*i} \chi^j \chi^k \\ &+ \frac{1}{4} \frac{\partial^4 K_{NM}|}{\partial A^i \partial A^j \partial A^{*k} \partial A^{*\ell}} \chi^i \chi^j \bar\chi^k \bar\chi^\ell \\ &- \frac{\partial^2 K_{NM}|}{\partial A^i \partial A^{*j}} \partial_m A^i \partial^m A^{*j} \;-\; i \frac{\partial^2 K_{NM}|}{\partial A^i \partial A^{*j}} \bar\chi^j \bar\sigma^m \partial_m \chi^i \\ &\left. - \, i \frac{\partial^3 K_{NM}|}{\partial A^i \partial A^j \partial A^{*k}} \bar\chi^k \bar\sigma^m \chi^i \partial_m A^j \right\} . \end{aligned} \qquad (22.7)$$

Equation (22.7) is also true for the full polynomial K. The expression simplifies in the notation of a Kähler manifold, where

$$\begin{aligned} g_{ij*} &= \frac{\partial}{\partial A^i} \frac{\partial}{\partial A^{*j}} K| \\ g_{ij*,k} &= \frac{\partial}{\partial A^k} g_{ij*} \;=\; g_{mj*} \Gamma^m_{ik} \\ g_{ij*,k*} &= \frac{\partial}{\partial A^{*k}} g_{ij*} \;=\; g_{im*} \Gamma^{m*}_{j*k*} \;. \end{aligned} \qquad (22.8)$$

One finds

$$\begin{aligned} K \;=&\; \ldots \;+\; \theta\theta\bar\theta\bar\theta \left\{ g_{ij*} F^i F^{*j} \;-\; \frac{1}{2} g_{im*} \Gamma^{m*}_{j*k*} F^i \bar\chi^j \bar\chi^k \right. \\ &- \frac{1}{2} g_{mi*} \Gamma^m_{jk} F^{*i} \chi^j \chi^k \;+\; \frac{1}{4} g_{ij*,k\ell*} \chi^i \chi^k \bar\chi^j \bar\chi^\ell \\ &- g_{ij*} \partial_m A^i \partial^m A^{*j} \;-\; i \, g_{ij*} \bar\chi^j \bar\sigma^m \partial_m \chi^i \\ &\left. - \, i \, g_{mk*} \Gamma^m_{ij} \bar\chi^k \bar\sigma^m \chi^i \partial_m A^j \right\} . \end{aligned} \qquad (22.9)$$

We now have all we need to write the full Lagrangian in terms of component fields. Substituting (22.3) and (22.9) into (22.1), we find

$$\begin{aligned}\mathcal{L} \;=\;& g_{ij*}\, F^i F^{*j} \;+\; \frac{1}{4}\, g_{ij*,k\ell*}\, \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\ &-\; F^i \left\{ \frac{1}{2}\, g_{im*}\, \Gamma^{m*}_{j*k*}\, \bar{\chi}^j \bar{\chi}^k \;-\; \frac{\partial P}{\partial A^i} \right\} \\ &-\; F^{*i} \left\{ \frac{1}{2}\, g_{mi*}\, \Gamma^m_{jk}\, \chi^j \chi^k \;-\; \frac{\partial P^*}{\partial A^{*i}} \right\} \\ &-\; g_{ij*}\, \partial_m A^i \partial^m A^{*j} \;-\; i g_{ij*}\, \bar{\chi}^j \bar{\sigma}^m D_m \chi^i \\ &-\; \frac{1}{2}\, \frac{\partial^2 P}{\partial A^i \partial A^j}\, \chi^i \chi^j \;-\; \frac{1}{2}\, \frac{\partial^2 P^*}{\partial A^{*i} \partial A^{*j}}\, \bar{\chi}^i \bar{\chi}^j \;.\end{aligned} \tag{22.10}$$

Here $D_m \chi^i = \partial_m \chi^i + \Gamma^i_{jk} \partial_m A^j \chi^k$ is a covariant spacetime derivative, assuming χ^i transforms like a contravariant vector under the transformations (C.1) on a Kähler manifold.

The auxiliary fields in this expression may be eliminated by their Euler equations,

$$g_{ij*}\, F^i \;-\; \frac{1}{2}\, g_{kj*}\, \Gamma^k_{m\ell}\, \chi^m \chi^\ell \;+\; \frac{\partial P^*}{\partial A^{*j}} \;=\; 0 \;. \tag{22.11}$$

Substituting into (22.10), we obtain the final form of the

component Lagrangian,

$$\begin{aligned}\mathcal{L} \;=\;& -\; g_{ij*}\, \partial_m A^i \partial^m A^{*j} \;-\; i g_{ij*}\, \bar{\chi}^j \bar{\sigma}^m D_m \chi^i \\ &+\; \frac{1}{4}\, R_{ij*k\ell*}\, \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\ &-\; \frac{1}{2}\, D_i D_j P\, \chi^i \chi^j \;-\; \frac{1}{2}\, D_{i*} D_{j*} P^*\, \bar{\chi}^i \bar{\chi}^j \\ &-\; g^{ij*}\, D_i P\, D_{j*} P^* \;,\end{aligned} \tag{22.12}$$

where

$$\begin{aligned} D_i P \;&=\; \frac{\partial}{\partial A^i}\, P \\ D_i D_j P \;&=\; \frac{\partial^2}{\partial A^i \partial A^j}\, P \;-\; \Gamma^k_{ij}\, \frac{\partial}{\partial A^k}\, P \;.\end{aligned} \tag{22.13}$$

Equation (22.12) describes the most general supersymmetric coupling of chiral multiplets. We have used Kähler notation to illustrate the geometrical nature of the result. Invariance under Kähler transformations is manifest.

Each term in the Lagrangian (22.12) has a natural interpretation in the language of Kähler geometry. The scalar fields should be thought of as the coordinates of a Kähler manifold, and the fermions as tensors in the tangent space. The Lagrangian (22.12) is a supersymmetric version of the sigma model, expressed in geometrical form.

In superspace notation, the appearance of the Kähler geometry can be traced to the invariance of the Lagrangian (22.1) under the superfield Kähler transformation:

$$K(\Phi,\Phi^+) \;\rightarrow\; K(\Phi,\Phi^+) \;+\; F(\Phi) \;+\; F^+(\Phi^+)\;. \qquad (22.14)$$

This invariance will play an important role in what follows.

References

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Equations

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar\theta\, K(\Phi^i,\Phi^{+j}) \;+\; \left[\int d^2\theta\, P(\Phi^i) + \text{ h.c.}\right]. \qquad \mathbf{(22.1)}$$

$$\begin{aligned}
\mathcal{L} \;=\; &-\; g_{ij*}\,\partial_m A^i \partial^m A^{*j} \;-\; i g_{ij*}\,\bar\chi^j \bar\sigma^m D_m \chi^i \\
&+\; \frac{1}{4} R_{ij*k\ell*}\,\chi^i\chi^k\bar\chi^j\bar\chi^\ell \\
&-\; \frac{1}{2} D_iD_jP\,\chi^i\chi^j \;-\; \frac{1}{2} D_{i*}D_{j*}P^*\,\bar\chi^i\bar\chi^j \\
&-\; g^{ij*}\,D_iP\,D_{j*}P^*\,.
\end{aligned} \qquad \mathbf{(22.12)}$$

$$\begin{aligned}
D_iP \;&=\; \frac{\partial}{\partial A^i}\,P \\
D_iD_jP \;&=\; \frac{\partial^2}{\partial A^i \partial A^j}\,P \;-\; \Gamma^k_{ij}\,\frac{\partial}{\partial A^k}\,P\,.
\end{aligned} \qquad \mathbf{(22.13)}$$

Exercises

(1) Check that the Lagrangian (22.12) is invariant (up to a total derivative) under the following supersymmetry transformations:

$$\delta_\xi A^i \;=\; \sqrt{2}\,\xi\chi$$

$$\begin{aligned}
\delta_\xi \chi^i \;&=\; i\sqrt{2}\,\sigma^m\bar\xi\,\partial_m A^i \;-\; \Gamma^i_{jk}\,\delta_\xi A^j\,\chi^k \\
&\quad -\; \sqrt{2}\,g^{ij*}\,\frac{\partial P^*}{\partial A^{*j}}\,\xi\,.
\end{aligned}$$

(2) Let $K = A^{*i}A^i$ and $P = \lambda_i A^i + \frac{1}{2}\,m_{ij}A^iA^j + \frac{1}{3}g_{ijk}A^iA^jA^k$. Show that (22.12) reduces to the renormalizable Lagrangian given in (5.13).

XXIII. GENERAL CHIRAL SUPERGRAVITY MODELS

Having discussed the role of Kähler geometry in flat space, we will now compute the most general coupling of chiral superfields to supergravity. As in flat space, we will find that the component Lagrangian has a natural interpretation in the language of Kähler geometry.

Motivated by our discussion in Chapter XXI, we take our superspace Lagrangian to be

$$\mathcal{L} \;=\; \frac{1}{\kappa^2}\int d^2\Theta\, 2\mathcal{E}\left[\frac{3}{8}\,(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\,\exp\left\{-\frac{\kappa^2}{3}K(\Phi,\Phi^+)\right\} + \kappa^2\,P(\Phi)\right] + \text{h.c.}\,, \qquad \mathbf{(23.1)}$$

where $K(\Phi,\Phi^+)$ is a hermitian function of the superfields Φ^i and Φ^{+j}, and $P(\Phi)$ is the superpotential. The exponential form is suggested by the relation between K and Ω below (21.21). Expanding in κ^2, it is not hard to see that K is the flat-space Kähler potential,

$$\mathcal{L} \;=\; -\,\frac{6}{\kappa^2}\int d^2\Theta\;\mathcal{E}R \;+\; \int d^2\Theta\, 2\mathcal{E}\left[-\frac{1}{8}\,(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\,K(\Phi,\Phi^+) \;+\; P(\Phi)\right] + \ldots \;+\; \text{h.c.} \qquad (23.2)$$

In this chapter, we will see that K is a Kähler potential in curved space as well.

The Lagrangian (23.1) is manifestly invariant under supergravity transformations. Its component form can be found using the techniques introduced for the minimal case in Chapter XXI. The steps are virtually identical; there are just a few extra terms that follow from the general nature of K. At the end of the computation, one finds precisely equation (21.21), where K is now an arbitrary real function of the scalar fields A^i, the lowest component of the superfield $K(\Phi,\,\Phi^+)$.

Equation (21.21) gives the component Lagrangian in terms of K and its derivatives. It can be written more compactly if we use g_{ij*} and $R_{ij*k\ell*}$, the metric and curvature of a Kähler manifold. In this form the geometric invariance of the Lagrangian is manifest. Comparing (C.10), (C.18) and (21.21), we find

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}\,e\,\mathcal{R} \;-\; e\,g_{ij*}\,\partial_m A^i\,\partial^m A^{*j} \\ & -\; i\,e\,g_{ij*}\,\bar{\chi}^j\bar{\sigma}^m\mathcal{D}_m\chi^i \;+\; e\,\epsilon^{k\ell mn}\,\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathcal{D}}_m\psi_n \\ & -\; \frac{1}{2}\,\sqrt{2}\,e\,g_{ij*}\,\partial_n A^{*j}\,\chi^i\sigma^m\bar{\sigma}^n\psi_m\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\sqrt{2}\, e\, g_{ij*}\, \partial_n A^i\, \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
&+ \frac{1}{4} e\, g_{ij*}\, [i\epsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m \; + \; \psi_m \sigma^n \bar{\psi}^m]\, \chi^i \sigma_n \bar{\chi}^j \\
&- \frac{1}{8} e\, [g_{ij*} g_{k\ell*} \; - \; 2R_{ij*k\ell*}]\, \chi^i \chi^k\, \bar{\chi}^j \bar{\chi}^\ell \\
&- e\, \exp(K/2) \Big\{ P^*\, \psi_a \sigma^{ab} \psi_b \; + \; P\, \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
&+ \frac{i}{2}\sqrt{2}\, D_i P\, \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2}\sqrt{2}\, D_{i*} P^*\, \bar{\chi}^i \bar{\sigma}^a \psi_a \\
&+ \frac{1}{2} \mathcal{D}_i D_j P\, \chi^i \chi^j \; + \; \frac{1}{2} \mathcal{D}_{i*} D_{j*} P^*\, \bar{\chi}^i \bar{\chi}^j \Big\} \\
&- e\, e^K\, [\, g^{ij*} (D_i P)(D_j P)^* \; - \; 3P^* P\,]\,. \qquad \textbf{(23.3)}
\end{aligned}
$$

The covariant derivatives are defined as follows:

$$
\begin{aligned}
\mathcal{D}_m \chi^i &= \partial_m \chi^i \; + \; \chi^i \omega_m \; + \; \Gamma^i_{jk}\, \partial_m A^j\, \chi^k \\
&\quad - \frac{1}{4} (K_j \partial_m A^j - K_{j*} \partial_m A^{*j})\, \chi^i \\
\tilde{\mathcal{D}}_m \psi_n &= \partial_m \psi_n \; + \; \psi_n\, \omega_m \\
&\quad + \frac{1}{4} (K_j \partial_m A^j - K_{j*} \partial_m A^{*j})\, \psi_n \\
D_i P &= P_i \; + \; K_i P \\
\mathcal{D}_i D_j P &= P_{ij} \; + \; K_{ij} P \; + \; K_i D_j P \; + \; K_j D_i P \\
&\quad - K_i K_j P \; - \; \Gamma^k_{ij} D_k P\,. \qquad \textbf{(23.4)}
\end{aligned}
$$

The covariant derivatives contain the Christoffel symbols for the Kähler geometry, and the spin connection (17.12) for spacetime. Note that they also contain a U(1) connection proportional to $\mathrm{Im}(K_j \partial_m A^j)$. The meaning of the U(1) connection will become clear as we proceed.

The Lagrangian (23.3) is invariant under supergravity transformations because it was derived from a superspace formalism. It is useful, however, to verify the invariance directly, using the following supergravity transformations:

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta \sigma_a \bar{\psi}_m \; + \; \bar{\zeta} \bar{\sigma}_a \psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\, \zeta \chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2}\, \sigma_m \bar{\zeta}\, \hat{D}_m A^i \; - \; \Gamma^i_{jk}\, \delta_\zeta A^j \chi^k \\
&\quad + \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \chi^i \\
&\quad - \sqrt{2}\, e^{K/2}\, g^{ij*}\, D_{j*} P^*\, \zeta \\
\delta_\zeta \psi_m &= 2\mathcal{D}_m \zeta \; - \; \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \psi_m \\
&\quad - \frac{i}{2} \sigma_{mn} \zeta\; g_{ij*}\, \chi^i \sigma^n \bar{\chi}^j \\
&\quad + i\, e^{K/2}\, P\, \sigma_m \bar{\zeta}\,, \qquad \textbf{(23.5)}
\end{aligned}
$$

where $\mathcal{D}_m \zeta$ includes the U(1) connection,

$$
\mathcal{D}_m \zeta \; = \; \partial_m \zeta + \zeta\, \omega_m + \frac{1}{4} (K_j \partial_m A^j - K_{j*} \partial_m A^{*j})\, \zeta\,. \quad (23.6)
$$

Note that the transformation for the field χ^i indicates that supersymmetry is spontaneously broken whenever $\langle D_i P\rangle \neq 0$. In this case, χ^i shifts by a constant and assumes the role of the Goldstone fermion.

To check the Kähler invariance, let us first examine the component Lagrangian (23.3). Under a Kähler transformation,

$$K(A,A^*) \rightarrow K(A,A^*) + F(A) + F^*(A^*) , \qquad (23.7)$$

the metric, Christoffel symbols and curvature terms are all invariant. The U(1) connection is not:

$$\mathcal{D}_m\chi^i \rightarrow \mathcal{D}_m\chi^i - \frac{i}{2}\partial_m(\mathrm{Im}\,F)\,\chi^i$$

$$\tilde{\mathcal{D}}_m\psi_n \rightarrow \tilde{\mathcal{D}}_m\psi_n + \frac{i}{2}\partial_m(\mathrm{Im}\,F)\,\psi_n . \qquad (23.8)$$

The Kähler invariance is restored if Kähler transformations are accompanied by Weyl rotations of the spinor fields,

$$\chi^i \rightarrow \exp+\frac{i}{2}(\mathrm{Im}\,F)\;\chi^i$$

$$\psi_n \rightarrow \exp-\frac{i}{2}(\mathrm{Im}\,F)\;\psi_n . \qquad \mathbf{(23.9)}$$

With this rule, the kinetic terms in (23.3) are invariant under the combined Kähler-Weyl transformations. The Kähler-

Weyl invariance insures that the kinetic terms are invariant under field redefinitions (such as isometries) that induce Kähler transformations of the Kähler potential K.

The superpotential contributions to the component Lagrangian contain explicit factors of K, so their invariance is not automatic under such Kähler-Weyl transformations. For example, the scalar potential

$$\mathcal{V} = e^K\,[\,g^{ij*}(D_iP)(D_jP)^* - 3P^*P\,] \qquad \mathbf{(23.10)}$$

is not invariant unless

$$P \rightarrow e^{-F}\,P \qquad (23.11)$$

as well. With this choice, the D_iP transform covariantly,

$$D_iP \rightarrow e^{-F}\,D_iP , \qquad (23.12)$$

and the full Lagrangian is invariant. Note that (23.11) does not in general preserve a polynomial structure in the superpotential (see Exercise 21.7).

In mathematical language, the transformations (23.9) and (23.11) imply that the spinors and the superpotential are not ordinary functions, but rather sections of appropriate line bundles over the Kähler manifold. If the manifold is nontrivial, the combined Kähler-Weyl invariance is necessary for the Lagrangian to be globally well-defined. Locally, however, we can simply think of the geometrical notation as giving a convenient shorthand that is useful for describing the full set of supergravity couplings.

The Kähler-Weyl invariance of the matter couplings can also be seen from the superspace Lagrangian (23.1). Now, however, the superfield Kähler transformation

$$K(\Phi,\Phi^{+}) \;\rightarrow\; K(\Phi,\Phi^{+}) \;+\; F(\Phi) \;+\; F^{+}(\Phi^{+}) \qquad (23.13)$$

must be accompanied by a super-Weyl transformation of the vielbein.

A super-Weyl transformation is defined to be a superfield rescaling of the vielbein, consistent with the torsion constraints (14.25). In the exercises, it is shown that the most general such transformation is of the form

$$\begin{aligned} \delta E_M{}^a &= (\Sigma + \bar\Sigma)\, E_M{}^a \\ \delta E_M{}^\alpha &= (2\bar\Sigma - \Sigma)\, E_M{}^\alpha \;+\; \frac{i}{2}\, E_M{}^b\, (\epsilon\sigma_b)^\alpha{}_{\dot\alpha}\, \bar{\mathcal{D}}^{\dot\alpha}\bar\Sigma\ , \end{aligned} \qquad (23.14)$$

where Σ and $\bar\Sigma$ are chiral superfields,

$$\mathcal{D}_\alpha \bar\Sigma \;=\; \bar{\mathcal{D}}_{\dot\alpha}\Sigma \;=\; 0\ . \qquad (23.15)$$

This implies

$$\begin{aligned} \delta R &= -\,2\,(2\Sigma - \bar\Sigma)\, R \;-\; \frac{1}{4}\,\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar\Sigma \\ \delta G_{\alpha\dot\alpha} &= -\,(\Sigma + \bar\Sigma)\, G_{\alpha\dot\alpha} \;+\; i\,\mathcal{D}_{\alpha\dot\alpha}\,(\bar\Sigma - \Sigma)\ . \end{aligned} \qquad (23.16)$$

The transformations (23.14) and (23.16) determine the super-Weyl transformations of the supergravity multiplet.

The super-Weyl transformations of the matter fields are also parametrized by Σ and $\bar\Sigma$. They are defined in such a way as to preserve the appropriate constraints. For example, a super-Weyl transformation of a chiral superfield is given by

$$\delta\Phi \;=\; w\,\Sigma\,\Phi\ , \qquad (23.17)$$

while that of a hermitian vector superfield is just

$$\delta V \;=\; w'\,(\Sigma + \bar\Sigma)\, V\ . \qquad (23.18)$$

In these expressions, w and w' are called the Weyl weights of the respective superfields.

Equations (23.14) - (23.18) allow one to find the super-Weyl scalings of the various component fields. These, in turn, can be written as variations of superfields in the new

Θ variables. After a small computation, one finds

$$\begin{aligned} \delta\mathcal{E} &= 6\Sigma\,\mathcal{E} + \frac{\partial}{\partial\Theta^\alpha}(S^\alpha\mathcal{E}) \\ \delta\Phi &= w\Sigma\,\Phi - S^\alpha\frac{\partial}{\partial\Theta^\alpha}\Phi \\ \delta(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)U &= (\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)\,[\,(w'-4)\Sigma + (w'+2)\bar{\Sigma}\,]\,U \\ &\quad - S^\alpha\frac{\partial}{\partial\Theta^\alpha}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)U\,, \end{aligned} \tag{23.19}$$

where

$$S^\alpha = \Theta^\alpha(2\bar{\Sigma}-\Sigma)| + \Theta\Theta\,\mathcal{D}^\alpha\Sigma|\,, \tag{23.20}$$

and U is an arbitrary hermitian superfield of weight w'.

The transformations (23.19) induce a variation of the superspace Lagrangian. For $w=0$, we find

$$\delta\mathcal{L} = \int d^2\Theta\,2\mathcal{E}\left[\frac{3}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)(\Sigma+\bar{\Sigma})e^{-K/3} + 6\Sigma P\right] + \text{h.c.} \tag{23.21}$$

This is precisely a Kähler transformation,

$$\delta\mathcal{L} = \int d^2\Theta\,2\mathcal{E}\left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R)(F+F^*)e^{-K/3} - FP\right] + \text{h.c.}\,, \tag{23.22}$$

where P is scaled to $e^{-F}P$ in accord with (23.11). Comparing the two transformations, we see that (23.21) cancels (23.22) if $F = 6\Sigma$. With this choice, the superspace Lagrangian is invariant under combined Kähler-Weyl transformations. It is a useful exercise to show that superspace Kähler-Weyl transformations induce local Weyl rotations (23.9) of the component fields.

An arbitrary super-Weyl transformation can be used to change the form of the Lagrangian (23.1). In particular, a super-Weyl transformation with a finite parameter $-6\Sigma = \log P$ simplifies that component expressions by rescaling the superpotential to one. Of course, this is an allowed transformation only if the expectation value $\langle P\rangle$ is nonzero. This is not an innocent assumption: it gives a nonvanishing contribution to the cosmological constant. This contribution can be cancelled only if supersymmetry is spontaneously broken.

The transformation with $-6\Sigma = \log P$ changes the Kähler potential K to a new potential $G = K + \log P + \log P^+$. Since this is a Kähler transformation, the geometry is left invariant. Therefore, to find the new Lagrangian, we simply replace P by 1, K by G, D_iP by G_i, and $\mathcal{D}_iD_jP$ by $G_{ij} + G_iG_j - \Gamma^k_{ij}G_k$. This gives

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}e\,\mathcal{R} \;-\; e\,g_{ij*}\,\partial_m A^i\,\partial^m A^{*j} \\
& - i\,e\,g_{ij*}\,\bar{\chi}^j\bar{\sigma}^m\mathcal{D}_m\chi^i \;+\; e\,\epsilon^{k\ell mn}\,\bar{\psi}_k\bar{\sigma}_\ell\tilde{\mathcal{D}}_m\psi_n \\
& - \frac{1}{2}\sqrt{2}\,e\,g_{ij*}\,\partial_n A^{*j}\,\chi^i\sigma^m\bar{\sigma}^n\psi_m \\
& - \frac{1}{2}\sqrt{2}\,e\,g_{ij*}\,\partial_n A^i\,\bar{\chi}^j\bar{\sigma}^m\sigma^n\bar{\psi}_m \\
& + \frac{1}{4}e\,g_{ij*}\,[i\epsilon^{k\ell mn}\psi_k\sigma_\ell\bar{\psi}_m \;+\; \psi_m\sigma^n\bar{\psi}^m]\,\chi^i\sigma_n\bar{\chi}^j \\
& - \frac{1}{8}e\,[g_{ij*}g_{k\ell*} \;-\; 2R_{ij*k\ell*}]\,\chi^i\chi^k\,\bar{\chi}^j\bar{\chi}^\ell \\
& - e\,e^{G/2}\Big\{\psi_a\sigma^{ab}\psi_b \;+\; \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b \\
& + \frac{i}{2}\sqrt{2}\,G_i\,\chi^i\sigma^a\bar{\psi}_a + \frac{i}{2}\sqrt{2}\,G_{i*}\,\bar{\chi}^i\bar{\sigma}^a\psi_a \\
& + \frac{1}{2}[G_{ij} \;+\; G_iG_j \;-\; \Gamma^k_{ij}G_k]\,\chi^i\chi^j \\
& + \frac{1}{2}[G^*_{i*j*} \;+\; G_{i*}G_{j*} \;-\; \Gamma^{k*}_{i*j*}G_{k*}]\,\bar{\chi}^i\bar{\chi}^j\Big\} \\
& - e\,e^G\,[\,g^{ij*}G_iG_{j*} \;-\; 3\,]\;,
\end{aligned}
\tag{23.23}
$$

Let us now examine the physical content of (23.23). We first note that the kinetic terms are properly normalized if $g_{ij*} = \delta_{ij*} + \ldots$. We shall always assume this to be true. We

then remark that the potential

$$
\mathcal{V} \;=\; e^G\,[\,g^{ij*}G_iG_{j*} \;-\; 3\,]\;, \tag{23.24}
$$

is extremized if $\langle \partial\mathcal{V}/\partial A^i\rangle = 0$, and that the resulting cosmological constant is zero if $\langle\mathcal{V}\rangle = 0$. Taken together, these conditions are satisfied if

$$
\begin{aligned}
\langle\, G^iG_i\,\rangle \;&=\; 3 \\
\langle\, G^i\nabla_kG_i \;+\; G_k\,\rangle \;&=\; 0\;,
\end{aligned}
\tag{23.25}
$$

where $G^i = g^{ij*}G_{j*}$ and $\nabla_kG_i = \partial_kG_i - \Gamma^j_{ki}G_j$.

The scalar mass matrix is found from the second variation of $\mathcal{V}$. It is of the form

$$
\begin{pmatrix} M^2_{ij*} & M^2_{ij} \\ M^2_{i*j*} & M^2_{i*j} \end{pmatrix} \tag{23.26}
$$

with

$$
\begin{aligned}
M^2_{ij*} \;&=\; \langle\,[\nabla_iG_k\nabla_{j*}G^k \;-\; R_{ij*k\ell*}\,G^kG^{\ell*} \;+\; g_{ij*}]\,e^G\,\rangle \\
M^2_{ij} \;&=\; \langle\,[\nabla_iG_j \;+\; \nabla_jG_i]\,e^G\,\rangle
\end{aligned}
\tag{23.27}
$$

where we have repeatedly used (23.25).

The spinor mass matrix can be found from (23.23) as well. Focussing on the quadratic terms, we find

$$
\begin{aligned}
& - \langle e^{G/2} \rangle \Big\{ \psi_a \sigma^{ab} \psi_b \ + \ \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
& + \frac{i}{2} \sqrt{2} \langle G_i \rangle \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} \langle G_{i*} \rangle \bar{\chi}^i \bar{\sigma}^a \psi_a \\
& + \frac{1}{2} \langle \nabla_i G_j \ + \ G_i G_j \rangle \chi^i \chi^j \\
& + \frac{1}{2} \langle \nabla_{i*} G_{j*} \ + \ G_{i*} G_{j*} \rangle \bar{\chi}^i \bar{\chi}^j \Big\} \ . \qquad (23.28)
\end{aligned}
$$

The mass of the gravitino is easily seen to be $\langle e^{G/2} \rangle$.

The masses of the spinors χ^i are a little more subtle because of the mixing between the gravitino $\sigma_n \bar{\psi}_n$ and the spinor $G_i \chi^i$. When $\langle G_i \rangle \neq 0$, this mixing must be removed in order to find the physical mass matrix. Of course, it is always possible to diagonalize the coupling by redefining the fields. It is more instructive, however, to recognize that the mixing has an important physical origin.

To see this, let us consider the supergravity transformation of the field $\eta = G_i \chi^i$, given by (23.5) with the appropriate substitutions,

$$
\begin{aligned}
\delta \eta \ &= \ -\sqrt{2}\, e^{G/2}\, G^i G_i \, \zeta \ + \ \ldots \\
&= \ -3\sqrt{2}\, m_\psi \, \zeta \ + \ \ldots \ , \qquad (23.29)
\end{aligned}
$$

where we have used the fact that the cosmological constant is zero, $\langle G^i G_i \rangle = 3$. We see that η transforms by a shift. This indicates that η is a Goldstone fermion, and supersymmetry is spontaneously broken.

Exactly as in ordinary gauge theory, the Goldstone fermion can be gauged away through a supersymmetric analog of the Higgs effect. In this "unitary gauge," all terms proportional to $G_i \chi^i$ vanish identically. This removes the gravitino-Goldstino mixing, and allows one to read off the mass matrix for the spinors χ^i, subject to the constraint $G_i \chi^i = 0$.

Of course, it is also possible to find the spinor mass matrix in a gauge-independent manner, by diagonalizing the terms quadratic in the fermion fields. The necessary field redefinition is suggested by the above arguments. We find

$$
\tilde{\psi}_n \ = \ \psi_n \ + \ \frac{1}{3} \sqrt{2}\, m_\psi^{-1}\, \partial_n \eta \ + \ \frac{i}{6} \sqrt{2}\, \sigma_n \bar{\eta} \ . \qquad (23.30)
$$

With this choice, the mixings are eliminated, and the mass terms are diagonal,

$$
- \ m_\psi \Big\{ \tilde{\psi}_a \sigma^{ab} \tilde{\psi}_b \ + \ \bar{\tilde{\psi}}_a \bar{\sigma}^{ab} \bar{\tilde{\psi}}_b
$$

$$+\ \frac{1}{2}\langle \nabla_i G_j \ +\ \frac{1}{3} G_i G_j \rangle\, \chi^i \chi^j$$

$$+\ \frac{1}{2}\langle \nabla_{i*} G_{j*} \ +\ \frac{1}{3} G_{i*} G_{j*} \rangle\, \bar\chi^i \bar\chi^j \Big\}\,. \qquad (23.31)$$

The spinor mass matrix is just

$$m_{ij} \ =\ \langle \nabla_i G_j \ +\ \frac{1}{3} G_i G_j \rangle\, m_\psi\,. \qquad (23.32)$$

Squaring, we have

$$m^2_{ij*} \ =\ g^{k\ell*}\, \langle \nabla_i G_k \ +\ \frac{1}{3} G_i G_k \rangle \langle \nabla_{j*} G_{\ell*} \ +\ \frac{1}{3} G_{j*} G_{\ell*} \rangle\, m^2_\psi$$

$$=\ \langle \nabla_i G_k\, \nabla_{j*} G^k \ -\ \frac{1}{3} G_i G_{j*} \rangle\, m^2_\psi\,. \qquad (23.33)$$

We are now in a position to derive a mass sum rule for the physical fields. Combining (23.26), (23.27) and (23.33), we find

$$\mathrm{Str}\, M^2 \ =\ \sum_{\text{spins } J} (-1)^{2J}\,(2J+1)\ \mathrm{Tr}\, M^2$$

$$=\ \langle\, 2 g^{ij*} M^2_{ij*} \ -\ 2 g^{ij*} m^2_{ij*} \ -\ 4\, e^G \rangle$$

$$=\ 2(n-1)\, m^2_\psi \ -\ 2\,\langle R_{ij*}\, G^i G^{j*} \rangle\, m^2_\psi\,, \qquad \mathbf{(23.34)}$$

for n scalar fields A^i. From the mass sum rule, we see that the boson-fermion mass splittings are proportional to m_ψ.

When the cosmological constant is zero, the gravitino mass serves as the order parameter for the spontaneous breaking of supergravity.

References

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G. Girardi, R. Grimm, M. Müller, and J. Wess, *Z. Phys.* *C26*, 427 (1984).

Equations

$$\mathcal{L} \ =\ \frac{1}{\kappa^2} \int d^2\Theta\, 2\mathcal{E} \left[\frac{3}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\, \exp\Big\{ -\frac{\kappa^2}{3} K(\Phi, \Phi^+) \Big\} \right.$$

$$\left. +\ \kappa^2\, P(\Phi) \right] \ +\ \text{h.c.}\,, \qquad \mathbf{(23.1)}$$

$$\mathcal{L} \ =\ -\frac{1}{2}\, e\, \mathcal{R} \ -\ e\, g_{ij*}\, \partial_m A^i\, \partial^m A^{*j}$$

$$-\ i\, e\, g_{ij*}\, \bar\chi^j \bar\sigma^m \mathcal{D}_m \chi^i \ +\ e\, \epsilon^{k\ell mn}\, \bar\psi_k \bar\sigma_\ell \tilde{\mathcal{D}}_m \psi_n$$

$$-\ \frac{1}{2} \sqrt{2}\, e\, g_{ij*}\, \partial_n A^{*j}\, \chi^i \sigma^m \bar\sigma^n \psi_m$$

$$
\begin{aligned}
&- \frac{1}{2}\sqrt{2}\, e\, g_{ij*}\, \partial_n A^i\, \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
&+ \frac{1}{4} e\, g_{ij*} \left[i\epsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m \;+\; \psi_m \sigma^n \bar{\psi}^m \right] \chi^i \sigma_n \bar{\chi}^j \\
&- \frac{1}{8} e \left[g_{ij*} g_{k\ell*} \;-\; 2R_{ij*k\ell*} \right] \chi^i \chi^k\, \bar{\chi}^j \bar{\chi}^\ell \\
&- e\, \exp(K/2) \Big\{ P^*\, \psi_a \sigma^{ab} \psi_b \;+\; P\, \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
&+ \frac{i}{2}\sqrt{2}\, D_i P\, \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2}\sqrt{2}\, D_{i*} P^*\, \bar{\chi}^i \bar{\sigma}^a \psi_a \\
&+ \frac{1}{2} \mathcal{D}_i D_j P\, \chi^i \chi^j \;+\; \frac{1}{2} \mathcal{D}_{i*} D_{j*} P^*\, \bar{\chi}^i \bar{\chi}^j \Big\} \\
&- e\, e^K \left[g^{ij*} (D_i P)(D_j P)^* \;-\; 3P^* P \right] . \qquad \textbf{(23.3)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_m \chi^i &= \partial_m \chi^i \;+\; \chi^i \omega_m \;+\; \Gamma^i_{jk}\, \partial_m A^j\, \chi^k \\
&\quad - \frac{1}{4} (K_j \partial_m A^j - K_{j*} \partial_m A^{*j})\, \chi^i \\
\tilde{\mathcal{D}}_m \psi_n &= \partial_m \psi_n \;+\; \psi_n\, \omega_m \\
&\quad + \frac{1}{4} (K_j \partial_m A^j - K_{j*} \partial_m A^{*j})\, \psi_n \\
D_i P &= P_i \;+\; K_i P \\
\mathcal{D}_i D_j P &= P_{ij} \;+\; K_{ij} P \;+\; K_i D_j P \;+\; K_j D_i P \\
&\quad - K_i K_j P \;-\; \Gamma^k_{ij} D_k P . \qquad \textbf{(23.4)}
\end{aligned}
$$

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta \sigma_a \bar{\psi}_m \;+\; \bar{\zeta} \bar{\sigma}_a \psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\, \zeta \chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2}\, \sigma_m \bar{\zeta}\, \hat{D}_m A^i \;-\; \Gamma^i_{jk}\, \delta_\zeta A^j \chi^k \\
&\quad + \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \chi^i \\
&\quad - \sqrt{2}\, e^{K/2}\, g^{ij*}\, D_{j*} P^*\, \zeta \\
\delta_\zeta \psi_m &= 2\mathcal{D}_m \zeta \;-\; \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \psi_m \\
&\quad - \frac{i}{2} \sigma_{mn} \zeta\;\; g_{ij*}\, \chi^i \sigma^n \bar{\chi}^j \\
&\quad + i\, e^{K/2}\, P\, \sigma_m \bar{\zeta}\, , \qquad \textbf{(23.5)}
\end{aligned}
$$

$$
\begin{aligned}
\chi^i &\;\to\; \exp + \frac{i}{2} (\mathrm{Im}\, F)\, \chi^i \\
\psi_n &\;\to\; \exp - \frac{i}{2} (\mathrm{Im}\, F)\, \psi_n \, . \qquad \textbf{(23.9)}
\end{aligned}
$$

$$
\mathcal{V} \;=\; e^K \left[g^{ij*} (D_i P)(D_j P)^* \;-\; 3P^* P \right] . \qquad \textbf{(23.10)}
$$

$$
\begin{aligned}
\operatorname{Str} M^2 &= \sum_{\text{spins } J} (-1)^{2J}\,(2J+1)\,\operatorname{Tr} M^2 \\
&= \langle\, 2g^{ij*}\, M^2_{ij*} \;-\; 2g^{ij*}\, m^2_{ij*} \;-\; 4\, e^{G} \,\rangle \\
&= 2(n-1)\, m_\psi^2 \;-\; 2\,\langle\, R_{ij*}\, G^i G^{j*} \,\rangle\, m_\psi^2 \;,
\end{aligned}
\qquad \textbf{(23.34)}
$$

Exercises

(1) Given variations $\delta E_M{}^A$ and $\delta\phi_{MB}{}^A$ of the vielbein and the connection, show that the most general variation of the torsion $T_{CB}{}^A$ is given by

$$
\begin{aligned}
\delta T_{CB}{}^A &= \mathcal{D}_C H_B{}^A \;-\; (-)^{bc}\, \mathcal{D}_B H_C{}^A \\
&\quad + \Omega_{CB}{}^A \;-\; (-)^{bc}\, \Omega_{BC}{}^A \\
&\quad + T_{CB}{}^D\, H_D{}^A \;-\; H_C{}^D\, T_{DB}{}^A \\
&\quad + (-)^{bc}\, H_B{}^D\, T_{DC}{}^A \;,
\end{aligned}
$$

where $H_A{}^B = E_A{}^M\, \delta E_M{}^B$ and $\Omega_{CB}{}^A = E_C{}^M\, \delta\phi_{MB}{}^A$.

(2) Use the results of Exercise 1 to show that the most general Weyl rescaling of the vielbein, consistent with the torsion constraints, is of the form (23.14).

(3) Find the Weyl rescalings of R and $G_{\alpha\dot\alpha}$. Check your results against (23.16).

(4) Show that the conditions (23.25) imply that the scalar potential (23.24) is extremized with vanishing cosmological constant.

(5) Compute the scalar mass matrix (23.27).

(6) For infinitesimal η, show that $\zeta = \eta\sqrt{2}/6m_\psi$ transforms η to zero.

(7) Show that the matrix (23.32) has a zero eigenvalue, with eigenvector proportional to G^i.

(8) Verify the mass sum rule (23.34).

(9) Show that $\operatorname{Str} M^2 = 0$ for the minimal chiral model, where $G = -3\log(1 - \frac{1}{3}A_i{}^* A_i)$. This is an important property of this model because most radiative corrections are proportional to $\operatorname{Str} M^2$.

XXIV. GAUGE INVARIANT MODELS

In Chapter XXII we studied the most general coupling of chiral superfields in flat space,

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar\theta\, K(\Phi^i,\Phi^{+j}) \;+\; \left[\int d^2\theta\, P(\Phi^i) + \text{ h.c.}\right] . \tag{24.1}$$

We found that K has a natural interpretation as the Kähler potential for a Kähler manifold $\mathcal{M}$. We also noted that the action (24.1) is invariant under the Kähler transformations

$$K(\Phi^i,\Phi^{+j}) \;\to\; K(\Phi^i,\Phi^{+j}) \;+\; F(\Phi^i) \;+\; F^+(\Phi^{+j})\,, \tag{24.2}$$

where F is an analytic function of the superfields Φ^i. In this chapter we will gauge the analytic isometries of the Kähler geometry, and in this way generalize (24.1) to include vector fields. We will take advantage of the fact that K transforms by a Kähler transformation under each of the analytic isometries of $\mathcal{M}$.

The analytic isometries of a Kähler manifold are generated by holomorphic Killing vectors,

$$X^{(b)} \;=\; X^{i(b)}(a^j)\,\frac{\partial}{\partial a^i}$$

$$X^{*(b)} \;=\; X^{*i(b)}(a^{*j})\,\frac{\partial}{\partial a^{*i}}\,, \tag{24.3}$$

where the index (b) runs over the dimension d of the isometry group G. As shown in Appendix D, this implies that Killing's equation reduces to the statement that there exist d real scalar functions $D^{(a)}(a,a^*)$ such that

$$\begin{aligned} g_{ij*}X^{*j(a)} &= i\frac{\partial}{\partial a^i}\,D^{(a)} \\ g_{ij*}X^{i(a)} &= -\,i\frac{\partial}{\partial a^{*j}}\,D^{(a)}\,. \end{aligned} \tag{24.4}$$

The $D^{(a)}$ are known as Killing potentials. They are defined up to constants $c^{(a)}$, $D^{(a)} \to D^{(a)} + c^{(a)}$. In what follows, we shall see that the freedom to redefine the potentials is related to the Fayet-Iliopoulos D term introduced in Chapter VIII.

The Killing vectors $X^{(a)}$ and $X^{*(a)}$ generate independent representations of the isometry group G. They obey the Lie bracket relations

$$\begin{aligned} [X^{(a)},X^{(b)}] &= -\,f^{abc}X^{(c)} \\ [X^{*(a)},X^{*(b)}] &= -\,f^{abc}X^{*(c)} \\ [X^{(a)},X^{*(b)}] &= 0\,, \end{aligned} \tag{24.5}$$

where the f^{abc} are the structure constants of G. In Appendix D it is shown that the Killing potentials $D^{(a)}$ can be chosen to transform in the adjoint representation,

$$\left[X^{i(a)}\frac{\partial}{\partial a^i} + X^{*i(a)}\frac{\partial}{\partial a^{*i}}\right] D^{(b)} = - f^{abc} D^{(c)} . \qquad \textbf{(24.6)}$$

This fixes the constants $c^{(a)}$ for non-Abelian groups. For each U(1) factor, however, there is an undetermined constant c.

Under an isometry in G, the variations of K and P are determined by the Killing vectors $X^{(a)}$,

$$\begin{aligned} \delta K &= \left[\epsilon^{(a)} X^{(a)} + \epsilon^{*(a)} X^{*(a)}\right] K \\ \delta P &= \epsilon^{(a)} X^{(a)} P . \end{aligned} \qquad (24.7)$$

The variation of the superpotential must vanish for the action to be invariant. The variation of the Kähler potential, however, does not need to vanish. As shown in Appendix D, it can be cast in the following form,

$$\delta K = \epsilon^{(a)} F^{(a)} + \epsilon^{*(a)} F^{*(a)} - i\left(\epsilon^{(a)} - \epsilon^{*(a)}\right) D^{(a)} , \qquad \textbf{(24.8)}$$

where $F^{(a)} = X^{(a)} K + i D^{(a)}$ is an analytic function of the coordinates. For real parameters $\epsilon^{(a)}$, (24.8) reduces to a Kähler transformation. For complex $\epsilon^{(a)}$, it is not of Kähler form; there is a change in K proportional to the Killing potential $D^{(a)}$.

The fact that (24.8) reduces to a Kähler transformation for real $\epsilon^{(a)}$ reflects the fact that the action (24.1) is invariant under the rigid isometries of the manifold $\mathcal{M}$. For local motions, however, the story is more complicated. This is because the parameter $\epsilon^{(a)}$ must be promoted to a chiral superfield in superspace. In this case, the variation of the action (24.1) is

$$\begin{aligned} \delta \mathcal{L} &= \int d^2\theta\, d^2\bar\theta\ \delta K \\ &= \int d^2\theta\, d^2\bar\theta\ \Big[\Lambda^{(a)} F^{(a)} + \Lambda^{+(a)} F^{+(a)} \\ &\qquad - i\left(\Lambda^{(a)} - \Lambda^{+(a)}\right) D^{(a)}\Big] \\ &= - i \int d^2\theta\, d^2\bar\theta\ \left(\Lambda^{(a)} - \Lambda^{+(a)}\right) D^{(a)} , \end{aligned} \qquad (24.9)$$

where $\Lambda^{(a)}$ is a chiral superfield with lowest component $\epsilon^{(a)}$, and the $D^{(a)}$ are hermitian functions of the chiral superfields Φ^i and Φ^{+j}.

In the rest of this chapter, we will see how to construct a supersymmetric gauge theory, invariant under the isometries parametrized by $\Lambda^{(a)}$. We will add a term to the action whose variation exactly cancels (24.9), using the formalism developed in Appendices E and F. We will find that the counterterm involves the vector superfield $V = V^{(a)} T^{(a)}$, where the $T^{(a)}$ are the hermitian generators of the isometry group G.

Since $\epsilon^{(a)}$ is complex, we must study the complexification of G, which we call $\mathcal{G}$. An arbitrary element of $\mathcal{G}$ can be written in the form

$$g = e^{\frac{1}{2}v^{(a)}T^{(a)}} e^{-\frac{i}{2}u^{(a)}T^{(a)}} , \qquad (24.10)$$

where $u^{(a)}$ and $v^{(a)}$ are real, and as above, the $T^{(a)}$ are the hermitian generators of G. Equation (24.10) splits g into the product of a hermitian and a unitary matrix, which can always be done.

Given the complexification $\mathcal{G}$ of G, the coset space $\mathcal{G}/G$ is constructed by identifying elements g and $g' \in \mathcal{G}$ if $g = g'u'$, for some $u' \in G$. Thus a point of the coset can be represented by

$$v = e^{\frac{1}{2}v^{(a)}T^{(a)}} . \qquad (24.11)$$

The matrix v is an element of $\mathcal{G}$, and the $v^{(a)}$ are coordinates of $\mathcal{G}/G$.

The group $\mathcal{G}$ acts naturally on the cosets $\mathcal{G}/G$ by *left* multiplication on v, $v' = g_0 v$. To find the transformation of v, it is useful to examine two cases, the first with $g_0 = u_0 \in G$, and the second with $g_0 = v_0 \in \mathcal{G}$ (but not in G). For a transformation parametrized by u_0, we have

$$v \rightarrow u_0 v = u_0 v u_0^{-1} u_0 \equiv v'u' , \qquad (24.12)$$

where $v' = u_0 v u_0^{-1}$ and $u' = u_0$. In terms of the coordinates $v^{(a)}$, this implies

$$e^{\frac{1}{2}v'^{(a)}T^{(a)}} = u_0\, e^{\frac{1}{2}v^{(a)}T^{(a)}} u_0^{-1} , \qquad \mathbf{(24.13)}$$

and we see that the $v^{(a)}$ transform linearly under elements $u_0 \in H$. In contrast, for a transformation v_0, we have

$$v \rightarrow v_0 v \equiv v'u' . \qquad (24.14)$$

Taking the hermitian conjugate, we find

$$v \rightarrow v v_0 \equiv u'^{\dagger} v' . \qquad (24.15)$$

Combining the two expressions, we see that

$$v'^2 = v_0 v^2 v_0 . \qquad (24.16)$$

In terms of the coordinates $v^{(a)}$, this implies

$$e^{v'^{(a)}T^{(a)}} = v_0\, e^{v^{(a)}T^{(a)}} v_0 , \qquad \mathbf{(24.17)}$$

a manifestly nonlinear transformation law. Note that the $v^{(a)}$ can be transformed to zero if we take $v_0 = e^{-\frac{1}{2}v^{(a)}T^{(a)}}$.

For infinitesimal variations, the transformations (24.13) and (24.17) can be combined to give

$$\delta \exp\left(v^{(a)}T^{(a)}\right) = -\frac{i}{2}\left[u_0^{(b)} + iv_0^{(b)}\right] T^{(b)} \exp\left(v^{(a)}T^{(a)}\right) + \frac{i}{2}\exp\left(v^{(a)}T^{(a)}\right)\left[u_0^{(b)} - iv_0^{(b)}\right] T^{(b)}$$
$$= -\,i\,\epsilon^* e^{v^{(a)}T^{(a)}} + i\,e^{v^{(a)}T^{(a)}}\epsilon\,, \qquad (24.18)$$

where we have set $\epsilon = \epsilon^{(b)}T^{(b)}$, with $\epsilon^{(b)} = \frac{1}{2}(u_0^{(b)} - iv_0^{(b)})$. If we identify $\epsilon^{(b)}$ with the lowest component of a chiral superfield $\Lambda^{(b)}$, and $v^{(a)}$ with the lowest component of vector superfield $V^{(a)}$, we see that the transformation (24.18) is precisely the lowest component of the gauge transformation (7.15):

$$\delta e^V = -\,i\,\Lambda^+ e^V + i\,e^V \Lambda\,. \qquad (24.19)$$

Comparing (24.13) and (24.17) with (E.17) and (E.23), we see that the transformation law of a vector superfield is just a nonlinear realization, corresponding to the coset $\mathcal{G}/G$. As in Appendix F, we can exploit this fact to construct a fully gauge invariant theory. Recall that previously we found

$$\delta\mathcal{L} = -\,i\int d^2\theta\, d^2\bar\theta\, \left(\Lambda^{(a)} - \Lambda^{+(a)}\right) D^{(a)}\,. \qquad (24.20)$$

To cancel this variation, we need to find a function

$\Gamma(\Phi^i, \Phi^{+j}, V^{(a)})$ such that

$$\delta\Gamma = i\left[\Lambda^{(a)} - \Lambda^{+(a)}\right] D^{(a)}\,. \qquad (24.21)$$

Then

$$\mathcal{L} = \int d^2\theta d^2\bar\theta \left[K(\Phi^i, \Phi^{+j}) + \Gamma(\Phi^i, \Phi^{+j}, V^{(a)})\right] + \left[\int d^2\theta\; P(\Phi^i) + \text{h.c.}\right] \qquad (24.22)$$

will be a fully gauge invariant action.

To find the counterterm Γ, we first restrict to its lowest component $\Gamma(a^i, a^{*j}, v^{(a)})$. We then write the variation $\delta\Gamma$ in terms of differential operators,

$$\delta\Gamma = \epsilon^{(a)}X^{(a)}\Gamma + \epsilon^{*(a)}X^{*(a)}\Gamma + \delta v^{(a)}\frac{\delta\Gamma}{\delta v^{(a)}}$$
$$\equiv \frac{1}{2}\left(\epsilon^{(a)} + \epsilon^{*(a)}\right)\mathcal{P}^{(a)}\Gamma + \frac{1}{2}\left(\epsilon^{(a)} - \epsilon^{*(a)}\right)\mathcal{O}^{(a)}\Gamma\,, \qquad (24.23)$$

where $\mathcal{P}^{(a)}$ and $\mathcal{O}^{(a)}$ include the variations of the coordinates a^i and a^{*j}, as well as the appropriate variations of the $v^{(a)}$. For (24.23) to agree with (24.21), we must demand

$$\mathcal{P}^{(a)}\Gamma = 0$$

$$\mathcal{O}^{(a)}\Gamma = 2i\,D^{(a)} . \tag{24.24}$$

Furthermore, we also require that Γ satisfy the boundary condition

$$\Gamma(a^i, a^{*j}, 0) = 0 . \tag{24.25}$$

With these ingredients, it is not hard to integrate (24.24). Following the steps of Appendix F, we find

$$\begin{aligned} \Gamma &= \frac{e^{\frac{i}{2}v^{(a)}O^{(a)}} - 1}{\frac{i}{2}v^{(b)}O^{(b)}}\, v^{(c)}D^{(c)} \\ &= \int_0^1 d\alpha\, e^{\frac{i}{2}\alpha v^{(a)}O^{(a)}} v^{(c)}D^{(c)} , \end{aligned} \tag{24.26}$$

In this expression, the operator $O^{(a)}$ is the same as $\mathcal{O}^{(a)}$, but without the variations of the $v^{(a)}$,

$$O^{(a)} = X^{(a)} - X^{*(a)} . \tag{24.27}$$

It is a useful exercise to check that Γ indeed obeys (24.24).

Having found the counterterm Γ, we are now ready to write the gauge invariant action in superspace. We first promote Γ to a superfield, replacing a^i, a^{*j} and $v^{(a)}$ by superfields Φ^i, Φ^{+j} and $V^{(a)}$. In a symbolic notation, we have

$$\Gamma(\Phi^i, \Phi^{+j}, V^{(a)}) = \int_0^1 d\alpha\, e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}}\, V^{(b)}D^{(b)} , \tag{24.28}$$

where the differentiations $O^{(a)}$ are performed *before* the fields are replaced by superfields. Substituting this expression into (24.22), we obtain the complete gauge invariant action in superspace:

$$\begin{aligned} \mathcal{L} = & \int d^2\theta\, d^2\bar\theta\; K(\Phi^i, \Phi^{+j}) \\ & + \int_0^1 d\alpha \int d^2\theta\, d^2\bar\theta\; e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}}\, V^{(b)}D^{(b)} \\ & + \left[\int d^2\theta\; P(\phi^i) + \text{h.c.} \right] . \end{aligned} \tag{24.29}$$

The action (24.29) is manifestly supersymmetric because it is written in superspace form. By construction, it is also invariant under the local isometries in G,

$$\begin{aligned} \delta\Phi^i &= \Lambda^{(a)} X^{i(a)}(\Phi^j) \\ \delta e^V &= -\,i\,\Lambda^{+(a)}T^{(a)}e^V + i\,e^V\Lambda^{(a)}T^{(a)} . \end{aligned} \tag{24.30}$$

Note that the explicit appearance of the Killing potentials in (24.29) implies that their global existence is necessary for gauging of the isometry group G.

To write this action in components, we add the kinetic term for the vector multiplet,

$$\mathcal{L} \;=\; \frac{1}{16kg^2}\int d^2\theta \,\mathrm{Tr}\, WW + \text{ h.c.}\;, \tag{24.31}$$

and then pass to the WZ gauge. It is a straightforward exercise to eliminate the auxiliary fields and cast the remaining terms into geometrical form. We find

$$\begin{aligned}\mathcal{L} \;=\; & - g_{ij*}\,\mathcal{D}_m A^i \mathcal{D}^m A^{*j} \;-\; i\,\lambda^{(a)}\sigma^m \mathcal{D}_m \bar{\lambda}^{(a)} \;-\; \frac{1}{2}\,g^2 D^{(a)2} \\ & - i\,g_{ij*}\,\chi^i\,\sigma^m \mathcal{D}_m \bar{\chi}^j \;-\; \frac{1}{4}\,F_{mn}{}^{(a)} F^{mn(a)} \\ & + g\sqrt{2}\,g_{ij*}\,[X^{i(a)}\,\bar{\chi}^j \bar{\lambda}^{(a)} \;+\; X^{*j(a)}\,\chi^i \lambda^{(a)}] \\ & - \frac{1}{2} D_i D_j P\,\chi^i \chi^j \;-\; \frac{1}{2} D_{i*} D_{j*} P^*\,\bar{\chi}^i \bar{\chi}^j \\ & - g^{ij*}\,D_i P\,D_{j*} P^* \\ & + \frac{1}{4}\,R_{ij*k\ell*}\,\chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell\;, \end{aligned} \tag{24.32}$$

where

$$\begin{aligned}\mathcal{D}_m A^i \;&=\; \partial_m A^i \;-\; g\,v_m{}^{(a)} X^{i(a)} \\ \mathcal{D}_m \chi^i \;&=\; \partial_m \chi^i \;+\; \Gamma^i{}_{jk} \mathcal{D}_m A^j\,\chi^k \;-\; g\,v_m{}^{(a)}\,\frac{\partial X^{i(a)}}{\partial A^j}\,\chi^j \\ \mathcal{D}_m \lambda^{(a)} \;&=\; \partial_m \lambda^{(a)} \;-\; g\,f^{abc} v_m{}^{(b)} \lambda^{(c)} \\ D_i P \;&=\; \frac{\partial P}{\partial A^i} \\ D_i D_j P \;&=\; \frac{\partial^2 P}{\partial A^i \partial A^j} \;-\; \Gamma^k{}_{ij} \frac{\partial P}{\partial A^k}\;, \end{aligned} \tag{24.33}$$

and we have rescaled $V \to 2gV$. The action (24.32) is invariant under the following gauge transformations,

$$\begin{aligned}\delta A^i \;&=\; \epsilon^{(a)} X^{i(a)} \\ \delta \chi^i \;&=\; \epsilon^{(a)}\,\frac{\partial X^{i(a)}}{\partial A^j}\,\chi^j \\ \delta \lambda^{(a)} \;&=\; f^{abc} \epsilon^{(b)} \lambda^{(c)} \\ \delta v_m{}^{(a)} \;&=\; g^{-1}\,\partial_m \epsilon^{(a)} + f^{abc} \epsilon^{(b)} v_m{}^{(c)}\;. \end{aligned} \tag{24.34}$$

The covariant derivatives (24.33) are fully gauge covariant, as is evident from the transformations (24.34).

The Lagrangian (24.32) includes the following scalar potential,

$$\mathcal{V} \;=\; \frac{1}{2}\,g^2\,D^{(a)2} \;+\; g^{ij*}\,D_i P\,D_{j*} P^*\;. \tag{24.35}$$

The first term is the sigma-model generalization of the "D-term" introduced in Chapter VII. Equation (24.35) implies that supersymmetry is spontaneously broken if either $\langle D^{(a)}\rangle \neq 0$ or $\langle D_i P\rangle \neq 0$, for some value of a or i.

For U(1) factors in the gauge group G, the relations (24.4) and (24.6) do not completely determine the Killing potentials. They leave the D's undetermined up to additive constants c,

$$D \rightarrow D + c . \tag{24.36}$$

Therefore, by choosing the constants appropriately, it is always possible to arrange for supersymmetry to be spontaneously broken. This is the sigma-model version of the Fayet-Iliopoulos mechanism for supersymmetry breaking.

To illustrate the generality of the formalism developed above, we conclude this chapter with two examples. We first consider $\mathbf{C}^n$, and gauge the $\mathrm{U}(n)$ rotations about the origin. We take $K = a^{*i}a^i + d$, so $g_{ij*} = \delta_{ij*}$ and $R_{ij*k\ell*} = 0$. The Killing vectors $X^{i(a)}$ are simply $-iT^{(a)i}{}_j a^j$; the Killing potentials are $D^{(a)} = a^{*i}T^{(a)i}{}_j a^j$. Promoting a^i and a^{*i} to superfields Φ^i and Φ^{+i}, we find

$$\Gamma(\Phi^i, \Phi^{+j}, V^{(a)}) = \int d^2\theta\, d^2\bar\theta \;\; \Phi^+ \left[e^V - 1 \right] \Phi . \tag{24.37}$$

Using this result, it is obvious that (24.29) reduces to the usual superspace Lagrangian for a $\mathrm{U}(n)$ gauge theory.

For our second example, we consider $CP^1 = S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. This is a Kähler manifold as well as a homogeneous space. For simplicity, we use projective coordinates a

and a^*. In these coordinates, we take $K = \log(1 + aa^*)$ and $P = 0$ (see Exercise 6 of Appendix D). We choose to gauge the entire isometry group $G = \mathrm{SU}(2)$, so the functions $D^{(a)}$ are as follows:

$$D^{(1)} = \frac{1}{2}\,\frac{a + a^*}{(1 + a^*a)}\,, \qquad D^{(2)} = -\,\frac{i}{2}\,\frac{a - a^*}{(1 + a^*a)}\,,$$

$$D^{(3)} = -\,\frac{1}{2}\left(\frac{1 - a^*a}{1 + a^*a}\right) . \tag{24.38}$$

From here one can work out the Lagrangian, in superspace and in components. We shall work in components, starting with the Lagrangian (24.32). Because we have gauged the full SU(2), we are able to go to the "unitary gauge" where $a = a^* = 0$. This gauge exhibits the particle content of the theory:

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} F_{mn}{}^{(a)} F^{mn(a)} - i\lambda^{(a)}\sigma^m \mathcal{D}_m \bar\lambda^{(a)} - i\chi\sigma^m\mathcal{D}_m\bar\chi \\ & - \frac{1}{2} g^2 v^+_m v^{-m} - \frac{1}{8} g^2 - i\, g\,(\chi\lambda_- - \bar\chi\bar\lambda_-) \\ & - \frac{1}{2}\chi\chi\bar\chi\bar\chi\,, \end{aligned} \tag{24.39}$$

where

$$v^{\pm}_m = \frac{1}{2}\sqrt{2}\left(v_m{}^{(1)} \pm i\, v_m{}^{(2)}\right)$$

$$\lambda_\pm = \frac{1}{2}\sqrt{2}\left(\lambda^{(1)} \pm i\,\lambda^{(2)}\right)$$
$$\mathcal{D}_m\chi = \partial_m\chi - i\,g\,v_m{}^{(3)}\chi\,. \qquad (24.40)$$

The SU(2) symmetry implies that $D^{(a)2}$ is a constant. The constant is positive, so supersymmetry is spontaneously broken. The mass spectrum is as follows. The charged vector mesons $v_m^\pm$ are massive; they have eaten the scalars a and a^*. The massless vector meson $v_m{}^{(3)}$ is the gauge field corresponding to the unbroken U(1) symmetry. Its supersymmetry partner is the massless Goldstone spinor $\lambda^{(3)}$. The Majorana spinors χ and λ_- are massive; they have combined to form one massive Dirac spinor. Finally, λ_+ is both massless and charged. The CP^1 model has spontaneously broken supersymmetry, no leftover Higgs, and a massless Weyl spinor in a complex representation of the unbroken gauge group. This model is remarkable because the particle spins (as well as their masses) violate supersymmetry. No model with unbroken supersymmetry has the same spin spectrum. Nevertheless, the numbers of bosonic and fermionic degrees of freedom balance on mass shell.

References

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Equations

$$g_{ij*}X^{*j(a)} = i\frac{\partial}{\partial a^i}D^{(a)}$$
$$g_{ij*}X^{i(a)} = -\,i\frac{\partial}{\partial a^{*j}}D^{(a)}\,. \qquad \mathbf{(24.4)}$$

$$[X^{(a)},X^{(b)}] = -\,f^{abc}X^{(c)}$$
$$[X^{*(a)},X^{*(b)}] = -\,f^{abc}X^{*(c)}$$
$$[X^{(a)},X^{*(b)}] = 0\,, \qquad \mathbf{(24.5)}$$

$$\left[X^{i(a)}\frac{\partial}{\partial a^i} + X^{*i(a)}\frac{\partial}{\partial a^{*i}}\right]D^{(b)} = -\,f^{abc}D^{(c)}\,. \qquad \mathbf{(24.6)}$$

$$\delta K = \epsilon^{(a)}F^{(a)} + \epsilon^{*(a)}F^{*(a)} - i\left(\epsilon^{(a)} - \epsilon^{*(a)}\right)D^{(a)}\,, \qquad \mathbf{(24.8)}$$

$$e^{\frac{1}{2}v'^{(a)}T^{(a)}} = u_0\,e^{\frac{1}{2}v^{(a)}T^{(a)}}u_0^{-1}\,, \qquad \mathbf{(24.13)}$$

$$e^{v'^{(a)}T^{(a)}} = v_0\,e^{v^{(a)}T^{(a)}}v_0\,, \qquad \mathbf{(24.17)}$$

$$\Gamma = \frac{e^{\frac{i}{2}v^{(a)}O^{(a)}} - 1}{\frac{i}{2}v^{(b)}O^{(b)}}\, v^{(c)}D^{(c)}$$

$$= \int_0^1 d\alpha\; e^{\frac{i}{2}\alpha v^{(a)}O^{(a)}} v^{(c)}D^{(c)}\,, \qquad (24.26)$$

$$O^{(a)} = X^{(a)} - X^{*(a)}\,. \qquad (24.27)$$

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\; K(\Phi^i, \Phi^{+j}) + \int_0^1 d\alpha \int d^2\theta\, d^2\bar\theta\; e^{\frac{i}{2}\alpha V^{(a)}O^{(a)}}\; V^{(b)}D^{(b)} + \left[\int d^2\theta\; P(\phi^i) + \text{h.c.}\right]. \qquad (24.29)$$

$$\delta\Phi^i = \Lambda^{(a)}\, X^{i(a)}(\Phi^j)$$
$$\delta e^V = -\,i\,\Lambda^{+(a)}T^{(a)}e^V + i\,e^V\Lambda^{(a)}T^{(a)}\,. \qquad (24.30)$$

$$\mathcal{L} = -\,g_{ij*}\,\mathcal{D}_m A^i \mathcal{D}^m A^{*j} - i\,\lambda^{(a)}\sigma^m \mathcal{D}_m \bar\lambda^{(a)} - \frac{1}{2}g^2 D^{(a)2} - i\,g_{ij*}\,\chi^i\sigma^m\mathcal{D}_m\bar\chi^j - \frac{1}{4}F_{mn}{}^{(a)}F^{mn(a)}$$

$$+\, g\sqrt{2}\,g_{ij*}\,[X^{i(a)}\,\bar\chi^j\bar\lambda^{(a)} + X^{*j(a)}\,\chi^i\lambda^{(a)}] - \frac{1}{2}D_iD_jP\,\chi^i\chi^j - \frac{1}{2}D_{i*}D_{j*}P^*\,\bar\chi^i\bar\chi^j - g^{ij*}\,D_iP\,D_{j*}P^* + \frac{1}{4}R_{ij*k\ell*}\,\chi^i\chi^k\bar\chi^j\bar\chi^\ell\,, \qquad (24.32)$$

$$\mathcal{D}_m A^i = \partial_m A^i - g\,v_m{}^{(a)}X^{i(a)}$$
$$\mathcal{D}_m \chi^i = \partial_m\chi^i + \Gamma^i{}_{jk}\mathcal{D}_m A^j\,\chi^k - g\,v_m{}^{(a)}\,\frac{\partial X^{i(a)}}{\partial A^j}\,\chi^j$$
$$\mathcal{D}_m\lambda^{(a)} = \partial_m\lambda^{(a)} - g\,f^{abc}v_m{}^{(b)}\lambda^{(c)}$$
$$D_iP = \frac{\partial P}{\partial A^i}$$
$$D_iD_jP = \frac{\partial^2 P}{\partial A^i\partial A^j} - \Gamma^k{}_{ij}\frac{\partial P}{\partial A^k}\,, \qquad (24.33)$$

$$\delta A^i = \epsilon^{(a)}X^{i(a)}$$
$$\delta\chi^i = \epsilon^{(a)}\,\frac{\partial X^{i(a)}}{\partial A^j}\,\chi^j$$
$$\delta\lambda^{(a)} = f^{abc}\epsilon^{(b)}\lambda^{(c)}$$
$$\delta v_m{}^{(a)} = g^{-1}\,\partial_m\epsilon^{(a)} + f^{abc}\epsilon^{(b)}v_m{}^{(c)}\,. \qquad (24.34)$$

$$\mathcal{V} \;=\; \frac{1}{2} g^2 D^{(a)2} \;+\; g^{ij^*} D_i P\, D_{j*} P^* \;. \qquad (24.35)$$

Exercises

(1) Prove that the Killing potentials can always be chosen to satisfy (24.6). This can be done by first differentiating the left-hand side with respect to a^i, and then using the relations introduced above to obtain the a^i derivative of the right-hand side of (24.6). The proof can be completed by repeating the procedure, this time differentiating with respect to a^{*i}.

(2) Show that

$$X^{i(a)} \frac{\partial}{\partial a^i} D^{(b)} \;+\; X^{*i(b)} \frac{\partial}{\partial a^{*i}} D^{(a)} \;=\; 0 \,.$$

(3) Verify that the differential operators

$$X^{(a)} \;=\; -i a^j T^{(a)k}{}_j \frac{\partial}{\partial a^k}$$

$$X^{*(a)} \;=\; i a^{*j} T^{(a)j}{}_k \frac{\partial}{\partial a^{*k}}$$

are indeed Killing vectors, where the commutation relations of the $T^{(a)}{}_j{}^k$ are given in (7.14). Show that their Lie brackets close into (24.5).

(4) Let $\mathcal{M}$ be the complex plane. In this exercise you will gauge translations in the y-direction on $\mathcal{M}$. (Note that one could have chosen to gauge translations in the x-direction, but because of (24.6), one cannot gauge both simultaneously.) As above, take $K = a^*a + d$, so $g_{aa^*} = 1$ and $R_{aa^*aa^*} = 0$. For D take the Killing potential $D = m(a + a^*)$. Find the Lagrangian and the mass spectrum in the unitary gauge.

(8) Show that (24.29) reduces to (24.32) in the WZ gauge.

XXV. GAUGE INVARIANT SUPERGRAVITY MODELS

Having discussed the geometrical interpretation of supersymmetric theories, we are now ready to write down the general coupling of matter fields to supergravity. The Lagrangian we derive is the starting point for the phenomenological study of supergravity theories. We present the Lagrangian in superspace (25.1), in two-component spinor notation (25.12), and as a service to the reader, in a more conventional form with four-component spinors (25.24). Readers interested only in the results should feel free to skip to the relevant part of the chapter.

The supergravity extension of the gauge invariant superspace Lagrangian is easy to find using the material from the previous chapters. As in (24.17), one first adds the counterterm Γ to the Kähler potential K. Then, as in (23.1), one exponentiates the result to find

$$\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \left[\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp\left\{ -\frac{1}{3}\Big[K(\Phi, \Phi^+) + \Gamma(\Phi, \Phi^+, V) \Big] \right\} + \frac{1}{16 g^2} H_{(ab)}(\Phi)\, W^{(a)} W^{(b)} + P(\Phi) \right] + \text{h.c.}\,, \qquad \textbf{(25.1)}$$

where $\kappa^2 = 1$ and

$$W_\alpha \equiv W_\alpha^{(a)} T^{(a)} = -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\, e^{-V} \mathcal{D}_\alpha e^V \qquad \textbf{(25.2)}$$

is the curved-space generalization of the supersymmetric Yang-Mills field strength. In this expression, K is an arbitrary hermitian function of the superfields Φ^i and Φ^{+j}, P is the superpotential, and Γ is the counterterm (24.22), which is necessary for gauge invariance, as we will see below. The analytic function $H_{(ab)}$ is included for generality. Under a gauge transformation, it must transform as necessary to render (25.1) invariant. In what follows, we shall take $H_{(ab)} = \delta_{ab}$; the Lagrangian with nontrivial $H_{(ab)}$ is presented in Appendix G.

The supergravity invariance of (25.1) is manifest because of the superspace formalism. The gauge invariance, however, is a little more subtle. To check it, let us recall the gauge transformations for K, Γ and P,

$$\begin{aligned} \delta K &= \Lambda^{(a)} F^{(a)} + \Lambda^{+(a)} F^{+(a)} - i\left[\Lambda^{(a)} - \Lambda^{+(a)}\right] D^{(a)} \\ \delta\Gamma &= i\left[\Lambda^{(a)} - \Lambda^{+(a)}\right] D^{(a)} \\ \delta P &= \Lambda^{(a)} X^{(a)} P\,, \end{aligned} \qquad (25.3)$$

as given in Chapter XXIV. Here

$$F^{(a)} = X^{(a)}K + iD^{(a)} \tag{25.4}$$

is an analytic function of the Φ^i, and $\Lambda^{(a)}$ is the superfield gauge parameter. When applied to the Lagrangian (25.1), the transformations (25.3) induce a variation of the following form,

$$\delta\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\left[\Lambda^{(a)}F^{(a)} + \Lambda^{+(a)}F^{+(a)}\right] e^{-(K+\Gamma)/3} + \Lambda^{(a)}X^{(a)}P \right] + \text{h.c.} \tag{25.5}$$

In Chapter XXIII, such a variation is cancelled by a super-Weyl transformation, where the Weyl weight of Φ^i is taken to be zero. Setting the weight of $V^{(a)}$ to be zero as well, we find

$$\delta\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \left[\frac{3}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)[\Sigma + \bar{\Sigma}]\, e^{-(K+\Gamma)/3} + 6\Sigma P \right] + \text{h.c.} \tag{25.6}$$

under a super-Weyl transformation with superfield parameter Σ. Comparing (25.5) to (25.6), we see that the variation is cancelled if

$$\Sigma = \frac{1}{6}\Lambda^{(a)}F^{(a)}$$

$$\delta P = \Lambda^{(a)}X^{(a)}P = -\Lambda^{(a)}F^{(a)}P\,. \tag{25.7}$$

The condition on δP is a nontrivial condition on the superpotential that is necessary for the gauge invariance of the theory.

The superspace Lagrangian presented above can be expressed in components using the techniques developed in the previous chapters. One first passes to the WZ gauge, where

$$\Gamma = V^{(a)}D^{(a)} + \frac{1}{2}g_{ij*}X^{i(a)}X^{*j(b)}V^{(a)}V^{(b)} \tag{25.8}$$

and

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\left\{\mathcal{D}_\alpha V - \frac{1}{2}[V, \mathcal{D}_\alpha V]\right\} \tag{25.9}$$

One then works out the Θ expansions for

$$(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\exp\left\{-\frac{1}{3}\Big[K + \Gamma\Big]\right\} \tag{25.10}$$

and

$$W^{(a)}W^{(a)}\,. \tag{25.11}$$

After eliminating the auxiliary fields, and rescaling and redefining the other fields as in Chapters XXI and XXIII, one finds the component Lagrangian in terms of the physical fields. This Lagrangian is the starting point for phenomenological studies of supergravity theories:

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2} e \mathcal{R} - e\, g_{ij*} \tilde{\mathcal{D}}_m A^i \tilde{\mathcal{D}}^m A^{*j} - \frac{1}{2} e\, g^2 D^{(a)2} \\
& - \frac{1}{4} e F_{mn}{}^{(a)} F^{mn(a)} - i e \bar{\lambda}^{(a)} \bar{\sigma}^m \tilde{\mathcal{D}}_m \lambda^{(a)} \\
& - i e\, g_{ij*} \bar{\chi}^j \bar{\sigma}^m \tilde{\mathcal{D}}_m \chi^i + e \epsilon^{k\ell mn} \bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathcal{D}}_m \psi_n \\
& + \sqrt{2} e g\, g_{ij*} X^{*j(a)} \chi^i \lambda^{(a)} + \sqrt{2} e g\, g_{ij*} X^{i(a)} \bar{\chi}^j \bar{\lambda}^{(a)} \\
& - \frac{1}{2} e g D^{(a)} \psi_m \sigma^m \bar{\lambda}^{(a)} + \frac{1}{2} e g D^{(a)} \bar{\psi}_m \bar{\sigma}^m \lambda^{(a)} \\
& - \frac{1}{2} \sqrt{2} e\, g_{ij*} \tilde{\mathcal{D}}_n A^{*j} \chi^i \sigma^m \bar{\sigma}^n \psi_m \\
& - \frac{1}{2} \sqrt{2} e\, g_{ij*} \tilde{\mathcal{D}}_n A^i \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
& + \frac{i}{4} e [\psi_m \sigma^{ab} \sigma^m \bar{\lambda}^{(a)} + \bar{\psi}_m \bar{\sigma}^{ab} \bar{\sigma}^m \lambda^{(a)}][F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)}] \\
& + \frac{1}{4} e\, g_{ij*} [i \epsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m] \chi^i \sigma_n \bar{\chi}^j \\
& - \frac{1}{8} e [g_{ij*} g_{k\ell*} - 2 R_{ij*k\ell*}] \chi^i \chi^k \bar{\chi}^j \bar{\chi}^\ell \\
& + \frac{1}{8} e\, g_{ij*} \bar{\chi}^j \bar{\sigma}^m \chi^i \bar{\lambda}^{(a)} \bar{\sigma}_m \lambda^{(a)} \\
& - \frac{3}{16} e \lambda^{(a)} \sigma^m \bar{\lambda}^{(a)} \lambda^{(b)} \sigma_m \bar{\lambda}^{(b)} \\
& - e \exp(K/2) \Big\{ P^* \psi_a \sigma^{ab} \psi_b + P \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \\
& + \frac{i}{2} \sqrt{2} D_i P \chi^i \sigma^a \bar{\psi}_a + \frac{i}{2} \sqrt{2} D_{i*} P^* \bar{\chi}^i \bar{\sigma}^a \psi_a \\
& + \frac{1}{2} \mathcal{D}_i D_j P \chi^i \chi^j + \frac{1}{2} \mathcal{D}_{i*} D_{j*} P^* \bar{\chi}^i \bar{\chi}^j \Big\} \\
& - e e^K [g^{ij*} (D_i P)(D_j P)^* - 3 P^* P] . \qquad \textbf{(25.12)}
\end{aligned}
$$

In this expression, the scalars A^i and the spinors χ^i and $\lambda^{(a)}$ are matter fields, while the vectors $v_m{}^{(a)}$ are the gauge fields for the gauge group G. The field ψ_m is the gravitino, and $e_m{}^a$ is the graviton. In (25.12), K, P and D are functions of the scalar fields. As before, the metric g_{ij*} is Kähler.

The Lagrangian (25.12) contains derivatives covariant with respect to gauge transformations, as well as spacetime and Kähler coordinate transformations,

$$
\begin{aligned}
\tilde{\mathcal{D}}_m A^i \;&=\; \partial_m A^i - g\, v_m{}^{(a)} X^{i(a)} \\
\tilde{\mathcal{D}}_m \chi^i \;&=\; \partial_m \chi^i + \chi^i \omega_m + \Gamma^i_{jk} \tilde{\mathcal{D}}_m A^j \chi^k \\
& \quad - g\, v_m{}^{(a)} \frac{\partial X^{i(a)}}{A^j} \chi^j \\
& \quad - \frac{1}{4} (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j}) \chi^i
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{i}{2} g\, v_m{}^{(a)}\, \mathrm{Im}\, F^{(a)}\, \chi^i \\
\tilde{\mathcal{D}}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} + \lambda^{(a)} \omega_m - g\, f^{abc} v_m{}^{(b)} \lambda^{(c)} \\
&\quad + \frac{1}{4} (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \lambda^{(a)} \\
&\quad + \frac{i}{2} g\, v_m{}^{(b)}\, \mathrm{Im}\, F^{(b)}\, \lambda^{(a)} \\
\tilde{\mathcal{D}}_m \psi_n &= \partial_m \psi_n + \psi_n \omega_m \\
&\quad + \frac{1}{4} (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \psi_n \\
&\quad + \frac{i}{2} g\, v_m{}^{(a)}\, \mathrm{Im}\, F^{(a)} \psi_n \\
D_i P &= P_i + K_i P \\
\mathcal{D}_i D_j P &= P_{ij} + K_{ij} P + K_i D_j P + K_j D_i P \\
&\quad - K_i K_j P - \Gamma^k_{ij} D_k P\, . \qquad \textbf{(25.13)}
\end{aligned}
$$

The covariant derivatives contain the Christoffel symbols for the Kähler geometry and the spin connection (17.12) for spacetime. They also contain the vector potential $v_m{}^{(a)}$. Note that the covariant derivatives contain a coupling between $\mathrm{Im}\, F^{(a)}$ and the vector potential. This is a reflection of the fact that the gauge transformations are accompanied by super-Weyl rotations of the component fields.

The above Lagrangian is invariant under the gauge group

G. The gauge transformations of the component fields are given by

$$
\begin{aligned}
\delta A^i &= \epsilon^{(a)}\, X^{i(a)} \\
\delta \chi^i &= \epsilon^{(a)} \frac{\partial X^{i(a)}}{A^j} \chi^j + \frac{i}{2} \epsilon^{(a)}\, \mathrm{Im}\, F^{(a)}\, \chi^i \\
\delta \lambda^{(a)} &= f^{abc} \epsilon^{(b)}\, \lambda^{(c)} - \frac{i}{2} \epsilon^{(b)}\, \mathrm{Im}\, F^{(b)}\, \lambda^{(a)} \\
\delta v_m{}^{(a)} &= g^{-1} \partial_m \epsilon^{(a)} + f^{abc} \epsilon^{(b)}\, v_m{}^{(c)} \\
\delta \psi_n &= - \frac{i}{2} \epsilon^{(a)}\, \mathrm{Im}\, F^{(a)}\, \psi_n\, , \qquad \textbf{(25.14)}
\end{aligned}
$$

It is also automatically invariant under supergravity transformations because it was derived from a superspace formalism. It is instructive, however, to verify the invariance directly, using the following transformations laws,

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta \sigma_a \bar{\psi}_m + \bar{\zeta} \bar{\sigma}_a \psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\, \zeta \chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2}\, \sigma_m \bar{\zeta}\, \hat{\tilde{D}}_m A^i - \Gamma^i_{jk}\, \delta_\zeta A^j \chi^k \\
&\quad + \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \chi^i \\
&\quad - \sqrt{2}\, e^{K/2}\, g^{ij*}\, D_{j*} P^*\, \zeta \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta \sigma_m \bar{\lambda}^{(a)} + \bar{\zeta} \bar{\sigma}_m \lambda^{(a)})
\end{aligned}
$$

$$\delta_\zeta \lambda^{(a)} \;=\; \hat{F}_{ab}{}^{(a)} \sigma^{ab} \zeta \;-\; i\, g\, D^{(a)} \zeta$$

$$\begin{aligned} \delta_\zeta \psi_m \;=\;& 2 \tilde{\mathcal{D}}_m \zeta \;-\; \frac{i}{2} \sigma_{mn} \zeta \; g_{ij*}\, \chi^i \sigma^n \bar{\chi}^j \\ &+\; \frac{i}{2} (g_{mn} \;+\; \sigma_{mn}) \zeta\, \lambda^{(a)} \sigma^n \bar{\lambda}^{(a)} \\ &-\; \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \psi_m \\ &+\; i\, e^{K/2}\, P\, \sigma_m \bar{\zeta}\,. \end{aligned} \qquad \mathbf{(25.15)}$$

Here $\tilde{\mathcal{D}}_m \zeta$ is defined to be

$$\tilde{\mathcal{D}}_m \zeta \;=\; \partial_m \zeta \;+\; \zeta\, \omega_m \;+\; \frac{1}{4} (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \zeta\,, \qquad (25.16)$$

while the supercovariant expressions $\hat{\tilde{D}}_m A^i$ and $\hat{F}_{mn}$ are given by

$$\begin{aligned} \hat{\tilde{D}}_m A^i \;&=\; \hat{D}_m A^i \;-\; g\, v_m{}^{(a)} X^{i(a)} \\ &=\; \tilde{\mathcal{D}}_m A^i \;-\; \frac{1}{2} \sqrt{2}\; \psi_m \chi^i \end{aligned}$$

$$\begin{aligned} \hat{F}_{mn}{}^{(a)} \;&=\; \hat{D}_m v_n{}^{(a)} \;-\; \hat{D}_n v_m{}^{(a)} \\ &=\; F_{mn}{}^{(a)} \;-\; \frac{i}{2} [\, \psi_m \sigma_n \bar{\lambda}^{(a)} + \bar{\psi}_m \bar{\sigma}_n \lambda^{(a)} \\ &\qquad -\; \psi_n \sigma_m \bar{\lambda}^{(a)} - \bar{\psi}_n \bar{\sigma}_m \lambda^{(a)} \,] \end{aligned} \qquad (25.17)$$

The action (25.12) differs from that of Chapter XXIII by the addition of the gauge supermultiplets. The additional

fields change the form of the scalar potential from (23.10) to

$$\mathcal{V} \;=\; \frac{1}{2} e\, g^2\, D^{(a)2} \;+\; e^K [\, g^{ij*} (D_i P)(D_j P)^* \;-\; 3 P^* P \,]\,. \qquad \mathbf{(25.18)}$$

They also change the trace formula from (23.34) to

$$\begin{aligned} \mathrm{Str}\, M^2 \;&=\; \sum_{\text{spins } J} (-1)^{2J} (2J+1)\, \mathrm{Tr}\, M^2 \\ &=\; (n-1) [\, 2 m_\psi^2 \;-\; g^2 \langle D^{(a)2} \rangle \,] \\ &\quad +\; 2 g^2 \langle g^{ij*} D^{(a)}_{ij*} D^{(a)} \rangle \;-\; 2 \langle R_{ij*} G^i G^{j*} \rangle\, m_\psi^2\,, \end{aligned} \qquad \mathbf{(25.19)}$$

From the form of the transformation laws, we see the condition for spontaneous supersymmetry breaking is either

$$\langle D_i P \rangle \;\neq\; 0 \qquad (25.20)$$

or

$$\langle D^{(a)} \rangle \;\neq\; 0\,, \qquad (25.21)$$

for some value of i or (a). Depending on the relative magnitudes of (25.20) and (25.21), an appropriate linear combination of χ^i and $\lambda^{(a)}$ plays the role of the Goldstone fermion.

Note that the Lagrangian (25.12) explicitly contains the Killing potentials $D^{(a)}$. Their existence is both necessary and sufficient to gauge the group G. If the group G contains a U(1) factor, we know from previous arguments that the $D^{(a)}$ are not uniquely defined. There is an arbitrary integration constant associated with each U(1) factor,

$$D \;\to\; D \,+\, \xi\,. \tag{25.22}$$

In the globally supersymmetric case, shifts of these constants give rise to the Fayet-Iliopoulos D-term $\mathcal{L}_{\rm FI}$. The same is true in supergravity. By shifting the functions D, we find the gauge invariant supergravity version of $\mathcal{L}_{\rm FI}$:

$$\mathcal{L}_{\rm FI} \;=\; -\,\frac{1}{2}\,eg^2\,\xi^2 \;-\; eg^2\xi\,D \;-\; \frac{1}{2}\,eg\,\xi\,(\psi_m\sigma^m\bar\lambda \;-\; \bar\psi_m\bar\sigma^m\lambda)\,. \tag{25.23}$$

Note that the shift (25.22) changes the spinor covariant derivatives as well as the transformation laws (25.15). New terms proportional to ξ are induced in all expressions involving the Killing potentials D.

In the rest of this chapter, we will present the Lagrangian (25.12) in four-component notation, following the conventions described in Appendix A. Care should be exercised in comparing this formula to those in the references; conventions vary throughout the literature. With this said, we

write the Lagrangian as follows,

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}\,e\mathcal{R} \;-\; e\,g_{ij*}\,\tilde{\mathcal{D}}_m A^i\,\tilde{\mathcal{D}}^m A^{*j} \;-\; \frac{1}{2}\,e\,g^2\,D^{(a)2} \\
& -\frac{1}{4}\,e\,F_{mn}{}^{(a)}F^{mn(a)} \;-\; i\,e\,\bar\lambda^{(a)}_L\gamma^m\tilde{\mathcal{D}}_m\lambda^{(a)}_L \\
& -\,i\,e\,g_{ij*}\,\bar\chi^j_L\gamma^m\tilde{\mathcal{D}}_m\chi^i_L \;+\; e\,\epsilon^{k\ell mn}\,\bar\psi_{Lk}\gamma_\ell\tilde{\mathcal{D}}_m\psi_{Ln} \\
& +\,e\,g\,\sqrt{2}\,g_{ij*}\,X^{*j(a)}\;\bar\chi^i_R\lambda^{(a)}_L \;+\; e\,g\,\sqrt{2}\,g_{ij*}\,X^{i(a)}\;\bar\chi^j_L\lambda^{(a)}_R \\
& +\,\frac{1}{2}\,e\,g\,D^{(a)}\,\bar\psi_{Lm}\gamma^m\lambda^{(a)}_L \;-\; \frac{1}{2}\,e\,g\,D^{(a)}\,\bar\psi_{Rm}\gamma^m\lambda^{(a)}_R \\
& -\,\frac{1}{2}\,\sqrt{2}\,e\,g_{ij*}\,\tilde{\mathcal{D}}_n A^{*j}\,\bar\chi^i_R\gamma^m\gamma^n\psi_{Lm} \\
& -\,\frac{1}{2}\,\sqrt{2}\,e\,g_{ij*}\,\tilde{\mathcal{D}}_n A^{i}\,\bar\chi^j_L\gamma^m\gamma^n\psi_{Rm} \\
& +\,\frac{i}{4}\,e\,[\,\bar\psi_{Lm}\sigma^{ab}\gamma^m\lambda^{(a)}_L + \bar\psi_{Rm}\sigma^{ab}\gamma^m\lambda^{(a)}_R\,]\,[\,F_{ab}{}^{(a)} + \hat F_{ab}{}^{(a)}\,] \\
& +\,\frac{1}{4}\,e\,g_{ij*}\,[i\epsilon^{k\ell mn}\bar\psi_{Lk}\gamma_\ell\psi_{Lm} \;-\; \bar\psi_{Lm}\gamma^n\psi^m_L]\,\bar\chi^i_R\gamma_n\chi^j_R \\
& -\,\frac{1}{8}\,e\,[g_{ij*}g_{k\ell*} \;-\; 2R_{ij*k\ell*}]\;\bar\chi^i_R\chi^k_L\,\bar\chi^j_L\chi^\ell_R \\
& -\,\frac{1}{8}\,e\,g_{ij*}\;\bar\chi^j_L\gamma^m\chi^i_L\,\bar\lambda^{(a)}_R\gamma_m\lambda^{(a)}_R \\
& +\,\frac{3}{16}\,e\,\bar\lambda^{(a)}_L\gamma^m\lambda^{(a)}_L\,\bar\lambda^{(b)}_R\gamma_m\lambda^{(b)}_R \\
& -\,e\,\exp(K/2)\,\Big\{P^*\,\psi_{Ra}\sigma^{ab}\psi_{Lb} \;+\; P\,\bar\psi_{La}\sigma^{ab}\psi_{Rb}
\end{aligned}$$

$$+ \frac{i}{2}\sqrt{2}\, D_{i*}P^* \,\bar\chi^i_L\gamma^a\psi_{La} + \frac{i}{2}\sqrt{2}\, D_i P\, \bar\chi^i_R\gamma^a\psi_{Ra}$$

$$+ \frac{1}{2}\mathcal{D}_i D_j P\, \bar\chi^i_R\chi^j_L + \frac{1}{2}\mathcal{D}_{i*} D_{j*}P^*\, \bar\chi^i_L\chi^j_R \Big\}$$

$$- e\, e^K [\, g^{ij*}(D_iP)(D_jP)^* - 3P^*P\,] , \qquad (25.24)$$

where $\chi^i_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\chi^i$, and similarly for $\lambda^{(a)}$ and ψ_m. The covariant derivatives are defined as follows:

$$\tilde{\mathcal{D}}_m A^i = \partial_m A^i - g\, v_m{}^{(a)} X^{i(a)}$$

$$\begin{aligned}\tilde{\mathcal{D}}_m \chi^i_L = {} & \partial_m\chi^i_L - \omega_m\,\chi^i_L + \Gamma^i_{jk}\,\tilde{\mathcal{D}}_m A^j\,\chi^k_L \\ & - g\, v_m{}^{(a)} \frac{\partial X^{i(a)}}{A^j}\,\chi^j_L \\ & - \frac{1}{4}(K_j\tilde{\mathcal{D}}_m A^j - K_{j*}\tilde{\mathcal{D}}_m A^{*j})\,\chi^i_L \\ & - \frac{i}{2} g\, v_m{}^{(a)}\,\mathrm{Im}\, F^{(a)}\,\chi^i_L\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{D}}_m \lambda^{(a)}_L = {} & \partial_m\lambda^{(a)}_L - \omega_m\,\lambda^{(a)}_L - g\, f^{abc} v_m{}^{(b)}\lambda^{(c)}_L \\ & + \frac{1}{4}(K_j\tilde{\mathcal{D}}_m A^j - K_{j*}\tilde{\mathcal{D}}_m A^{*j})\,\lambda^{(a)}_L \\ & + \frac{i}{2} g\, v_m{}^{(b)}\,\mathrm{Im}\, F^{(b)}\,\lambda^{(a)}_L\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{D}}_m \psi_{Ln} = {} & \partial_m\psi_{Ln} - \omega_m\,\psi_{Ln} \\ & + \frac{1}{4}(K_j\tilde{\mathcal{D}}_m A^j - K_{j*}\tilde{\mathcal{D}}_m A^{*j})\,\psi_{Ln} \\ & + \frac{i}{2} g\, v_m{}^{(a)}\,\mathrm{Im}\, F^{(a)}\,\psi_{Ln}\end{aligned}$$

$$D_iP = P_i + K_iP$$

$$\begin{aligned}\mathcal{D}_i D_j P = {} & P_{ij} + K_{ij}P + K_i D_j P + K_j D_i P \\ & - K_iK_j P - \Gamma^k_{ij} D_k P\, . \end{aligned} \qquad (25.25)$$

The Lagrangian (25.24) is invariant under the supergravity transformations

$$\delta_\zeta e_m{}^a = i(\bar\zeta_L\gamma_a\psi_{Lm} + \bar\zeta_R\gamma_a\psi_{Rm})$$

$$\delta_\zeta A^i = \sqrt{2}\,\bar\zeta_R\chi^i_L$$

$$\begin{aligned}\delta_\zeta \chi^i_L = {} & i\sqrt{2}\,\gamma_m\zeta_R\,\hat{\tilde{D}}_m A^i - \Gamma^i_{jk}\,\delta_\zeta A^j\chi^k_L \\ & + \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\,\chi^i_L \\ & - \sqrt{2}\, e^{K/2} g^{ij*} D_{j*}P^*\,\zeta_L\end{aligned}$$

$$\delta_\zeta v_m{}^{(a)} = i(\bar\zeta_L\gamma_m\lambda^{(a)}_L + \bar\zeta_R\gamma_m\lambda^{(a)}_R)$$

$$\delta_\zeta\lambda^{(a)}_L = \hat{F}_{ab}{}^{(a)}\sigma^{ab}\zeta_L - i\, g\, D^{(a)}\,\zeta_L$$

$$\begin{aligned}\delta_\zeta\psi_{Lm} = {} & 2\tilde{\mathcal{D}}_m\zeta_L - \frac{i}{2}\sigma_{mn}\zeta_L\, g_{ij*}\,\bar\chi^i_R\gamma^n\chi^j_R \\ & + \frac{i}{2}(g_{mn} + \sigma_{mn})\zeta_L\,\bar\lambda^{(a)}_R\gamma^n\lambda^{(a)}_R \\ & - \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\,\psi_{Lm}\end{aligned}$$

$$+ \; i\, e^{K/2}\, P\, \gamma_m \zeta_R \,, \qquad (25.26)$$

where $\tilde{\mathcal{D}}_m \zeta_L$ is given by

$$\tilde{\mathcal{D}}_m \zeta_L \;=\; \partial_m \zeta_L \;-\; \omega_m\, \zeta_L \;+\; \frac{1}{4}\,(K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \zeta_L \,. \qquad (25.27)$$

References

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E. Cremmer, S. Ferrara, L. Girardello, and A. van Proeyen, *Nucl. Phys.* *B212*, 413 (1983).

Equations

$$\mathcal{L} \;=\; \int d^2\Theta\, 2\mathcal{E} \left[\frac{3}{8}\,(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp \left\{ -\frac{1}{3} \Big[K(\Phi, \Phi^+) + \Gamma(\Phi, \Phi^+, V) \Big] \right\} + \frac{1}{16 g^2}\, H_{(ab)}(\Phi)\, W^{(a)} W^{(b)} \;+\; P(\Phi) \right] \;+\; \text{h.c.} \,, \qquad \mathbf{(25.1)}$$

$$W_\alpha \;\equiv\; W_\alpha^{(a)} T^{(a)} \;=\; -\,\frac{1}{4}\,(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)\, e^{-V} \mathcal{D}_\alpha e^{V} \,. \qquad \mathbf{(25.2)}$$

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}\, e\, \mathcal{R} \;-\; e\, g_{ij*}\, \tilde{\mathcal{D}}_m A^i\, \tilde{\mathcal{D}}^m A^{*j} \;-\; \frac{1}{2}\, e\, g^2\, D^{(a)2} \\
& - \frac{1}{4}\, e\, F_{mn}{}^{(a)} F^{mn(a)} \;-\; i\, e\, \bar{\lambda}^{(a)} \bar{\sigma}^m \tilde{\mathcal{D}}_m \lambda^{(a)} \\
& - i\, e\, g_{ij*}\, \bar{\chi}^j \bar{\sigma}^m \tilde{\mathcal{D}}_m \chi^i \;+\; e\, \epsilon^{k\ell mn}\, \bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathcal{D}}_m \psi_n \\
& + \sqrt{2}\, e\, g\, g_{ij*}\, X^{*j(a)}\, \chi^i \lambda^{(a)} \;+\; \sqrt{2}\, e\, g\, g_{ij*}\, X^{i(a)}\, \bar{\chi}^j \bar{\lambda}^{(a)} \\
& - \frac{1}{2}\, e\, g\, D^{(a)}\, \psi_m \sigma^m \bar{\lambda}^{(a)} + \frac{1}{2}\, e\, g\, D^{(a)}\, \bar{\psi}_m \bar{\sigma}^m \lambda^{(a)} \\
& - \frac{1}{2}\, \sqrt{2}\, e\, g_{ij*}\, \tilde{\mathcal{D}}_n A^{*j}\, \chi^i \sigma^m \bar{\sigma}^n \psi_m \\
& - \frac{1}{2}\, \sqrt{2}\, e\, g_{ij*}\, \tilde{\mathcal{D}}_n A^i\, \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
& + \frac{i}{4}\, e\, [\, \psi_m \sigma^{ab} \sigma^m \bar{\lambda}^{(a)} + \bar{\psi}_m \bar{\sigma}^{ab} \bar{\sigma}^m \lambda^{(a)} \,]\, [\, F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)} \,] \\
& + \frac{1}{4}\, e\, g_{ij*}\, [\, i \epsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m \;+\; \psi_m \sigma^n \bar{\psi}^m \,]\, \chi^i \sigma_n \bar{\chi}^j \\
& - \frac{1}{8}\, e\, [\, g_{ij*} g_{k\ell*} \;-\; 2 R_{ij*k\ell*} \,]\, \chi^i \chi^k\, \bar{\chi}^j \bar{\chi}^\ell \\
& + \frac{1}{8}\, e\, g_{ij*}\, \bar{\chi}^j \bar{\sigma}^m \chi^i\, \bar{\lambda}^{(a)} \bar{\sigma}_m \lambda^{(a)} \\
& - \frac{3}{16}\, e\, \lambda^{(a)} \sigma^m \bar{\lambda}^{(a)}\, \lambda^{(b)} \sigma_m \bar{\lambda}^{(b)}
\end{aligned}$$

$$
\begin{aligned}
&- e\,\exp(K/2)\Big\{P^*\,\psi_a\sigma^{ab}\psi_b \;+\; P\,\bar\psi_a\bar\sigma^{ab}\bar\psi_b \\
&+ \frac{i}{2}\sqrt{2}\,D_iP\,\chi^i\sigma^a\bar\psi_a + \frac{i}{2}\sqrt{2}\,D_{i*}P^*\,\bar\chi^i\bar\sigma^a\psi_a \\
&+ \frac{1}{2}\mathcal{D}_i D_jP\,\chi^i\chi^j \;+\; \frac{1}{2}\mathcal{D}_{i*}D_{j*}P^*\,\bar\chi^i\bar\chi^j\Big\} \\
&- e\,e^K\left[g^{ij*}(D_iP)(D_jP)^* \;-\; 3P^*P\right]. \qquad (25.12)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{D}}_mA^i &= \partial_mA^i \;-\; g\,v_m{}^{(a)}X^{i(a)} \\
\tilde{\mathcal{D}}_m\chi^i &= \partial_m\chi^i \;+\; \chi^i\omega_m \;+\; \Gamma^i_{jk}\,\tilde{\mathcal{D}}_mA^j\,\chi^k \\
&\quad - g\,v_m{}^{(a)}\frac{\partial X^{i(a)}}{A^j}\,\chi^j \\
&\quad - \frac{1}{4}\,(K_j\tilde{\mathcal{D}}_mA^j - K_{j*}\tilde{\mathcal{D}}_mA^{*j})\,\chi^i \\
&\quad - \frac{i}{2}\,g\,v_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\,\chi^i \\
\tilde{\mathcal{D}}_m\lambda^{(a)} &= \partial_m\lambda^{(a)} \;+\; \lambda^{(a)}\omega_m - g\,f^{abc}v_m{}^{(b)}\lambda^{(c)} \\
&\quad + \frac{1}{4}\,(K_j\tilde{\mathcal{D}}_mA^j - K_{j*}\tilde{\mathcal{D}}_mA^{*j})\,\lambda^{(a)} \\
&\quad + \frac{i}{2}\,g\,v_m{}^{(b)}\,\mathrm{Im}\,F^{(b)}\,\lambda^{(a)} \\
\tilde{\mathcal{D}}_m\psi_n &= \partial_m\psi_n \;+\; \psi_n\,\omega_m \\
&\quad + \frac{1}{4}\,(K_j\tilde{\mathcal{D}}_mA^j - K_{j*}\tilde{\mathcal{D}}_mA^{*j})\,\psi_n \\
&\quad + \frac{i}{2}\,g\,v_m{}^{(a)}\,\mathrm{Im}\,F^{(a)}\,\psi_n
\end{aligned}
$$

$$
\begin{aligned}
D_iP &= P_i \;+\; K_iP \\
\mathcal{D}_i D_jP &= P_{ij} \;+\; K_{ij}\,P \;+\; K_i\,D_jP \;+\; K_j\,D_iP \\
&\quad - K_iK_j\,P \;-\; \Gamma^k_{ij}D_kP\,. \qquad (25.13)
\end{aligned}
$$

$$
\begin{aligned}
\delta A^i &= \epsilon^{(a)}\,X^{i(a)} \\
\delta\chi^i &= \epsilon^{(a)}\,\frac{\partial X^{i(a)}}{A^j}\,\chi^j \;+\; \frac{i}{2}\,\epsilon^{(a)}\,\mathrm{Im}\,F^{(a)}\,\chi^i \\
\delta\lambda^{(a)} &= f^{abc}\epsilon^{(b)}\,\lambda^{(c)} \;-\; \frac{i}{2}\,\epsilon^{(b)}\,\mathrm{Im}\,F^{(b)}\,\lambda^{(a)} \\
\delta v_m{}^{(a)} &= g^{-1}\partial_m\epsilon^{(a)} \;+\; f^{abc}\epsilon^{(b)}\,v_m{}^{(c)} \\
\delta\psi_n &= -\,\frac{i}{2}\,\epsilon^{(a)}\,\mathrm{Im}\,F^{(a)}\,\psi_n\,, \qquad (25.14)
\end{aligned}
$$

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta\sigma_a\bar\psi_m \;+\; \bar\zeta\bar\sigma_a\psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\,\zeta\chi^i \\
\delta_\zeta\chi^i &= i\sqrt{2}\,\sigma_m\bar\zeta\,\hat{\tilde{D}}_mA^i \;-\; \Gamma^i_{jk}\,\delta_\zeta A^j\chi^k \\
&\quad + \frac{1}{4}\,(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\,\chi^i \\
&\quad - \sqrt{2}\,e^{K/2}\,g^{ij*}\,D_{j*}P^*\,\zeta \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta\sigma_m\bar\lambda^{(a)} \;+\; \bar\zeta\bar\sigma_m\lambda^{(a)})
\end{aligned}
$$

$$\delta_\zeta \lambda^{(a)} \;=\; \hat{F}_{ab}{}^{(a)} \sigma^{ab} \zeta \;-\; i\, g\, D^{(a)}\, \zeta$$

$$\begin{aligned} \delta_\zeta \psi_m \;=\; & 2\tilde{\mathcal{D}}_m \zeta \;-\; \frac{i}{2} \sigma_{mn} \zeta \; g_{ij*}\, \chi^i \sigma^n \bar{\chi}^j \\ & +\; \frac{i}{2} (g_{mn} \;+\; \sigma_{mn}) \zeta\, \lambda^{(a)} \sigma^n \bar{\lambda}^{(a)} \\ & -\; \frac{1}{4} (K_j \delta_\zeta A^j - K_{j*} \delta_\zeta A^{*j})\, \psi_m \\ & +\; i\, e^{K/2}\, P\, \sigma_m \bar{\zeta}\, . \end{aligned} \qquad \textbf{(25.15)}$$

$$\mathcal{V} \;=\; \frac{1}{2} e\, g^2\, D^{(a)2} \;+\; e^K \left[\, g^{ij*} (D_i P)(D_j P)^* \;-\; 3 P^* P\, \right] . \qquad \textbf{(25.18)}$$

$$\begin{aligned} \mathrm{Str}\, M^2 \;=\; & \sum_{\text{spins } J} (-1)^{2J} \, (2J+1)\; \mathrm{Tr}\, M^2 \\ =\; & (n-1) \left[\, 2 m_\psi^2 \;-\; g^2 \langle D^{(a)2} \rangle \,\right] \\ & +\; 2 g^2 \langle g^{ij*} D^{(a)}_{ij*} D^{(a)} \rangle \;-\; 2 \langle R_{ij*}\, G^i G^{j*} \rangle\, m_\psi^2 \, , \end{aligned} \qquad \textbf{(25.19)}$$

Exercises

(1) Show that

$$W_\alpha \;=\; -\, \frac{1}{4} (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \;-\; 8R)(e^{-V} \mathcal{D}_\alpha e^V)$$

is gauge covariant under the following non-Abelian gauge transformation,

$$e^{V'} \;=\; e^{-i\Lambda^+}\, e^V\, e^{i\Lambda} \, ,$$

where

$$\bar{\mathcal{D}}_{\dot\alpha} \Lambda \;=\; \mathcal{D}_\alpha \Lambda^+ \;=\; 0 \, .$$

(2) Verify that

$$e^{-V} \mathcal{D}_\alpha e^V \;=\; \mathcal{D}_\alpha V \;-\; \frac{1}{2} [V, \mathcal{D}_\alpha V]$$

in the WZ gauge.

(3) For an Abelian group, the components of W were given in (19.28). Use the results of Exercise 7 in Chapter XIX to show

$$\begin{aligned} (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \;-\; 8R) [V, \mathcal{D}_\alpha V] \Big| \;&=\; 0 \\ \mathcal{D}_\beta (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \;-\; 8R) [V, \mathcal{D}_\alpha V] \Big| \;&=\; 8\, (\sigma^{ab} \epsilon)_{\beta\alpha}\, v_a v_b \\ \mathcal{D}^\beta \mathcal{D}_\beta (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \;-\; 8R) [V, \mathcal{D}_\alpha V] \Big| \;&=\; -16 i\, [v_{\alpha\dot\alpha}, \bar{\lambda}^{\dot\alpha}] \\ &\quad -\; 8 i\, [v^c, v_{\alpha\dot\alpha} \bar{\psi}_c{}^{\dot\alpha}] \, . \end{aligned}$$

Then find all the components of W for a non-Abelian group G.

(4) Compute

$$
\begin{aligned}
(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\,\Phi^{+}V = {} & 2\Theta\,\{\, iA^*(2\lambda + \sigma^c\bar\sigma^a\psi_c\, v_a) \\
& - \sqrt{2}\,\sigma^a\bar\chi\, v_a \,\} \\
& + \Theta\Theta\Big\{ i2\sqrt{2}\,\bar\chi\bar\lambda + 4i\,\hat{D}_c A^*\, v_c \\
& + A^*\Big[-2D + 2ie_c{}^m\mathcal{D}_m v^c \\
& - \frac{4}{3}v_a b^a + \bar\psi_c\bar\sigma^c\lambda \\
& - \bar\lambda\bar\sigma^c\psi_c - \psi^d\sigma^c\bar\psi_d\, v_c \Big]\Big\}
\end{aligned}
$$

and

$$
(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\,\Phi^{+}V^2 = 2\Theta\Theta\, A^* v_a v^a \,.
$$

(5) Check that

$$
(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} - 8R)\,\Phi^{+}V^3 = 0
$$

in the WZ gauge.

XXVI. LOW-ENERGY THEOREMS

In the study of chiral dynamics, nonlinear realizations of chiral symmetries have proven to be useful tools for constructing low-energy effective Lagrangians. In this chapter we shall see that similar techniques can be used to describe the low-energy effects of spontaneously broken supersymmetry. The resulting low-energy theorems describe the effective couplings of Goldstone and matter fields at energies far below the scale of the symmetry breaking.

The fact that the low-energy theorems hold for supersymmetry might seem surprising, for the usual proofs in chiral dynamics rely on the finite volume of a compact group. For the case of supersymmetry, the anticommuting nature of the group parameters makes such volumes vanish. Nevertheless, we shall see that alternative proofs can be supplied which validate the supersymmetric versions of the low-energy theorems.

In chiral dynamics, the low-energy theorems apply when a group G is spontaneously broken to a subgroup H. The subgroup H is linearly represented on the physical fields, while the remaining generators of G have a nonlinear realization in terms of the coset parameters for G/H. The coset parameters can be interpreted as Goldstone bosons associated with the spontaneous breaking of G down to H.

The nonlinear realizations of G are determined up to field redefinitions. They are often parametrized in certain canonical forms known as standard realizations. These realizations linearize on the subgroup H. Any linear representation of H can be promoted to a standard realization of G. Conversely, any realization of G that linearizes on H can be decomposed into a set standard realizations and Goldstone fields.

For the case of supersymmetry, the Lorentz group plays the role of the subgroup H. The remaining generators generate pure supersymmetry transformations. In Chapter XI we used this construction to find a nonlinear realization for the Goldstone fermion λ,

$$\begin{aligned}
\delta_\xi \lambda_\alpha(x) &= \frac{1}{\kappa}\,\xi_\alpha \;-\; i\,v_\xi^m(x)\,\partial_m \lambda_\alpha(x) \\
\delta_\xi \bar\lambda_{\dot\alpha}(x) &= \frac{1}{\kappa}\,\bar\xi_{\dot\alpha} \;-\; i\,v_\xi^m(x)\,\partial_m \bar\lambda_{\dot\alpha}(x) \\
v_\xi^m(x) &= \kappa\,[\lambda(x)\sigma^m\bar\xi \;-\; \xi\sigma^m\bar\lambda(x)]\,, \qquad \textbf{(26.1)}
\end{aligned}$$

where κ is a constant that parametrizes the supersymmetry breaking scale, analogous to f_π in chiral dynamics. These transformations can be lifted to superfield form using the techniques introduced in Chapter IV. The relevant construction is given in (4.11); for the case at hand, it gives a superfield Λ whose lowest component is the Goldstino λ:

$$\Lambda_\alpha(x,\theta,\bar\theta) \;=\; \exp(\theta Q + \bar\theta\bar Q) \;\times\; \lambda_\alpha(x)$$
$$\bar\Lambda_{\dot\alpha}(x,\theta,\bar\theta) \;=\; \exp(\theta Q + \bar\theta\bar Q) \;\times\; \bar\lambda_{\dot\alpha}(x)\,. \qquad (26.2)$$

The superfield Λ is built out of λ, its derivatives, and the constant κ:

$$\Lambda_\alpha(x,\theta,\bar\theta) \;=\; \lambda_\alpha(x) \;+\; \frac{1}{\kappa}\theta_\alpha \;+\; ...$$
$$\bar\Lambda_{\dot\alpha}(x,\theta,\bar\theta) \;=\; \bar\lambda_{\dot\alpha}(x) \;+\; \frac{1}{\kappa}\bar\theta_{\dot\alpha} \;+\; .... \qquad (26.3)$$

It is a short exercise to show that the transformations (4.10) reduce to (26.1) when applied to the lowest component of Λ.

The Goldstone superfield Λ can also be defined as the solution to a certain set of constraints. These conditions can be found with the help of the identity

$$D_\alpha \;\exp(\theta Q + \bar\theta\bar Q) \times \;=\; \exp(\theta Q + \bar\theta\bar Q)\; Q_\alpha \;\times$$
$$\bar D_{\dot\alpha} \;\exp(\theta Q + \bar\theta\bar Q) \times \;=\; \exp(\theta Q + \bar\theta\bar Q)\; \bar Q_{\dot\alpha} \;\times\;. \qquad (26.4)$$

Applying (26.4) to (26.2), and using (26.1), we find

$$D_\beta\Lambda_\alpha \;=\; \frac{1}{\kappa}\epsilon_{\alpha\beta} \;+\; i\kappa\,\sigma_{\beta\dot\beta}{}^m\,\bar\Lambda^{\dot\beta}\,\partial_m\Lambda_\alpha$$

$$\bar D_{\dot\beta}\Lambda_\alpha \;=\; -\,i\kappa\,\Lambda^\beta\,\sigma_{\beta\dot\beta}{}^m\partial_m\Lambda_\alpha\,. \qquad \mathbf{(26.5)}$$

These constraints are consistent with the D algebra (4.7). Their solution is the superfield Λ as defined in (26.2).

To derive the low-energy theorems, we need the supersymmetric analogs of standard realizations. We shall define a standard realization of supersymmetry to have the following transformation law,

$$\delta_\xi f(x) \;=\; -\,i\,v_\xi^n(x)\,\frac{\partial}{\partial x^n}\,f(x)\,, \qquad \mathbf{(26.6)}$$

where v_ξ^n is given in (26.1). In the exercises, you will show that (26.6) closes into the supersymmetry algebra. The field f is free to carry an arbitrary set of Lorentz or internal symmetry indices.

As with the Goldstone fermion λ, we would like to promote f to a superfield F whose variation reduces to (26.6) when restricted to its lowest component. Using the construction of Chapter IV, we find

$$F(x,\theta,\bar\theta) \;=\; \exp(\theta Q + \bar\theta\bar Q) \;\times\; f(x)$$
$$\;=\; f(x) \;-\; i\,v_\theta^n(x)\,\frac{\partial}{\partial x^n}\,f(x) \;+\; .... \qquad (26.7)$$

In (26.7), the superfield F carries the same indices as f. Its component fields are built out of λ, f and their derivatives. It is also possible to derive (26.7) from the constraint equations

$$D_\alpha F \;=\; i\kappa\,(\sigma^m\bar\Lambda)_\alpha\,\partial_m F$$

$$\bar D_{\dot\alpha} F \;=\; -i\kappa\,(\Lambda\sigma^m)_{\dot\beta}\,\partial_m F\;. \qquad (26.8)$$

In the case of chiral dynamics, it is well-known how to convert any nonlinear realization into a standard realization. As shown in Appendix E, one simply applies a finite group transformation with the field-dependent parameter that would transform the Goldstone fields to zero. This procedure also works for supersymmetry. To see this, let $\tilde f$ be an arbitrary nonlinear realization of supersymmetry, and let $\tilde F$ be its superfield extension. A standard realization F' is obtained by taking

$$F'(x,\theta,\bar\theta,\lambda) \;=\; e^{\xi Q+\bar\xi\bar Q}\;\tilde F(x,\theta,\bar\theta)\big|_{\xi=-\kappa\lambda}\;, \qquad (26.9)$$

where the Q's are the differential operators (4.4), and the substitution $\xi = -\kappa\lambda$ is made after all the differentiations are performed. We can also write (26.9) in a more explicit form, avoiding derivatives on λ, by changing arguments as follows:

$$F'(x,\theta,\bar\theta,\lambda) \;=\; \exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\;\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta}+\bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right]$$

$$\times\;\tilde F(x,\theta,\bar\theta)\Big|_{x=y}\;. \qquad (26.10)$$

In this expression, we are able to separate the exponents because of the fact that

$$\left[\left(\lambda(y)\frac{\partial}{\partial\theta}+\bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right),\;v_\theta^m(y)\right] \;=\; 0\;. \qquad (26.11)$$

To show that f' is a standard realization, we must compute the change in F' from a supersymmetry transformation. This is most easily done using (26.10). The variation of v_θ^m follows from (26.1):

$$\delta_\xi v_\theta^m(y) \;=\; \xi\sigma^m\bar\theta \;-\; \theta\sigma^m\bar\xi \;-\; i\,v_\xi^n(y)\,\frac{\partial}{\partial y^n}v_\theta^m(y)\;. \qquad (26.12)$$

Using (26.1), (26.12) and (4.4), we can then compute the change in F',

$$\delta_\xi F'(x,\theta,\bar\theta,\lambda) \;=\; \left\{\left[i(\xi\sigma^m\bar\theta \;-\; \theta\sigma^m\bar\xi)\,\frac{\partial}{\partial x^m}\right.\right.$$

$$\left.\left.-\;i\,v_\xi^n(y)\,\frac{\partial}{\partial y^n}\right]\;\exp\left[iv_\theta^m(y)\frac{\partial}{\partial x^m}\right]\right\}$$

$$\times\;\exp\left[-\kappa\left(\lambda(y)\frac{\partial}{\partial\theta}\;+\;\bar\lambda(y)\frac{\partial}{\partial\bar\theta}\right)\right]\;\tilde F(x,\theta,\bar\theta)$$

$$
\begin{aligned}
& + \exp\left[i\, v_\theta^m(y) \frac{\partial}{\partial x^m}\right] \left\{ \left[- \xi \frac{\partial}{\partial \theta} - \bar{\xi} \frac{\partial}{\partial \bar{\theta}} \right.\right. \\
& \qquad \left. - i\, v_\xi^n(y) \frac{\partial}{\partial y^n} \right] \\
& \quad \times \exp\left[-\kappa \left(\lambda(y) \frac{\partial}{\partial \theta} + \bar{\lambda}(y) \frac{\partial}{\partial \bar{\theta}} \right) \right] \Bigg\} \tilde{F}(x,\theta,\bar{\theta}) \\
& + \exp\left[i v_\theta^m(y) \frac{\partial}{\partial x^m} \right] \exp\Big[- \kappa \Big(\lambda(y) \frac{\partial}{\partial \theta} \\
& \qquad + \bar{\lambda}(y) \frac{\partial}{\partial \bar{\theta}} \Big) \Big] \left\{ \xi \frac{\partial}{\partial \theta} + \bar{\xi} \frac{\partial}{\partial \bar{\theta}} \right. \\
& \qquad \left. - i\, (\xi \sigma^m \bar{\theta} - \theta \sigma^m \bar{\xi}) \frac{\partial}{\partial x^m} \right\} \tilde{F}(x,\theta,\bar{\theta}) \Big|_{x=y} \\
& = - i\, v_\xi^n(x) \frac{\partial}{\partial x^n} F'(x,\theta,\bar{\theta},\lambda) \, . \qquad (26.13)
\end{aligned}
$$

Taking the lowest component, we see that f' indeed transforms as a standard realization,

$$
\delta_\xi f'(x,\lambda) \;=\; - i\, v_\xi^n(x) \frac{\partial}{\partial x^n} f'(x,\lambda) \, . \qquad (26.14)
$$

As above, the fields f' and F' can carry any Lorentz or internal symmetry indices.

With these results, we are now in a position to supersymmetrize any Lorentz-invariant Lagrangian. The first step is

to find a Lagrangian for the Goldstone spinor Λ. Two obvious choices are

$$
\mathcal{L}_0 \;=\; -\frac{\kappa^2}{2} \int d^2\theta d^2\bar{\theta} \;\; \Lambda^2 \bar{\Lambda}^2 \qquad \mathbf{(26.15)}
$$

and

$$
\mathcal{L}_1 \;=\; -\frac{1}{2} \int d^2\theta d^2\bar{\theta} \;\; (\Lambda^2 + \bar{\Lambda}^2) \, . \qquad (26.16)
$$

It is not hard to show that the highest component of $\Lambda^2 + \bar{\Lambda}^2$ is a total spacetime derivative, so (26.16) is unsuitable for a supersymmetric action. In contrast, (26.15) is perfectly fine, and coincides with (11.11) when expanded in terms of component fields. We shall take it to be the Lagrangian for the Goldstone fermion.

The next step is to construct the matter superfields. We start with the original matter fields, which have well-defined transformations with respect to the Lorentz and internal symmetry groups. We assign the fields supersymmetry transformations via (26.6), and promote them to superfields via (26.7). In this way we build a superfield out of each matter field in the original theory.

The final step is to construct the supersymmetric matter coupling. We start with the original Lagrangian $\tilde{\mathcal{L}}$, and replace all the matter fields by their corresponding superfields.

This gives a superfield Lagrangian whose lowest component is the original Lagrangian. We then turn this lowest component into a highest component by multiplying the superfield expression by $\Lambda^2\bar{\Lambda}^2$,

$$\Lambda^2\bar{\Lambda}^2 \;=\; \frac{1}{\kappa^4}\,\theta^2\bar{\theta}^2 \;+\; \ldots . \qquad (26.17)$$

This gives a fully supersymmetric Lagrangian

$$\mathcal{L} \;=\; \kappa^4 \int d^2\theta d^2\bar{\theta}\;\Lambda^2\bar{\Lambda}^2\;\tilde{\mathcal{L}} \qquad \mathbf{(26.18)}$$

whose λ-independent part is just the original Lagrangian $\tilde{\mathcal{L}}$.

As usual in the theory of nonlinear realizations, it is always possible to include higher-derivative terms in the effective action. For example, a contribution of the form

$$\mathcal{L}_2 \;=\; \int d^2\theta d^2\bar{\theta}\;(\bar{D}^2\Lambda^2)(D^2\bar{\Lambda}^2)$$

$$\sim\; (\partial_m\lambda\sigma^{mn}\partial_n\lambda)(\partial_k\bar{\lambda}\bar{\sigma}^{k\ell}\partial_\ell\bar{\lambda}) \;+\; \ldots \qquad (26.19)$$

adds a higher-derivative interaction to $\mathcal{L}$. The coefficients of such terms are not determined by symmetry, and must be regarded as parameters of the theory. The leading term in the derivative expansion is the only term that is unique. At high energies, where the higher-order terms become important, the predictive power of the theorems breaks down.

The Lagrangian (26.18) describes the low-energy interactions in a theory where supersymmetry is spontaneously broken at some scale much greater than the energies involved in the low-energy effective theory. For example, the formalism would apply to the situation where all the supersymmetric partners of the physical fields are very heavy (except for the Goldstino). In this case, the low-energy scattering amplitudes are determined by the Goldstino couplings. The only signals of supersymmetry are the nonlinear couplings of the Goldstino to the physical fields.

To illustrate this construction, let us consider the case of a free scalar field $a(x)$ and a free spinor field $\psi(x)$,

$$\tilde{\mathcal{L}} \;=\; -\frac{1}{2}\,\partial_m a\,\partial^m a \;-\; \frac{1}{2}\,m^2\,a^2$$

$$-\,i\,\psi\sigma^m\partial_m\bar{\psi} \;-\; \frac{1}{2}\,\mu\,(\psi^2+\bar{\psi}^2)\,. \qquad (26.20)$$

We supersymmetrize the Lagrangian by assigning transformations to a and ψ via (26.6), and lifting them to superfields A and Ψ that satisfy the constraints (26.8). We then replace the fields in (26.20) by A and Ψ, to find the superfield Lagrangian $\mathcal{L}$,

$$\mathcal{L} \;=\; \mathcal{L}_0 \;+\; \int d^2\theta d^2\bar{\theta}\;\Lambda^2\bar{\Lambda}^2\;\Big[-\frac{1}{2}\,\partial_m A\,\partial^m A \;-\; \frac{1}{2}\,m^2\,A^2$$

$$-i\,\Psi\sigma^m\partial_m\bar\Psi \;-\; \frac{1}{2}\,\mu\,(\Psi^2+\bar\Psi^2)\Big] \;. \qquad (26.21)$$

The Lagrangian (26.21) should be expanded in terms of the Goldstone spinor λ. A helpful trick is to replace the $d^2\theta d^2\bar\theta$ by $D^2\bar D^2/16$, and use the constraints (26.5) and (26.8) to compute the D and $\bar D$ derivatives. To second order in λ, the resulting Lagrangian is of the form

$$\mathcal{L} \;=\; -\frac{1}{2\kappa^2} \;-\; i\,\lambda\sigma^m\partial_m\bar\lambda \;+\; \frac{i}{\kappa^2}\,\lambda\sigma^m\bar\lambda\,\partial_m\tilde{\mathcal{L}}$$

$$+\; \frac{i}{\kappa^2}\,(\lambda\sigma^m\partial^n\bar\lambda - \partial^n\lambda\sigma^m\bar\lambda)\,T_{mn} \;+\; .... \qquad \mathbf{(26.22)}$$

At low energies, we see that the Goldstino couples to the energy-momentum tensor T_{mn}, independent of the details of the symmetry breaking. This is the low-energy theorem for supersymmetry.

References

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S. Samuel and J. Wess, *Nucl. Phys. B221*, 153 (1983).

Equations

$$\delta_\xi\lambda_\alpha(x) \;=\; \frac{1}{\kappa}\,\xi_\alpha \;-\; i\,v_\xi^m(x)\,\partial_m\lambda_\alpha(x)$$

$$\delta_\xi\bar\lambda_{\dot\alpha}(x) \;=\; \frac{1}{\kappa}\,\bar\xi_{\dot\alpha} \;-\; i\,v_\xi^m(x)\,\partial_m\bar\lambda_{\dot\alpha}(x)$$

$$v_\xi^m(x) \;=\; \kappa\,[\lambda(x)\sigma^m\bar\xi \;-\; \xi\sigma^m\bar\lambda(x)]\;. \qquad \mathbf{(26.1)}$$

$$D_\beta\Lambda_\alpha \;=\; \frac{1}{\kappa}\,\epsilon_{\alpha\beta} \;+\; i\kappa\,\sigma_{\alpha\dot\beta}{}^{m}\,\bar\Lambda^{\dot\beta}\,\partial_m\Lambda_\alpha$$

$$\bar D_{\dot\beta}\Lambda_\alpha \;=\; -\,i\kappa\,\Lambda^\beta\,\sigma_{\beta\dot\beta}{}^{m}\partial_m\Lambda_\alpha\;. \qquad \mathbf{(26.5)}$$

$$\delta_\xi f(x) \;=\; -\,i\,v_\xi^n(x)\,\frac{\partial}{\partial x^n}\,f(x)\;. \qquad \mathbf{(26.6)}$$

$$D_\alpha F \;=\; i\kappa\,(\sigma^m\bar\Lambda)_\alpha\,\partial_m F$$

$$\bar D_{\dot\alpha} F \;=\; -i\kappa\,(\Lambda\sigma^m)_{\dot\beta}\,\partial_m F\;. \qquad \mathbf{(26.8)}$$

$$\mathcal{L}_0 \;=\; -\frac{\kappa^2}{2}\int d^2\theta d^2\bar\theta\;\;\Lambda^2\bar\Lambda^2\;. \qquad \mathbf{(26.15)}$$

$$\mathcal{L} \;=\; \kappa^4\int d^2\theta d^2\bar\theta\;\Lambda^2\bar\Lambda^2\;\tilde{\mathcal{L}}\;. \qquad \mathbf{(26.18)}$$

$$\mathcal{L} = -\frac{1}{2\kappa^2} - i\,\lambda\sigma^m\partial_m\bar{\lambda} + \frac{i}{\kappa^2}\lambda\sigma^m\bar{\lambda}\,\partial_m\tilde{\mathcal{C}}$$

$$+ \frac{i}{\kappa^2}(\lambda\sigma^m\partial^n\bar{\lambda} - \partial^n\lambda\sigma^m\bar{\lambda})\,T_{mn} + \ldots \qquad \mathbf{(26.22)}$$

Exercises

(1) Show that (26.2) satisfies the constraints (26.5).

(2) Check that the transformation law (26.6) for a standard realization closes into the supersymmetry algebra.

(3) Verify that (26.7) is a solution to the constraints (26.8).

(4) Show that (26.15) coincides with (11.11) when expanded in terms of component fields.

(5) The Lagrangian (26.21) is supersymmetric because the derivative of a superfield is still a superfield. However, the derivative of a standard realization is not a standard realization. Use the techniques introduced here and in Appendix E to find a "covariant derivative" Δ that preserves the transformation properties of a stan-

dard realization. The Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\;\Lambda^2\bar{\Lambda}^2\;\Big[-\frac{1}{2}\Delta_m A\,\Delta^m A - \frac{1}{2}m^2A^2$$

$$-i\,\Psi\sigma^m\Delta_m\bar{\Psi} - \frac{1}{2}\mu\,(\Psi^2 + \bar{\Psi}^2)\Big]$$

is another possible extension of (26.20). It differs from (26.21) by higher-derivative terms in λ. The derivative Δ is natural to use when gauging an internal symmetry if the vector superfields belong to a standard realization.

(6) Show that (26.21) reduces to (26.22) in terms of components fields.

APPENDIX C
KAHLER GEOMETRY

The matter couplings of chiral multiplets are conveniently described in the language of Kähler geometry. It is useful, therefore, to introduce the notion of a Kähler manifold. A Kähler manifold is a special type of analytic Riemann manifold, subject to certain conditions that we will discuss below. Since the manifold is analytic, it can be parametrized in terms of complex coordinates a^i and a^{*i}, where $i = 1, ..., n$. Under an analytic coordinate transformation,

$$a^i \;=\; a^i(a') \qquad\qquad a^{*i} \;=\; a^{*i}(a^{*\prime})\,, \tag{C.1}$$

the differentials and derivatives transform as follows:

$$\begin{aligned} da'^i \;&=\; \frac{\partial a'^i}{\partial a^j}\, da^j & da^{*\prime i} \;&=\; \frac{\partial a^{*\prime i}}{\partial a^{*j}}\, da^{*j} \\ \frac{\partial}{\partial a'^i} \;&=\; \frac{\partial a^j}{\partial a'^i}\, \frac{\partial}{\partial a^j} & \frac{\partial}{\partial a^{*\prime i}} \;&=\; \frac{\partial a^{*j}}{\partial a^{*\prime i}}\, \frac{\partial}{\partial a^{*j}}\,. \end{aligned} \tag{C.2}$$

These transformations preserve the analytic nature of the coordinates. They also define the transformations of covariant and contravariant vector fields,

$$\begin{aligned} V'_i(a', a^{*\prime}) \;&=\; \frac{\partial a^j}{\partial a'^i}\, V_j(a, a^*) \\ V'^i(a', a^{*\prime}) \;&=\; \frac{\partial a'^i}{\partial a^j}\, V^j(a, a^*) \\ V'_{i*}(a', a^{*\prime}) \;&=\; \frac{\partial a^{*j}}{\partial a^{*\prime i}}\, V_{j*}(a, a^*) \\ V'^{i*}(a', a^{*\prime}) \;&=\; \frac{\partial a^{*\prime i}}{\partial a^{*j}}\, V^{j*}(a, a^*)\,. \end{aligned} \tag{C.3}$$

The first condition on a Kähler manifold is that it be endowed with a hermitian metric g_{ij*}. The metric must be positive definite and invertible, which allows us to raise and lower the indices i and j^*:

$$\begin{aligned} V_i \;&=\; g_{ij*}\, V^{j*} & V_{j*} \;&=\; g_{ij*}\, V^i \\ V^i \;&=\; g^{ij*}\, V_{j*} & V^{j*} \;&=\; g^{ij*}\, V_i\,. \end{aligned} \tag{C.4}$$

The second requirement is that the covariant derivative must respect the analytic structure. This implies that $\Gamma^{k*}_{ij} = \Gamma^{k*}_{i*j} = 0$, so the covariant derivative is of the following form:

$$\begin{aligned} \nabla_i V_j \;&=\; \frac{\partial}{\partial a^i}\, V_j \;-\; \Gamma^k_{ij}\, V_k \\ \nabla_{i*} V_j \;&=\; \frac{\partial}{\partial a^{*i}}\, V_j \;-\; \Gamma^k_{i*j}\, V_k\,. \end{aligned} \tag{C.5}$$

The third condition is that the connection be compatible with the hermitian metric. This imposes the additional restriction

$$\nabla_k \, g_{ij*} = 0 \qquad \nabla_{k*} \, g_{ij*} = 0 \,. \tag{C.6}$$

The transformation law for the connection is chosen to assure that covariant derivatives of tensors transform as tensors. This implies

$$\begin{aligned} \Gamma'^{k}_{ij} &= \frac{\partial a^{\ell}}{\partial a'^{i}} \frac{\partial a^{m}}{\partial a'^{j}} \frac{\partial a'^{k}}{\partial a^{n}} \Gamma^{n}_{\ell m} + \frac{\partial^2 a^{n}}{\partial a'^{i} \partial a'^{j}} \frac{\partial a'^{k}}{\partial a^{n}} \\ \Gamma'^{k}_{i*j} &= \frac{\partial a^{*\ell}}{\partial a^{*\prime i}} \frac{\partial a^{m}}{\partial a'^{j}} \frac{\partial a'^{k}}{\partial a^{n}} \Gamma^{n}_{\ell * m} \,. \end{aligned} \tag{C.7}$$

The first of the equations (C.7) tells us that it is consistent to set the torsion to zero, leaving only the symmetric part of the connection,

$$\Gamma^{k}_{ij} = \Gamma^{k}_{ji} \,. \tag{C.8}$$

The second equation implies that it is also permissible to demand

$$\Gamma^{k}_{i*j} = 0 \,. \tag{C.9}$$

Equations (C.8) and (C.9) are the two remaining postulates that define a Kähler manifold.

On a Kähler manifold, the conditions discussed above imply that the only nonvanishing components of the connection are Γ^{i}_{jk} and its complex conjugate Γ^{i*}_{j*k*}. Equation (C.6) can be solved to give

$$\Gamma^{k}_{ij} = g^{k\ell *} \frac{\partial}{\partial a^{i}} g_{j\ell *} \,. \tag{C.10}$$

Since

$$\Gamma^{k}_{ij} = \Gamma^{k}_{ji} \,, \tag{C.11}$$

the metric must obey the following integrability condition:

$$\frac{\partial}{\partial a^{k}} g_{ij*} = \frac{\partial}{\partial a^{i}} g_{kj*} \,. \tag{C.12}$$

A similar relation holds for the conjugate derivatives,

$$\frac{\partial}{\partial a^{*k}} g_{ij*} = \frac{\partial}{\partial a^{*j}} g_{ik*} \,. \tag{C.13}$$

Equations (C.12) and (C.13) imply that the metric is the derivative of a scalar function K,

$$g_{ij*} = \frac{\partial}{\partial a^{i}} \frac{\partial}{\partial a^{*j}} K(a, a^{*}) \,. \tag{C.14}$$

The function K is called the Kähler potential; its derivatives determine the metric and the connection. Kähler manifolds are often *defined* through (C.14), in which case the conditions on the connection are then deduced.

The Kähler potential completely specifies the Kähler geometry. Note that the metric g_{ij*} is invariant under analytic shifts of K,

$$K(a,a^*) \rightarrow K(a,a^*) + F(a) + F^*(a^*) . \qquad (C.15)$$

Such a shift is called a Kähler transformation of the Kähler potential.

The curvature of a Kähler manifold can be defined as the commutator of two covariant derivatives:

$$[\nabla_i, \nabla_j] V_k = R^{\ell}_{ijk} V_\ell$$

$$[\nabla_i, \nabla_{j*}] V_k = R^{\ell}_{ij*k} V_\ell . \qquad (C.16)$$

The upper index on R can be lowered with the help of the metric, giving

$$R_{ijk\ell*} = g_{m\ell*} R^m_{ijk}$$

$$R_{ij*k\ell*} = g_{m\ell*} R^m_{ij*k} . \qquad (C.17)$$

In Exercise 3 we will see that only $R_{ij*k\ell*}$ and its complex conjugate are nonvanishing. From the definition of the covariant derivative, we find

$$R_{ij*k\ell*} = g_{m\ell*} \frac{\partial}{\partial a^{*j}} \Gamma^m_{ik}$$

$$= \frac{\partial}{\partial a^i} \frac{\partial}{\partial a^{*j}} g_{k\ell*} - g^{mn*} \left(\frac{\partial}{\partial a^{*j}} g_{m\ell*} \right) \left(\frac{\partial}{\partial a^i} g_{kn*} \right) . \qquad (C.18)$$

Using (C.16), (C.17) and (C.18), it is not hard to show that the curvature obeys the following symmetries,

$$R_{ij*k\ell*} = - R_{ij*\ell*k} = - R_{j*ik\ell*} = R_{j*i\ell*k} . \qquad (C.19)$$

This is all the Kähler geometry we need to discuss the general couplings of chiral fields.

References

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K. Itoh, T. Kugo and H. Kunitomo, *Nucl. Phys.* *B263*, 295 (1986).

Exercises

(1) Verify the transformation law (C.7) for the connection Γ.

(2) Impose the Kähler conditions (C.8) and (C.9), and solve for the connection in terms of the metric.

(3) Show that $R_{ij*k\ell*}$ is the only nonvanishing component of the curvature on a Kähler manifold, and solve for the curvature in terms of the metric.

(4) Compute the curvature, Ricci tensor and curvature scalar for the manifold with Kähler potential $K = -3\log(1 - \frac{1}{3}a^{*i}a^i)$.

(5) Show that in the language of differential forms, the Kähler condition (C.14) is equivalent to the statement that the fundamental form

$$\Omega = \frac{i}{2}\, g_{ij*}\, da^i\, da^{*j}$$

is closed,

$$d\Omega = 0\ .$$

APPENDIX D
ISOMETRIES AND KAHLER GEOMETRY

In this appendix we will discuss the isometries of Kähler manifolds. The techniques we introduce will prove useful in constructing gauge invariant matter couplings, in flat and curved space. Before specializing to Kähler manifolds, however, we first define the general notion of an isometry group. Consider, therefore, an arbitrary differentiable manifold $\mathcal{M}$, and a set of parametrized curves that fill the manifold without intersecting. Then construct the map $\phi_t : \mathcal{M} \to \mathcal{M}$ which takes each point $p \in \mathcal{M}$ a parameter distance t along the unique curve that passes through p. This map also induces a map on the tangent space. If the induced map leaves the metric invariant, ϕ_t is said to be an *isometry* of the manifold $\mathcal{M}$. The set of isometries forms a group, called the *isometry group* of $\mathcal{M}$.

Curves and vectors are closely related geometrical objects. Consider a curve λ, described by real coordinates $x^i = x^i(t)$, and a differentiable function $f : \mathcal{M} \to \mathbf{R}$. Then the *directional derivative* of f along the curve λ is given by

$$\frac{df}{dt} = \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} \equiv Xf , \qquad \text{(D.1)}$$

and the operator

$$X = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} \qquad \text{(D.2)}$$

maps any function f to its directional derivative along λ. In mathematical language, X is called a vector, and the dx^i/dt are its components. The operator X is the natural generalization of a tangent vector to curved space.

This definition of a vector can be applied to a space-filling set of curves as well. The components dx^i/dt become functions on $\mathcal{M}$, and $X \equiv (dx^i/dt)\partial/\partial x^i$ is known as a *vector field*.

Alternatively, given a set of continuous functions X^i on $\mathcal{M}$, it is always possible to define an associated set of *integral curves* $x^i(t)$ through the equations

$$\frac{dx^i}{dt} = X^i . \qquad \text{(D.3)}$$

The corresponding vector field is just $X = X^i\partial/\partial x^i$. Locally, such curves can never cross because the solutions to (D.3) are unique. They are also globally well-defined because (D.3) holds at each point of the manifold $\mathcal{M}$.

Thus we have seen that sets of space-filling curves are in one-one correspondence with vector fields X. The map ϕ_t defines a motion along the integral curve defined by X.

As with any map of a manifold onto itself, ϕ_t induces a map between vectors in the tangent space. The induced map allows us to compare vectors at different points along integral curves. To construct it explicitly, let x^i denote the coordinates at p, and x'^i the coordinates at p'. Then let Y be a vector field, with components $Y^i(x)$ at p. The components $Y^i(x)$ at p can be mapped to components $\tilde{Y}^i(x')$ at p' as follows,

$$\tilde{Y}^i(x') \; = \; \frac{\partial x'^i}{\partial x^j} \, Y^j(x(x')) \, . \qquad \text{(D.4)}$$

Equation (D.4) defines a map of vectors at p onto vectors at p'. It is sometimes called Lie transport. The Lie-transported vector field

$$\tilde{Y} \; \equiv \; \tilde{Y}^i(x') \, \frac{\partial}{\partial x'^i} \qquad \text{(D.5)}$$

is defined for all points p' along the integral curve. Infinitesimally, if $x'^i = x^i + X^i \delta t$, (D.4) reduces to

$$\tilde{Y}^i(x') \; = \; Y^i(x') \; + \; \frac{\partial X^i}{\partial x^j} \, Y^j \delta t \; - \; \frac{\partial Y^i}{\partial x^j} \, X^j \delta t \; , \qquad \text{(D.6)}$$

and the new field $\tilde{Y}$ is infinitesimally close to Y.

Since Y and $\tilde{Y}$ are both defined at the same points, it makes sense to take their difference, and construct the *Lie derivative* of Y with respect to X,

$$\begin{aligned} (\pounds_X Y)^i \; &\equiv \; \lim_{\delta t \to 0} \; \frac{Y^i(x') - \tilde{Y}^i(x')}{\delta t} \\ &= \; X^j \, \frac{\partial}{\partial x^j} \, Y^i \; - \; Y^j \, \frac{\partial}{\partial x^j} \, X^i \\ &\equiv \; [X, Y]^i \, . \end{aligned} \qquad \text{(D.7)}$$

The Lie derivative of two vector fields gives a third vector field on the manifold $\mathcal{M}$.

Using similar logic, the definition of the Lie derivative can be generalized to any other tensor field. For example, an expression analogous to (D.4) implies that the Lie derivative of a covariant vector field is as follows,

$$(\pounds_X Y)_i \; = \; X^j \, \frac{\partial}{\partial x^j} \, Y_i \; + \; Y_j \, \frac{\partial}{\partial x^i} \, X^j \, . \qquad \text{(D.8)}$$

In a similar fashion, the Lie derivative of the metric is given by

$$\begin{aligned} (\pounds_X g)_{ij} \; &= \; X^k \, \frac{\partial}{\partial x^k} \, g_{ij} \; + \; g_{ik} \, \frac{\partial}{\partial x^j} \, X^k \; + \; g_{jk} \, \frac{\partial}{\partial x^i} \, X^k \\ &= \; \nabla_i X_j \; + \; \nabla_j X_i \; , \end{aligned} \qquad \text{(D.9)}$$

where $X_i = g_{ij} X^j$ and $\nabla_i X_j = \partial_i X_j - \Gamma^k_{ij} X_k$ contains the torsion-free connection compatible with the metric g_{ij}.

A field is *invariant* under Lie transport if it has vanishing Lie derivative. If the metric is invariant, then

$$(\pounds_X g)_{ij} \;=\; \nabla_i X_j \;+\; \nabla_j X_i \;=\; 0 \qquad \text{(D.10)}$$

for some vector field X. In this case, X generates an *isometry* of the manifold $\mathcal{M}$. It is called a *Killing vector field*, and (D.10) is known as Killing's equation.

The Killing vectors generate the continuous symmetries of a manifold. These symmetries close into the isometry group. Indeed, it is not hard to show that the Lie bracket of two Killing vectors gives another,

$$[X^{(a)}, X^{(b)}] \;=\; -\, f^{abc}\, X^{(c)}\ , \qquad \text{(D.11)}$$

where the f^{abc} are the *structure constants* of the isometry group G.

Let us now assume that our manifold is Kähler, with metric g_{ij*} and complex coordinates a^i and a^{*i}. We shall focus our attention on the analytic isometries, those that preserve the analytic structure of the manifold. This requires that the associated Killing vectors be holomorphic vector fields,

$$X^{(b)} \;=\; X^{i(b)}(a)\, \frac{\partial}{\partial a^i}$$

$$X^{*(b)} \;=\; X^{*i(b)}(a^*)\, \frac{\partial}{\partial a^{*i}}\ . \qquad \text{(D.12)}$$

The index (b) labels the Killing vectors, and runs over the dimension d of the isometry group G.

Because the $X^{(a)}$ are holomorphic, Killing's equation (D.10) reduces to the following form:

$$\begin{aligned} \nabla_i X^{(a)}_j \;+\; \nabla_j X^{(a)}_i \;&=\; 0 \\ \nabla_{i*} X^{(a)}_j \;+\; \nabla_j X^{*(a)}_i \;&=\; 0\ . \end{aligned} \qquad \text{(D.13)}$$

On a Kähler manifold, the first equation is automatically satisfied because of the definition of the covariant derivative. The second is an integrability condition; it is locally equivalent to the statement that there exist d real scalar functions $D^{(a)}(a, a^*)$ such that

$$\begin{aligned} g_{ij*} X^{*j(a)} \;&=\; i\, \frac{\partial}{\partial a^i}\, D^{(a)} \\ g_{ij*} X^{i(a)} \;&=\; -\, i\, \frac{\partial}{\partial a^{*j}}\, D^{(a)}\ . \end{aligned} \qquad \text{(D.14)}$$

The $D^{(a)}$ are known as Killing potentials. They are defined up to constants $c^{(a)}$, $D^{(a)} \rightarrow D^{(a)} + c^{(a)}$. In Chapter XXIV it is shown that the freedom to redefine the potentials is related to the Fayet-Iliopoulos D term.

The relations (D.14) can be inverted to give the Killing vectors in terms of the Killing potentials,

$$X^{i(a)} \;=\; -\,i\,g^{ij*}\,\frac{\partial}{\partial a^{*j}}\,D^{(a)}$$
$$X^{*j(a)} \;=\; i\,g^{ij*}\,\frac{\partial}{\partial a^{i}}\,D^{(a)}\;. \qquad \text{(D.15)}$$

The requirement that the fields $X^{i(a)}$ be holomorphic places a constraint on the $D^{(a)}$. Solving this constraint is equivalent to solving (D.13). In general, it may be difficult to find the Killing potentials on a given Kähler manifold.

Because of the holomorphic structure, the Killing vectors $X^{(a)}$ and $X^{*(a)}$ generate independent representations of the isometry group G. They obey the Lie bracket relations

$$[X^{(a)},X^{(b)}] \;=\; -\;f^{abc}X^{(c)}$$
$$[X^{*(a)},X^{*(b)}] \;=\; -\;f^{abc}X^{*(c)}$$
$$[X^{(a)},X^{*(b)}] \;=\; 0\;, \qquad \text{(D.16)}$$

where the f^{abc} are the structure constants of G. The Killing potentials $D^{(a)}$ also transform under the isometry group. As shown in Exercise 3, they can be chosen to transform in the adjoint representation,

$$\left[X^{i(a)}\frac{\partial}{\partial a^{i}} \;+\; X^{*i(a)}\frac{\partial}{\partial a^{*i}}\right] D^{(b)} \;=\; -\;f^{abc}D^{(c)}\;. \qquad \text{(D.17)}$$

This fixes the constants $c^{(a)}$ for non-Abelian groups. For each U(1) factor, however, there is an undetermined constant c.

Let us now turn our attention to the variation of the Kähler potential under an isometry in G. Such an isometry is generated by the Killing vectors $X^{(a)}$ and $X^{*(a)}$:

$$\delta K \;=\; \left(\epsilon^{(a)}X^{(a)} \;+\; \epsilon^{*(a)}X^{*(a)}\right)K\;. \qquad \text{(D.18)}$$

Note that we have used a complex parameter $\epsilon^{(a)}$, and that the hermitian nature of the Kähler potential is preserved. It is straightforward to show that (D.18) can be rewritten as follows,

$$\delta K \;=\; \epsilon^{(a)}F^{(a)} \;+\; \epsilon^{*(a)}F^{*(a)} \;-\; i\left(\epsilon^{(a)} - \epsilon^{*(a)}\right)D^{(a)}\;, \qquad \text{(D.19)}$$

where the $F^{(a)} = X^{(a)}K + iD^{(a)}$ are analytic functions of the coordinates,

$$\frac{\partial F^{(a)}}{\partial a^{*j}} \;=\; g_{ij*}\,X^{i(a)} \;+\; i\,\frac{\partial D^{(a)}}{\partial a^{*j}} \;=\; 0\;, \qquad \text{(D.20)}$$

and we have used (D.14). For real parameters $\epsilon^{(a)}$, (D.19) reduces to a Kähler transformation. For complex parameters, however, it is not of Kähler form; there is a change in K proportional to the Killing potential $D^{(a)}$. In Chapter XXIV this plays an important role in the construction of gauge invariant actions.

References

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Exercises

(1) Show that the Lie bracket of two Killing vectors gives another.

(2) Demonstrate that the first equation in (D.13) is automatically satisfied on a Kähler manifold, and that the second is locally equivalent to (D.14).

(3) Prove that the Killing potentials can always be chosen to satisfy (D.17). This can be done by first differentiating the left-hand side with respect to a^i, and then using the relations introduced above to obtain the a^i derivative of the right-hand side of (D.17). The proof can be completed by repeating the procedure, this time differentiating with respect to a^{*i}.

(4) Show that

$$X^{i(a)}\frac{\partial}{\partial a^i} D^{(b)} + X^{*i(b)}\frac{\partial}{\partial a^{*i}} D^{(a)} = 0 .$$

(5) Consider the manifold with Kähler potential $K = a^{*i}a^i$. Verify that the differential operators

$$X^{(a)} = -ia^j T^{(a)k}{}_j \frac{\partial}{\partial a^k}$$

$$X^{*(a)} = ia^{*j} T^{(a)j}{}_k \frac{\partial}{\partial a^{*k}}$$

are indeed Killing vectors, where the $T^{(a)k}{}_j$ are given in (7.14). Show that their Lie brackets close into (D.16).

(6) Given the Kähler potential $K = \log(1 + aa^*)$, and the Killing potentials

$$D^{(1)} = \frac{1}{2}\frac{a + a^*}{(1 + a^*a)}, \quad D^{(2)} = -\frac{i}{2}\frac{a - a^*}{(1 + a^*a)},$$

$$D^{(3)} = -\frac{1}{2}\left(\frac{1 - a^*a}{1 + a^*a}\right),$$

find the Killing vectors $X^{(a)}$ using (D.15). Compute their commutators and identify the isometry group G.

APPENDIX E
NONLINEAR REALIZATIONS

Nonlinear realizations play an important role in theories with spontaneously broken symmetries. They were first studied in the context of chiral dynamics, where they were used to describe the pion and its interactions. They can also be applied to theories with spontaneously broken supersymmetry, where they are used to derive low-energy theorems for the Goldstone fermion. In this appendix we will develop the necessary formalism for the case of compact, connected, semisimple Lie groups. This will serve as a guide for our study of spontaneously broken supersymmetry, where similar results can be proved using different techniques.

We start by assuming that we have a manifold $\mathcal{M}$ and a group G of transformations that act on $\mathcal{M}$,

$$x' \;=\; g \cdot x \; , \tag{E.1}$$

where $g \in G$, and x, x' are points of $\mathcal{M}$. These transformations induce a *realization* of G on the coordinates in each neighborhood of $\mathcal{M}$. Such realizations clearly include the case of linear representations, but they also include more general realizations that cannot be reduced to linear transformations by appropriate coordinates on $\mathcal{M}$.

Given a particular realization, one would like to know whether or not it can be reduced to a linear transformation. For the case of compact, connected semisimple Lie groups, there is a simple answer: A realization can be linearized (in a given coordinate patch) if and only if it leaves a point in the patch invariant.

Now, a linear transformation always leaves the origin invariant, so the first direction is trivial. The other direction, however, is a little less obvious. Therefore, let us assume that we have a point $x_0 \in \mathcal{M}$ that is invariant under all the transformations in G,

$$g \cdot x_0 \;=\; x_0 \; . \tag{E.2}$$

We will explicitly construct a set of coordinates that linearize the transformation (E.1) in the neighborhood of x_0. Since x_0 is invariant, we assign it the coordinate $\vec{0}$. Away from x_0, we choose an arbitrary set of coordinates, denoting the coordinates of x by $\vec{x}$. In terms of these parameters, the transformation (E.1) has a power series expansion,

$$\vec{x}\,' \;=\; g \cdot \vec{x} \;\equiv\; D(g)\,\vec{x} \;+\; O(\vec{x}^{\,2}) \; , \tag{E.3}$$

where $D(g)$ is a matrix expression. [In Exercise 1 we will see that $D(g)$ is a matrix representation of G.] The constant term is absent because the origin is invariant. We now introduce new coordinates $\vec{y}$ at the point x as follows,

$$\vec{y} = \int d\mu(g)\, D^{-1}(g)\, g \cdot \vec{x} \,. \tag{E.4}$$

The integration is over the group G, and is well-defined for compact groups. The measure $d\mu(g)$ can be chosen to be left- and right-invariant,

$$d\mu(g_0 g) = d\mu(g g_0) = d\mu(g) \,, \tag{E.5}$$

and normalized so that

$$\int d\mu(g) = 1 \,. \tag{E.6}$$

With these conventions, it is easy to see that

$$\vec{y} = \vec{x} + O(\vec{x}^2) \,, \tag{E.7}$$

so (E.4) is an allowed change of coordinates.

Let us now study the action of G on the coordinates $\vec{y}$. We find

$$\begin{aligned} g_0 \cdot \vec{y} &= \int d\mu(g)\, D^{-1}(g)\, g \cdot g_0 \cdot \vec{x} \\ &= \int d\mu(g g_0)\, D(g_0) D^{-1}(g_0) D^{-1}(g)\, g \cdot g_0 \cdot \vec{x} \end{aligned}$$

$$\begin{aligned} &= D(g_0) \int d\mu(g)\, D^{-1}(g)\, g \cdot \vec{x} \\ &= D(g_0)\, \vec{y} \,, \end{aligned} \tag{E.8}$$

which demonstrates that the coordinates $\vec{y}$ do indeed linearize the transformation (E.1).

This construction relies heavily on the properties of group integration. Curiously enough, similar results hold even when group integration cannot be properly defined. For example, in Chapter XXVI we study the case of supersymmetry, in which a supergroup of transformations acts on superspace. The linearization condition still holds, even though the group volume is formally zero.

Given an arbitrary point $x_0 \in \mathcal{M}$, the transformations that leave the point invariant close into a group H, called the *stability group* of x_0. In general, H is a proper subgroup of G. We have just seen that the transformations in the stability group are precisely those that can be realized linearly in the neighborhood of the point x_0.

In preparation for what follows, let us now shift our attention to the submanifold $\mathcal{N}$ of $\mathcal{M}$ that can be reached by group transformations acting on the point x_0,

$$x = g \cdot x_0 \,. \tag{E.9}$$

Clearly, the points in $\mathcal{N}$ are in one-one correspondence with the coset space G/H. This space has a natural parametrization in terms of the group parameters. An arbitrary element of G can be written in the form

$$g \;=\; e^{-i\vec{\xi}\cdot\vec{X}} e^{-i\vec{u}\cdot\vec{T}} \;, \qquad \text{(E.10)}$$

where the parameters $\vec{u}$ and $\vec{\xi}$ are real. In this expression, the $\vec{T}$ are the (hermitian) generators of H, while the $\vec{X}$ are the generators of G in the orthogonal complement of H. Two elements g and $g' \in G$ correspond to the same point of G/H if they are related by a *right* H transformation: $g \sim g'$ if $g = g'u'$, for some u' of the form

$$u' \;=\; e^{-i\vec{u}'\cdot\vec{T}} \;. \qquad \text{(E.11)}$$

This implies that the cosets can be parametrized by the group elements

$$v \;=\; e^{-i\vec{\xi}\cdot\vec{X}} \;, \qquad \text{(E.12)}$$

and that the $\vec{\xi}$ are coordinates of the space G/H.

With these conventions, an element $g_0 \in G$ acts on the cosets by *left* multiplication,

$$g_0\, e^{-i\vec{\xi}\cdot\vec{X}} \;=\; e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}} \;, \qquad \text{(E.13)}$$

The coordinates $\vec{\xi}'$ are completely determined in terms of $\vec{\xi}$ and g_0,

$$g_0 : \quad \vec{\xi} \;\rightarrow\; \vec{\xi}'(\vec{\xi}, g_0) \;. \qquad \text{(E.14)}$$

The parameters $\vec{u}'$ can be computed as well; they too depend on g_0 and the coset parameters $\vec{\xi}$,

$$g_0 : \quad \vec{u} \;\rightarrow\; \vec{u}'(\vec{\xi}, g_0) \;. \qquad \text{(E.15)}$$

For elements $g_0 = u_0 \in H$, the transformations (E.14) and (E.15) can always be written in closed form. Then

$$v \;\rightarrow\; u_0 v \;=\; u_0 v u_0^{-1} u_0 \;\equiv\; v'u' \;, \qquad \text{(E.16)}$$

where $v' = u_0 v u_0^{-1}$ and $u' = u_0$. In terms of the coordinate $\vec{\xi}$, this implies

$$e^{-i\vec{\xi}'\cdot\vec{X}} \;=\; u_0\, e^{-i\vec{\xi}\cdot\vec{X}}\, u_0^{-1} \;, \qquad \text{(E.17)}$$

and we see explicitly that the $\vec{\xi}$ transform linearly under elements $u_0 \in H$,

$$u_0 : \quad \vec{\xi} \;\rightarrow\; \vec{\xi}' \;=\; D(u_0)\,\vec{\xi} \;. \qquad \text{(E.18)}$$

For transformations $g_0 \in G$ that are not in H, however, equations (E.14) and (E.15) cannot generally be written in closed form.

There is a special case, however, where these transformations can be made more explicit. This is when the structure relations of G admit the automorphism

$$\begin{aligned} \vec{T} &\rightarrow \vec{T} \\ \vec{X} &\rightarrow -\vec{X} \,, \end{aligned} \tag{E.19}$$

in which case G/H is called a symmetric space. To see how this works, consider a transformation v_0,

$$v \rightarrow v_0 v \equiv v' u' \,. \tag{E.20}$$

This can be rewritten by first taking the inverse, and then applying the automorphism (E.19),

$$v \rightarrow v v_0 \equiv u'^{-1} v' \,, \tag{E.21}$$

Combining the two expressions, we find

$$\begin{aligned} v'^2 &= v_0 v^2 v_0 \\ u' &= v'^{-1} v \,. \end{aligned} \tag{E.22}$$

In terms of the coordinates $\vec{\xi}$, this implies

$$e^{-2i\vec{\xi}'\cdot\vec{X}} = v_0\, e^{-2i\vec{\xi}\cdot\vec{X}} v_0 \,. \tag{E.23}$$

This is a manifestly nonlinear transformation law. Note that $\vec{\xi}$ can be transformed to zero if we take $v_0 = e^{i\vec{\xi}\cdot\vec{X}}$.

We will now show that we can use these results to promote any representation of H to a realization of G, with the help of the coset parameters $\vec{\xi}$. We start with a representation $\tilde{D}$, which acts linearly on a vector space spanned by $\vec{\psi}$,

$$u_0 : \quad \vec{\psi} \rightarrow \tilde{D}(u_0)\,\vec{\psi} \tag{E.24}$$

for $u_0 \in H$. Then, using (E.15), this transformation can immediately be extended to a realization of G,

$$g_0 : \quad \vec{\psi} \rightarrow \tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\,\vec{\psi} \,. \tag{E.25}$$

The variables $\vec{u}'$ parametrize an element of H, but they are functions of $\vec{\xi}$ and g_0. To show that (E.25) is indeed a realization of G, we compute

$$\begin{aligned} g_1\, e^{-i\vec{\xi}\cdot\vec{X}} &= e^{-i\vec{\xi}'\cdot\vec{X}}\, e^{-i\vec{u}'\cdot\vec{T}} \\ g_2\, e^{-i\vec{\xi}'\cdot\vec{X}} &= e^{-i\vec{\xi}''\cdot\vec{X}}\, e^{-i\vec{u}''\cdot\vec{T}} \\ g_2\, g_1\, e^{-i\vec{\xi}\cdot\vec{X}} &= e^{-i\vec{\xi}'''\cdot\vec{X}}\, e^{-i\vec{u}'''\cdot\vec{T}} \\ &= e^{-i\vec{\xi}''\vec{X}}\, e^{-i\vec{u}''\cdot\vec{T}}\, e^{-i\vec{u}'\cdot\vec{T}} \,. \end{aligned} \tag{E.26}$$

From this we see that

$$e^{-i\vec{u}'''\cdot\vec{T}} = e^{-i\vec{u}''\cdot\vec{T}}\, e^{-i\vec{u}'\cdot\vec{T}} \,, \tag{E.27}$$

which implies

$$\tilde{D}(e^{-i\vec{u}'''\cdot\vec{T}}) = \tilde{D}(e^{-i\vec{u}''\cdot\vec{T}})\,\tilde{D}(e^{-i\vec{u}'\cdot\vec{T}}) , \qquad \text{(E.28)}$$

since $\tilde{D}$ is a representation of H. In this way we can realize the group G on the space spanned by the vectors $\vec{\psi}$.

The transformation (E.25) plays a special role in the study of nonlinear realizations. It defines what is known as a *standard realization* of the group G. The realization is standard because *any* realization of G that linearizes on H can be reduced to this form with the help of the coset parameters $\vec{\xi}$.

To see this, we assume as before that we have a manifold $\mathcal{M}$ and a group G that acts on $\mathcal{M}$ as a group of transformations. We also assume that we can choose coordinates $(\vec{\xi},\vec{\chi})$ in some neighborhood $\mathcal{U}$ of $\mathcal{M}$, where the coordinates $\vec{\xi}$ parametrize the points in $\mathcal{U}$ that can be reached from $(0,\vec{\chi})$ by the action of G. Because of the construction (E.8), the parameters $\vec{\chi}$ can be chosen to transform linearly under H. The transformations of the $\vec{\xi}$ are completely determined by (E.14). Therefore, under an H-transformation, we have

$$u_0 \cdot (\vec{\xi},\vec{\chi}) = (D(u_0)\,\vec{\xi}, \tilde{D}(u_0)\,\vec{\chi}) , \qquad \text{(E.29)}$$

where $\vec{\xi}$ and $\vec{\chi}$ transform in the representations D and $\tilde{D}$, respectively.

Now, among the full set of G transformations, there is one that transforms $\vec{\xi}$ to zero,

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\chi}) = (\vec{0},\vec{\psi}) . \qquad \text{(E.30)}$$

This transformation also takes $\vec{\chi}$ to $\vec{\psi}$, which can be computed because we know the action of G on the manifold $\mathcal{M}$. The parameters $\vec{\psi}$ transform in the representation $\tilde{D}$ under H, as follows from (E.29).

We shall now construct a new coordinate system on $\mathcal{M}$ as follows. Start at a point x, parametrized by the coordinates $(\vec{\xi},\vec{\chi})$, and map it to the point $(\vec{0},\vec{\psi})$ as in (E.30),

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\chi}) = (\vec{0},\vec{\psi}) \qquad \text{(E.31)}$$

Then take the new coordinates at the original point x to be given by $(\vec{\xi},\vec{\psi})$. This defines an acceptable coordinate transformation on the manifold $\mathcal{M}$ because the Jacobian of the transformation $(\vec{\xi},\vec{\chi}) \to (\vec{\xi},\vec{\psi})$ is nonvanishing near the origin. In terms of the new coordinates, the transformation (E.31) can be written as $e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\psi}) = (\vec{0},\vec{\psi})$. This allows us to show that the new coordinates $\vec{\psi}$ transform as a standard realization:

$$g \cdot (\vec{\xi},\vec{\psi}) = g\, e^{-i\vec{\xi}\cdot\vec{X}} \cdot (\vec{0},\vec{\psi})$$

$$= e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}} \cdot (\vec{0},\vec{\psi})$$

$$= e^{-i\vec{\xi}'\cdot\vec{X}} \cdot (\vec{0}, \tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\,\vec{\psi})$$

$$= (\vec{\xi}', \tilde{D}(e^{-i\vec{u}'\cdot\vec{T}})\,\vec{\psi}) \,. \qquad \text{(E.32)}$$

Together with $\vec{\xi}$, they are the natural coordinates on $\mathcal{M}$ adapted to the action of G.

In physical applications, the coordinates $\vec{\xi}$ and $\vec{\psi}$ are x^m-dependent fields. The coset coordinates $\vec{\xi}$ play the role of the Goldstone bosons that arise from spontaneously breaking G to H. The standard realizations $\vec{\psi}$ describe the other fields that transform in representations of the unbroken group H. In this appendix, we have seen that any representation of H can be extended to a realization of G with the help of the Goldstone bosons $\vec{\xi}$.

To write down invariant Lagrangians we would like to have covariant derivatives that transform as standard realizations. Our general arguments tell us that such derivatives must exist. Constructing them involves a straightforward application of what we have just learned, as well as a nice illustration.

To find the covariant derivatives, we start from the manifold parametrized by $(\vec{\xi},\vec{\psi},\partial_m\vec{\xi},\partial_m\vec{\psi})$. As above, we apply

a group transformation with $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$. This gives

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot (\vec{\xi},\vec{\psi},\partial_m\vec{\xi},\partial_m\vec{\psi}) \;\equiv\; (\vec{0},\vec{\psi},\Delta_m\vec{\xi},\Delta_m\vec{\psi}) \;, \qquad \text{(E.33)}$$

and from our general prescription we know that $\Delta_m\vec{\xi}$ and $\Delta_m\vec{\psi}$ are covariant derivatives that transform as standard realizations.

To compute $\Delta_m\vec{\xi}$, we start from the formula (E.13),

$$g_0\, e^{-i\vec{\xi}\cdot\vec{X}} \;=\; e^{-i\vec{\xi}'\cdot\vec{X}} e^{-i\vec{u}'\cdot\vec{T}} \;, \qquad \text{(E.34)}$$

and differentiate with respect to x^m,

$$g_0\,\partial_m e^{-i\vec{\xi}\cdot\vec{X}} \;=\; \left(\partial_m e^{-i\vec{\xi}'\cdot\vec{X}}\right) e^{-i\vec{u}'\cdot\vec{T}} + e^{-i\vec{\xi}'\cdot\vec{X}} \left(\partial_m e^{-i\vec{u}'\cdot\vec{T}}\right) . \qquad \text{(E.35)}$$

As before, the parameters $\vec{\xi}'$ and $\vec{u}'$ depend on x^m through $\vec{\xi}$. We now choose $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$, which transforms $\vec{\xi}'$ and $\vec{u}'$ to zero at the point x^m. This gives

$$e^{i\vec{\xi}\cdot\vec{X}}\,\partial_m e^{-i\vec{\xi}\cdot\vec{X}} \;=\; \partial_m e^{-i\vec{\xi}'\cdot\vec{X}} \;+\; \partial_m e^{-i\vec{u}'\cdot\vec{T}}\,\Big|_{\vec{\xi}'=\vec{u}'=0}$$

$$=\; -i\,\partial_m\vec{\xi}'\cdot\vec{X} \;-\; i\,\partial_m\vec{u}'\cdot\vec{T}\,\Big|_{\vec{\xi}'=\vec{u}'=0}$$

$$\equiv\; -i\,\Delta_m\vec{\xi}\cdot\vec{X} \;-\; i\,\vec{V}_m\cdot\vec{T} \,. \qquad \text{(E.36)}$$

Equation (E.36) allows us to compute $\Delta_m\vec{\xi}$ as a function of the parameters $\vec{\xi}$. In Exercise 3 we will see that $\Delta_m\vec{\xi}$ indeed transforms as a standard realization.

Similar techniques can be used to find $\Delta_m \vec{\psi}$. One starts by differentiating (E.25),

$$g_0 \cdot \partial_m \vec{\psi} = \partial_m D(e^{-i\vec{u}'\cdot\vec{T}})\,\vec{\psi} + D(e^{-i\vec{u}'\cdot\vec{T}})\,\partial_m \vec{\psi} \,. \quad \text{(E.37)}$$

As above, one then takes $g_0 = e^{i\vec{\xi}\cdot\vec{X}}$, to find

$$e^{i\vec{\xi}\cdot\vec{X}} \cdot \partial_m \vec{\psi} = -i\left(\partial_m \vec{u}' \cdot \vec{T}\right)\vec{\psi} + \partial_m \vec{\psi}\,\Big|_{\vec{\xi}'=\vec{u}'=0} \,. \quad \text{(E.38)}$$

Comparing (E.38) with (E.36), we find

$$\Delta_m \vec{\psi} = \partial_m \vec{\psi} - i\left(\vec{V}_m \cdot \vec{T}\right)\vec{\psi} \,. \quad \text{(E.39)}$$

In Exercise 4 one is asked to show that $\Delta_m \vec{\psi}$ transforms as a standard realization.

References

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Exercises

(1) Show that the matrices D, defined in (E.3), form a representation of the group G.

(2) Demonstrate that $\Delta_m \vec{\psi}$ transforms as a standard realization. Start by eliminating g_0 between (E.34) and (E.35),

$$e^{-i\vec{\xi}'\cdot\vec{X}}\, e^{-i\vec{u}'\cdot\vec{T}}\, e^{i\vec{\xi}\cdot\vec{X}} \left(\partial_m e^{-i\vec{\xi}\cdot\vec{X}}\right)$$

$$= \left(\partial_m e^{-i\vec{\xi}'\cdot\vec{X}}\right) e^{-i\vec{u}'\cdot\vec{T}} + e^{-i\vec{\xi}'\cdot\vec{X}} \left(\partial_m e^{-i\vec{u}'\cdot\vec{T}}\right) .$$

Then multiply on the left by $e^{i\vec{\xi}'\cdot\vec{X}}$ and $e^{i\vec{u}'\cdot\vec{T}}$, to find

$$e^{i\vec{\xi}\cdot\vec{X}}\, \partial_m e^{-i\vec{\xi}\cdot\vec{X}}$$

$$= e^{i\vec{u}'\cdot\vec{T}} \left(e^{i\vec{\xi}'\cdot\vec{X}}\, \partial_m e^{-i\vec{\xi}'\cdot\vec{X}}\right) e^{-i\vec{u}'\cdot\vec{T}} + e^{i\vec{u}'\cdot\vec{T}}\, \partial_m e^{-i\vec{u}'\cdot\vec{T}} .$$

This shows that

$$(\Delta_m \vec{\xi} \cdot \vec{X})' = e^{-i\vec{u}'\cdot\vec{T}} \left(\Delta_m \vec{\xi} \cdot \vec{X}\right) e^{i\vec{u}'\cdot\vec{T}} ,$$

and

$$(\vec{V}_m \cdot \vec{T})' = e^{-i\vec{u}'\cdot\vec{T}} \left(\vec{V}_m \cdot \vec{T}\right) e^{i\vec{u}'\cdot\vec{T}} + e^{-i\vec{u}'\cdot\vec{T}}\, \partial_m e^{i\vec{u}'\cdot\vec{T}} .$$

(4) Use the transformation law for $\vec{V}_m$ to show that $\Delta_m \vec{\psi}$ transforms as a standard realization.

APPENDIX F
NONLINEAR REALIZATIONS AND INVARIANT ACTIONS

In this appendix we will continue our study of nonlinear realizations. We will use the methods introduced in Appendix E to show that an action invariant under a group H can be promoted to a new action invariant under a larger group $G \supset H$. The results derived here are used in Chapter XXIV to construct the gauge invariant matter couplings in superspace.

We start by assuming we have a Lagrangian $\mathcal{L}_H$ which is a function of certain fields A^i. The Lagrangian is invariant under a symmetry group H. The fields A^i are arbitrary, except that they have well-defined transformations under a group $G \supset H$. The Lagrangian $\mathcal{L}_H$, however, is not invariant under the full group G. Instead, it has a variation $\delta\mathcal{L}_H \neq 0$.

In this appendix we will construct a counterterm $\mathcal{L}_{CT}$ whose variation precisely cancels that of $\mathcal{L}_H$. We will build the counterterm out of the fields A^i, together with fields $\xi^{(\alpha)}$ that parametrize the coset G/H. We impose the condition that $\mathcal{L}_{CT}$ must vanish when $\xi^{(\alpha)} = 0$. In this way the Lagrangian

$$\mathcal{L}_G \ = \ \mathcal{L}_H \ + \ \mathcal{L}_{CT} \qquad \text{(F.1)}$$

is invariant under the full group G, and reduces to $\mathcal{L}_H$ for $\xi^{(\alpha)} = 0$.

As in Appendix E, let us split the transformations in G into two classes, those in H and those not. Under a transformation $u_0 \in H$, the Lagrangian $\mathcal{L}_H$ is assumed to be invariant,

$$\delta_H\,\mathcal{L}_H \ \equiv \ -iu_0^{(a)}\hat{T}^{(a)}\,\mathcal{L}_H \ = \ 0 \ , \qquad \text{(F.2)}$$

where the $\hat{T}^{(a)}$ are differential operators that act on the fields A^i and generate the transformations of H. Under a transformation $v_0 \in G$, $\mathcal{L}_H$ has an infinitesimal variation of the form

$$\delta_{G/H}\,\mathcal{L}_H \ \equiv \ -iv_0^{(\alpha)}\hat{X}^{(\alpha)}\,\mathcal{L}_H \ \equiv \ -v_0^{(\alpha)}D^{(\alpha)} \ , \qquad \text{(F.3)}$$

where the operators $\hat{X}^{(\alpha)}$ generate the transformations of G that are in the orthogonal complement of H. We see that we need to find a function $\mathcal{L}_{CT}(A^i,\xi^{(\alpha)})$ such that

$$\begin{aligned} u_0^{(a)}\hat{T}^{(a)}\,\mathcal{L}_{CT} \ &= \ 0 \\ v_0^{(\alpha)}\hat{X}^{(\alpha)}\,\mathcal{L}_{CT} \ &= \ iv_0^{(\alpha)}D^{(\alpha)} \ , \end{aligned} \qquad \text{(F.4)}$$

subject to the boundary condition

$$\mathcal{L}_{CT}(A^i,0) \ = \ 0 \ . \qquad \text{(F.5)}$$

In these expressions, the operators $\hat{T}^{(a)}$ and $\hat{X}^{(\alpha)}$ act on the fields A^i and on the parameters $\xi^{(\alpha)}$ in the counterterm Lagrangian.

We shall now find $\mathcal{L}_{CT}$ as follows. We first compute

$$\left(-iv_0^{(\alpha)}\hat{X}^{(\alpha)}\right)^n \mathcal{L}_{CT} = \left(-iv_0^{(\alpha)}\hat{X}^{(\alpha)}\right)^{n-1}\left(v_0^{(\gamma)}D^{(\gamma)}\right) . \tag{F.6}$$

This can be exponentiated to give

$$e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}}\mathcal{L}_{CT} = \mathcal{L}_{CT} + \frac{e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}} - 1}{-iv_0^{(\beta)}\hat{X}^{(\beta)}}\left(v_0^{(\gamma)}D^{(\gamma)}\right) , \tag{F.7}$$

where, on the right-hand side, the differential operators $\hat{X}^{(\alpha)}$ reduce to operators $\delta^{(\alpha)}A_i\,(\delta/\delta A_i)$ because the $D^{(\alpha)}$ do not contain the fields $\xi^{(\alpha)}$. We can now solve for $\mathcal{L}_{CT}$ by noting that $e^{-iv_0^{(\alpha)}\hat{X}^{(\alpha)}}$ transforms $\mathcal{L}_{CT}$ with parameter $v_0^{(\alpha)}$. In Appendix E we showed that such a transformation with parameter $v_0^{(\alpha)} = -\xi^{(\alpha)}$ maps $\xi^{(\alpha)}$ to zero. Therefore, in conjunction with the boundary condition (F.5), this implies

$$\begin{aligned}\mathcal{L}_{CT} &= \frac{e^{i\xi^{(\alpha)}\hat{X}^{(\alpha)}} - 1}{i\xi^{(\beta)}\hat{X}^{(\beta)}}\,\xi^{(\gamma)}D^{(\gamma)} \\ &= \int_0^1 d\alpha\,\exp\left(i\alpha\xi^{(\alpha)}\hat{X}^{(\alpha)}\right)\xi^{(\gamma)}D^{(\gamma)} , \end{aligned} \tag{F.8}$$

where the derivatives in $\hat{X}^{(\alpha)}$ do not act on the fields $\xi^{(\gamma)}$. It is a useful exercise to check that (F.8) satisfies (F.4), following the steps outlined in Exercises 2 and 3.

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Exercises

(1) Exponentiate (F.6) to find (F.7).

(2) Derive the conditions on $D^{(\alpha)}$ that follow from applying the group commutators on $\mathcal{L}_{CT}$,

$$\begin{aligned}\hat{X}^{(\alpha)}D^{(\beta)} - \hat{X}^{(\beta)}D^{(\alpha)} &= i\,f^{\alpha\beta\gamma}D^{(\gamma)} \\ \hat{T}^{(a)}D^{(\beta)} &= i\,f^{a\beta\gamma}D^{(\gamma)} .\end{aligned}$$

(3) Use the relations of Exercise 2 to show that (F.8) obeys (F.4).

APPENDIX G

GAUGE INVARIANT SUPERGRAVITY MODELS

In this Appendix, we will write the most general gauge invariant supergravity model in terms of component fields. We start with the superspace Lagrangian, as given in Chapter XXV,

$$\mathcal{L} = \int d^2\Theta\, 2\mathcal{E} \left[\frac{3}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp\left\{ -\frac{1}{3} \Big[K(\Phi, \Phi^+) + \Gamma(\Phi, \Phi^+, V) \Big] \right\} + \frac{1}{16 g^2} H_{(ab)}(\Phi)\, W^{(a)} W^{(b)} + P(\Phi) \right] + \text{h.c.} \qquad (G.1)$$

Then, using the techniques developed in Chapters XXI through XXV, we expand this Lagrangian in terms of component fields. This gives

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2} e\,\mathcal{R} - e\, g_{ij*}\, \tilde{\mathcal{D}}_m A^i\, \tilde{\mathcal{D}}^m A^{*j} - \frac{1}{2} e\, g^2\, D_{(a)} D^{(a)} \\
&- i\, e\, g_{ij*}\, \bar{\chi}^j \bar{\sigma}^m \tilde{\mathcal{D}}_m \chi^i + e\, \epsilon^{k\ell mn}\, \bar{\psi}_k \bar{\sigma}_\ell \tilde{\mathcal{D}}_m \psi_n \\
&- \frac{1}{4} e\, h^R{}_{(ab)}\, F_{mn}{}^{(a)} F^{mn(b)} + \frac{1}{8} e\, h^I{}_{(ab)}\, \epsilon^{mnk\ell}\, F_{mn}{}^{(a)} F_{k\ell}{}^{(b)} \\
&- \frac{i}{2} e \left[\lambda_{(a)} \sigma^m \tilde{\mathcal{D}}_m \bar{\lambda}^{(a)} + \bar{\lambda}_{(a)} \bar{\sigma}^m \tilde{\mathcal{D}}_m \lambda^{(a)} \right] \\
&+ \frac{1}{2} h^I{}_{(ab)}\, \tilde{\mathcal{D}}_m \left[e \lambda^{(a)} \sigma^m \bar{\lambda}^{(b)} \right] \\
&+ \sqrt{2}\, e\, g\, g_{ij*}\, X^{*j}_{(a)}\, \chi^i \lambda^{(a)} + \sqrt{2}\, e\, g\, g_{ij*}\, X^i_{(a)}\, \bar{\chi}^j \bar{\lambda}^{(a)} \\
&- \frac{i}{4} \sqrt{2}\, e\, g\, \partial_i h_{(ab)}\, D^{(a)}\, \chi^i \lambda^{(b)} + \frac{i}{4} \sqrt{2}\, e\, g\, \partial_{i*} h^*_{(ab)}\, D^{(a)}\, \bar{\chi}^i \bar{\lambda}^{(b)} \\
&- \frac{1}{4} \sqrt{2}\, e\, \partial_i h_{(ab)}\, \chi^i \sigma^{mn} \lambda^{(a)}\, F_{mn}{}^{(b)} - \frac{1}{4} \sqrt{2}\, e\, \partial_{i*} h^*_{(ab)}\, \bar{\chi}^i \bar{\sigma}^{mn} \bar{\lambda}^{(a)}\, F_{mn}{}^{(b)} \\
&- \frac{1}{2} e\, g\, D_{(a)}\, \psi_m \sigma^m \bar{\lambda}^{(a)} + \frac{1}{2} e\, g\, D_{(a)}\, \bar{\psi}_m \bar{\sigma}^m \lambda^{(a)} \\
&- \frac{1}{2} \sqrt{2}\, e\, g_{ij*}\, \tilde{\mathcal{D}}_n A^{*j}\, \chi^i \sigma^m \bar{\sigma}^n \psi_m - \frac{1}{2} \sqrt{2}\, e\, g_{ij*}\, \tilde{\mathcal{D}}_n A^i\, \bar{\chi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
&+ \frac{i}{4} e \left[\psi_m \sigma^{ab} \sigma^m \bar{\lambda}_{(a)} + \bar{\psi}_m \bar{\sigma}^{ab} \bar{\sigma}^m \lambda_{(a)} \right] \left[F_{ab}{}^{(a)} + \hat{F}_{ab}{}^{(a)} \right] \\
&+ \frac{1}{4} e\, g_{ij*} \left[i \epsilon^{k\ell mn} \psi_k \sigma_\ell \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m \right] \chi^i \sigma_n \bar{\chi}^j \\
&- \frac{1}{8} e \left[g_{ij*} g_{k\ell*} - 2 R_{ij*k\ell*} \right] \chi^i \chi^k\, \bar{\chi}^j \bar{\chi}^\ell \\
&+ \frac{1}{16} e \left[2\, g_{ij*}\, h^R{}_{(ab)} + h^{R(cd)-1}\, \partial_i h_{(bc)}\, \partial_{j*} h^*_{(ad)} \right] \bar{\chi}^j \bar{\sigma}^m \chi^i\, \bar{\lambda}^{(a)} \bar{\sigma}_m \lambda^{(b)} \\
&+ \frac{1}{8} e \left[\nabla_i \partial_j h_{(ab)} - \frac{1}{4} h^{R(cd)-1} \partial_i h_{(ac)} \partial_j h_{(bd)} \right] \chi^i \chi^j\, \lambda^{(a)} \lambda^{(b)} \\
&+ \frac{1}{8} e \left[\nabla_{i*} \partial_{j*} h^*_{(ab)} - \frac{1}{4} h^{R(cd)-1} \partial_{i*} h^*_{(ac)} \partial_{j*} h^*_{(bd)} \right] \bar{\chi}^i \bar{\chi}^j\, \bar{\lambda}^{(a)} \bar{\lambda}^{(b)} \\
&- \frac{1}{32} e\, h^{R(cd)-1} \partial_i h_{(ac)} \partial_j h_{(bd)}\, \chi^i \sigma^{mn} \chi^j\, \lambda^{(a)} \sigma_{mn} \lambda^{(b)}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{32}\, e\, h^{R(cd)-1} \partial_{i*} h^*_{(ac)} \partial_{j*} h^*_{(bd)}\, \bar\chi^i \bar\sigma^{mn} \bar\chi^j\, \bar\lambda^{(a)} \bar\sigma_{mn} \bar\lambda^{(b)} \\
&- \frac{1}{16}\, e\, g^{ij*}\, \partial_i h_{(ab)}\, \partial_{j*} h^*_{(cd)}\; \lambda^{(a)} \lambda^{(b)}\, \bar\lambda^{(c)} \bar\lambda^{(d)} \\
&- \frac{3}{16}\, e\, \lambda_{(a)} \sigma^m \bar\lambda^{(a)}\, \lambda_{(b)} \sigma_m \bar\lambda^{(b)} \\
&+ \frac{i}{4}\sqrt{2}\, e\, \partial_i h_{(ab)} \,[\, \chi^i \sigma^{mn} \lambda^{(a)}\, \bar\psi_m \bar\sigma_n \lambda^{(b)} \;+\; \frac{1}{4}\, \bar\psi_m \bar\sigma^m \chi^i\, \lambda^{(a)} \lambda^{(b)} \,] \\
&+ \frac{i}{4}\sqrt{2}\, e\, \partial_{i*} h^*_{(ab)} \,[\, \bar\chi^i \bar\sigma^{mn} \bar\lambda^{(a)}\, \psi_m \sigma_n \bar\lambda^{(b)} \;+\; \frac{1}{4}\, \psi_m \sigma^m \bar\chi^i\, \bar\lambda^{(a)} \bar\lambda^{(b)} \,] \\
&- e \exp(K/2) \Big\{ P^*\, \psi_a \sigma^{ab} \psi_b \;+\; P\, \bar\psi_a \bar\sigma^{ab} \bar\psi_b \\
&+ \frac{i}{2}\sqrt{2}\, D_i P\, \chi^i \sigma^a \bar\psi_a + \frac{i}{2}\sqrt{2}\, D_{i*} P^*\, \bar\chi^i \bar\sigma^a \psi_a \\
&+ \frac{1}{2}\, \mathcal{D}_i D_j P\, \chi^i \chi^j \;+\; \frac{1}{2}\, \mathcal{D}_{i*} D_{j*} P^*\, \bar\chi^i \bar\chi^j \Big\} \\
&- \frac{1}{4}\, g^{ij*}\, D_{j*} P^*\, \partial_i h_{(ab)}\; \lambda^{(a)} \lambda^{(b)} \;-\; \frac{1}{4}\, g^{ij*}\, D_i P\, \partial_{j*} h^*_{(ab)}\; \bar\lambda^{(a)} \bar\lambda^{(b)} \\
&- e\, e^K \,[\, g^{ij*} (D_i P)(D_j P)^* \;-\; 3 P^* P \,]\,, \qquad \text{(G.2)}
\end{aligned}
$$

where ${h^R}_{(ab)} = \mathrm{Re}\, H_{(ab)}$ and ${h^I}_{(ab)} = \mathrm{Im}\, H_{(ab)}$. The covariant derivatives are given by

$$
\begin{aligned}
\tilde{\mathcal{D}}_m A^i &= \partial_m A^i \;-\; g\, {v_m}^{(a)} X^i_{(a)} \\
\tilde{\mathcal{D}}_m \chi^i &= \partial_m \chi^i \;+\; \chi^i \omega_m \;+\; \Gamma^i_{jk}\, \tilde{\mathcal{D}}_m A^j\, \chi^k \\
&\quad -\; g\, {v_m}^{(a)} \frac{\partial X^i_{(a)}}{A^j}\, \chi^j \\
&\quad -\; \frac{1}{4}\, (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \chi^i \\
&\quad -\; \frac{i}{2}\, g\, {v_m}^{(a)}\, \mathrm{Im}\, F_{(a)}\, \chi^i \\
\tilde{\mathcal{D}}_m \lambda^{(a)} &= \partial_m \lambda^{(a)} \;+\; \lambda^{(a)} \omega_m \;-\; g\, f^{abc} {v_m}^{(b)} \lambda^{(c)} \\
&\quad +\; \frac{1}{4}\, (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \lambda^{(a)} \\
&\quad +\; \frac{i}{2}\, g\, {v_m}^{(b)}\, \mathrm{Im}\, F_{(b)}\, \lambda^{(a)} \\
\tilde{\mathcal{D}}_m \psi_n &= \partial_m \psi_n \;+\; \psi_n \omega_m \\
&\quad +\; \frac{1}{4}\, (K_j \tilde{\mathcal{D}}_m A^j - K_{j*} \tilde{\mathcal{D}}_m A^{*j})\, \psi_n \\
&\quad +\; \frac{i}{2}\, g\, {v_m}^{(a)}\, \mathrm{Im}\, F_{(a)}\, \psi_n \\
D_i P &= P_i \;+\; K_i P \\
\mathcal{D}_i D_j P &= P_{ij} \;+\; K_{ij} P \;+\; K_i D_j P \\
&\quad +\; K_j D_i P \;-\; K_i K_j P \;-\; \Gamma^k_{ij} D_k P \,. \qquad \text{(G.3)}
\end{aligned}
$$

In these expressions, the fields in the vector multiplet are defined to have *upper* gauge indices, such as ${v_m}^{(a)}$ and $\lambda^{(a)}$. The Killing vectors and Killing potentials have *lower* gauge indices, $X^i_{(a)}$ and $D_{(a)}$. These indices can be raised and lowered with $h^R{}_{(ab)}$ and its inverse. Using these conventions, one can check that the Lagrangian (G.2) is invariant under the following set of supergravity transformations:

$$
\begin{aligned}
\delta_\zeta e_m{}^a &= i(\zeta\sigma_a\bar{\psi}_m \;+\; \bar{\zeta}\bar{\sigma}_a\psi_m) \\
\delta_\zeta A^i &= \sqrt{2}\,\zeta\chi^i \\
\delta_\zeta \chi^i &= i\sqrt{2}\,\sigma_m\bar{\zeta}\,\hat{\tilde{D}}_m A^i \;-\; \Gamma^i_{jk}\,\delta_\zeta A^j\chi^k \\
&\quad + \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\,\chi^i \\
&\quad - \sqrt{2}\,e^{K/2}\,g^{ij*}\,D_{j*}P^*\,\zeta \\
&\quad + \frac{1}{4}\sqrt{2}\,\zeta\;g^{ij*}\,\partial_{j*}h^*_{(ab)}\;\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
\delta_\zeta v_m{}^{(a)} &= i(\zeta\sigma_m\bar{\lambda}^{(a)} \;+\; \bar{\zeta}\bar{\sigma}_m\lambda^{(a)}) \\
\delta_\zeta \lambda^{(a)} &= \hat{F}_{ab}{}^{(a)}\sigma^{ab}\zeta \;-\; i\,g\,D^{(a)}\,\zeta \\
&\quad + \frac{1}{2}\zeta\;h^{R(ab)-1}\,\partial_i h_{(bc)}\;\chi^i\lambda^{(c)} \\
&\quad - \frac{1}{2}\zeta\;h^{R(ab)-1}\,\partial_{i*}h^*_{bc}\;\bar{\chi}^i\bar{\lambda}^{(c)} \\
\delta_\zeta \psi_m &= 2\tilde{\mathcal{D}}_m\zeta \;-\; \frac{i}{2}\sigma_{mn}\zeta\;g_{ij*}\,\chi^i\sigma^n\bar{\chi}^j \\
&\quad + \frac{i}{2}(g_{mn} \;+\; \sigma_{mn})\zeta\,\lambda_{(a)}\sigma^n\bar{\lambda}^{(a)} \\
&\quad - \frac{1}{4}(K_j\delta_\zeta A^j - K_{j*}\delta_\zeta A^{*j})\,\psi_m \\
&\quad + i\,e^{K/2}\,P\,\sigma_m\bar{\zeta}\,. \qquad\qquad \text{(G.4)}
\end{aligned}
$$

The Lagrangian (G.2) is the starting point for phenomeno-

logical studies of supergravity theories.